→ Blasius
→ Prandtl
→ Ekman

Lecture VIII   Laminar Boundary Layers

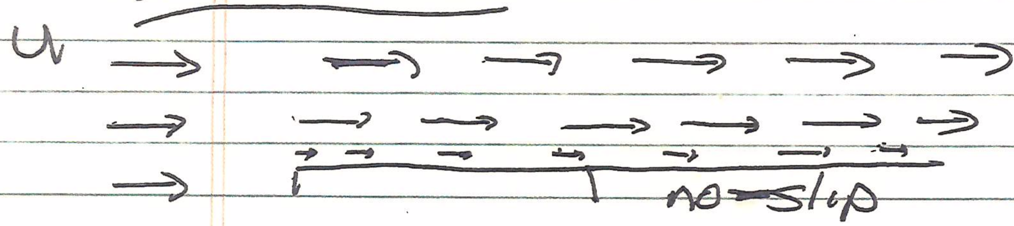
{ Acheson
Landau/Lifshitz }

viscous fluid

(1)

- Consider a flat plate immersed in flow

Blasius B.L.

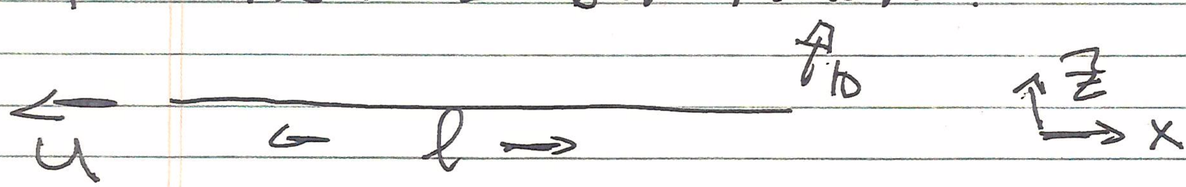


$Re > 1$ but not turbulent

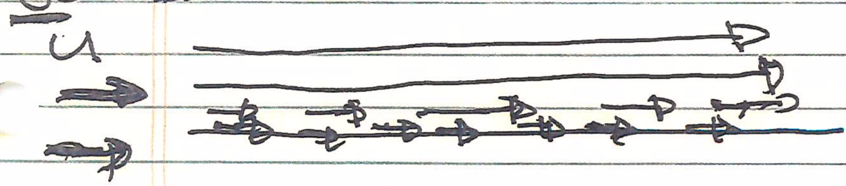
~ away from plate, flow is as imposed

~ near plate, no-slip B.C. $V_x(0) = 0$ forces boundary layer which connects:
- near plate → viscous
- mean flow U (i.e. potential)

For practical question: calculate drag on plate of width b, length ℓ for $Re > 1$ but laminar.



so =



then as before:

$$\underline{V}_T = U\hat{x} + \underline{V}$$

$$\partial_t \underline{V} + \underline{V}\cdot\underline{\nabla}\,\underline{V} - \nu \nabla^2 \underline{V} = -\frac{\underline{\nabla} p}{\rho}$$

$$U\,\partial_x \underline{V} - \nu(\partial_x^2 + \partial_z^2)\underline{V} = -\frac{\underline{\nabla} p}{\rho}$$

$$\partial_x^2 \sim \frac{1}{\ell_x^2} \sim \frac{1}{\ell^2} \qquad \text{(anisotropy)}$$

$$\partial_z^2 \sim \frac{1}{w^2} \qquad w^2 \ll \ell^2$$

so, noting P irrelevant,

$$U\,\partial_x V_x - \nu\,\partial_z^2 V_x \simeq 0$$

$$\Rightarrow \quad \boxed{w \sim (\nu x / U)^{1/2}} \quad \rightarrow \text{thickness B.L. at location } x \text{ in B.L. (and wake)}$$

width thickens downstream.

Now, for drag

$$F_d = -\eta b \int_0^{\ell} \frac{\partial V_x}{\partial z}\, dx$$

$$\simeq -\eta b \int_0^{\ell} \frac{\overbrace{\Delta V_x}^{\sim U}}{w(x)}\, dx$$

$$\therefore \quad F_d \sim -\eta b U \int_0^{\ell} \frac{1}{\left(\frac{\nu x}{U}\right)^{1/2}}\, dx$$

$$= -\mu b \frac{U^{3/2}}{\nu^{1/2}} \ell^{1/2}$$

$$= -\rho \nu^{1/2} U^{3/2} b \ell^{1/2}$$

$$= -\rho \frac{U^2 \ell b}{(Re)^{1/2}} \qquad Re \sim \frac{U \ell}{\nu}$$

so

$$-\boxed{\begin{aligned} F_d &\sim \rho U^2 A / \sqrt{Re} \\ &\sim \rho \nu^{1/2} U^{3/2} b \underline{\ell}^{1/2} \end{aligned}}$$

compare Stokes:

$$F_d \sim 6\pi \eta \underline{\ell} U$$

⇒ Blasius B.L. thickens with length along plate

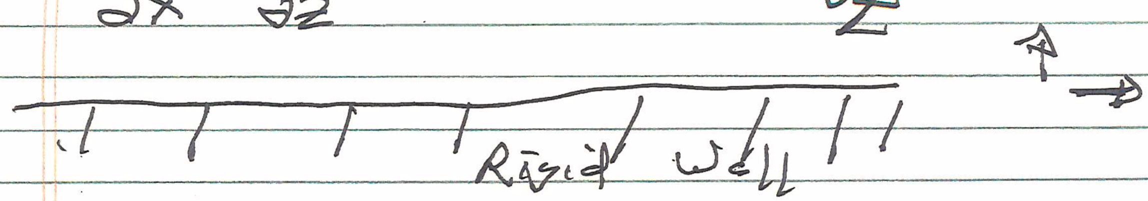
→ Now, Prandtl Boundary Layer Analysis

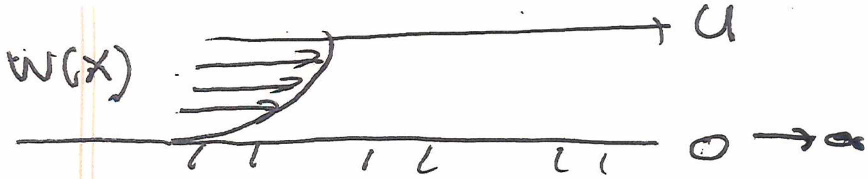
Similarity soln. of NL problem

$$u \partial_x u + v \partial_z u = -\frac{1}{\rho}\frac{dp}{dx} + \nu \frac{\partial^2 u}{\partial z^2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial z} = 0$$

↑ v → u





Key ideas:

→ variation in z more rapid than in x (like wake)

→ variation of U with z enough so ν non-negligible.

Now, need require:

$u \to U_0(x)$ as $z/w \to \infty$

so, for plate, $\left(p + \frac{1}{2}U^2\rho \sim \text{const} \right.$ at edge BL.$\left.\right)$

$U \to$ const. here

$p \to$ const.

$u\partial_x u + v\partial_z u = \nu \frac{\partial^2 u}{\partial z^2}$

NL

$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial z} = 0$

u, v not f'tun

→ z dependence is strong fct

know w(x) → scale

$(\nu x/U)^{1/2}$

$x, z \to \eta = \frac{z}{w(x)}$

Seek similarity solution

$u = u(\eta)$, $\eta = z/g(x)$

i.e. - reduce pde to ode

- guess $g(x) \to w = (\nu x/U_0)^{1/2}$

can show.

stretched coord.

2D, $\underline{\nabla} \cdot \underline{V} = 0$

$\underline{V} = \underline{\nabla}\phi \times \vec{y}$ stream function ψ

$\Rightarrow$ $\boxed{u = \partial\psi/\partial z \,, \quad v = -\partial\psi/\partial x}$

$u = U h(\eta)$

some fcty

then $\boxed{\dfrac{\partial\psi}{\partial z} = U h(\eta)}$

so

$$\psi = U g(x) \int_0^{\eta} h(s)\, ds + k(x)$$

want $\psi = 0$ at $\eta = 0$
(plate a streamline)

$\Rightarrow k(x) = 0$.

re-write:

$$\boxed{\psi = U g(x) f(\eta)} \,, \qquad f(\eta) = 0.$$

$\boxed{' \Rightarrow df/d\eta}$

so

$u = U g f'(\eta)$

$v = -U\left(g'(x) f + g f'(\eta) \dfrac{\partial\eta}{\partial x}\right)$

So

$$v = -\frac{\partial \psi}{\partial x} = U\left(\eta f'(\eta) - f\right) g'(x)$$

so now plug into basic flow eqn

$$u\,\partial_x u + v\,\partial_z u = \nu\,\partial_z^2 u$$

where

$g' = \frac{dg}{dx}$

$f' = \frac{df}{d\eta}$

(1)
$$u = U g f'$$

(2) (1)
$$\partial_x u = U g' f' + U g f''\left(\frac{-z}{g^2}\right) g'$$

(3)
$$v = U(\eta f' - f) g'$$

(4)
$$\partial_z v = U\left(\cancel{\frac{1}{g} f'} + \eta f'' \frac{1}{g} - \cancel{\frac{f'}{g}}\right) g'$$

$\nu\,\partial_z^2 u =$

similar crank.

(5)
$$= \nu U g\left(\frac{f'''}{g^2}\right)$$

assembling: (1)(2)

(1),(2) (3),(4)

$$\boxed{-U^2 f' f'' \frac{z g'}{g^2} + U^2(\eta f' - f) g' \frac{f''}{g} \overset{(5)}{=} \nu U \frac{f'''}{g^2}}$$

$$\eta = z/g$$

$$-u^2 \cancel{f'} f'' \frac{zg'}{g^2} + u^2 (\eta f' - f) gg' f''/g^2 = (\nu f'''/g^2) u$$

$$-ug' ff'' z + u \cancel{z \frac{ff''}{g}} gg' - u ff'' gg' = \nu f'''$$

$$[-ug' \cancel{ff''} z + uz \cancel{ff'' g'} - uff f'' gg' = \nu f''']$$

Finally:

$$\boxed{f''' + \frac{ugg'}{\nu} ff'' = 0}$$

here: $f = f(\eta)$

$$\eta = z/g(x)$$

<u>Now, idea of similarity solution is to reduce pde to ode in η.</u>

So, as g to be specified:

$$gg' = \nu/u \qquad g = g(x)$$

$$\left(\frac{g^2}{2}\right)' = \nu/u$$

$$g^2 = \frac{2\nu x}{u} + \cancel{c}$$

thickness must vanish at leading edge.

so $g(x) = \left(\frac{2\nu x}{U}\right)^{1/2}$

no surprise!

$$\Rightarrow \boxed{\begin{aligned} \psi &= (2\nu U x)^{1/2} f(\eta) \\ \eta &= z \Big/ \left(\frac{2\nu x}{U}\right)^{1/2} \end{aligned}} \longrightarrow \text{NL solution}$$

and $\boxed{f''' + f f'' = 0}$ requires numerics

with b.c.'s:

$f'(\infty) = 1$ $\qquad u \to U$ at $z \to \infty$

$f(0) = f'(0) = 0$ $\quad \to$ no slip

and

$$\mu \frac{\partial u}{\partial z} = \mu U \left(\frac{U}{2\nu x}\right)^{1/2} f''(0)$$

N.B. $\partial_x U + \partial_z V = 0$

$$\frac{U}{\ell} \sim \frac{V}{\left(\frac{\nu \ell}{U}\right)^{1/2}} \qquad \underbrace{\frac{V}{U} \sim 1/Re}_{V \text{ small.}}$$

~~and, define drag coefficient C_d via:~~ ~~defn.~~

$\cancel{C_d = \left[\frac{F}{L}\right] \Big/ \frac{1}{2}\rho U^2 \, 2\ell}$ 2D defn.

~~→ for 2D flow~~

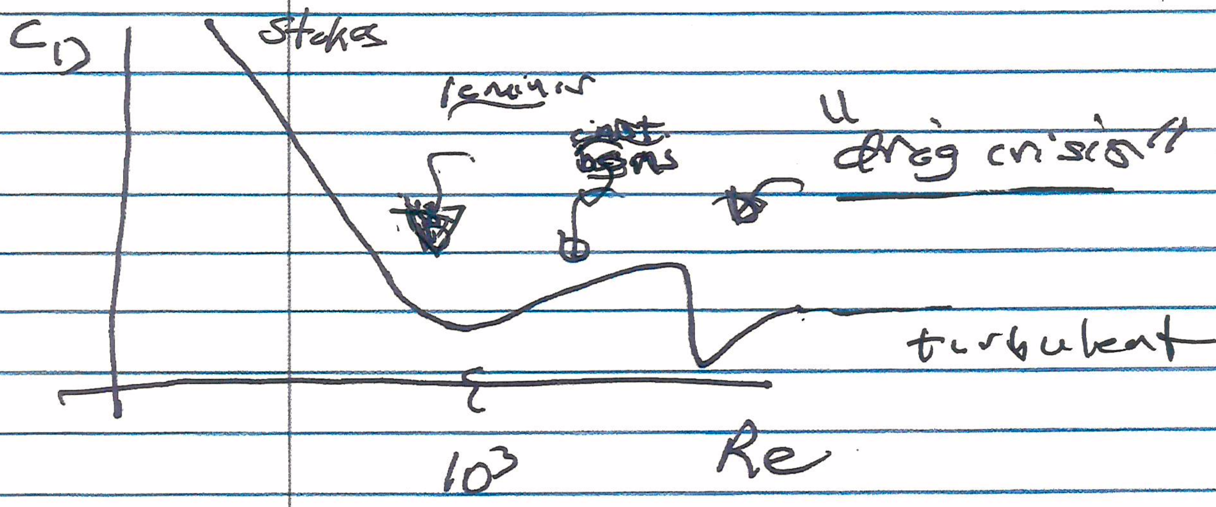
~~~ # / $\sqrt{Re}$~~

in general

$$C_d = F \Big/ \tfrac{1}{2}\rho U^2 S$$

surface area.

Comments on Drag:



~ $Re > 1$ ⇒ laminar BL

~ $Re > 10^3$ ⇒ instability, separation begin, reach body.
C_D rises
~~~

8b.

prior

→ turbulence onset ⇒ C_D drops

'drag crisis'

→ sep. pts kicked back. ~ B.L. energized

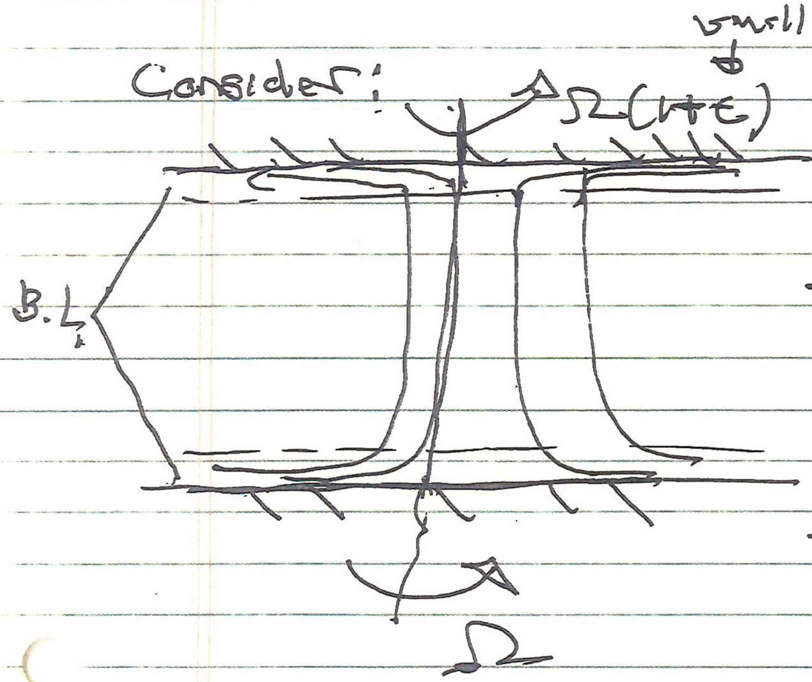
B.L. flow with turb. can beat viscous losses.

why golf balls have dimples

~ C_D indep. Re ?? ⇒ turbulence physics

(b) Ekman Layer – Rotating Fluid

Consider:



– two rotating boundaries, slight difference

– "suction" → up !

Now, recall for rotating fluid:

$$\frac{\partial \underline{v}}{\partial t} + 2\underline{\Omega} \times \underline{v} = -\frac{\underline{\nabla} P^*}{\rho} + \nu \nabla^2 \underline{v}$$

absorbs centrifugal

$$\underline{\nabla} \cdot \underline{v} = 0$$

For steady flow, $\nu = 0$

I = inviscid

$$\left\{\begin{matrix} u \\ v \\ w \end{matrix}\right\}$$

uniform $(0, 0, \Omega)$

$$-2\Omega v_I = -\frac{1}{\rho}\frac{\partial P_I}{\partial x}$$

$$2\Omega u_I = -\frac{1}{\rho}\frac{\partial P_I}{\partial y}$$

$$0 = -\frac{1}{\rho}\frac{\partial P_I}{\partial z}$$

$$\partial_x u_I + \partial_y v_I + \partial_z w_I = 0$$

$\Rightarrow$

p_I indep z, i.e. $p_I = p(x, y)$

so u_I, v_I indep z

and using u_I, v_I in $\nabla \cdot \underline{v} = 0$

$\Rightarrow$

$$\frac{\partial w_I}{\partial z} = 0 \quad \checkmark$$

so $\underline{v}$ indep z $\rightarrow$ T-P Thm!

$\rightsquigarrow$ Consider B. L. at $z = 0$:

Now viscous effects sensitive to z $\Rightarrow$

i.e. Boundary Layer depends z, so

for $z \sim 0$ BL:

$$-2\Omega v = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial z^2}$$

$$2\Omega u = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu \frac{\partial^2 v}{\partial z^2}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu \frac{\partial^2 w}{\partial z^2}$$

$$\partial_x u + \partial_y v + \partial_z w = 0$$

Now $\frac{\partial w}{\partial z} + \underline{\nabla}_\perp \cdot \underline{V}_\perp = 0$

so

$|w| \ll |\underline{U}_\perp|$ as $\Delta z \ll r_{tank}$.

$\Rightarrow |\nabla_\perp P| \gg |\nabla_z P|$

$\Rightarrow P \cong P(x, y)$, only

view flow eqns as for P

so

$$\frac{\partial P}{\partial x} \cong \frac{\partial P_I}{\partial x} \Rightarrow -\frac{1}{\rho}\frac{\partial P}{\partial x} \cong -2\Omega V_I$$

$$\frac{\partial P}{\partial y} \cong \frac{\partial P_I}{\partial y} \Rightarrow -\frac{1}{\rho}\frac{\partial P}{\partial y} \cong +2\Omega U_I$$

so re-write

$$-2\Omega(V - V_I) = \nu \frac{\partial^2 U}{\partial z^2} \qquad (1)$$

$$+2\Omega(U - U_I) = \nu \frac{\partial^2 V}{\partial z^2} \qquad (2)$$

(2) * i and add

$$\nu \frac{\partial^2 f}{\partial z^2} = \Big[\big(U - U_I(x,y)\big) + i\big(V - V_I(x,y)\big)\Big] 2\Omega i$$
$$= f(2\Omega i)$$

$f = (U - U_I) + i(V - V_I)$ $\quad U_I, V_I$ no z dependence!

so
$$\nu \frac{\partial^2 f}{\partial z^2} = 2\Omega i f$$

$\Rightarrow$
$$f = A e^{-(1+i) z_*} + B e^{(1+i) z_*}$$

$$z_* = z \Big/ \left(\frac{\nu}{\Omega}\right)^{1/2}$$

Ekman B.L
has thickness
$d \sim (\nu/\Omega)^{1/2}$

Now, to match:

$f \to 0$ as $z_* \to \infty$
$\begin{pmatrix} u \to u_I \\ v \to v_I \end{pmatrix}$

$\Rightarrow B = 0$ unphysical

And since $u = v = 0$ at $z = 0$
(boundary at rest in rotating frame) $\Rightarrow$ no slip

$$f = -(u_I + i v_I) e^{-(1+i) z_*}$$

$\Rightarrow$ finally,

$$\boxed{\begin{aligned} u &= u_I - e^{-z_*}\left(u_I \cos z_* + v_I \sin z_*\right) \\ v &= v_I - e^{-z_*}\left(v_I \cos z_* - u_I \sin z_*\right) \end{aligned}}$$

Now, further can note:

$$\frac{\partial w}{\partial z} = \left(\frac{\Omega}{\nu}\right)^{1/2} \frac{\partial w}{\partial z_*} = -\left(\partial_x u + \partial_y v\right)$$

upon plug in

$u_I = \nabla p \times \hat{z}$

$$= \left(\frac{\partial v_I}{\partial x} - \frac{\partial u_I}{\partial y}\right) e^{-z_*} \sin z_* - \cancel{\left(\frac{\partial u_I}{\partial x} + \frac{\partial v_I}{\partial y}\right)\left(1 + e^{-z_*} \cos z_*\right)}$$

$$w_E(x,y) = \int_0^\infty dz_* \left(\frac{\Omega}{\nu}\right)^{1/2} \frac{\partial w}{\partial z_*}$$

Ekman velocity (w_E)

$$= \frac{1}{2}\left(\frac{\nu}{\Omega}\right)^{1/2} \left(\frac{\partial v_I}{\partial x} - \frac{\partial u_I}{\partial y}\right)$$

$$\boxed{w_E(x,y) = \frac{1}{2} \frac{\nu}{\Omega} \omega_I}$$

[z component vorticity]

Now easy to show from B.C. that if lower boundary rotates at Ω_B relative to rotating frame:

$$w_E(x,y) = (\nu/\Omega)^{1/2} \left(\frac{1}{2}\omega_I - \Omega_B\right)$$

Similarly, for top at Ω_T:

$$w_E(x,y) = (\nu/\Omega)^{1/2}\left(\Omega_T - \tfrac{1}{2}\,\omega_I\right)$$

Now, for rapid rotation T-P Thm. says vertical uniformity $\Rightarrow$

$$w_{E_B} = w_{E_T}$$

$$\boxed{\omega_I/2 - \Omega_B = \Omega_T - \tfrac{1}{2}\,\omega_I}$$ global match

so $\boxed{\omega_I = \Omega_B + \Omega_T}$

so $\Omega_B = 0$

$\Omega_T = \epsilon\Omega$

$\therefore$ $\boxed{\omega_I = \epsilon\Omega}$

and trivially,

$$1/r\,\frac{d}{dr}\left(r u_{\theta_I}\right) = \epsilon\Omega$$

$$\boxed{u_{\theta_I} = \tfrac{1}{2}\,\epsilon\Omega\, r} \simeq \begin{cases}\text{avg. top} \\ \text{bttm rot freq.}\end{cases}$$

and $w_E \Rightarrow$ $\boxed{\tfrac{1}{2}(\nu\Omega)^{1/2}\epsilon = W_{Z_I}}$

no radial flow.