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Stochastic closure for nonlinear Rossby waves

By GREG HOLLOWAY†
AND MYRL C. HENDERSHOTT
Scripps Institution of Oceanography, La Jolla, California 92037

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An extension of the turbulence 'test-field model' (Kraichnan 1971*a*) is given for two-dimensional flow with Rossby-wave propagation. Such a unified treatment of waves and turbulence is necessary for flows in which the relative strength of nonlinear terms depends upon the length scale considered. We treat the geophysically interesting case in which long, fast Rossby waves propagate substantially without interaction while short Rossby waves are thoroughly dominated by advection. We recover the observations of Rhines (1975) that the tendency of two-dimensional flow to organize energy into larger scales of motion is inhibited by Rossby waves and that an initially isotropic flow develops anisotropy preferring zonal motion. The anisotropy evolves to an equilibrium functional dependence on the isotropic part of the flow spectrum. Theoretical results are found to be in quantitative agreement with numerical flow simulations.

1. Introduction

We consider statistically homogeneous, non-divergent, barotropic flow on a β-plane, i.e.

$$\partial_t \nabla^2\psi + \partial(\psi, \nabla^2\psi)/\partial(x,y) + \beta\,\partial_x\psi = 0, \tag{1}$$

where ψ is the stream function and $\zeta = \nabla^2\psi$ is the vertical component of the vorticity. The Jacobian term provides advection of vorticity while the β-term propagates Rossby waves. Some form of dissipation is required in (1) but, for the moment, we avoid stating our prejudices on dissipation. Now we impose that ζ be spatially periodic on a rectangular 'cell' and so be given by a discrete Fourier series

$$\zeta = \sum_{\mathbf{k}} \zeta_{\mathbf{k}}(t)\, e^{i\mathbf{k}\cdot\mathbf{x}}.$$

The Fourier transform of (1) is

$$\left(\frac{d}{dt} + i\omega_{\mathbf{k}} + \nu_k\right)\zeta_{\mathbf{k}} = \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} A_{\mathbf{k},\mathbf{p}}\,\zeta_{\mathbf{p}}\,\zeta_{\mathbf{q}}, \tag{2}$$

where $\omega_{\mathbf{k}} = -\beta k_x/k^2$, $A_{\mathbf{k},\mathbf{p}} = \mathbf{k}\times\mathbf{p}/p^2$ and ν_k is included as an explicit, but as yet unspecified, dissipation function $\nu_k = d(k)$. Our aim will be a statistical description of (2), averaging in principle over many realizations of the flow, to obtain the evolution, from statistically prescribed initial conditions, of ensemble-average covariances $\langle\zeta_{\mathbf{k}}\zeta_{-\mathbf{k}}\rangle$.

† Present address: National Center for Atmospheric Research, Advanced Study Program, P.O. Box 3000, Boulder, Colorado 80307.

If nonlinearity is sufficiently weak that a wave propagates over several wave periods without being significantly modified by interaction with other waves then, neglecting dissipation, we might proceed by a resonant wave interaction calculation (Kenyon 1964; Longuet-Higgins & Gill 1967). One supposes $\zeta_{\mathbf{k}}(t) = a_{\mathbf{k}}(\tau)\exp(-i\omega_{\mathbf{k}}t)$, where $\tau = \epsilon^2 t$ is the slow time scale of wave interaction, t is a fast time scale with units of the wave period and ϵ is a small number proportional to the amplitude of the motion. By integrating (2) over the fast time scale as $t \rightarrow \infty$ while supposing that $\epsilon \rightarrow 0$ such that $\epsilon^2 t$ remains finite, one obtains contributions to the slow rate of change of $|a_{\mathbf{k}}|^2$ from resonant wave triads satisfying

$$\mathbf{p}+\mathbf{q} = \mathbf{k}$$

and

$$\omega_{\mathbf{p}}+\omega_{\mathbf{q}} = \omega_{\mathbf{k}}. \tag{3}$$

Such a result is obtained only asymptotically for $\epsilon \rightarrow 0$. A problem is that the dispersion relation $\omega_{\mathbf{k}} = -\beta k_x/k^2$ provides very long periods for very short waves, so that no matter how weak the flow (i.e. how long the interaction time) the very short waves will interact significantly in times short compared with their periods. More troublesome yet are the components of zonal flow with $k_x = 0$ and hence indefinite periods. All interaction with the zonal flow is essentially strong regardless of its amplitude. Thus, for a continuous spectrum of Rossby waves, we may not consider ϵ to be small.

If, on the other hand, we take β negligibly small then (2) is the equation of 'two-dimensional turbulence'. The significant feature of this flow is that vorticity is advected without stretching in contrast with the tendency in three-dimensional flow to extend vortex lines. Thus two-dimensional advection preserves both mean energy density $\bar{E} = \frac{1}{2}\overline{|\nabla\psi|^2}$ and mean 'enstrophy' density $\bar{Z} = \overline{\zeta^2}$. (Overbars denote area averages over the periodic flow cell.) Now energy cannot cascade into ever finer scales since this would entail generation of enstrophy. Instead one speaks of a cascade of enstrophy into finer scales together with an 'inverse cascade' of energy into larger scales of motion (Kraichnan 1967). A problem arises when we attempt to relax periodicity by letting the period length become arbitrarily large. No matter how small β is, we shall admit long waves with periods small compared with any interaction time. In other words, on an unbounded β-plane we may not consider ϵ to be large.

Rhines (1975) suggests that we think of wavenumber space as divided into a wave regime and a turbulence regime with a 'soft' border between. The border can be defined approximately by the condition that wave phase speeds equal fluid particle speeds. In the turbulence regime, enstrophy will cascade towards high wavenumbers while energy cascades towards low wavenumbers. However, the energy cascade will encounter this waves–turbulence border beyond which further energy transfer can proceed only weakly by resonant wave interaction. The result is to accumulate energy near the waves–turbulence border. Rhines then argues that by inhibiting energy transfer to low wavenumbers the flow becomes limited in its enstrophy transfer to high wavenumbers, resulting in a high wavenumber spectrum which falls off more steeply than in a flow without Rossby waves. Finally, the anisotropic dispersion relation results in anisotropic evolution from an initially isotropic flow field. Numerical simulations show a marked tendency to prefer zonal (east–west) motion. Rhines suggests that this may be due to a stabilizing effect of β on zonal flow, possibly even resulting in evolution towards a state of steady zonal jets.

Numerical simulations provide qualitative support for Rhines' description. The energy spectrum does become peaked near a waves–turbulence border. The high wavenumber spectrum does fall off more steeply with β present though, as we shall see, this may be a consequence of Rhines' assumption of viscous-type dissipation $\nu_k = \nu k^2$. Finally, simulations do become anisotropic though an end state of steady zonal flow remains conjectural. However, this phenomenological admixture of turbulence and waves ideas does not admit quantitative calculation. Also, the description is essentially local in wavenumber space: turbulence-like dynamics bring energy up to a border where, somehow, a transition to wave-like dynamics is effected. Indirectly the suppression of energy transfer comes to be felt as a limitation on enstrophy transfer at high wavenumbers. But are there non-local, direct interactions of waves and turbulence? Does anisotropy find an equilibrium state short of steady zonal flow?

This paper attempts a more analytical treatment by extending a class of turbulence theories to include wave propagation. The derivation does not depend upon nonlinearity being either weak or strong. In the limit where waves dominate nonlinearity we just recover a resonant wave interaction approximation.

2. Markovian quasi-normal (MQN) closure

We employ a closure in which evolution of second moments $\langle \zeta_{\mathbf{k}}(t)\, \zeta_{-\mathbf{k}}(t)\rangle$ is expressed in terms of the values of the second moments $\langle \zeta_{\mathbf{p}}(t)\, \zeta_{-\mathbf{p}}(t)\rangle$ in other modes, with all quantities known only at the instant t. This requires that the statistics have evolved to a state of quasi-stationarity, by which we mean that second moments change slowly during a response time of higher moments (the time in which an arbitrary small perturbation to any higher moment will decay).

The label 'quasi-normal' here is a bit misleading. Although a formal appeal is sometimes made to expansion about a normal distribution, the resulting closure may describe statistics which are far from normally distributed. The wave interaction approximation (Hasselmann 1962; Benney & Saffman 1966) is a member of the MQN class, as are a variety of turbulence models, in particular the 'test-field model' (TFM), which has shown agreement with numerical simulations of turbulence (Kraichnan 1971*a*; Herring *et al.* 1974). The 'direct-interaction approximation' (DIA) for turbulence (Kraichnan 1958, 1959) is a more fundamental approach, couched in simultaneous equations for evolution of mode response functions $G_k(t,t')$ and time-displaced covariances $\langle \zeta_{\mathbf{k}}(t)\, \zeta_{-\mathbf{k}}(t')\rangle$. However, the stationary DIA may be abridged to an MQN form by assuming *ad hoc* that time-displaced covariances decay exponentially as $\langle \zeta_{\mathbf{k}}(t)\, \zeta_{-\mathbf{k}}(t')\rangle = \langle \zeta_{\mathbf{k}}(t)\, \zeta_{-\mathbf{k}}(t)\rangle \exp(-\eta_{\mathbf{k}}|t-t'|)$ for some $\eta_{\mathbf{k}}$. Relationships among the various MQN turbulence models and their relation to the DIA are reviewed by Leslie (1973, chaps. 7 and 11) and by Orszag (1974).

Different approaches may be employed in deriving or justifying a closure model. Here we give a heuristic sketch which differs somewhat in style from other accounts, e.g. Orszag (1974), but is equivalent in its consequences.

Ensemble averaging (2) and deleting for the moment all subscripts, coefficients and summations, we have an unclosed set of equations

$$d\langle\zeta\zeta\rangle/dt = \langle\zeta\zeta\rangle + \langle\zeta\zeta\zeta\rangle, \tag{4a}$$

$$d\langle\zeta\zeta\zeta\rangle/dt = \langle\zeta\zeta\zeta\rangle + \langle\zeta\zeta\rangle\langle\zeta\zeta\rangle + \langle\zeta\zeta\zeta\zeta\rangle^C, \tag{4b}$$

$$d\langle\zeta\zeta\zeta\zeta\rangle^C/dt = \langle\zeta\zeta\zeta\zeta\rangle^C + \langle\zeta\zeta\rangle\langle\zeta\zeta\zeta\rangle + \langle\zeta\zeta\zeta\zeta\zeta\rangle^C, \tag{4c}$$

etc. A superscript C denotes a cumulant, i.e. the remainder of a moment, say of the fourth moment $\langle\zeta\zeta\zeta\zeta\rangle$, after any contributions from non-vanishing lower moments (here products of second moments $\langle\zeta\zeta\rangle\langle\zeta\zeta\rangle$) have been removed. Terms appear on the right sides of (4) owing to the linear and nonlinear terms in (2). In each of ($4a$–c), nonlinearity introduces the next higher cumulant; hence the closure problem.

What is appropriately termed a 'quasi-normal' or fourth-cumulant discard hypothesis closes (4) by taking $\langle\zeta\zeta\zeta\zeta\rangle^C = 0$ for all time. However, direct integration of ($4a, b$) then produces quite unrealistic results for turbulence, evidenced in part by unrealizable, negative values for $\langle|\zeta_{\mathbf{k}}|^2\rangle$ (Ogura 1962). The wave interaction approximation is different: based on distinction of fast and slow time scales t and τ, integration of ($4b$) is considered only in the limit of large t and leads, without dissipation, to indefinitely large values of $\langle\zeta\zeta\zeta\rangle$ on a vanishingly small (in continuous wavenumber space measure) resonant interaction set (3). Substitution of these resonant members of ($4b$) into ($4a$) gives the wave interaction approximation and assures that $\langle|\zeta_{\mathbf{k}}|^2\rangle \geqslant 0$. Given plausible initial conditions, the evolution of $\langle\zeta\zeta\zeta\zeta\rangle^C$ turns out not to matter in the limit of vanishingly weak nonlinearity. On the other hand, Ogura's calculations clearly indicate the significant role of fourth cumulants in turbulence.

A plausible turbulence model, without waves, is obtained by observing that in ($4b$) products $\langle\zeta\zeta\rangle\langle\zeta\zeta\rangle$ cause $\langle\zeta\zeta\zeta\rangle$ to build up until checked at some level by a dissipation term $-\nu\langle\zeta\zeta\zeta\rangle$ obtained from (2). If dissipation is weak, $\langle\zeta\zeta\zeta\rangle$ will become quite large unless it is being relaxed statistically by $\langle\zeta\zeta\zeta\zeta\rangle^C$, which we therefore assume to be simply of the form $\langle\zeta\zeta\zeta\zeta\rangle^C = -\mu\langle\zeta\zeta\zeta\rangle$, where μ is some coefficient. Observe that μ is not a usual eddy viscosity since we have not altered ($4a$). Now, by quasi-stationarity, we integrate ($4b$) over a time large compared with $(\nu+\mu)^{-1}$ and substitute the result into ($4a$) to get

$$(d/dt+\nu)\langle\zeta\zeta\rangle = \theta\langle\zeta\zeta\rangle\langle\zeta\zeta\rangle$$

with $\theta = (\nu+\mu)^{-1}$. The object of an MQN turbulence theory is to evaluate μ or, equivalently, θ. The TFM is one such evaluation. The wave interaction approximation uses θ to isolate resonant wave triads: $\theta = \pi\delta(\omega_{\mathbf{p}}+\omega_{\mathbf{q}}-\omega_{\mathbf{k}})$ with δ the Dirac delta-function.

3. 'Test-field model' with waves

Restoring the algebraic detail which was deleted in the preceding section, we seek to model the evolution of second moments $Z_{\mathbf{k}}(t) = \langle\zeta_{\mathbf{k}}(t)\,\zeta_{-\mathbf{k}}(t)\rangle$ by an equation

$$\left(\frac{d}{dt}+2\nu_k\right)Z_{\mathbf{k}} = \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} \theta_{-\mathbf{k},\mathbf{p},\mathbf{q}}\left[a_{kpq}Z_{\mathbf{p}}Z_{\mathbf{q}} - 2b_{kpq}Z_{\mathbf{p}}Z_{\mathbf{k}}\right], \tag{5}$$

where
$$a_{kpq} = |\mathbf{k}\times\mathbf{p}|^2\left(\frac{1}{p^2}-\frac{1}{q^2}\right)^2$$

and
$$b_{kpq} = |\mathbf{k}\times\mathbf{p}|^2\left(\frac{1}{p^2}-\frac{1}{q^2}\right)\left(\frac{1}{p^2}-\frac{1}{k^2}\right)$$

are geometric coefficients obtained from (2). a_{kpq} and b_{kpq} depend only on the lengths k, p and q. Sums over wave vectors $\mathbf{k}$, $\mathbf{p}$ or $\mathbf{q}$ range over some finite set of modes, usually defined by requiring that the wave vector's length be less than some $k_{\max}$. It is this Fourier truncation of (1) in passing to (2) which has caused us to admit the explicit dissipation ν_k. However, the form of ν_k still need not concern us.

$\theta_{-\mathbf{k},\mathbf{p},\mathbf{q}}$ is the unknown array which becomes the focus of our attention. If $\theta_{-\mathbf{k},\mathbf{p},\mathbf{q}}$ is constrained to be invariant to permutations of its indices, then the sum of the right-hand side of (5) over $\mathbf{k}$ vanishes. Likewise the weighted sum of $1/k^2$ times the right-hand side of (5) vanishes. Thus the constraint to a fully symmetric $\theta_{-\mathbf{k},\mathbf{p},\mathbf{q}}$ assures that nonlinear interaction preserves both mean enstrophy and mean energy density.

The 'test-field model' (Kraichnan 1971*a*) completes the specification of $\theta_{-\mathbf{k},\mathbf{p},\mathbf{q}}$ up to a single universal empirical constant which could be evaluated, for example, in terms of an estimated Kolmogorov inertial range constant. We sketch the TFM, here appealing to a correspondence between (5) and stochastic models of Langevin type (Leith 1971).

For the moment we omit waves, considering just two-dimensional turbulence. Equation (5) attempts to *approximate* the statistical evolution of (2). But (5) is also an *exact* closure for a set of linear stochastic differential equations

$$(d/dt+\nu_k+\eta_{\mathbf{k}})\,\tilde{\zeta}_{\mathbf{k}} = f_{\mathbf{k}}, \tag{6}$$

where

$$\eta_{\mathbf{k}} = \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} \theta_{-\mathbf{k},\mathbf{p},\mathbf{q}}\, b_{kpq} Z_{\mathbf{p}}$$

and

$$\langle f_{\mathbf{k}}(t) f_{-\mathbf{k}}(t')\rangle = \Big(\sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} \theta_{-\mathbf{k},\mathbf{p},\mathbf{q}}\, a_{kpq} Z_{\mathbf{p}} Z_{\mathbf{q}}\Big)\,\delta(t-t').$$

Equation (6) is a Langevin equation describing a random variable $\tilde{\zeta}_{\mathbf{k}}$ evolving under the influence of a linear drag $\nu_k+\eta_{\mathbf{k}}$ and a random force $f_{\mathbf{k}}$.

Physically, $\theta_{-\mathbf{k},\mathbf{p},\mathbf{q}}$ has the role of a relaxation time of phase correlations among modes $\mathbf{k}$, $\mathbf{p}$ and $\mathbf{q}$, which we might estimate by some heuristic argument. However, the correspondence between (5) and (6) holds for any non-negative choice of $\theta_{-\mathbf{k},\mathbf{p},\mathbf{q}}$. As a linear equation, (6) is characterized by its Green's function $G_{\mathbf{k}}(t,t')$. Thus, without using further physical arguments, we may already obtain a form for $\theta_{-\mathbf{k},\mathbf{p},\mathbf{q}}$ in terms of the form of (6). This integral time scale for correlations among the stochastic model variables $\tilde{\zeta}_{-\mathbf{k}}$, $\tilde{\zeta}_{\mathbf{p}}$ and $\tilde{\zeta}_{\mathbf{q}}$ is

$$\theta_{-\mathbf{k},\mathbf{p},\mathbf{q}} = \int_{-\infty}^{t} G_{-\mathbf{k}}(t,t')\, G_{\mathbf{p}}(t,t')\, G_{\mathbf{q}}(t,t')\, dt'. \tag{7}$$

Equations (5)–(7) are a closure for (2), omitting waves. Unfortunately a spurious effect has been introduced which becomes evident in the stochastic model representation (6). Evolution of each stochastic variable $\tilde{\zeta}_{\mathbf{k}}$ depends only on the variance in other modes $\langle\tilde{\zeta}_{\mathbf{p}}\tilde{\zeta}_{-\mathbf{p}}\rangle$ while any phase information is lost. Very large scales of motion, approaching uniform translation, contribute to $\eta_{\mathbf{k}}$ and hence to a rapid decrease of $G_{\mathbf{k}}(t,t')$ in $t-t'$, in turn resulting in a small value for $\theta_{-\mathbf{k},\mathbf{p},\mathbf{q}}$. In reality large-scale motion advects triple phase correlations nearly coherently and so, in the limit of uniform translation, ought to have no effect on the value of $\theta_{-\mathbf{k},\mathbf{p},\mathbf{q}}$.

The TFM restores a proper Galilean invariance by adopting $\theta_{-\mathbf{k},\mathbf{p},\mathbf{q}}$ in the form (7) while estimating a modified $\hat{G}_{\mathbf{k}}(t,t')$ which suppresses the effect of large-scale advection. The heuristic argument (Kraichnan 1971*a*) is that $\theta_{-\mathbf{k},\mathbf{p},\mathbf{q}}$ represents the deformation of fluid parcels. In a Lagrangian description of the motion, deformation is accomplished only by pressure and viscous forces. Accordingly we seek that part $\hat{G}_{\mathbf{k}}$ of the overall $G_{\mathbf{k}}$ which is due to pressure effects. Pressure in incompressible Navier–Stokes flow is evaluated from the incompressibility condition, i.e. pressure is the agent which prevents a solenoidal flow field from developing a longitudinal part. Thus a measure of the pressure effect is had by computing the rate at which a solenoidal flow would generate a longitudinal flow in the absence of a pressure term. This measure of the pressure effect is assumed to be related to the actual triple-moment relaxation through an empirical constant of proportionality.

Kraichnan's derivation of the TFM for Navier–Stokes turbulence is given in velocity-component notation, which is inconvenient for many problems of quasi-geostrophic motion, e.g. Rossby waves. We therefore rederive the TFM for two-dimensional turbulence in the scalar notation of stream functions and velocity potentials. Consider an advecting field given by a vorticity ζ having the statistics of the turbulence. Let this field advect a passive test field given by a stream function ψ and velocity potential ϕ. $\chi = \nabla^2\psi$ and $\sigma = \nabla^2\phi$ are the vorticity and divergence of the test field. We seek the rate at which advection by ζ will mix the two parts χ and σ. This is

$$\frac{d}{dt}\chi_{\mathbf{k}} = \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} \frac{|k\times p|^2}{p^2q^2}\zeta_{\mathbf{p}}\sigma_{\mathbf{q}}$$

and

$$\frac{d}{dt}\sigma_{\mathbf{k}} = -\sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} \frac{|k\times p|^2}{p^2q^2}\zeta_{\mathbf{p}}\chi_{\mathbf{q}},$$

for which an MQN closure is

$$\begin{aligned}\frac{d}{dt}X_{\mathbf{k}} &= \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} 2\frac{|\mathbf{k}\times\mathbf{p}|^2}{p^2q^2}\operatorname{Re}\langle\zeta_p\sigma_q\chi_{-k}\rangle \\ &= \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} 2\hat{\theta}_{-\mathbf{k},\mathbf{p},\mathbf{q}}\frac{|\mathbf{k}\times\mathbf{p}|^4}{p^2q^2}\frac{Z_p}{p^2}\left(\frac{Y_q}{q^2}-\frac{X_k}{k^2}\right), \\ \frac{d}{dt}Y_{\mathbf{k}} &= -\sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} 2\frac{|\mathbf{k}\times\mathbf{p}|^2}{p^2q^2}\operatorname{Re}\langle\zeta_p\chi_q\sigma_{-k}\rangle \\ &= \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} 2\hat{\theta}_{-\mathbf{k},\mathbf{p},\mathbf{q}}\frac{|\mathbf{k}\times\mathbf{p}|^4}{p^2q^2}\frac{Z_p}{p^2}\left(\frac{X_q}{q^2}-\frac{Y_k}{k^2}\right),\end{aligned}$$

where $X_{\mathbf{k}} = \langle\chi_{\mathbf{k}}\chi_{-\mathbf{k}}\rangle$, $Y_{\mathbf{k}} = \langle\sigma_{\mathbf{k}}\sigma_{-\mathbf{k}}\rangle$ and where $\hat{\theta}_{-\mathbf{k},\mathbf{p},\mathbf{q}}$ is another unknown array. We observe that these equations describe the solution of another Langevin model:

$$(d/dt+\hat{\mu}_{\mathbf{k}})\,\tilde{\sigma}_{\mathbf{k}} = \tilde{f}_{\mathbf{k}},$$

where

$$\hat{\mu}_{\mathbf{k}} = \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} \hat{\theta}_{-\mathbf{k},\mathbf{p},\mathbf{q}}\hat{b}_{kpq}Z_{\mathbf{p}}$$

with

$$\hat{b}_{kpq} = |\mathbf{k}\times\mathbf{p}|^4/k^2p^4q^2.$$

The argument is that the rate $\hat{\mu}_{\mathbf{k}}$ is a measure of the rate $\mu_{\mathbf{k}}$ of self-deformation of the ζ field. Including viscosity, $\mu_{\mathbf{k}}$ is written as

$$\mu_{\mathbf{k}} = \nu_k + g^2\sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} \theta_{-\mathbf{k},\mathbf{p},\mathbf{q}}\hat{b}_{kpq}Z_{\mathbf{p}}, \tag{8}$$

where g is a phenomenological constant of order unity and $\theta_{-\mathbf{k},\mathbf{p},\mathbf{q}}$ is again given by (7) but with a modified Green's function $\hat{G}_{\mathbf{k}}(t,t')$ depending on $\mu_{\mathbf{k}}$ in place of $\eta_{\mathbf{k}}$.

Now the reintroduction of waves is straightforward. Since wave propagation results in deformation of fluid parcels, this linear effect enters the modified Green's function as

$$(d/dt+i\omega_{\mathbf{k}}+\mu_{\mathbf{k}})\,\hat{G}_{\mathbf{k}}(t,t') = \begin{cases}0 & \text{for} \quad t<t', \\ \delta(t-t') & \text{for} \quad t\geqslant t'.\end{cases}$$

We are therefore obliged to consider complex $\hat{G}_{\mathbf{k}}(t,t')$ and hence, by (7), complex $\theta_{-\mathbf{k},\mathbf{p},\mathbf{q}}$. However, in closing (2)–(5) one adds complex conjugates, thus retaining only the real part of (7). In its quasi-stationary form, $\theta_{-\mathbf{k},\mathbf{p},\mathbf{q}}$ is

$$\theta_{-\mathbf{k},\mathbf{p},\mathbf{q}} = \mathrm{Re}\int^{t} \hat{G}_{\mathbf{k}}(t,t')\,\hat{G}_{\mathbf{p}}(t,t')\,\hat{G}_{\mathbf{q}}(t,t')\,dt'$$

$$\rightarrow \mu_{\mathbf{k},\mathbf{p},\mathbf{q}}/(\mu^2_{\mathbf{k},\mathbf{p},\mathbf{q}}+\omega^2_{-\mathbf{k},\mathbf{p},\mathbf{q}}) \quad \text{as} \quad t\rightarrow\infty, \tag{9}$$

where $\mu_{\mathbf{k},\mathbf{p},\mathbf{q}} = \mu_{\mathbf{k}}+\mu_{\mathbf{p}}+\mu_{\mathbf{q}}$ and $\omega_{-\mathbf{k},\mathbf{p},\mathbf{q}} = \omega_{-\mathbf{k}}+\omega_{\mathbf{p}}+\omega_{\mathbf{q}}$. Equations (5), (8) and (9) together with an estimate of g comprise the TFM closure for (2).

4. Comparison with numerical experiment

We have integrated (2) over a set of wave vectors lying in an annulus between $|\mathbf{k}| = 1$ and $|\mathbf{k}| = k_{\max} = 62$. The amplitude of the motion is scaled such that a nominal 'eddy turnover time' $2\pi/\zeta_{\mathrm{r.m.s.}} \approx 1$, where $\zeta_{\mathrm{r.m.s.}}$ is the root-mean-square vorticity. It is observed that a unit of time corresponds to significant evolution of the flow. Integrations are then carried to $t = 6{\cdot}4$, well into the quasi-stationary decay of the flow. Initial conditions consist of a random phase $\zeta_{\mathbf{k}}$ in an isotropic wavenumber band near $k_1 = 11$. As the flow evolves and energy migrates into larger scales, the characteristic energetic wavenumber k_1 decreases to $k_1 \approx 5$ at $t = 6{\cdot}4$.

The relative role of β can be expressed by defining $k_\beta = \beta/\zeta_{\mathrm{r.m.s.}}$, a wavenumber for which the Rossby-wave period of the faster westward-propagating waves ($\approx 2\pi k/\beta$) is of the order of the eddy turnover time. However, we caution against too literal an interpretation of k_β as a waves–turbulence border. Rather, k_β is the lower wavenumber of a transitional region, roughly from k_β to perhaps $3k_\beta$, within which most triad interactions are characterized by $\mu^2_{\mathbf{k},\mathbf{p},\mathbf{q}} \approx \omega^2_{-\mathbf{k},\mathbf{p},\mathbf{q}}$, where p and q are typically greater than k_1, for which we find below that $\mu \approx \zeta_{\mathrm{r.m.s.}}/2\pi$. We have investigated three choices of β: $\beta = 0$ ($k_\beta = 0$), $\beta = 12{\cdot}5$ ($k_\beta = 1{\cdot}5 \rightarrow 2{\cdot}5$) and $\beta = 25$ ($k_\beta = 3 \rightarrow 5$), where we indicate in parentheses that k_β increases as $\zeta_{\mathrm{r.m.s.}}$ decays from an initial value $\zeta_{\mathrm{r.m.s.}} \approx 8$ to a final value $\zeta_{\mathrm{r.m.s.}} \approx 5$. The cases may be described as 'no β' (for reference), 'small β' and 'moderate β'. We have not investigated the case of large β in part because we intend shortly to approximate (5) in an expansion about isotropy, i.e. about $\beta = 0$, and in part because the extension of resonant interaction theories to waves of small but finite amplitude warrants a more careful study on its own.

Lastly we must provide a truncation-induced dissipation ν_k. At present there appears to be no fundamental justification for the form of the dissipation operator. We have chosen

$$\nu_k = \frac{\nu_0 k^2_{\max}}{(k^2_{\max}-k^2_d)^2}\times\begin{cases}0 & \text{for} \quad k < k_d,\\ (k^2-k^2_d)^2 & \text{for} \quad k \geqslant k_d,\end{cases}$$

a form which is discussed, though hardly justified, by Holloway (1976). With $k_{\max} = 62$ we have taken $k_d = 45$ and $\nu_0 = 0{\cdot}005$, a conservatively large damping rate which results in a falling off of the spectrum in the dissipation range $k_d < k < k_{\max}$.

In turbulence theory it is usually most convenient to assume isotropic statistics, summing the modal $Z_{\mathbf{k}}$ in concentric shells to obtain a scalar enstrophy spectrum $Z(k)$. Then (5) gives the evolution of $Z(k)$ in terms of integrals over $Z(k)$. With β present we cannot assume isotropy. Rather we follow Herring (1975), resolving $Z_{\mathbf{k}}$ into angular Fourier harmonics

$$Z_{\mathbf{k}} = \sum_n Z_n(k)\, e^{in\phi_{\mathbf{k}}},$$

where $\phi_{\mathbf{k}}$ is the angle between $\mathbf{k}$ and the k_x axis. Odd harmonics $\phi_{\mathbf{k}}$, $3\phi_{\mathbf{k}}$, etc., vanish since $Z_{\mathbf{k}} = Z_{-\mathbf{k}}$. The result is an open set of coupled nonlinear integral equations for the even harmonics $Z_n(k)$. It is difficult to see how consistently to truncate this set, both in the number of harmonics and in the order of coupling, except for truncation at the lowest order in the departure from isotropy. Thus we limit the representation to

$$2\pi k Z_{\mathbf{k}} = Z(k)\,(1 - R(k)\cos 2\phi_{\mathbf{k}}), \tag{10}$$

where a term in $\sin 2\phi_{\mathbf{k}}$ vanishes by symmetry about the k_x axis. In the appendix we substitute (10) into (5), integrating over $\phi_{\mathbf{k}}$ and linearizing in the departure from isotropy, i.e. at order $R(k)$ or at order β^2, the lowest order in β. The result is the following pair of equations for the isotropic enstrophy spectrum $Z(k)$ and the anisotropy $R(k)$:

$$\left(\frac{d}{dt} + 2\nu_k\right) Z(k) = \frac{k(\Delta k)^2}{4\pi^2} \int\!\!\int_{\Delta} \left\{ \frac{dp\,dq}{\sin x}\, \frac{I_1(\epsilon)}{\mu(k,p,q)} \times \left[a_{kpq}\frac{Z(p)Z(q)}{pq} - 2b_{kpq}\frac{Z(p)\,Z(k)}{pk} \right] \right\} = 2Z(k)\,(\xi(k) - \eta(k)), \tag{11}$$

$$\frac{d}{dt} R(k) = S(k) + \int_1^{k_{\max}} dp\, K(k,p)\, R(p) - v(k)\, R(k), \tag{12}$$

where

$$S(k) = \frac{k(\Delta k)^2}{2\pi^2 Z(k)} \int\!\!\int_{\Delta} \frac{dp\,dq}{\sin x}\, \frac{-I_2(\epsilon,\alpha)}{\mu(k,p,q)} \left[a_{kpq}\frac{Z(p)\,Z(q)}{pq} - 2b_{kpq}\frac{Z(p)\,Z(k)}{pk} \right],$$

$$K(k,p) = \frac{2k(\Delta k)^2}{2\pi^2 Z(k)} \int_{\Delta q} \frac{dq}{\sin x}\, \frac{I_3(\epsilon,\alpha)}{\mu(k,p,q)} \cos 2z \left[a_{kpq}\frac{Z(p)\,Z(q)}{pq} - b_{kpq}\frac{Z(p)\,Z(k)}{pk} \right]$$

and

$$v(k) = \frac{k(\Delta k)^2}{2\pi^2 Z(k)} \int\!\!\int_{\Delta} \frac{dp\,dq}{\sin x} \left\{ \frac{I_1(\epsilon)}{\mu(k,p,q)}\, \frac{a_{kpq}}{2}\, \frac{Z(p)\,Z(q)}{pq} - \left(\frac{I_1(\epsilon)}{2} - I_3(\epsilon,\alpha) \right) \frac{2b_{kpq}}{\mu(k,p,q)}\, \frac{Z(p)\,Z(k)}{pk} \right\}.$$

The various symbols are defined in the appendix. Equation (11) describes the evolution of the isotropic enstrophy spectrum in terms of a forcing rate $\xi(k)$ and a damping rate $\eta(k)$. Anisotropy does not appear in (11). However, β enters the coefficient $I_1(\epsilon)$, namely

$$I_1(\epsilon) = 4\pi\epsilon/(\epsilon^2 + 2\epsilon)^{\frac{1}{2}}, \quad \text{where} \quad \epsilon(k,p,q) = (2k^2/\beta^2)\,(\mu(k,p,q))^2$$

when $k < p, q$. Equation (12) describes the evolution of the anisotropy in terms of a source term $S(k)$, a transfer-of-anisotropy term with kernel $K(k,p)$ and a return-to-isotropy coefficient $v(k)$. $S(k)$, $K(k,p)$ and $v(k)$ are given by integrals over the isotropic spectrum with β entering the coefficients I_1, I_2 and I_3.

Figure 1 shows the isotropic enstrophy spectra at $t = 6{\cdot}4$ in the three numerical experiments: $\beta = 0$, $\beta = 12{\cdot}5$ and $\beta = 25$. Also shown are calculated values of the TFM distortion rate $\mu(k)$ according to equation (A 5) for the particular choice $g = 0{\cdot}7$ of the TFM parameter. Subsequently we shall investigate other choices $g = 0{\cdot}6$ and

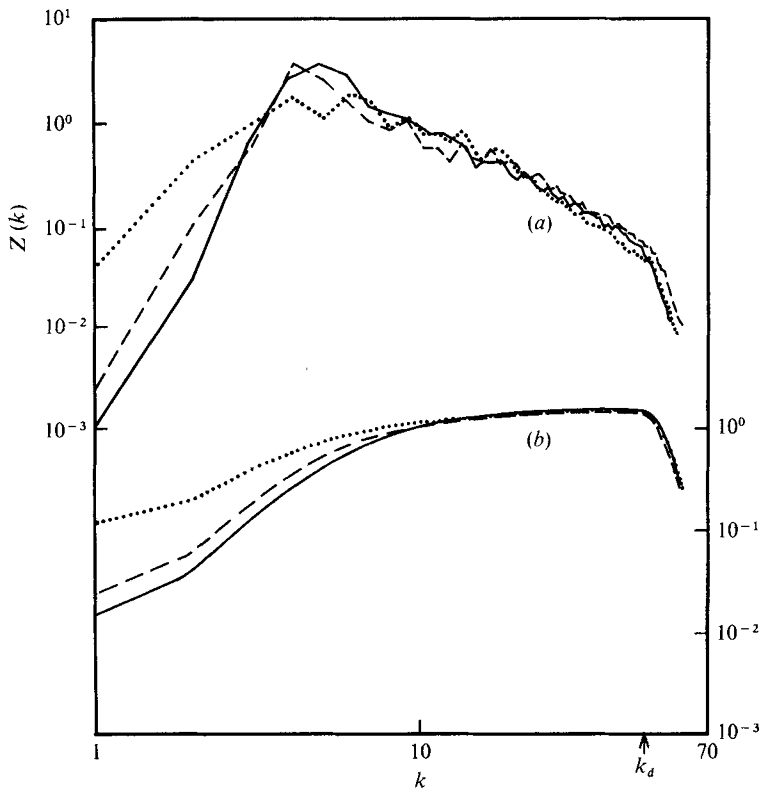

FIGURE 1. (*a*) Observed isotropic enstrophy spectra at $t = 6{\cdot}4$: ·······, $\beta = 0$; – – –, $\beta = 12{\cdot}5$; ——, $\beta = 25$. (*b*) Theoretical deformation rates $\mu(k)$ calculated from the observed enstrophy spectra; $g = 0{\cdot}7$.

$g = 1{\cdot}0$ to be compared with a previous estimate $g = 0{\cdot}65$ (Herring *et al.* 1974) obtained in a direct simulation of two-dimensional turbulence. Two immediate observations are that, with β present, less enstrophy is found in lower wavenumbers, and that the high wavenumber spectra ($k > 10$) are quite similar in the three experiments, falling roughly along a power law with slope between k^{-1} and k^{-2}. $Z(k) \propto k^{-1}$ would correspond to a Kolmogorov-type enstrophy-cascading subrange. Over $10 < k < k_d$ the theoretical $\mu(k)$ is rather flat with a value of the order of $\zeta_{\text{r.m.s.}}/2\pi \approx 1$, supporting our earlier definition $k_\beta = \beta/\zeta_{\text{r.m.s.}}$.

To isolate the role of β, we consider in figure 2 just the experiment with $\beta = 12{\cdot}5$. In figure 2(*a*) we reproduce the distortion rate $\mu(k)$ along with the forcing rate $\xi(k)$ and damping rate $\eta(k)$. $\xi(k) > \eta(k)$ at very low and at very high wavenumbers indicates the transfer of enstrophy into these wavenumbers while $\xi(k) < \eta(k)$ near $k \approx k_1 \approx 5$ indicates removal of energy from these wavenumbers. $\xi(k)$ and $\eta(k)$ very nearly cancel over $10 < k < k_d$. In figure 2(*b*) we recompute each of these rates for the same isotropic enstrophy spectrum, but now omitting β. The result is that all rates increase at all wavenumbers. However, the effect on spectral evolution is most evident for small wavenumbers $k \lesssim k_\beta$, where the difference $\xi(k) - \eta(k)$ approximately triples in value. Over $10 < k < k_d$, both $\xi(k)$ and $\eta(k)$ have increased but remain nearly in cancellation. Finally, and at first surprisingly, $\xi(k) - \eta(k)$ increases significantly in

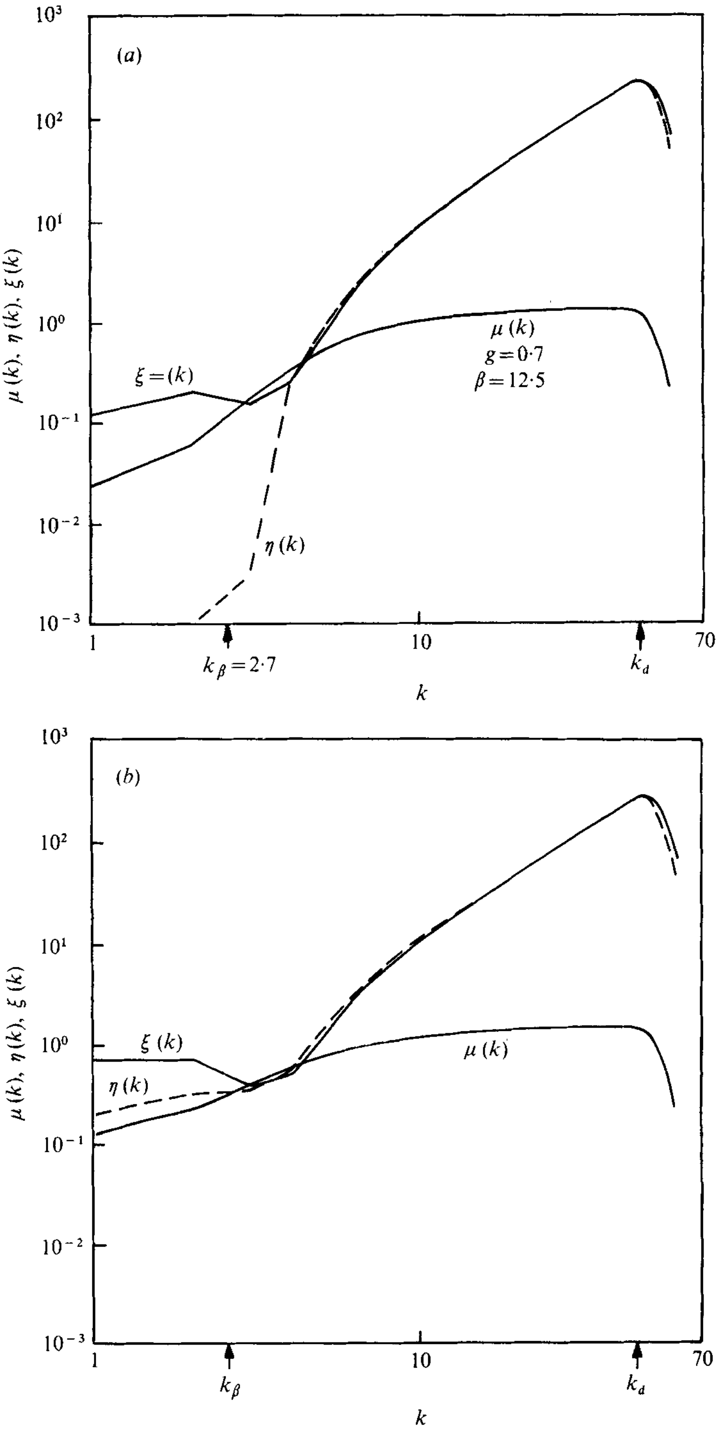

FIGURE 2. (a) For the case $\beta = 12{\cdot}5$, theoretical curves show the deformation rate $\mu(k)$, the damping rate $\eta(k)$ and the forcing rate $\xi(k)$. (b) The role of β in (a) is revealed by recomputing $\mu(k)$, $\eta(k)$ and $\xi(k)$ with β set to zero.

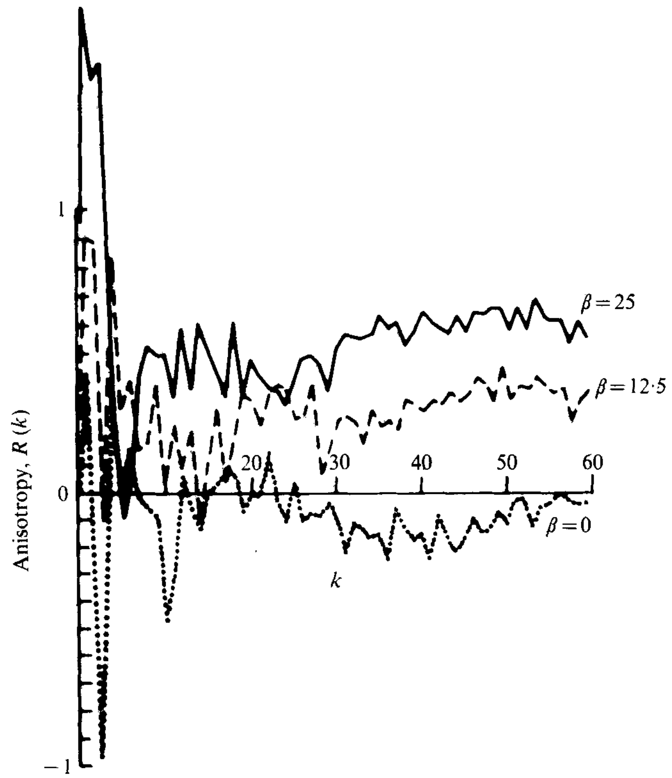

FIGURE 3. Observed anisotropy at $t = 6{\cdot}4$ for $\beta = 0$, $\beta = 12{\cdot}5$ and $\beta = 25$.

the dissipation range $k > k_d$ though Rossby-wave propagation is totally negligible on these scales. We return to the discussion of these results in the next section.

Figure 3 shows the anisotropy in the three experiments at $t = 6{\cdot}4$. With β present, strong zonal ($R(k) > 0$) anisotropy develops in $k \lesssim k_\beta$. Although the zonal tendency is less near k_1, the anisotropy increases slightly with k over $k > k_1$. When the flow has become quasi-stationary, we expect a balance among the source, transfer and return terms on the right-hand side of (12). Thus an equilibrium anisotropy profile is

$$R(k) = \frac{S(k)}{v(k)} + \frac{\int K(k,p)\,R(p)\,dp}{v(k)}.$$

This $R(k)$ is shown in figure 4 for $\beta = 12{\cdot}5$ and various choices of the TFM parameter g. Good agreement is found near $g = 0{\cdot}65$. Also shown are the contributions due to $S(k)/v(k)$ and $\int K(k,p)\,R(p)\,dp/v(k)$ in the case $g = 0{\cdot}7$. Poorer agreement is found in figure 5, where $\beta = 25$, but this is to be expected since both the low-order representation (10) and the linearizing approximations made in obtaining (12) will fail as $R(k)$ becomes a substantial fraction of unity. For larger values of β one must return to the use of (5), (8) and (9) directly.

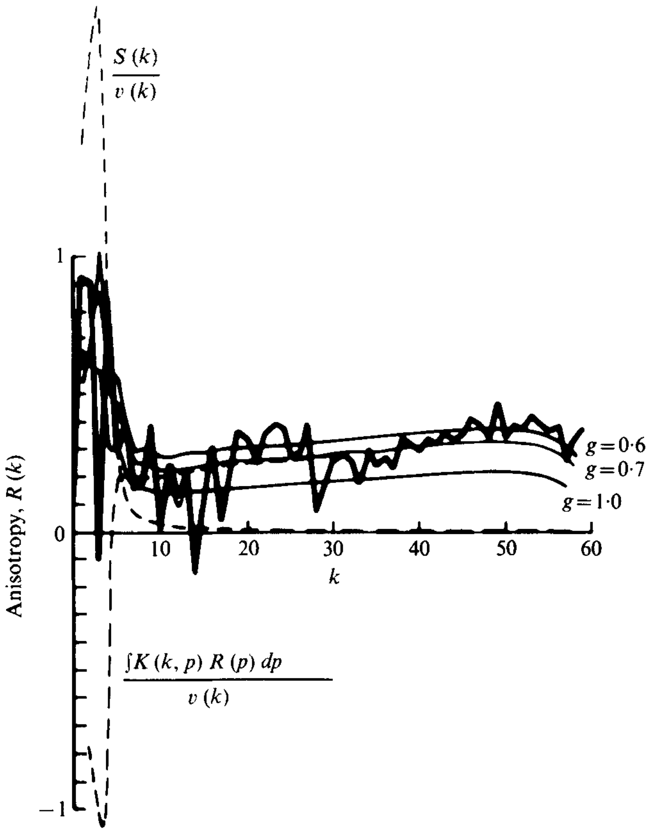

FIGURE 4. For the case $\beta = 12{\cdot}5$, theoretical curves for the equilibrium anisotropy are compared with the observed anisotropy for three choices of the TFM parameter g. Dashed curves show the roles of generation $S(k)$ and transfer $K(k, p)$ of anisotropy compared with a return-to-isotropy coefficient $v(k)$.

5. Discussion

A principal feature of this paper is that we avoid any fundamental distinction between wave-like and turbulent dynamics. The result provides a continuous description ranging from waves of arbitrarily small amplitude to fully developed turbulence, as given by (8) and (9), namely

$$\mu_{\mathbf{k}} = \nu_k + g^2 \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} \theta_{-\mathbf{k},\mathbf{p},\mathbf{q}} \hat{b}_{kpq} Z_{\mathbf{p}},$$

$$\theta_{-\mathbf{k},\mathbf{p},\mathbf{q}} = \mu_{\mathbf{k},\mathbf{p},\mathbf{q}}/(\mu^2_{\mathbf{k},\mathbf{p},\mathbf{q}} + \omega^2_{-\mathbf{k},\mathbf{p},\mathbf{q}}).$$

For vanishingly small amplitudes, and omitting dissipation, $\mu_{\mathbf{k}}$ becomes vanishingly small while $\theta_{-\mathbf{k},\mathbf{p},\mathbf{q}}$ isolates resonant wave triads:

$$\theta_{-\mathbf{k},\mathbf{p},\mathbf{q}} \rightarrow \pi\delta(\omega_{-\mathbf{k}} + \omega_{\mathbf{p}} + \omega_{\mathbf{q}}) \quad \text{as} \quad \mu^2/\omega^2 \rightarrow 0.$$

The appearance of a transition from turbulence-like to wave-like dynamics may seem to occur abruptly since $\mu_{\mathbf{k}}$, the measure of nonlinearity, increasingly draws its contributions only from nearly resonant wave triads. In the limit of weak wave interaction, dependence on the phenomenological constant g vanishes as one expects.

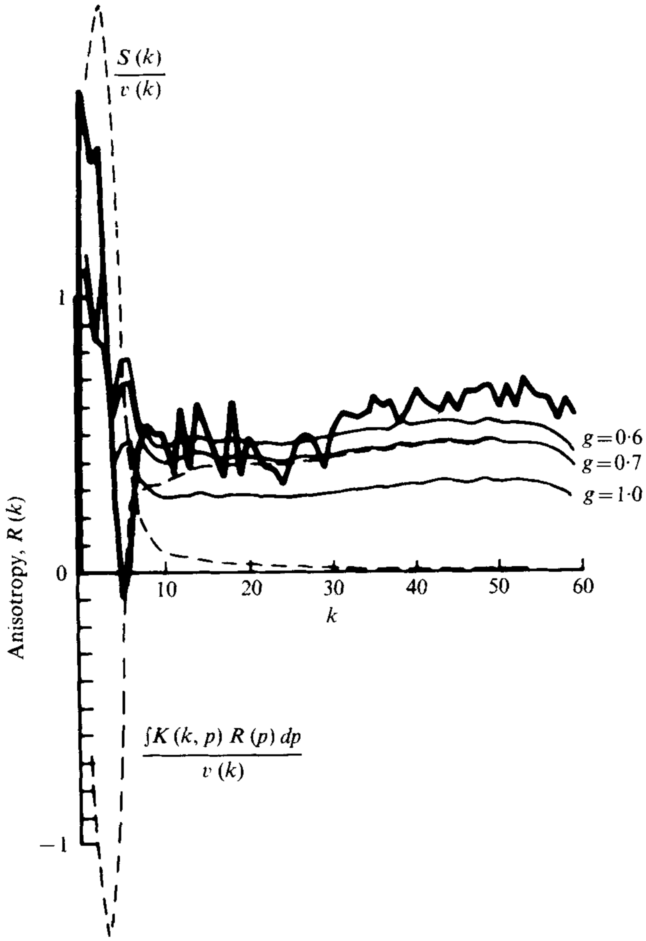

FIGURE 5. For the case $\beta = 25$, the theoretical equilibrium anisotropy is compared with the observed anisotropy. Here the approximations made in obtaining (12) begin to fail as $R \approx 1$.

When advection thoroughly dominates wave propagation,

$$\theta_{-\mathbf{k},\mathbf{p},\mathbf{q}} \rightarrow 1/(\mu_{\mathbf{k}}+\mu_{\mathbf{p}}+\mu_{\mathbf{q}}) \quad \text{as} \quad \mu^2/\omega^2 \rightarrow \infty$$

and we recover the 'test-field model' for two-dimensional turbulence. The fact that this is the TFM is non-essential. We could have introduced waves at (7), obtaining the same wave interaction limit but a less accurate turbulence limit.

We have examined the consequences of this closure for the case of barotropic Rossby waves, for which $\omega_{\mathbf{k}} = -\beta k_x/k^2$. A wavenumber $k_\beta = \beta/\zeta_{\text{r.m.s.}}$, where $\zeta_{\text{r.m.s.}}$ is the root-mean-square vorticity, may be thought of as a lower wavenumber in a transitional zone, roughly from k_β to $3k_\beta$, over which the character of the flow changes from more wave-like when $k < k_\beta$ to more turbulence-like when $k > 3k_\beta$. But neither regime is pure. If k_β is small compared with a characteristic energetic wavenumber k_1, then any departure from isotropic statistics will be expected to be small and we can simplify the theory by considering the flow field to consist of an isotropic part plus a lowest wave vector harmonic of anisotropy. The resulting approximations (11) and (12) are found to be in some agreement with numerical simulations and to recover, in part, the observations of Rhines (1975).

One effect of β is to suppress isotropic enstrophy, or energy, transfer into large scales of motion $k < k_\beta$. This effect is represented in the coefficient I_1 in (11). For $k \gg k_\beta$, $I_1 \rightarrow 4\pi$ as in turbulence, whereas, for $k \ll k_\beta$, I_1 vanishes as $I_1 \propto k$. In a subrange $k_\beta < k_1 \ll k \ll k_d$ the effect of β is slight. Asymptotically as $k_d/k_1 \rightarrow \infty$, one may expect approach to an enstrophy-cascade subrange $Z(k) \propto k^{-1}$. Kraichnan (1971*b*) argues though that, in a subrange as steep as k^{-1}, enstrophy transfer across very high wavenumbers is directly influenced by shearing on larger scales up to k_1, leading to a logarithmic correction $Z(k) \propto k^{-1}(\ln k/k_1)^{-\frac{1}{3}}$. Now if $k_1 \approx k_\beta$, wave propagation renders the large scales less effective in shearing the fine scales, suppressing somewhat the logarithmic correction. Although numerical simulations with $k_d/k_1 \approx 10$ cannot realize such subranges, we may observe in figure 1 that the spectra with β present do exhibit flatter shapes over $10 < k < k_d$, implying that β is inhibiting some enstrophy transfer across high wavenumbers. This is supported by comparing theoretical rates in figures 2(*a*) and (*b*), which show that, for a given isotropic spectrum, β suppresses transfer into the dissipation range $k > k_d$. We infer that Rhines' observation that β produces a steeper high wavenumber spectrum can be attributed to his use of a molecular form for the eddy viscosity, i.e. $\nu_k = \nu_0 k^2$, leading to a much broader dissipation range over which a steeper spectrum is expected.

Another important effect of β is to induce anisotropy. For sufficiently small departures from isotropy, this effect is described by (12) in terms of a source, or forcing, term $S(k)$, a transfer term $\int K(k,p)\, R(p)\, dp$ by which anisotropy in the wavenumber p induces anisotropy in the wavenumber k, and a relaxation, or return-to-isotropy, term $v(k)\, R(k)$. When these three effects are balanced, an equilibrium profile results as shown in figures 4 and 5. Dashed curves in these figures show the separate roles of the source term $S(k)$ and transfer term $\int K(k,p)\, R(p)\, dp$ scaled by $v(k)$, for the choice $g = 0{\cdot}7$. In fact, β acts as a source of anisotropy across the entire spectrum. However, $v(k) \approx 2\xi(k)$ increases rapidly with k, approximately as k^2 as seen in figure 2. Thus the direct effect of β results in large departures from isotropy in $k < k_\beta$ with negligible anisotropy over $k > k_\beta$.

Transfer of anisotropy alters this in three ways. First, small scales of motion are directly sheared by the large scales, producing substantial zonal anisotropy in high wavenumbers despite the return tendency. Second, actual relaxation of anisotropy is not measured only by the return-to-isotropy coefficient $v(k)$. For $k \approx p > k_1$, the positive kernel $K(k,p)$ tends to maintain anisotropy so that a continuous spectrum of anisotropy exhibits a weaker return tendency than one would suppose from $v(k)$ alone. A third effect is due to negative $K(k,p)$ in $k < p \approx k_1$. Indeed, it is a distinguishing feature of flow in two dimensions that the back-reaction from anisotropy in higher wavenumbers is to induce oppositely signed anisotropy (here meridional) in low wavenumbers. As the figures show, this meridional tendency acts effectively to limit the growth of zonal anisotropy in $k < k_\beta$.

A few further comments should be made. Our use of the Rossby-wave dispersion relation was really incidental. $\omega_{\mathbf{k}}$ may be chosen arbitrarily so long as $\omega_{-\mathbf{k}} = -\omega_{\mathbf{k}}$, suggesting that the formalism provides a more general closure for nonlinear waves. An objection may be that Rossby waves are unusual since they derive from a first-order wave equation, i.e. allow only westward phase propagation. More commonly we should encounter a second-order wave equation

$$\left(\frac{d^2}{dt^2}+\omega_{\mathbf{k}}^2\right)\phi_{\mathbf{k}} = \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} n_{-\mathbf{k},\mathbf{p},\mathbf{q}}\,\phi_{\mathbf{p}}\phi_{\mathbf{q}}.$$

However, this factorizes into a pair of first-order equations in the variables

$$\phi_{\mathbf{k}}^{+} = (d/dt-i\omega_{\mathbf{k}})\,\phi_{\mathbf{k}}$$

and

$$\phi_{\mathbf{k}}^{-} = (d/dt+i\omega_{\mathbf{k}})\,\phi_{\mathbf{k}},$$

whence $\phi_{\mathbf{k}} = (i/2\omega_{\mathbf{k}})(\phi_{\mathbf{k}}^{+}-\phi_{\mathbf{k}}^{-})$. Closure consists of coupled master equations for three species of second moments: $\langle\phi_{\mathbf{k}}^{+}\phi_{-\mathbf{k}}^{-}\rangle$, $\langle\phi_{\mathbf{k}}^{-}\phi_{-\mathbf{k}}^{+}\rangle$ and $\mathrm{Re}\,\langle\phi_{\mathbf{k}}^{+}\phi_{-\mathbf{k}}^{+}\rangle$. These master equations, in turn, correspond to components in the closure of a Langevin equation for a vector random variable $(\tilde{\phi}_{\mathbf{k}}^{+},\tilde{\phi}_{\mathbf{k}}^{-})$.

Another point is that, if we strictly omit dissipation ν_k in the truncated set of equations (2), then methods of classical statistical mechanics may be applied to predict evolution to an inviscid equilibrium state

$$Z_{\mathbf{k}} = 1/(\gamma_1+\gamma_2 k^2),$$

which is Kraichnan's (1967) equipartition solution for inviscid, spectrally truncated, two-dimensional turbulence. It may be verified that this is the stable, stationary solution of (5), omitting dissipation. γ_1 and γ_2 are determined from the average energy and enstrophy density and the truncation wavenumber. We note especially that the inviscid equilibrium solution is independent of β, so that the effects of β can be associated only with disequilibrium processes, i.e. the response to the disequilibrium initial state and the role of dissipation. Thus, for example, omitting dissipation in (2), we should expect much the same early departure from isotropy but, after some very long time, a complete return to isotropy. Indeed, substitution of the equilibrium isotropic spectrum $Z(k) = 2\pi k/(\gamma_1+\gamma_2 k^2)$ into (12) leads to $R(k) = 0$.

One can imagine extending closure theories to include such effects as underlying irregular topography or the resolution of vertical degrees of freedom (baroclinic modes). Beginnings of such theories may be found in Herring (1977), Holloway (1977) and Salmon (1977). For the case of unforced, inviscid, spectrally truncated equations of flow in two layers with β, irregular bottom topography and lateral boundaries, absolute-equilibrium solutions are given by Salmon, Holloway & Hendershott (1976). A rather surprising feature of these absolute-equilibrium solutions is their physically realistic appearance despite quite unrealistic dynamics, i.e. no average energy transfer among modes. Equilbrium calculations show, for example, the prevalence of steady anticyclonic circulation around hills and the prevalence, in two-layer flow, of barotropic motion on scales larger than the internal Rossby radius of deformation. The fact that such features persist qualitatively in realistic flows, i.e. far from equilibrium, indicates the statistical-mechanical tendency towards entropy maximization. Compared with these equilibrium calculations, turbulence theoretical closures can be characterized as theories of the disequilibrium statistical mechanics of quasi-geostrophic motions as, in the present case, for Rossby waves.

Though the results of closures for more complicated situations are not yet available, we may still note some of the more immediate connexions with the present work.

Thus, in barotropic flow, β is equivalent to a uniform bottom slope. The tendency towards zonal anisotropy on the β-plane can be identified as a tendency towards flow along contours of f/H, where f is the rotation rate and H the depth of fluid. However, for topographic elements of finite extent, an important difference arises: on the unbounded β-plane anisotropy is due solely to disequilibrium processes. Over topography, these disequilibrium processes enhancing contour flow of arbitrary sign act in the presence of an equilibrium tendency towards contour flow of a definite sign, i.e. anticyclonic around hills. Another effect of topography is to prevent the transfer of anisotropy from the large to fine scales of motion, plausibly by providing a negative contribution to the transfer kernel $K(k,p)$ and so producing a strong isotropizing effect on small scales. In saying this we assume no mean zonal flow, hence no resonant forcing of topographic Rossby waves. Finally, if we consider flow in two layers, the effect of β is to propagate both faster barotropic and slower baroclinic Rossby waves. It was noted that absolute equilibrium is characterized by barotropic motion on large scales with uncorrelated (mixed barotropic and baroclinic) motion on small scales. Extension of the present barotropic theory then suggests that disequilibrium flow in two layers will develop an anisotropic correlation between, say, the stream functions in the two layers. For flow initially isotropic and uncorrelated between layers, we expect zonal flow on scales larger than the internal deformation radius to become well correlated (barotropic) while the larger-scale meridional flow remains uncorrelated (mixed barotropic and baroclinic). These concluding remarks are only qualitative. In quantitative predictions and comparisons, we can expect to assess both the validity of Markovian quasi-normal kinds of closures as well as their usefulness in realistic geophysical fluid dynamics.

This study was prompted by the previous investigations of Dr P. B. Rhines. Much of this development was substantially clarified in conversations with Dr J. R. Herring and Dr U. Frisch. Computations were performed at the National Center for Atmospheric Research, which is sponsored by the National Science Foundation. This work was supported under Grant NSF-ID074-23117.

Appendix. Low-order representation of anisotropy

We have supposed that for sufficiently small β, in the sense that k_β is small compared with k_1, a departure from isotropy will remain small, so that we may adopt a low-order representation

$$\frac{2\pi k}{(\Delta k)^2} Z_{\mathbf{k}} = Z(k)\,(1 - R(k)\cos 2\phi_{\mathbf{k}}), \tag{A 1}$$

where $\Delta k = 1$ is the discrete wave-vector separation. The consistency of (A 1) depends upon a calculation showing that $R(k)$ is in fact small compared with unity. We substitute (A 1) into (5) and (8), approximating summation over $\mathbf{p}$ by integration

$$\int_1^{k_{\max}} \int_0^{2\pi} p\,dp\,d\phi_{\mathbf{p}}.$$

Subsequently we separate $Z(k)$ and $R(k)$ by integration

$$\int_0^{2\pi} d\phi_{\mathbf{k}}.$$

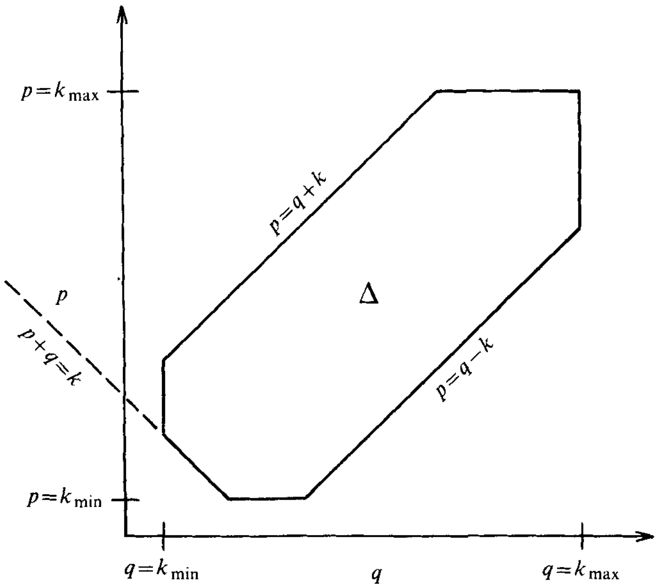

FIGURE 6. The domain of integration Δ.

First it is convenient to change the integrals around a bit. Let x, y and z be the interior angles of a triangle opposite sides k, p and q. Integration is rewritten as

$$\iint_{\Delta} \frac{\partial(\tilde{p}, z)}{\partial(p, q)} dp\, dq, \quad \frac{\partial(\tilde{p}, z)}{\partial(p, q)} = \frac{1}{p \sin x},$$

where the domain of integration Δ, in which k, p and q can form the sides of a triangle, is shown in figure 6. Integration over Δ is considered to be performed twice, once for each of the following choices of triad:

$$\phi_{\mathbf{p}} = \phi_{\mathbf{k}} \pm z, \quad \phi_{\mathbf{q}} = \phi_{\mathbf{k}} \mp y. \tag{A 2}$$

In isotropic turbulence these choices do not matter and a single integration over Δ may be doubled.

Equations (5) and (8) become

$$\left(\frac{d}{dt} + 2\nu_k\right) [Z(k)\,(1 - R(k) \cos 2\phi_{\mathbf{k}})] = \frac{k(\Delta k)^2}{2\pi} \iint_{\Delta} \frac{dp\,dq}{\sin x} \theta_{-\mathbf{k},\mathbf{p},\mathbf{q}} \times \left[a_{kpq} \frac{Z(p)\,Z(q)}{pq} (1 - R(p) \cos 2\phi_{\mathbf{p}})\,(1 - R(q) \cos 2\phi_{\mathbf{q}}) - 2b_{kpq} \frac{Z(p)\,Z(k)}{pk} (1 - R(p) \cos 2\phi_{\mathbf{p}})\,(1 - R(k) \cos 2\phi_{\mathbf{k}})\right] \tag{A 3}$$

and

$$\mu_{\mathbf{k}} = \frac{g^2}{2} \frac{(\Delta k)^2}{2\pi} \iint_{\Delta} \frac{dp\,dq}{\sin x} \theta_{-\mathbf{k},\mathbf{p},\mathbf{q}} \hat{b}_{kpq} \frac{Z(p)}{p} (1 - R(p) \cos 2\phi_{\mathbf{p}}). \tag{A 4}$$

The separation of (A 3) into isotropic and anisotropic parts remains complicated by the $\phi_{\mathbf{k}}$ dependence of $\theta_{-\mathbf{k},\mathbf{p},\mathbf{q}}$, which is due primarily to the $\phi_{\mathbf{k}}$ dependence of $\omega^2_{-\mathbf{k},\mathbf{p},\mathbf{q}}$. The $\phi_{\mathbf{k}}$ dependence of $\mu_{\mathbf{k}}$ is weaker, being derived from $\theta_{-\mathbf{k},\mathbf{p},\mathbf{q}}$ and $R(p)$ in (A 4).

Thus on substitution from (A 4) into (A 3) we may consistently retain only an isotropic estimate of $\mu_{\mathbf{k}}$, i.e.

$$\mu(k) = \frac{g^2}{2}\frac{(\Delta k)^2}{2\pi}\int\int_{\Delta}\frac{dp\,dq}{\sin x}\left(\frac{1}{2\pi}\int_0^{2\pi}d\phi_{\mathbf{k}}\,\theta_{-\mathbf{k},\mathbf{p},\mathbf{q}}\right)\hat{b}_{kpq}\frac{Z(p)}{p}. \qquad \text{(A 5)}$$

Finally, the smallness of k_β compared with k_1 means that triads all of whose components lie in $k \lesssim k_\beta$ will be relatively ineffective and we shall approximate the $\phi_{\mathbf{k}}$ dependence of $\omega^2_{-\mathbf{k},\mathbf{p},\mathbf{q}}$ by the $\phi_{\mathbf{k}}$ dependence in that mode $\mathbf{k}$, $\mathbf{p}$ or $\mathbf{q}$ which has the least modulus k, p or q. Call this mode l. Then we take for $\theta_{-\mathbf{k},\mathbf{p},\mathbf{q}}$

$$\theta_{-\mathbf{k},\mathbf{p},\mathbf{q}} = \frac{\mu(k)+\mu(p)+\mu(q)}{(\mu(k)+\mu(p)+\mu(q))^2+(\beta^2/2l^2)\,(1+\cos 2\phi_{\mathbf{k}}\cos 2\alpha \pm \sin 2\phi_{\mathbf{k}}\sin 2\alpha)},$$

where α is the interior angle opposite l when $\mathbf{l} \neq \mathbf{k}$ and $\alpha = 0$ when $\mathbf{l} = \mathbf{k}$. Since this $\theta_{-\mathbf{k},\mathbf{p},\mathbf{q}}$ remains symmetric in its indices, we retain the property that nonlinear interactions preserve energy and enstrophy.

Integrations over $\phi_{\mathbf{k}}$ are of three kinds:

$$I_1(\epsilon) = \int_0^{2\pi}\frac{d\sigma}{1+(1+\cos\sigma\cos 2\alpha \pm \sin\sigma\sin 2\alpha)/\epsilon} = \frac{4\pi\epsilon}{(\epsilon^2+2\epsilon)^{\frac{1}{2}}},$$

$$I_2(\epsilon,\alpha) = \int_0^{2\pi}\frac{\cos\sigma\,d\sigma}{1+(1+\cos\sigma\cos 2\alpha \pm \sin\sigma\sin 2\alpha)/\epsilon}$$
$$= 4\pi\epsilon\cos 2\alpha\left(1-\frac{\epsilon+1}{(\epsilon^2+2\epsilon)^{\frac{1}{2}}}\right),$$

$$I_3(\epsilon,\alpha) = \int_0^{2\pi}\frac{\cos^2\sigma\,d\sigma}{1+(1+\cos\sigma\cos 2\alpha \pm \sin\sigma\sin 2\alpha)/\epsilon}$$
$$= 2\pi\epsilon\left[\frac{(2\epsilon^2+4\epsilon+1)\cos 4\alpha+1}{(\epsilon^2+2\epsilon)^{\frac{1}{2}}} - 2(\epsilon+1)\cos 4\alpha\right],$$

where $\epsilon(k,p,q) = (\mu(k)+\mu(p)+\mu(q))^2/(\beta^2/2l^2)$ and the integral is the sum over the sign choices $\pm$. With these expressions, (A 3) yields (11) and (12), where terms of order R^2 or order $\beta^2 R$ are dropped. Further discussion and interpretation of this derivation, including calculation up to order $\beta^2 R$, are given by Holloway (1976).

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