Electric and magnetic phenomena are described by Maxwell's equations

$$\vec{\nabla}\cdot\vec{E} = 4\pi\rho \qquad (1)$$

$$\vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t} = \frac{4\pi}{c}\vec{j} \qquad (2)$$

$$\vec{\nabla}\times\vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0 \qquad (3)$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad (4)$$

(given in Gaussian units - more on this later) plus the Lorentz force equation

$$\vec{F} = q\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right) \qquad (5)$$

In these $\vec{E}=\vec{E}(\vec{x},t)$ and $\vec{B}=\vec{B}(\vec{x},t)$ are fields - which I am displaying explicitly. Just to make sure we agree on notation,

$\rho = \rho(\vec{x},t)$ = electric charge density

$\vec{j} = \vec{j}(\vec{x},t)$ = electric current density

c = speed of light $\simeq 2.99\times 10^8$ m/s

$\vec{v}$ = velocity of charge-q particle

$\vec{F}$ = force due $\vec{E}$ & $\vec{B}$ fields on said particle

And $\vec{E}=(E_x,E_y,E_z)$ etc are 3-vectors (we'll distinguish 3-vectors from other d-vectors)

$$\vec{\nabla} = \left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right) = (\partial_x, \partial_y, \partial_z)$$

so that on a function $f(x,y,z)$, $\vec{\nabla}f = \vec{\text{grad}}(f)$ is a vector $(\partial_x f, \partial_y f, \partial_z f)$ ("gradient")

on a vector $\vec{E}$, $\vec{\nabla}\cdot\vec{E} = \text{div}(\vec{E})$ is a scalar (ie, pure number) $= \partial_x E_x + \partial_y E_y + \partial_z E_z$ (divergence)

and on a vector $\vec{E}$, $\vec{\nabla}\times\vec{E} = \vec{\text{curl}}(\vec{E})$ is a vector $= (\partial_y E_z - \partial_z E_y,\ \partial_z E_x - \partial_x E_z,\ \partial_x E_y - \partial_y E_x)$ (curl)

We start our discussion of Electrodynamics by exploring two key aspects of eqs (1)-(5):

(i) They are invariant under Lorentz transformations.

(ii) The fundamental dynamical variables are fields.

We will look at those together, moving back and forth between them. We will make contact with the more familiar aspects of special relativity (eg, boosts on point particles) only at the end, for completeness.

Space-time

Fields are functions of space and time. This in itself does not require we think of space and time as part of the same continuum "space-time". It is the invariance of Eqs (1)-(5) under Lorentz transformations, and that these mix space and time, that lead us to consider space and time on an al<u>most</u> equal footing.

Warm-up: <u>Rotations and space</u>.

Points in space are accounted for with coordinates $\vec{x} = (x^1, x^2, x^3) \in \mathbb{R}^3$

There is much arbitrariness in how this is done. Given a coordinate system $\vec{x}$ can define a new one $\vec{x}' = \vec{x}'(\vec{x})$ (3 functions of 3 variables), with the obvious constraint that the map be 1-to-1, invertible (if you know about manifolds and differential geometry, this can be done in patches). But we'd like to focus on coordinates we can assign with a meter stick; call them "Cartesian".

Given one such coordinate system, others are obtained by

- translations $\vec{x}' = \vec{x} + \vec{a}$ $\quad \vec{a}$ = fixed vector
- rotations $\vec{x}' = R\,\vec{x}$ $\quad R$ = orthogonal matrix

Here is another look at this. In our 'ruled' space distance is given by

$$ds^2 = dx^2 + dy^2 + dz^2 = \sum_{i=1}^{3} (dx^i)^2 \qquad (6)$$

OK, recall distance between 2-points, P_1 and P_2, is given by

$$\int_{P_1}^{P} ds \text{ along a straight line } \left(\text{or, equivalently, } \min_{\text{all paths}} \int_{P_1}^{P_2} ds \right).$$

Question: what is the set of transformations $\vec{x} \to \vec{y}(\vec{x})$ that preserve the form of (6),

$$ds^2 = \sum_{i=1}^{3} (dx^i)^2 = \sum_{i=1}^{3} (dy^i)^2 \quad ? \qquad (7)$$

Since $\vec{y} = \vec{y}(\vec{x})$ we have

$$\sum_{i=1}^{3} (dy^i)^2 = \sum_{i=1}^{3} \left(\sum_{j=1}^{3} \frac{\partial y^i}{\partial x^j} dx^j \right)^2 = \sum_{j,k} \left(\sum_{i=1}^{3} \frac{\partial y^i}{\partial x^j} \frac{\partial y^i}{\partial x^k} \right) dx^j dx^k$$

One can show $\displaystyle\sum_{i=1}^{3} \frac{\partial y^i}{\partial x^j} \frac{\partial y^i}{\partial x^k} = \delta_{jk} = \begin{cases} 1 & \text{if } j=k \\ 0 & j \neq k \end{cases}$ ("Kronecker delta") $\qquad (7')$

only if $\vec{y} = \vec{y}(\vec{x})$ is a linear transformation. $\displaystyle y^i = \sum_{j=1}^{3} R^i{}_j x^j + a^i$

with $R^i{}_j$ a 3x3 matrix of numbers and a^i a 3-vector of numbers (by "numbers" we mean constants, independent of $\vec{x}$).
Moreover it is required that

$$\sum_{i=1}^{3} R^i{}_j R^i{}_k = \delta_{jk} \qquad (8)$$

which follows directly from (7').

Note: the reason for the peculiar upper/lower indices will become clear soon.

Condition (8) defines "orthogonal" matrices.

It is convenient to introduce a <u>metric tensor</u> η_{ij} so that

$$ds^2 = \sum_{i=1}^{3} \sum_{j=1}^{3} \eta_{ij} dx^i dx^j \qquad (9)$$

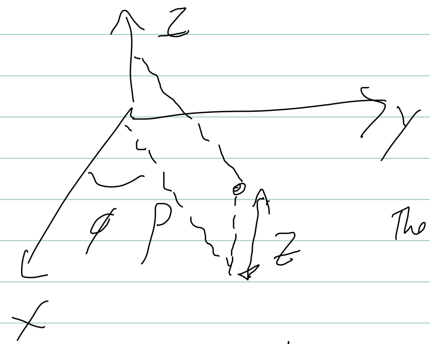
Of course, $\eta_{ij} = \delta_{ij}$ in our Cartesian system. But we have already seen that if we go nutty with coordinate choices, $y^i = y^i(x^j)$ then

$$ds^2 = \sum_{k,l}^{3} g_{kl} dy^k dy^l \quad \text{with } g_{kl} = \sum_{i,j}^{3} \frac{\partial x^i}{\partial y^k} \frac{\partial x^j}{\partial y^l} \eta_{ij} \qquad (10)$$

(we used $x^i = x^i(\vec{y})$ as the inverse of $\vec{y} = \vec{y}(\vec{x})$). This can be convenient! We can relate the metric tensors in coordinate systems that are not Cartesian.

Examples follow:

✱ Cylindrical coordinates.



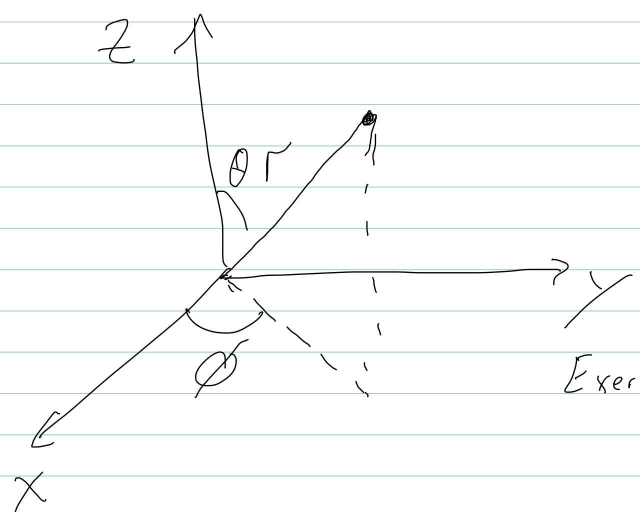
$x = \rho\cos\phi$
$y = \rho\sin\phi$
$z = z$ ← ok, I should give them different labels, but why bother....

Then $ds^2 = (d\rho\cos\phi - \rho\sin\phi\, d\phi)^2 + (d\rho\sin\phi + \rho\cos\phi\, d\phi)^2 + (dz)^2$

$$= d\rho^2 + \rho^2 d\phi^2 + dz^2 \qquad (11)$$

and $g_{k\ell} = \mathrm{diag}(1, \rho^2, 1)$ (or $g_{\rho\rho} = 1$, $g_{\phi\phi} = \rho^2$, $g_{zz} = 1$, all others vanish).

✱ Spherical coordinates



$x = r\sin\theta\cos\phi$
$y = r\sin\theta\sin\phi$
$z = r\cos\theta$

Exercise: verify $ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2$ (12)

We will make this useful momentarily.

But lets go back to rotations: consider $x^i \to R^i{}_j x^j$ (this is often going to be shorthand, or shortspeak, for let $y^i = R^i{}_j x^j$ in a function of $\vec{y}$):

$$ds^2 = \sum_{i,j=1}^3 \eta_{ij}\, dx^i dx^j \longrightarrow \sum_{i,j=1}^3 \eta_{ij}\Big(\sum_{k=1}^3 R^i{}_k\, dx^k\Big)\Big(\sum_{\ell=1}^3 R^j{}_\ell\, dx^\ell\Big) = \sum_{k,\ell=1}^3 \eta'_{k\ell}\, dx^k dx^\ell$$

where $$\eta'_{k\ell} = \sum_{i,j=1}^3 R^i{}_k R^j{}_\ell\, \eta_{ij} \qquad (13)$$

The invariance condition (7) is now $\eta'_{ij} = \eta_{ij}$

Notice that since $\eta_{ij} = \delta_{ij}$, condition (13) is just the same as (8). But it tells us something interesting: orthogonal transformations are those that leave the metric tensor invariant under transformations $\eta \to \eta'$ given in (13)

Einstein convention:

Repeated indices are presumed summed over their understood range unless otherwise stated.

So $y^i = R^i{}_j x^j$ stands for $y^i = \sum_{j=1}^{3} R^i{}_j x^j$

and $ds^2 = \eta_{ij}\, dx^i dx^j$ stands for $ds^2 = \sum_{i,j=1}^{3} \eta_{ij}\, dx^i dx^j$

Sometimes we even imply the indices, $y = Rx$ means $y^i = R^i{}_j x^j$

And

$R^T \eta R = \eta$ characterizes transformations that leave metric invariant

and $\eta = \mathbb{1}$ means $R^T R = \mathbb{1}$, a more familiar condition for orthogonal matrix.

Physics:

Let's consider a rotation on Maxwell's equations. Start from Gauss's law (1):

$$\vec{\nabla} \cdot \vec{E} = 4\pi\rho$$

or rather

$$\partial_i E^i = 4\pi\rho$$

where $\partial_i = \frac{\partial}{\partial x^i}$, and the lower index on ∂_i, upper on x^i, will be explained later.

Consider a coordinate change $y^i = y^i(x^j) = R^i{}_j x^j$

We want to show that there is a matrix function of R, say $D(R)^i{}_j$ such that (1) is form invariant if

$$E'^i(\vec{y}, t) = D^i{}_j E^j(\vec{x}, t) = D^i{}_j E^j(R^{-1}\vec{y}, t)$$

Since t is going along for a ride, I will omit below. "Form invariant" or plainly "invariant" means

$$\partial'_i E'^i = \frac{\partial}{\partial y^i} E'^i(y) = 4\pi\rho(\bar{y})$$

To see this is the case, and infer $D(R)$, compute the divergence:

$$\partial'_i E'^i = D^i{}_j\, \partial'_i E^j(R^{-1}y) = D^i{}_j\, \partial'_i (R^{-1}y)^k \left(\partial_k E^j(x)\right)\Big|_{x=R^{-1}y} = D^i{}_j (R^{-1})^k{}_i \left(\partial_k E^j(x)\right)\Big|_{x=R^{-1}y}$$

We want this to equal $4\pi\rho(\vec{x})\big|_{\vec{x}=R^{-1}\vec{y}} = \partial_i E^i(\vec{x})\big|_{\vec{x}=R^{-1}\vec{y}}$. Comparing we see that we need

$$(R^{-1})^k{}_i D^i{}_j = \delta^k_j \Rightarrow D^i{}_j = R^i{}_j \quad \text{or in matrix notation} \quad D = R$$

(So indeed, D is a function of R, namely, $D(R) = R$).

Re-cap: Eq (1) is invariant under the change of coordinates ("rotations")

$$\vec{y} = R\vec{x}$$

if in the new coordinate system $\vec{E}'(\vec{y}) = R\vec{E}(\vec{x})$ and $\rho'(\vec{y}) = \rho(\vec{x})$.

Or simply, (1) is invariant under $\vec{E}'(\vec{y}) = R\vec{E}(R^{-1}y)$ and $\rho'(\vec{y}) = \rho(R^{-1}\vec{y})$.

We say that $\vec{E}$ is a $\boxed{\text{vector}}$ because it transforms under rotations just like $\vec{x}$ does

(namely $\vec{x} \to R\vec{x}$, $\vec{E} \to R\vec{E}$).

We say that ρ is a $\boxed{\text{scalar}}$: it transforms under rotations just as ds^2

(namely $\rho \to \rho$).

Exercise: show that with c a scalar, and $\vec{B}$, $\vec{j}$, $\vec{v}$ and $\vec{F}$ vectors eqs (2)-(5) are also invariant.

Maxwell equations are invariant under rotations. This may seem trivial, particularly since we have written them in an explicitly covariant notation. That is, once we know that

(i) Dot products are scalars (ie, do not transform)

(ii) $\vec{\nabla}$ is a vector

(iii) cross products of vectors are vectors

we can "see" that each equation is invariant because both sides of the equality transform the same way, eg

$$\underbrace{\underbrace{\underbrace{\vec{\nabla}}_{\text{vector}}\underbrace{\times}_{\times\text{ prod}}\underbrace{\vec{B}}_{\text{vector}}}_{\text{vector}} - \underbrace{\frac{1}{c}\frac{\partial\vec{E}}{\partial t}}_{\text{vector}}}_{\text{vector}} = \underbrace{\frac{4\pi}{c}\vec{j}}_{\text{vector}}$$

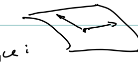
Our aim is to show that Eqs (1)-(5) are invariant under a bigger set of transformations, namely Lorentz transformations (plus translations in space-time, but these are already explicit). The problem is much simpler if we can make the symmetry explicit, as we just showed for rotations in Ampère's law (Eq (2)) rather than going through explicit computations as we did with Gauss's law (and I proposed as an Exercise for Eqs (2)-(5)).

Before leaving rotations, let's use the technology we developped to derive a couple of useful equations.

When going to curvilinear coordinates (eg, spherical) we have to be more careful in defining vectors. It is not necessarily true that new coordinates $y^i = y^i(\vec{x})$ transform as vectors. What is always true is that the infinitesimal displacement between two points is a vector, and the vector transformation is given by

$$dy^i = \frac{\partial y^i}{\partial x^j} dx^j$$

dx^i defines a tangent vector to a curved surface:



which corresponds to $dy^i = R^i{}_j\, dx^j$ for transformations $\vec{y} = \vec{y}(\vec{x})$ of the form $\vec{y} = R\vec{x}$.

The dot-product of two vectors $\vec{a}$ & $\vec{b}$ is $\vec{a}\cdot\vec{b} = \eta_{ij}\, a^i b^j$

Recall $ds^2 = g_{ij}\, dy^i dy^j$ has

$$g_{ij} = \frac{\partial x^k}{\partial y^i}\frac{\partial x^l}{\partial y^j}\eta_{kl} \qquad (14)$$

So if $a'^i = \frac{\partial y^i}{\partial x^k} a^k$ for vectors, then

$$\vec{a}'\cdot\vec{b}' = g_{ij}\, a'^i b'^j = \underline{\left(\frac{\partial x^k}{\partial y^i}\frac{\partial x^l}{\partial y^j}\eta_{kl}\right)}\left(\frac{\partial y^i}{\partial x^m}a^m\right)\left(\frac{\partial y^j}{\partial x^n}b^n\right) = \eta_{kl}\, a^k b^l = \vec{a}\cdot\vec{b}$$

Consider the gradient $\partial_i\phi$ of a scalar function, ie, $\phi'(\vec{y}) = \phi(\vec{x})$.

$$\partial'_i\phi' = \frac{\partial\phi'(y)}{\partial y^i} = \frac{\partial}{\partial y^i}\phi(x) = \frac{\partial x^j}{\partial y^i}\frac{\partial\phi}{\partial x^j}$$

Aha! This is NOT a vector! $\partial'_i\phi' = \frac{\partial x^j}{\partial y^i}\partial_j\phi$ instead of $a'^i = \frac{\partial y^i}{\partial x^j}a^j$

In the language of differential geometry these are "1-forms" (spoken "one forms").
In old physics language

a^i = contra-variant vector
$\partial_i\phi$ = co-variant vector

Any ω_i that transforms as $\omega'_i = \frac{\partial x^j}{\partial y^i}\omega_j$ is a 1-form (or a co-variant vector)

Note that $\omega'_i a'^i = \omega_i a^i$, ie, the "contraction" of a 1-form and a vector is a scalar (sometimes this is used to <u>define</u> 1-forms).

Note furthermore that $a_i \equiv g_{ij}a^j$ transforms as (and therefore <u>IS</u>) a 1-form

THIS IS WHY WE HAVE DIFFERENTIATED BETWEEN UPPER AN LOWER INDICES.

Given a 1-form a_i can I make a vector a^i? Yes! Let g^{ij} denote the inverse matrix to g_{ij}, so that

$$g^{ij}\, g_{jk} = \delta^i_k \qquad\qquad (g^{-1} g = \mathbb{1})$$

Then

$$a^i = g^{ij} a_j$$

is a vector.

Exercise: Prove the above assertion.

Invariant integrals.

Consider $\int_V d^3x \equiv \int_V dx\,dy\,dz = \int_V \prod_{i=1}^{3} dx^i$

In changing variables to curvilinear coordinates $\vec{y} = \vec{y}(\vec{x})$

$$\int_V d^3x = \int_V d^3y\ J$$

$$J = \text{Jacobian} = \left| \det\left(\frac{\partial x^i}{\partial y^j} \right) \right|$$

Recall, if $ds = g_{ij}\, dy^i dy^j = \eta_{ij}\, dx^i dx^j$ then (eq (14)): $g_{ij} = \frac{\partial x^k}{\partial y^i} \frac{\partial x^l}{\partial y^j} \eta_{kl}$

That is $g \equiv \det g_{ij} = J^2 \det \eta$ (and $\det \eta = 1$, but let's keep it explicit, for now)

So $\int_V d^3x \sqrt{\eta} = \int_V d^3y \sqrt{g}$ is the invariant integration volume.

Examples:

(i) Cylindrical: $g = \det\begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \rho^2 \Rightarrow$ volume $= d\rho\, d\theta\, dz \cdot \rho = \rho\, d\rho\, d\theta\, dz$

(ii) Spherical: $g = \det\begin{pmatrix} 1 & & \\ & r^2 & \\ & & r^2\sin^2\theta \end{pmatrix} = r^4 \sin^2\theta$ vol $= dr\, d\theta\, d\phi\ r^2 \sin\theta = r^2 dr\ d(\cos\theta)\ d\phi$

Divergence in curvilinear coordinates

Looks a priori messy, but neat trick:

Consider $\int_V d^3y \sqrt{g}\, a^i \partial_i \phi$ clearly invariant under coordinate transformations

Take ϕ to vanish at spatial ∞ and integrate by parts (in fact we will want ϕ to have local support):

$$\underbrace{\int_V d^3y \sqrt{g}\, a^i \partial_i \phi}_{\text{invariant}} = -\int_V d^3y\ \phi\ \partial_i(\sqrt{g}\, a^i) = -\underbrace{\int_V d^3y \sqrt{g}\,\phi}_{\text{invariant}} \Rightarrow \underbrace{\left[\frac{1}{\sqrt{g}}\partial_i(\sqrt{g}\, a^i)\right]}_{\text{must be invariant}}$$

Moreover, comparing to Cartesian coordinates, this invariant is what we called $\text{div}(\vec{a})$

So

$$\text{div}(\vec{a}) = \frac{1}{\sqrt{g}}\, \partial_i\left(\sqrt{g}\, a^i\right)$$

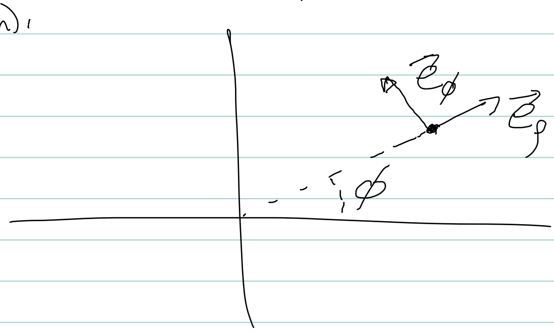
Examples:

(i) Cylindrical: $\text{div}(\vec{a}) = \frac{1}{\rho}\partial_\rho(\rho a^\rho) + \partial_\phi a^\phi + \partial_z a^z$

(ii) Spherical $\text{div}(\vec{a}) = \frac{1}{r^2}\partial_r(r^2 a^r) + \frac{1}{\sin\theta}\partial_\theta(\sin\theta\, a^\theta) + \partial_\phi a^\phi$

Note: This result differs from many textbooks (eg, Jackson or Garg). The reason is their meaning for a^i is different — mine is better :)

Suppose you want to write $\vec{a} = A^i \vec{e}_i$, where $\vec{e}_i$ are orthonormal vectors in the Carthesian sense. For example, for cylindrical coordinates (omitting the z-direction):



$$|\vec{e}_\rho| = |\vec{e}_\phi| = 1$$

$$\vec{e}_\phi \cdot \vec{e}_\rho = 0$$

Then, for example $\vec{a}\cdot\vec{a} = g_{ij}\, a^i a^j = A^i A^j\, \vec{e}_i \cdot \vec{e}_j = A^i A^j \delta_{ij}$. Since g_{ij} is diagonal (a condition for this to work) we find $A^i = \sqrt{g_{ii}}\, a^i$ (no sum on i) and

$$\text{div}(a) = \frac{1}{\sqrt{g}}\partial_i\left(\sqrt{g}\,\frac{1}{\sqrt{g_{ii}}}A^i\right)$$

cyl: $\frac{1}{\rho}\partial_\rho(\rho A^\rho) + \frac{1}{\rho}\partial_\phi A^\phi + \partial_z A^z$

spher: $\frac{1}{r^2}\partial_r(r^2 A^r) + \frac{1}{r\sin\theta}\partial_\theta(\sin\theta\, A^\theta) + \frac{1}{r\sin\theta}\partial_\phi A^\phi$

Lesson: make sure you know what your symbols mean (especially when you use formulas from the back flap of a textbook).

Laplacian

A simple extension of the previous exercise: use $a^i = g^{ij}\partial_j f$ where f is a scalar. Then

$$\nabla^2(f) = \vec{\nabla}\cdot\vec{\nabla}f = \frac{1}{\sqrt{g}}\partial_i(\sqrt{g}\,g^{ij}\partial_j f)$$

In Carthesian coordinates $\nabla^2 f = \eta^{ij}\partial_i\partial_j f = (\partial_1^2+\partial_2^2+\partial_3^2)f$

— — —

Exercise:

Cylindrical: $\nabla^2 f = \frac{1}{\rho}\partial_\rho(\rho\,\partial_\rho f) + \frac{1}{\rho^2}\partial_\phi^2 f + \partial_z^2 f$

Spherical: $\nabla^2 f = \frac{1}{r^2}\partial_r(r^2\partial_r f) + \frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta f) + \frac{1}{r^2\sin^2\theta}\partial_\phi^2 f$

— — —

Tensors, invariants and the peculiar cross product.

Restrict attention to rotations, $y^i = R^i{}_j x^j$.

Vectors $a'^i = R^i{}_j a^j$ (so $b'_i a'^i = R_i{}^n R^i{}_j b_n a^j = \delta^n_j b_n a^j = b_j a^j$)

1-forms $b'_i = R_i{}^n b_n$

2-index tensors: $T'^{ij} = R^i{}_m R^j{}_n T^{mn}$

$$Q'_{ij} = R_i{}^m R_j{}^n Q_{mn}$$

Note that $g_{im}g_{jn}T^{mn}$ transforms just like Q_{ij}

Tensors are defined by their transformation properties, not by having indices.

We can form a 2-index tensor from two vectors by a "tensor product"

$$T^{ij} = a^i b^j$$

Similarly $T_{ij} = a_i b_j$ and $T^i{}_j = a^i b_j$ are tensors. The latter, obviously has $T'^i{}_j = R^i{}_m R_j{}^n T^m{}_n$.

3-index tensors: $T'^{ijk} = R^i{}_l R^j{}_m R^k{}_n T^{lmn}$, etc. The generalization is

obvious: $T'^{i_1 \cdots i_p}{}_{j_1 \cdots j_q} = R^{i_1}{}_{m_1} \cdots R^{i_p}{}_{m_p} R_{j_1}{}^{n_1} \cdots R_{j_q}{}^{n_q} T^{m_1 \cdots m_p}{}_{n_1 \cdots n_q}$

Definition: $S^{lm} = S^{ml}$ is a symmetric tensor.

$a^{lm} = -a^{ml}$ is an anti-symmetric tensor

with generalizations to higher index tensors, eg $S^{i_1 \cdots i_n}$ is completely symmetric if it is invariant under permutations of the indices.

Useful: If $S^{lm} = S^{ml}$ and $a_{lm} = -a_{ml}$ $\Rightarrow$ $S^{lm} a_{lm} = 0$.

Proof: $S^{lm} a_{lm} = -S^{lm} a_{ml}$ (anti-symmetry of a_{ml})

$= -S^{ml} a_{lm}$ (dummy variables, change labels)

$= -S^{lm} a_{lm}$ (symmetry of S^{lm})

$\Rightarrow 2 S^{lm} a_{lm} = 0$ $\Rightarrow S^{lm} a_{lm} = 0$

Invariant tensor: Def. $T'^{i_1 \cdots i_p}{}_{j_1 \cdots j_q} = T^{i_1 \cdots i_p}{}_{j_1 \cdots j_q}$

We have already encountered one:

$$\eta'_{ij} = R_i{}^m R_j{}^n \eta_{mn} = \eta_{mn}$$

(where, of course, $\eta_{mn} = \delta_{mn}$).

If you like math, this generalizes to spaces in any number of dimensions: δ_{ij} is always an invariant tensor.

In 3-dimensions, there is another interesting tensor. Let

$$\epsilon_{ijk} = \begin{cases} +1 & (ijk) \text{ an even permutation of } (123) \\ -1 & (ijk) \text{ an odd permutation of } (123) \\ 0 & \text{otherwise} \end{cases}$$

This is the completely anti-symmetric 3-index tensor, or Levi-Civita tensor.

That is $\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$ $\epsilon_{213} = \epsilon_{132} = \epsilon_{321} = -1$ $\epsilon_{111} = \epsilon_{112} = \cdots = \epsilon_{333} = 0$.

Consider $T_{ijk} \equiv R_i{}^l R_j{}^m R_k{}^n \epsilon_{lmn}$

First note that if any two indices in T_{ijk} are equal then it vanishes,

eg $T_{11k} = \underbrace{R_1{}^l R_1{}^m}_{\text{symmetric under } l \leftrightarrow m} R_k{}^n \underbrace{\epsilon_{lmn}}_{\text{anti-symmetric under } l \leftrightarrow m} \Rightarrow 0$

It is then easy to see T_{ijk} is completely antisymmetric

Now $T_{123} = R_1{}^l R_2{}^m R_3{}^n \epsilon_{lmn} = \det(R)$

or, since T_{ijk} is completely antisymmetric, like ϵ_{ijk}, and $\epsilon_{123} = +1$,

$$T_{ijk} = \det(R)\,\epsilon_{ijk}$$

Furthermore $\eta_{ij} = R_i{}^k R_j{}^l \eta_{kl} = R_i{}^k \eta_{kl} (R^T)^l{}_j = (R\eta R^T)_{ij}$

$$\Rightarrow \det \eta = \det(R\eta R^T) = \det R \det\eta \det R^T = (\det R)^2 \det\eta$$

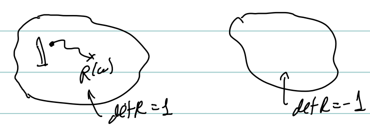
$$\Rightarrow \det R = \pm 1$$

$\Rightarrow$ $\boxed{\text{Under rotations + } \det(R) = +1 \quad \epsilon_{ijk} \text{ is an invariant tensor.}}$

Importantly, it flips sign under rotations with $\det(R) = -1$.

Let $R(\omega)$ be a continuous function $[0,1] \to \{3\times 3 \text{ orthogonal matrices}\}$ such that $R(0) = \mathbb{1}$. Then $\det(R(\omega))$ is a continuous function $[0,1] \to \mathbb{R}$ which can only take values -1 and $+1$, and has $\det(R(0)) = \det \mathbb{1} = +1 \Rightarrow \det(R(\omega)) = +1$

In words rotations that can be reached from $\mathbb{1}$ continuously all have $\det R = +1$.



If R_1 & R_2 both have $\det R = 1$ then so does $R_3 = R_1 R_2$: $\det(R_3) = \det(R_1 R_2) = \det R_1 \cdot \det R_2 = (+1)^2 = +1$

but if both have $\det R = -1$ then $R_3 = R_1 R_2$ has $\det R_3 = +1$.

In fact every rotation R' with $\det(R') = -1$ can be written as $R' = (-\mathbb{1})\cdot R$ where $\det(R) = +1$.

$(-\mathbb{1})$, of course, is a "spatial inversion", or "reflection", or "parity transformation"

Mathy stuff: The set of rotations $\{3\times 3 \text{ real matrices} \mid R^T R = \mathbb{1} \ (\text{or } R\eta R^T = \eta)\}$ form a group, called $O(3)$, for "orthogonal group in 3 dimensions".

Obvious extension: $O(N)$ ... N dimensions.

The subset of matrices with $\det R = +1$ form a subgroup, $SO(3)$, for special orthogonal.

The subset with $\det R = -1$ does not form a group. (Question: why?)

Cross Product:

For proper rotations (ie, with $\det(R) = +1$) we have ϵ_{ijk} is an invariant tensor.

$\Rightarrow \omega_i = \epsilon_{ijk} a^j b^k$ transforms as a 1-form and $\omega^i = \eta^{ij}\omega_j$ transforms as a vector

Also,

(i) $a_{ij} = \epsilon_{ijk} b^k$ is a 2-index antisymmetric tensor

(ii) If a^{ij} is a 2-index anti-symmetric tensor, then $\omega_k = \frac{1}{2}\epsilon_{kij} a^{ij}$ is a 1-form

and $a^{ij} = \epsilon^{ijk}\omega_k$

That is, there is a 1-to-1, invertible, correspondance between vectors and anti-symmetric 2-index tensors

Exercise: Show $\epsilon^{ijk}\epsilon_{mjk} = 2\delta^i_m$ and then $a^{ij} = \epsilon^{ijk}\left(\frac{1}{2}\epsilon_{kmn} a^{mn}\right)$.

In Carthesian coordinates we do not distinguish ω^i from ω_i since $\omega^i = \eta^{ij}\omega_j = \delta^{ij}\omega_j$

So $\omega_i = \epsilon_{ijk} a^j b^k$ is a vector $\vec{\omega}$ made out of $\vec{a}$ & $\vec{b}$ with components

$$\omega_1 = a_2 b_3 - a_3 b_2, \quad \omega_2 = a_3 b_1 - a_1 b_3, \quad \omega_3 = a_1 b_2 - a_2 b_1$$

denoted by $\vec{\omega} = \vec{a} \times \vec{b}$

Space inversions: vectors vs pseudo-vectors (also called "axial" vectors).

Let $P = -\mathbb{1}$ be a space inversion ("P" is for parity)

Vectors $\vec{a} \to R\vec{a}$ transform as $\vec{a} \to P\vec{a} = -\vec{a}$ under space inversions.

Pseudovectors, however, transform as $\vec{\omega} \to +\vec{\omega}$ under space inversions.

This is the statement that if $\vec{a} \to R\vec{a}$ & $\vec{b} \to R\vec{b}$

then

$$\vec{a} \times \vec{b} \longrightarrow \det(R)\, R\, (\vec{a} \times \vec{b})$$

Exercise: show this

For "improper" rotations (ie, those with $\det(R) = -1$) this means $\vec{a} \times \vec{b} \to -R(\vec{a} \times \vec{b})$

In particular, under space inversions, $\vec{a} \times \vec{b} \to +(\vec{a} \times \vec{b})$

The cross product vector × vector is pseudovector

vector × pseudovector is vector

pseudo × pseudo is pseudovector.

Tensors in curvilinear coordinates

The generalization is straightforward.
Recall if $y^i = y^i(x)$ then

$$a'^i = \frac{\partial y^i}{\partial x^j} a_j \qquad \left(\text{just as } dy^i = \frac{\partial y^i}{\partial x^j} dx^j\right)$$

and

$$\omega'_i = \frac{\partial x^j}{\partial y^i} \omega_j$$

A 2-index tensor has $\quad T'^{ij} = \frac{\partial y^i}{\partial x^m}\frac{\partial y^j}{\partial x^n} T^{mn}$

$$T'^i{}_j = \frac{\partial y^i}{\partial x^m}\frac{\partial x^n}{\partial y^j} T^m{}_n$$

$$T'_{ij} = \frac{\partial x^m}{\partial y^i}\frac{\partial x^n}{\partial y^j} T_{mn}$$

etc.

Exercise: check that these are consistent with $T^m{}_n = g_{nk} T^{mk} = g^{ml} T_{ln}$
That is, we can "raise" and "lower" indices using the metric and its inverse to consistently make other tensors.

Recall that if $ds^2 = g_{mn} dy^m dy^n = \eta_{ij} dx^i dx^j$ then $g_{mn} = \frac{\partial x^i}{\partial y^m}\frac{\partial x^j}{\partial y^n}\eta_{ij}$

$\Rightarrow$ the metric tensor $\underline{\text{is}}$ a tensor (not just in name).

It is not generally invariant. That $\eta_{ij} = \delta_{ij}$ is invariant under rotations is the statement that there is a special set of coordinate transformations (rotations + translations) that leave η_{ij} invariant.

What about ϵ_{ijk}?
By the same calculation as above

$$\frac{\partial x^l}{\partial y^i}\frac{\partial x^m}{\partial y^j}\frac{\partial x^n}{\partial y^k}\epsilon_{lmn} = \det\left(\frac{\partial x}{\partial y}\right)\epsilon_{ijk} = \sqrt{g}\,\epsilon_{ijk}$$

That is, $\epsilon_{ijk} T'^{ijk} = \epsilon_{ijk} \frac{\partial y^i}{\partial x^l}\frac{\partial y^j}{\partial x^m}\frac{\partial y^k}{\partial x^n} T^{lmn} = \det\left(\frac{\partial y}{\partial x}\right)\epsilon_{lmn} T^{lmn} = \frac{1}{\sqrt{g}}\epsilon_{lmn} T^{lmn}$

so that $\sqrt{g}\,\epsilon_{ijk} T'^{ijk} = \sqrt{\eta}\,\epsilon_{lmn} T^{lmn}$

$\Rightarrow$ In any frame there is a 3-form, completely anti-symmetric, given by $\omega_{ijk} = \sqrt{g}\,\epsilon_{ijk}$

This is called a (metric) volume form

Curl and Stoke's Theorem

We first review in Cartesian coordinates

$\vec{\text{curl}}(\vec{A}) = \vec{\nabla}\times\vec{A}$ has $(\vec{\nabla}\times\vec{A})^i = \epsilon^{ijk}\,\partial_j A_k$ (Balancing upper lower indices in an ad-hoc way).

eg $(\vec{\nabla}\times\vec{A})_x = \partial_y A_z - \partial_z A_y$

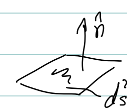
which we prefer to write as $(\vec{\nabla}\times\vec{A})_1 = \partial_2 A_3 - \partial_3 A_2$

If $R^i{}_j$ is a rotation matrix (rigid, in the sense that it's $\vec{x}$ independent) then curl$(\vec{A})$ is a vector (actually if $\vec{A}$ is a vector, curl$(\vec{A})$ is a pseudovector).

Exercise: Show this. (Hint $A_i'(\vec{x}) = R_i{}^j A_j(R^{-1}\vec{x})$).

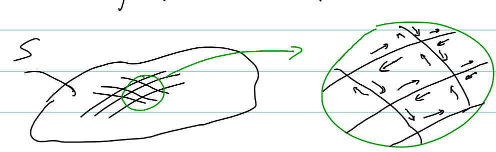
Stoke's theorem states that

$$\int_S (\vec{\nabla}\times\vec{A})\cdot\hat{n}\, d^2s = \oint_{\partial S} \vec{A}\cdot d\vec{l}$$



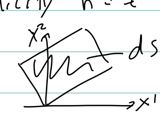
when $\int_S$ is a surface integral over S, a sum over infinitesimal surfaces normal (locally) to $\hat{n}$ with area d^2s, and $\oint_{\partial S}$ is a line integral over the boundary ∂S of the whole surface S, with tangent line element $d\vec{l}$.

Sketch of Proof:



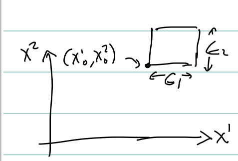
Note that if it is true over an infinitesimal area element then it is true over the whole of S bounded by ∂S (the Σ elements of $\sum_N (\vec{\nabla}\times\vec{A})\cdot\hat{n}\,\Delta s \to \int_S (\vec{\nabla}\times\vec{A})\cdot\hat{n}\, d^2s$ while the line integrals cancel except at the boundry ∂S, so $\sum_{\text{paths}} \vec{A}\cdot\Delta\vec{l} \to \oint_{\partial S}\vec{A}\cdot d\vec{l}$

So need to show infinitesimally: now an infinitesimal surface element is flat and has normal $\hat{n}$ everywhere. Take for simplicity $\hat{n} = \hat{z}$ (ie, 3^{rd} direction). Then $d^2s = dx\,dy$. If the boundaries are not aligned



make a rotation to align them: since $\vec{\nabla}\times\vec{A}$ is a vector $(\vec{\nabla}\times\vec{A})\cdot\hat{n}$ won't change.

So we consider



$$\int_{x_0^1}^{x_0^1+\epsilon^1} dx^1 \int_{x_0^2}^{x_0^2+\epsilon^2} dx^2\,(\partial_1 A_2 - \partial_2 A_1) = \int_{x_0^2}^{x_0^2+\epsilon^2} dx^2\left(A_2(x_0^1+\epsilon^1, x^2) - A_2(x_0^1, x^2)\right) - \int_{x_0^1}^{x_0^1+\epsilon^1} dx^1\left(A_1(x^1, x_0^2+\epsilon^2) - A_1(x^1, x_0^2)\right)$$

$$= \oint_{\square} \vec{A}\cdot d\vec{l}$$

While this proof may not seem quite general, the fact that we can always find a rotation to put $\hat{n}$ in the $\hat{z}$ direction, and that this just corresponds to a change of variables makes it a truly general argument!

Note: I was a bit careless about direction of $\hat{n}$ vs orientation of loop (but I did it right).

The curl in curvilinear coordinates is tricky. The reason is that when we see that it involves ϵ_{ijk} we may think it involves the volume form $\sqrt{g}\,\epsilon_{ijk}$ when going curvilinear.

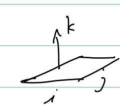
But since Stoke's theorem holds and should be generalized, and this involves a surface integral of the (normal component of) the curl, it is actually the "volume" form of the 2-dimensional space S that should be involved. If the metric restricted to the surface S (at a point on the surface element) is h_{ij} then we want $\frac{1}{\sqrt{h}}\epsilon^{ij}\partial_i a_j$ for the component of curl($\vec{a}$) along the normal.

This is simple in orthogonal coordinates (with $g_{ij}=0$ if $i\neq j$), because the surface element $dx^i dx^j$ ($i\neq j$) has a normal along ϵ^{kij} (ie, the 3rd direction).

So the 3 components of curl($\vec{a}$) are $\epsilon^{kij}\frac{1}{\sqrt{h}}\partial_i a_j$. And as before, expressing this in terms of components of normalized metric, so that $(A^i)^2 = g_{ii}(a^i)^2 = g^{ii}(a_i)^2 = \frac{1}{g_{ii}}(a_i)^2$

and noting that in the ij plane $\sqrt{h} = \sqrt{g_{ii}g_{jj}}$ we have

$$\text{curl}(\vec{A})^k = \epsilon^{kij}\frac{1}{\sqrt{g_{ii}\,g_{jj}}}\,\partial_i\left(\sqrt{g_{jj}}\,A^j\right)$$



Examples

Cylindrical: $\epsilon^{\rho\theta z}\frac{1}{\sqrt{g_{\theta\theta}\,g_{zz}}}\left(\partial_\theta(\sqrt{g_{zz}}A^z) - \partial_z(\sqrt{g_{\theta\theta}}A^\theta)\right) = (+1)\frac{1}{\sqrt{\rho^2\cdot 1}}\left(\partial_\theta(\sqrt{1}A^z) - \partial_z(\sqrt{\rho^2}A^\theta)\right) = \frac{1}{\rho}\partial_\theta A^z - \partial_z A^\theta$

a bit faster... $\epsilon^{\theta z\rho}\frac{1}{\sqrt{1\cdot 1}}\left(\partial_z\sqrt{1}A^\rho - \partial_\rho\sqrt{1}A^z\right) = \left(\partial_z A^\rho - \partial_\rho A^z\right)$, and $\epsilon^{z\rho\theta}\frac{1}{\rho}\left(\partial_\rho(\rho A^\theta) - \partial_\theta A^\rho\right)$

or $\text{curl}(\vec{A}) = \left(\frac{1}{\rho}\partial_\theta A^z - \partial_z A^\theta,\ \partial_z A^\rho - \partial_\rho A^z,\ \frac{1}{\rho}\partial_\rho(\rho A^\theta) - \frac{1}{\rho}\partial_\theta A^\rho\right)$

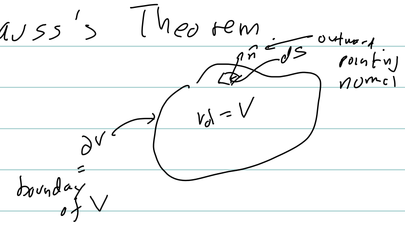
Spherical: $[\text{curl}(\vec{A})]^r = \frac{1}{r^2\sin\theta}\left[\partial_\theta(r\sin\theta A^\phi) - \partial_\phi(rA^\theta)\right] = \frac{1}{r\sin\theta}\left[\partial_\theta(\sin\theta A^\phi) - \partial_\phi A^\theta\right]$

Exercise: Compute remaining components

ANS $[\text{curl}(\vec{A})]^\theta = \frac{1}{r\sin\theta}\partial_\phi A^r - \frac{1}{r}\partial_r(rA^\phi)$

$[\text{curl}(\vec{A})]^\phi = \frac{1}{r}\partial_r(rA^\theta) - \frac{1}{r}\partial_\theta A^r$

Gauss's Theorem.



$$\int_{\partial V} \vec{A}\cdot\hat{n}\, ds = \int_V \vec{\nabla}\cdot\vec{A}\, dV$$

This is easy to prove for an infinitesimal cube, and then extended to finite volumes by summation, as in Stoke's case

$\Rightarrow$ left as exercise (but countless textbooks have it; still you should be able to construct the proof).

Explicit sample calculations using Stoke's & Gauss's theorem are given as part of Homework #1, and in problem session.

Back to Space-Time (it's about time).

Coordinates in space-time: $(x^0 = ct, \underbrace{x^1, x^2, x^3}_{\vec{x}\ (\text{or } \vec{r})})$

A point in space-time is called "an event"

Basic property of space-time: invariance of the interval.

Interval between $(x^0, \vec{x})$ and $(x^0+dx^0, \vec{x}+d\vec{x})$ $\quad ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$

Notation: greek indices $\mu, \nu, \ldots$ range 0–3 (while latin indices $i,j,\ldots$ range 1–3)

Einstein summation convention as before, applies to any type of index.

Metric $\eta_{\mu\nu}$ is 4x4, diagonal with $\eta_{\mu\nu} = \begin{cases} +1 & \mu=\nu=0 \\ -1 & \mu=\nu=1,2,3 \end{cases}$

So $\quad ds^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$ $\quad$ ie, the more familiar looking $c^2(dt)^2 - (d\vec{r})^2$

Lorentz transformations: $x'^\mu = \Lambda^\mu{}_\nu x^\nu$, are defined to be those that leave ds^2 form invariant. As with rotations this means

$\Lambda^\mu{}_\nu$ is a Lorentz transformation $\Longleftrightarrow \eta_{\mu\nu} \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma = \eta_{\rho\sigma}$ $\quad$ or $\quad \Lambda^T \eta \Lambda = \eta$ $\quad$ for short

ie, leave the metric invariant.

Vectors & tensors

A (contravariant) vector transforms as $a^\mu \to a'^\mu = \Lambda^\mu{}_\nu a^\nu$

Indices can be lowered with $\eta_{\mu\nu}$, eg $a_\mu = \eta_{\mu\nu} a^\nu$

The inverse metric η^{-1} is denoted $\eta^{\mu\nu}$, ie, $\eta^{\mu\lambda}\eta_{\lambda\nu} = \delta^\mu_\nu$

A low index vector (or covariant vector, or really, a 1-form) transforms so that $\omega'_\mu a'^\mu = \omega_\mu a^\mu$ (invariant)

$\omega'_\mu = \omega_\nu \Lambda^{-1\,\nu}{}_\mu$ $\qquad$ Since $\eta^{-1}\Lambda^T \eta \Lambda = \mathbb{1}$ $\rightarrow$ $\eta^{\kappa\rho} \Lambda^\mu{}_\rho \eta_{\mu\nu} \Lambda^\nu{}_\sigma = \delta^\kappa_\rho \Rightarrow (\Lambda^{-1})^\kappa{}_\nu = \Lambda_\nu{}^\kappa$

or $\quad \omega'_\mu = \Lambda_\nu{}^\mu \omega_\nu$

Indices can be raised with η^{-1}: $\quad a^\mu = \eta^{\mu\nu} a_\nu$

Shorthand: $\quad a^2 = \eta_{\mu\nu} a^\mu a^\nu = a^\mu a_\mu = \eta^{\mu\nu} a_\mu a_\nu$ $\quad ; \quad a\cdot b = \eta_{\mu\nu} a^\mu b^\nu = a_\mu b^\mu = a^\mu b_\mu = \eta^{\mu\nu} a_\mu b_\nu$

Tensors: $\quad T'^{\mu_1\cdots\mu_p}{}_{\nu_1\cdots\nu_q} = \Lambda^{\mu_1}{}_{\kappa_1} \cdots \Lambda^{\mu_p}{}_{\kappa_p} \Lambda_{\nu_1}{}^{\sigma_1} \cdots \Lambda_{\nu_q}{}^{\sigma_q}\, T^{\kappa_1\cdots\kappa_p}{}_{\sigma_1\cdots\sigma_q}$

eg $\quad T'^{\mu\nu} = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma T^{\rho\sigma}$. $\quad$ Again η, η^{-1} lower, raise indices, eg $T^\mu{}_\nu = \eta_{\nu\rho} T^{\mu\rho}$

Many generalizations from the discussion of rotations are straightforward, e.g.:

- A field is a function of spacetime. A <u>scalar</u> field $\phi(x^\mu)$ (or simply $\phi(x)$) satisfies

$$\phi'(x') = \phi(x) \qquad \text{under} \quad x' = \Lambda x$$

This is often written as

$$\phi'(x) = \phi(\Lambda^{-1}x)$$

A vector field satisfies

$$A'^\mu(x) = \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}x)$$

and

$$B'_\mu(x) = \Lambda_\mu{}^\nu B_\nu(\Lambda^{-1}x)$$

- The gradient is a (co-variant) vector: $\partial_\mu\phi$ transforms as B_μ above.

Some things are a little different.
- ϵ_{ijk} is not an invariant tensor. Neither is $\epsilon_{\mu\nu\sigma}$.
  But $\epsilon_{\mu\nu\lambda\sigma}$ is invariant under transformations with $\det(\Lambda)=+1$.

As before $\det(\Lambda)=\pm1$

Exercise: show this.

But now there are 4 connected components of the group of Lorentz transformations:
- Space inversion: as before $\vec{x}' = -\vec{x}$ (and $t'=t$) gives $\det(\Lambda) = -1$

but now
- Time reflection: $t' = -t$ and $\vec{x}' = \vec{x}$ also gives $\det(\Lambda) = -1$.

Are these disconnected? To understand the meaning of the question go back to rotations for a moment. Are

$$P = \begin{pmatrix} -1 & & \\ & -1 & \\ & & -1 \end{pmatrix} \quad \text{and} \quad P_z = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}$$

in disconnected components, both with $\det(R) = -1$? The answer is no:

$$P_z = R_z P = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} -1 & & \\ & -1 & \\ & & -1 \end{pmatrix} \quad \text{with } R_z \text{ a } 180° \text{ rotation about the } z \text{ axis}$$

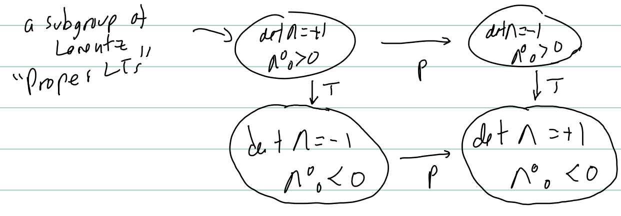
Back to 4D & Lorentz Transformations:

Since $\eta_{\mu\nu} = \eta_{\rho\sigma} \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu$, $\quad (\Lambda^0{}_0)^2 - \Sigma(\Lambda^i{}_0)^2 = 1 \Rightarrow \Lambda^0{}_0 = \pm\sqrt{1+\Sigma(\Lambda^i{}_0)^2}$

$\Rightarrow$ cannot smoothly connect $\Lambda^0{}_0 \leq -1$ to $\Lambda^0{}_0 \geq +1$

And for fixed sign of $\Lambda^0{}_0$, a space-inversion gives a flip in sign of $\det\Lambda$

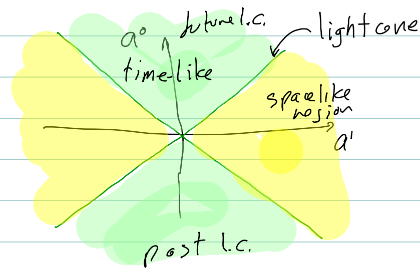
So there are 4-connected components



T = time reflection
P = space inversion

Since a^2 is invariant, if $a^2 > 0$ then $a^0 > |\vec{a}|$ or $a^0 < -|\vec{a}|$

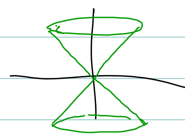
Light-cone diagram



Remarks (i) This diagram is for an arbitrary vector, not necesarily coordinates

(ii) We still call the regions by their coordinate analogs (ie, spacelike and timelike regions). We also say a^μ is timelike/lightlike/spacelike according to whether $a^2>0$, $a^2=0$, $a^2<0$.

(iii) The 2dim image above is limited: should draw 4dim, but I can't. Sorry :) But it should be clear that the light cone is a cone, a 3dim hypersurface. We can at least draw a 2dim hypersurface in a 3dim spacetime:



This should make it clear that the spacelike region is connected, while future and past light cones are not (they "touch" at the origin).

Now, if $\Lambda: a^0 \to -a^0$, $a^i \to a^i$ and $a^2>0$, it maps future light cone $\longleftrightarrow$ past light cone. Cannot go from one to the other continuously (going through origin is not allowed, since $\Lambda=0$ is not invertible, and does not leave η invariant).

Explicit form of Lorentz Transformations.

Find explicit solutions to $\quad \eta_{\mu\nu} = \eta_{\rho\sigma} \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu$

That is

$\mu\nu=00 \qquad (\Lambda^0{}_0)^2 - \sum_{i=1}^3 (\Lambda^i{}_0)^2 = 1 \qquad (1)$

$\mu\nu=0i \qquad \Lambda^0{}_0 \Lambda^0{}_i - \sum_k \Lambda^k{}_0 \Lambda^k{}_i = 0 \qquad (2)$

$\mu\nu=ij \qquad \Lambda^0{}_i \Lambda^0{}_j - \sum_k \Lambda^k{}_i \Lambda^k{}_j = -\delta_{ij} \qquad (3)$

Observations, including some solutions:

(i) Rotations: $\Lambda^0{}_0 = 1 \quad \Lambda^0{}_i = \Lambda^i{}_0 = 0 \quad , \Lambda^i{}_j = R^i{}_j$ with R a rotation ($R \in O(3)$)

ex, rotations about z axis ($z = x^3$)

$$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c & -s & 0 \\ 0 & s & c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{aligned} c &= \cos\theta \\ s &= \sin\theta \end{aligned}$$

$x'^\mu = \Lambda^\mu{}_\nu x^\nu$ is

$ct' = ct \quad z' = z$

$x' = \cos\theta\, x - \sin\theta\, y$

$y' = \sin\theta\, x + \cos\theta\, y$

(ii) Boosts: $\Lambda^0{}_0 = \cosh\eta \quad \sum_{\mu=1}^3 (\Lambda^i{}_0)^2 = \sinh^2\eta$ solves (1)

eg. in x direction ($x = x^1$)

$$\Lambda = \begin{pmatrix} \cosh\eta & \sinh\eta & 0 & 0 \\ \sinh\eta & \cosh\eta & 0 & 0 \\ 0 & 0 & 1 & \\ 0 & 0 & & 1 \end{pmatrix}$$

or

$ct' = \cosh\eta\, ct + \sinh\eta\, x$

$x' = \sinh\eta\, ct + \cosh\eta\, x$

$y' = y$

$z' = z$

Setting $x = 0$, the origin of un-primed system is moving with velocity $v = \beta c = c \tanh\eta$ as measured in primed system $x' = (c\sinh\eta)t = \left(\frac{c\sinh\eta}{\cosh\eta}\right)t'$

$1 - \tanh^2\eta = \frac{1}{\cosh^2\eta} \Rightarrow \gamma \equiv \cosh\eta = \frac{1}{\sqrt{1-\beta^2}}$ and $\sinh\eta = \tanh\eta \cdot \cosh\eta = \gamma\beta$

(iii) Counting: how many independent parameters for LT's?

Warm-up: for rotations first $R^T R = \mathbb{1}$ R is $3\times 3 = 9$ entries, the condition is on a symmetric matrix, ie on $\frac{3\times 4}{2} = 6$ components $\Rightarrow 9-6 = 3$ independent parameters $\rightarrow$ Euler Angles! ✓✓

Similarly $\Lambda^T \eta \Lambda = \eta$ puts $\frac{4\times 5}{2} = 10$ constraints on $4\times 4 = 16$ matrix $\Rightarrow 16 - 10 = 6$ independent parameters $\Rightarrow$ 3 Euler angles + 3 boosts

(iv) Product of two transformations is a transformation

While obvious physically, this can be expressed and shown mathematically:

Exercise: If Λ_1 & Λ_2 are LT's show that $\Lambda_1\Lambda_2$ is an LT. (ie, satisfies $\Lambda^T\eta\Lambda=\eta$).

This makes LTs a group (called $O(3,1)$).

One can use this to build LTs out of infinitesimal ones

$$\tilde\Lambda = 1 + \epsilon \qquad \text{with } \epsilon \text{ infinitesimal}$$

$$\tilde\Lambda^T\eta\tilde\Lambda=\eta \Rightarrow (1+\epsilon)^T\eta(1+\epsilon)=\eta \Rightarrow \epsilon^T\eta+\eta\epsilon=0$$

Or lowering indices $\epsilon_{\mu\nu}=\eta_{\mu\lambda}\epsilon^\lambda{}_\nu$ this is $\epsilon_{\mu\nu}+\epsilon_{\nu\mu}=0 \Rightarrow$ anti-symmetric

A 4×4 anti-symmetric matrix has $\frac{4\times3}{2}=6$ independent parameters $\rightarrow$ same counting as above!

Now $\tilde\Lambda^N=(1+\epsilon)(1+\epsilon)\cdots(1+\epsilon)$ is a transformation

and one can show

$$\Lambda = \lim_{N\to\infty}\left(1+\tfrac{1}{N}\omega\right)^N = \exp(\omega)$$

is a general expression for LT's. See eg Jackson.

Exercise: What is the analog in the case of rotations?

Tensors & pseudo tensors.

Tensors & pseudo tensors transform the same way under proper LTs:

$$T'^{\mu_1\cdots}{}_{\nu_1\cdots} = \Lambda^{\mu_1}{}_{\kappa_1}\cdots\ \Lambda_{\nu_1}{}^{\rho_1}\cdots\ T^{\kappa_1\cdots}{}_{\rho_1\cdots}$$

Tensors still transform this way under any (not neccessarily proper) LT. But for a pseudo-tensor under space inversion

$$T'^{\mu_1\cdots}{}_{\nu_1\cdots} = -\ \mathcal{P}^{\mu_1}{}_{\kappa_1}\cdots\ \mathcal{P}_{\nu_1}{}^{\rho_1}\ T^{\kappa_1\cdots}{}_{\rho_1\cdots}$$

where $\mathcal{P}^{\mu_1}{}_{\kappa_1}=\mathrm{diag}(1,-1,-1,-1)$, and a pseudo-tensor under time-reversal

$$T'^{\mu_1\cdots}{}_{\nu_1\cdots} = -\ \mathcal{T}^{\mu_1}{}_{\kappa_1}\cdots\mathcal{T}_{\nu}{}^{\rho_1}\ T^{\kappa_1\cdots}{}_{\rho_1\cdots}$$

where $\mathcal{T}=\mathrm{diag}(-1,1,1,1)$.

Ex: $\epsilon'^{\mu\nu\lambda\sigma}=\epsilon^{\mu\nu\lambda\sigma}$ is an invariant pseudo-tensor (under both P & T).

Particular examples:

A vector $V^\mu = (V^0, V^i) = (V^0, \vec{V})$

has $V'^0 = V^0$ and $\vec{V}' = -\vec{V}$ under P

$V'^0 = -V^0$ and $\vec{V}' = \vec{V}$ under T

An "axial" vector is a pseudo-vector under parity: $A^\mu = (A^0, \vec{A})$ has

$A'^0 = -A^0$ and $\vec{A}' = \vec{A}$ under P