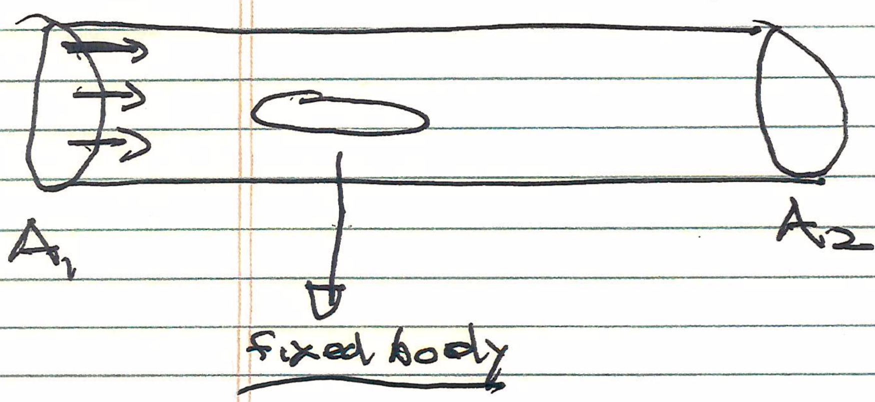
Lecture VII Wakes: Stability and Structure

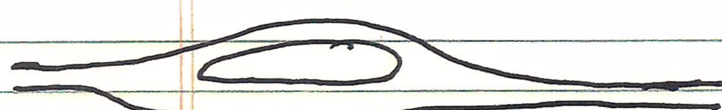
→ Recall D'Alembert's Paradox



Drag Force = Difference in Momentum Flux thru ends

$$F_d = \int_{A_1} da_1 \left(P_1 + \rho V_1^2\right) - \int_{A_2} da_2 \left(P_2 + \rho V_2^2\right)$$

– For [redacted] ideal fluid, upstream/downstream symmetry of flow



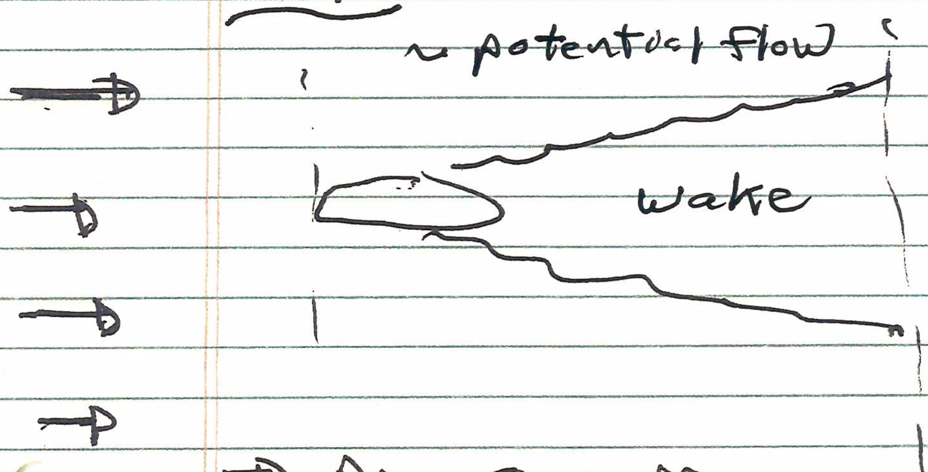
i.e. $P_1 = P_2$
$V_1 = V_2$

so $F_d = 0$

i.e. effect of flow is enhanced inertia (induced mass).

– for viscous fluid (i.e. no slip boundary condition) → upstream/downstream asymmetry

→ wake

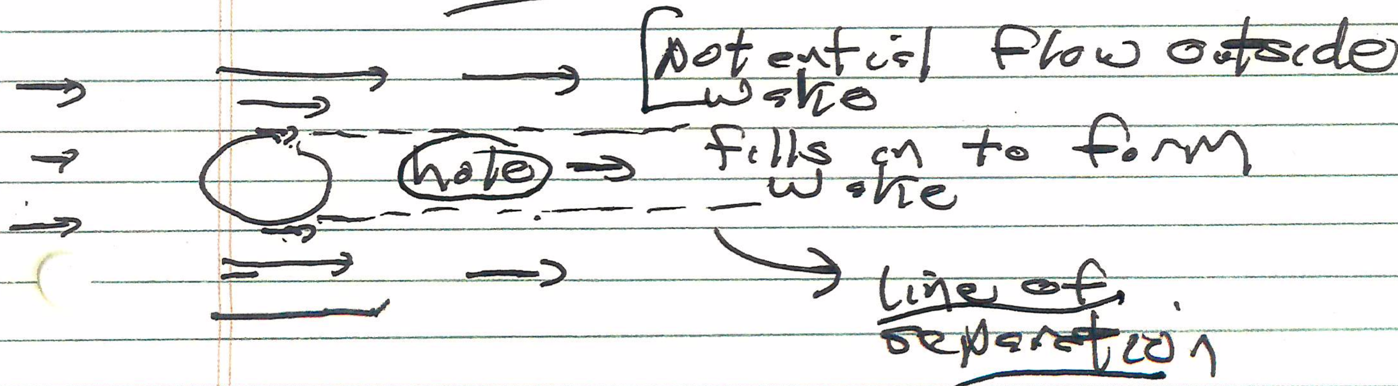


⇒ Drag occurs

Aim is to understand structure and dynamics of wake.

→ Recall: No-slip B.C.'s

⇒ Separation occurs



→ wake is rotational / vortical for $Re > Re_{crit}$

— how does hole fill in?

①

⇒ Separation is unstable

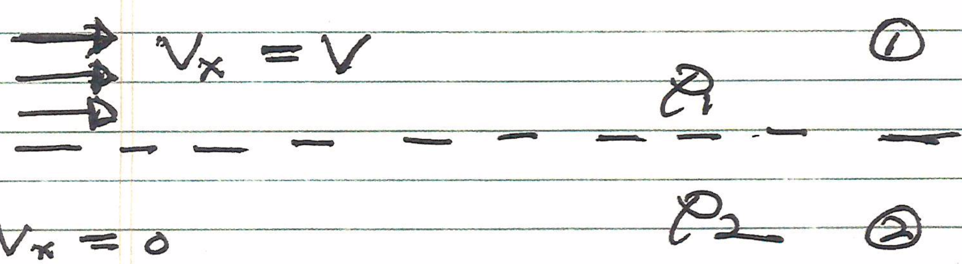
⇒ Kelvin – Helmholtz Instability

$\rho_1 = \rho_2$ $\qquad \omega = (k\,\Delta V)/2$

KH

→ free energy → ∇V

— simplification: interface



$(k_x d \ll 1)$

— $\nabla V = 0$, except interface

— $\omega = \partial V_x / \partial z = 0$, except interface

— can treat as potential flow in regions ①, ② and match at interface

— interface ripples → dynamic b.c.

etc. → as before

→ Wake Structures:

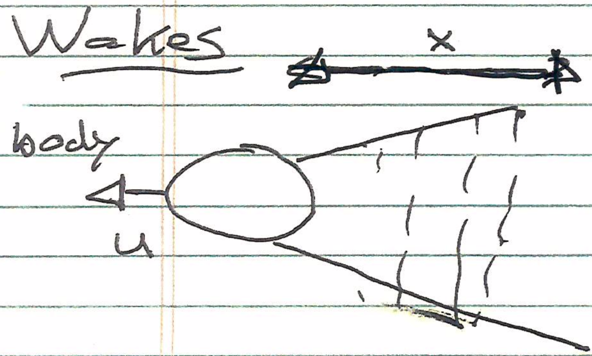
→ Physics ideas → wake flow created by response to separation.

– Link: Drag – asymmetry – wake flow

– width: Laminar, Turbulent scalings...

– Deficit and punchline re: N-S vs Euler → implications for potential flow

Wakes



region of wake limited in angular extent

– region behind body of departure from potential flow. Wake rotational. Rotational/vortical flow expands/spreads onto potential. Not reverse

– wake breaks upstream–downstream symmetry of ideal flow

sot

– wake is consequence of a body experiencing drag.

– $\boxed{\text{Message: A little viscosity forces a global adjustment in flow structure!}}$

Note:

* – in general, wake multi-component

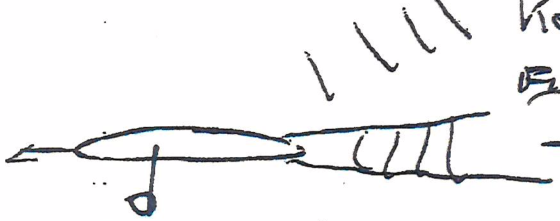
Kelvin → waves

Fanagen's wake:

→ center-line

due to screw bubbles

b.c. (skin friction)

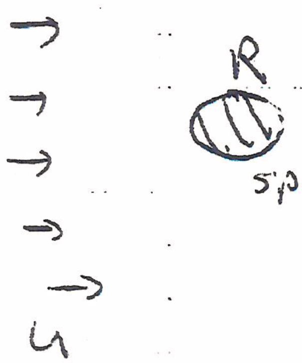


e.g. Fanagen

Kelvin

– here, consider spherical cow of wake problems.

ρ, ν



R

sphere

u

so $F_d \sim \rho u^2 R^2 \mathcal{C}(Re)$

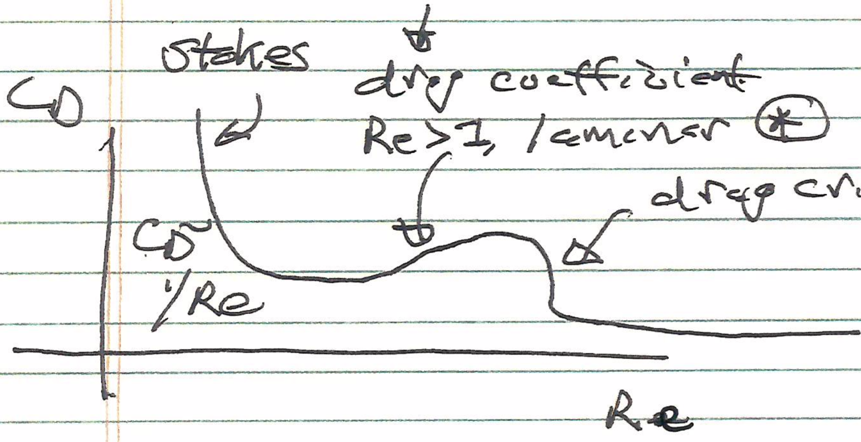
drag coeff

→ no surface effects.

→ How calculate wake structure?

Force of Drag $\equiv$ { Net Rate of Momentum Loss from Flow } i.e. $F_d \sim C_D \rho A U^2$

drag coefficient



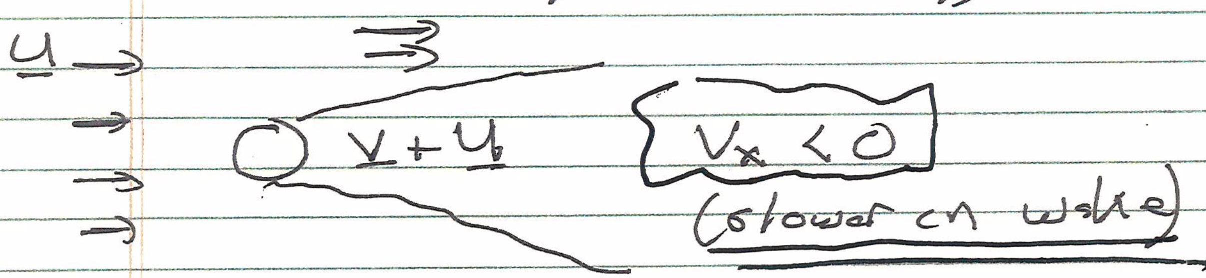
$C_D \sim r^0$
turbulence

i.e. in laminar wake, flow not turbulent but inertia relevant

Furthermore:

- distance behind body $x \gg R$, → wake.

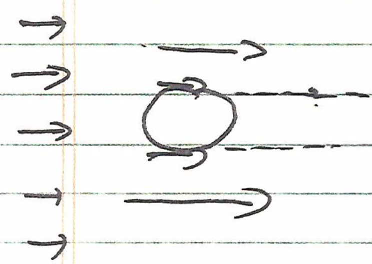
- if body speed U, then in frame where body stationary,



→ $\underline{v}$ differs zero in limited region

→ How limited? → as laminar, & signal propagation is diffusion only.

→ How does wake form?



– no slip boundary condition 'slows down' fluid flowing past body

∴

– separation, discontinuity

but → ~~left~~ results, mixing fluid into hole

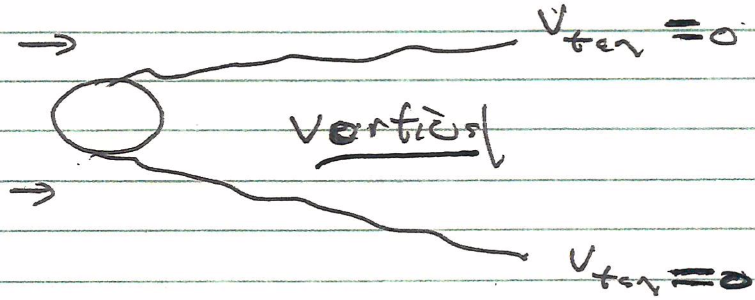
→ viscosity smooths out discontinuity

n.b. if turbulent wake turbulent mixing smooths discontinuity faster than viscous mixing.

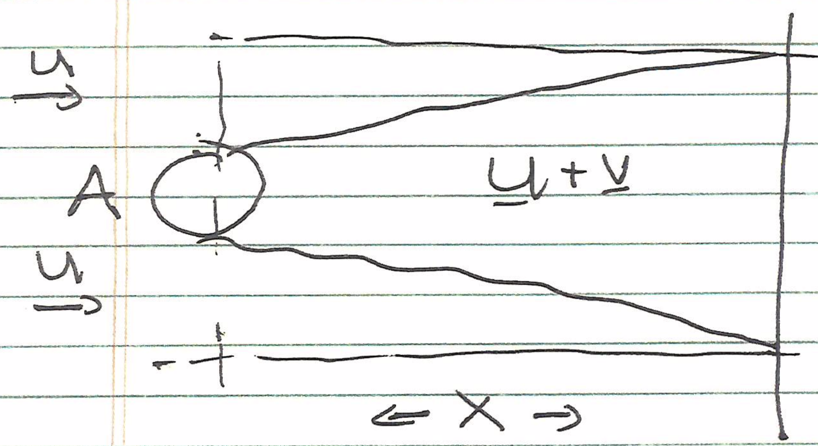
→ boundary of wake is traced by fluid particles

– passing close to body

– scattered by diffusion and turbulent mixing ⇒ expansion i.e.



Now, to calculate rate of loss of momentum from flow, return to D'Alembert construction, but with asymmetry



Drag = difference influx, outflux

$$F_d = \int_{A_0} d\underline{a} \cdot \underline{\underline{\Pi}}_{tot.}(0) - \int_{A_W} d\underline{a} \cdot \underline{\underline{\Pi}}_{Total}(x)$$

$$\underline{\underline{\Pi}} = p + \rho(\underline{U}+\underline{V})(\underline{U}+\underline{V})$$ ↳ momentum flux

$$\equiv P_{Tot.}$$

i.e. $O(v)$

$$\Pi^{(0)}_{x,x} = P_0 + \rho U^2 \qquad \oint [p' \, da_n + \rho U_n V_i] \, da_n$$

$$\Pi_{xx}(x) = P_0 + p' + \rho U^2 + \rho U V_x \quad \text{h.o.}$$

$\underline{\underline{\text{so}}}$

$$F_d \cong -\int_{A_w} da\, \rho U v_x$$

Now, can take conical symmetry, so

$$F_d \cong -\pi w(x)^2 \rho U v_x$$

$v_x < 0$
$F_d > 0$
$\rightarrow x$

<u>What is width</u>?

Now, need $w(x)$ to get v_x!

→ Observe

- problem now reduced to one of scale.
- <u>wakes self-similar.</u>

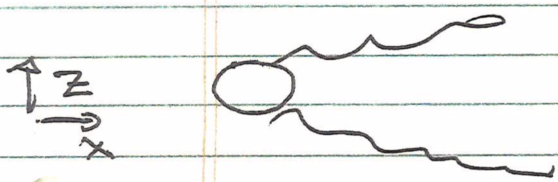
  $w \sim x^\alpha$, α?
- wakes can be laminar or turbulent

(i) Laminar, $\frac{UR}{\nu} > 1$

but not $\gg 1$

$$\frac{\partial \underline{V}}{\partial t} + \underline{V}\cdot\nabla\underline{V} - \nu\nabla^2\underline{V} = -\frac{\nabla P}{\rho}$$

$$\underline{U}\cdot\underline{\nabla} V + \cancel{\underline{V}\cdot\nabla\underline{V}} - \nu\nabla^2\underline{V} = -\frac{\nabla P}{\rho}$$



$\nabla\cdot\underline{V} = 0$

narrow wake

$$U\,\partial_x V_z - \nu(\cancel{\partial_x^2} + \partial_z^2)\, V_z = -\frac{\partial_z P}{\rho}$$

$$U\,\partial_x V_x - \nu\,\partial_z^2 V_x = -\frac{\partial_x P}{\rho}$$

B-O-E

$\partial_x \sim 1/x$ downstream distance

$\partial_z \sim 1/w$ $\perp$ scale.

Obvious: $V \simeq \frac{1}{\left(\frac{\nu x}{U}\right)^{1/2}} \exp\left[\frac{-z^2}{\nu x/U}\right]$

$$U\,\partial_x V_z - \nu\,\partial_z^2 V_z = -\frac{\partial_z P}{\rho}$$

$$\left(\frac{U}{X} - \frac{\nu}{W^2}\right) V_z \sim -\frac{P}{W\rho}$$

$$\left(\frac{U}{X} - \frac{\nu}{W^2}\right) V_x \sim -\frac{P}{X\rho}$$

$$\underline{\nabla} \cdot \underline{V} = 0 \quad \Rightarrow \quad \frac{V_x}{X} \sim \frac{V_z}{W}$$

as P negligible (will show) $\Rightarrow$

$$\frac{U}{X} \sim \frac{\nu}{W^2}$$

$$W \sim \left(\frac{\nu X}{U}\right)^{1/2} \quad \Big] \text{ Blasius}$$

$\rightarrow$ diffusive spreading of momentum by ν

$$\Rightarrow \sim (\nu t)^{1/2} \quad \text{with} \quad t \sim x/U$$

so

$$\boxed{W \sim (X/R)^{1/2} \left(\frac{\nu R}{U}\right)^{1/2} \quad \boxed{W/R \sim \left(\frac{X}{R}\right)^{1/2} 1/Re^{1/2}}}$$

– akin Blasius ✓

check: $\tilde{z} \rightarrow$

if $\quad \frac{P}{\rho W} \sim \frac{\nu V_z}{W^2}$ $\quad$ P likely important in V_z eqn.

but $\quad \frac{V_x}{X} \sim \frac{\cancel{V_z}}{W}$

$$P \sim \rho \nu \frac{V_x}{X}$$

$$\frac{P}{\rho X} \sim \nu \frac{V_x}{X^2}$$

but eqn. $\quad \underbrace{\left(\frac{U}{X} - \frac{\nu}{W^2}\right)}_{O(1/W^2)} V_x \sim \underbrace{\frac{-P}{\rho X}}_{O(1/X^2)}$

so drop P ✓ ✓

N. B. $\quad \frac{P}{\rho W} \sim \frac{\nu}{W^2} V_z$ $\quad$ ok

→ Some Observations re: Wakes

— $F_x = -\rho U \int_{wake} V_x \, dy\, dz$

⟹

$Q = \rho \int_{wake} V_x \, dy\, dz$ → Wake deficit

mass to difference with, without body

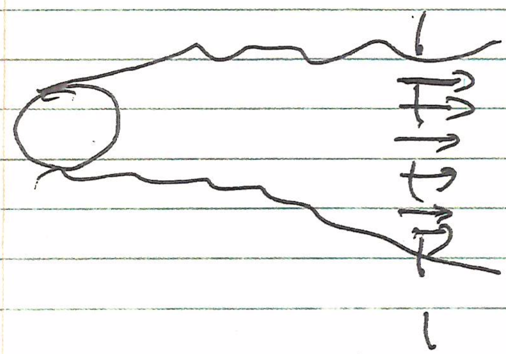
$F = -UQ$

↓

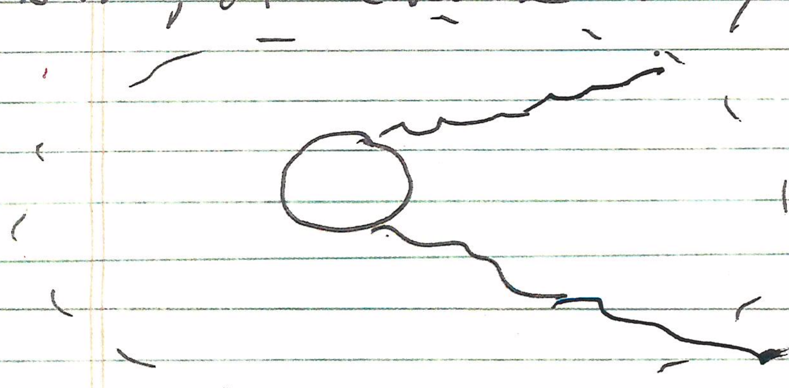
net mass flow thru wake area

Q is x independent:

i.e. $Q \sim F_x / U$ → const



but, if encircle body:



have:

$$\oint_{tot} \underline{v} \cdot d\underline{a} = 0$$

i.e. no water lost

but

$$0 = \underbrace{\int_{wake} \underline{v}\, dy\, dz}_{Q} + \int_{pot\ flow} \underline{v} \cdot d\underline{a}$$

$$Q \sim V_x A$$

Q finite $\Rightarrow$

$$V_x \sim \frac{1}{A} \sim 1/r^2$$

so

$$\int_{pot\ flow} \underline{v} \cdot d\underline{a} = -Q$$

$$V_x A \sim -Q$$

$$V_x \sim 1/r^2 \rightarrow$$ potential flow monopole

So

- Euler equation

$$v \sim 1/r^3 \rightarrow \text{dipole}$$

but for N-S. eqn.

$$v \sim 1/r^2 \rightarrow \text{monopole!}$$

⇒

- global adjustment in potential flow outside wake induced by viscosity and the wake.

- Message: [A little ν forces a global adjustment in flow structure!

important!

Effects viscosity not limited to layers, etc.

words on what is turbulence.

by analogy with K.T. gases

$\underline{v}\cdot\nabla\underline{v} \to -\nu_T \nabla^2 \underline{v}$

$\nu_T \sim \tilde{v}\, l_{mix}$

ii) Turbulent Wake — $Re \sim UR/\nu \gg 1$ (drop)

$\underline{U}\cdot\nabla\underline{v} + \underline{v}\cdot\nabla\underline{v} - \cancel{\nu\nabla^2\underline{v}} = -\frac{\nabla p}{\rho}$

ignore

$\Rightarrow \frac{U}{x} v_x \sim \frac{\tilde{v}_y}{W} v_x$

$\nabla\cdot\underline{v} = 0$

$W \sim \frac{\tilde{v}_y x}{U}$

$\sim \tilde{v}_x x / U$

[wave spreads by advection, not diffusion]

$\boxed{W \sim \frac{\tilde{v}_y\, x}{U}}$

$\boxed{\tilde{v}_y \sim \text{turbulent velocity integral scale}}$

Take wake turbulence isotropic, → turb.

so $\tilde{v}_x \sim \tilde{v}_y \sim \tilde{v}_z$ {Fair? Test}

— $W \sim x\tilde{v}_x/U$ —

$\boxed{\begin{array}{l} U\cdot\nabla v + v\cdot\nabla v \\ \frac{U}{x} v_x + \frac{\tilde{v}_x}{W} v_x \end{array}}$

but from drag:

— $\tilde{v}_x \sim F_d/\rho U W^2$ —

$\Longrightarrow$

$$W \sim x \frac{F_d}{\rho u^2 W^2} \sim x \left(F_d / \rho u^2 W^2 \right)$$

$$W^3 \sim F_d\, x / \rho u^2$$

$$\Rightarrow \boxed{W \sim \left(F_d / \rho u^2 \right)^{1/3} x^{1/3} \sim \left(C_D R^2 \right)^{1/3} \underline{x^{1/3}}}$$

!

then, comparing widths:

laminar: $W/R \sim (x/R)^{1/2}\, Re^{-1/2}$

$Re \sim UR/\nu$

turbulent: $W/R \sim (x/R)^{1/3}\, C_D^{1/3}$

interestingly laminar wake expands with downstream length more rapidly!

[redacted]

Why?

⇒ turbulence can relax ΓV behind object (due separation) rapidly, and faster than ν. Thus surrounding flow penetrates the dead water region more rapidly, less wake expansion.

Also observe: Wake Re drops with x.

⇒

$$Re \sim \frac{W V_y}{\nu} \sim \frac{\cancel{W} V_x}{\nu} \sim \frac{\cancel{\Gamma}}{\nu} \quad \frac{F_d}{\rho U W^{\cancel{2}}}$$

Re: y direction (opr)

$\frac{W V_y}{\nu}$: wake flow Re

$$Re \sim F_d / \rho U W \nu$$

$$\sim \frac{U^{\cancel{2}} R^2 \cancel{\rho_0} C_D}{\nu \cancel{\rho_0} \cancel{U} (C_D R^2)^{1/3} x^{1/3}}$$

$C_D \sim 1$

$$\sim \left(\frac{UR}{\nu}\right)\left(R/x\right)^{1/3}$$

[redacted]

$$\Rightarrow \quad \boxed{Re(x) \sim Re_o \, (R/x)^{1/3}}$$

and $Re(x) \to 1$ at

$$x_L \sim R \, (Re_o)^3$$

distance behind boat where turbulent wake transitions to laminar.

i.e. akin l_d: transition from turbulent mixing to viscous mixing

N.B. [In wake, vortical/rotational region can expand into irrotational region, but never reverse!

i.e. would really violate H-Thm ...

Later discussion

Wakes - Supplement (Ch. 4?)

→ Revisit turbulent wake, using turbulent viscosity, i.e.

$$w \sim (\nu x/u)^{1/2} \qquad (\nu \to D_T)$$

$$\to (D_T x/u)^{1/2}$$

i.e. width of turbulent wake set by turbulent diff., following Blasius Law?

but $D_T \sim w\tilde{v}$ ⇒ turbulent viscosity at mixing length level.

$$\sim w\left(F_d/\rho u w^2\right)$$

$$\sim F_d/\rho u w \quad \sim \quad \text{const}/w$$

⇒

$$w \sim \left(F_d x/\rho u^2 w\right)^{1/2}$$

$$w^{3/2} \sim \left(F_d/\rho u^2\right)^{1/2} x^{1/2} \sim (C_D R^2)^{1/2} x^{1/2}$$

$$\sim C_D^{1/2} R\, x^{1/2}$$

$$w \sim (C_D)^{1/3} R^{2/3} x^{1/3}$$

$\Rightarrow$ $\boxed{W/R \sim C_D^{1/3} \, (X/R)^{1/3}}$ agrees ✓.

Now, $D_T \sim \tilde{\nu} W$

$$\sim \frac{(\tilde{\nu} W^2)}{W}$$

$$\sim \frac{\rho U \tilde{\nu} W^2}{\rho U W}$$

$$\sim \frac{Q}{W} \sim Q/R\,(x/R)^{1/3}$$

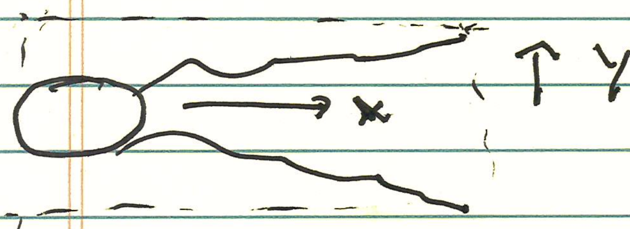
∴ – Point is that turbulent viscosity mixing drops downstream, relative to constant viscous mixing.

– follows from $\tilde{\nu} W \sim \overset{\to \text{const.}}{Q} / W$

– explains why turbulent wake spreads more slowly than laminar wake.

→ Wakes, cont'd – Lift.

Recall, showed:



$$\underline{p} = p(\underline{x})\,\delta_{ij}$$
$$\underline{u} = u\,\hat{x}$$

$$\underline{F} = \int d\underline{a}_1 \cdot \left(p + \rho u^2\right)_{ij} - \int_{\text{wake}} d\underline{a}_2 \cdot \left(p + \cancel{p} + \rho\left(u\hat{x} + \underline{v}\right)^2\right)_{ij}$$

$$= -\int_{\text{wake}} d\underline{a} \cdot \rho\,\underline{u}\,\underline{v}$$

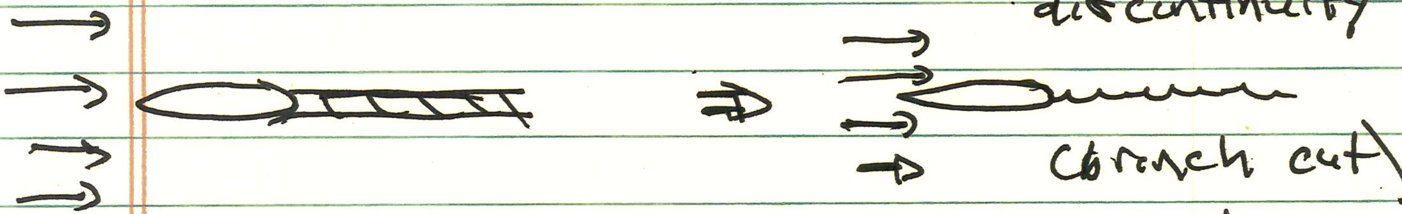
$$F_x = -\int_{\text{wake}} da\ \rho\, u\, v_x \quad \overset{\Pi_{xx}}{\rightarrow} \quad \underline{\text{drag}}$$

but

$$F_y = -\int_{\text{wake}} da\ \rho\, u\, v_y \quad \overset{\Pi_{xy}}{\rightarrow} \quad \underline{\text{Lift}}$$

i.e. consider:

i.e. wing



streamlined body → narrow wake.

Need calculate F_y → lift!?

$$F_y = -\int_{\text{wake}} dy\, dz\, \rho U v_y$$

Now,

– need integrate $\int_{-\infty}^{+\infty} dy$ – pot flow

– wake thin, v_y slow, so $\int dy$ is basically due pot flow.
(i.e. wake as branch cut)

so

$$\int_{-\infty}^{+\infty} v_y\, dy \cong \int_{y_2}^{\infty} v_y\, dy + \int_{-\infty}^{y_1} v_y\, dy$$

as now integrating outside wake → U → potential flow!

$$\cong \int_{y_1}^{\infty} \frac{\partial \phi}{\partial y} dy + \int_{-\infty}^{y_2} \frac{\partial \phi}{\partial y} dy$$

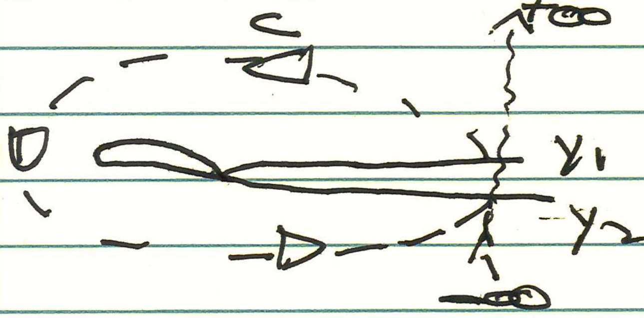
$$\cong -\phi_1 + \phi_2$$

so

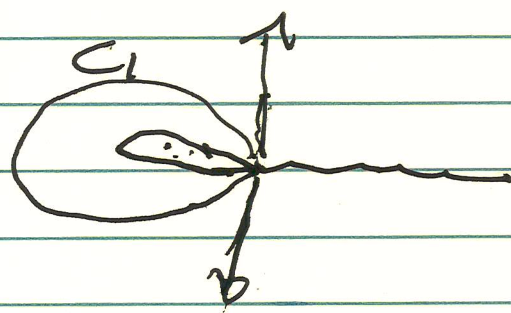
$$\boxed{F_y = -\rho U \int dz\,(\phi_2 - \phi_1)}$$

Lift force.

i.e



→



Furthermore can note:

$$\oint \nabla\phi \cdot d\ell = \oint (V_y\,dy + V_x\,dx)$$

so along C_1;

$$\oint \underline{v}\cdot d\underline{\ell} = \phi_2 - \phi_1 \equiv \Gamma$$

again, treat wake as discontinuity

$$= \int \underline{\omega}\cdot da.$$

thus

so

Zhukovski's Thm

$$\boxed{F_y = F_{Lift} = -\rho U \int \Gamma \, dz}$$

→ Lift force set by circulation about wing.

→ contrast to grade school story.

→ key to matter of flight is calculating Γ

Long wing,

$F_y = -\rho U \Gamma$, (or really F/ℓ).

$F_y = C_L \rho S U^2$

C_L: lift coeff — shape etc.

$S \sim W$ wing

- Calculating Γ exploits surface of discontinuity.

- For wing, width a, attack α

$\Gamma = \pi \alpha a U$, $C_L = -\rho U \Gamma / \frac{1}{2}\rho U^2 a$

So $C_y = C_L = 2\pi\alpha$.