Convection Patterns I

So Far:

- basics of dynamics (information) < dimension, attractors / Lyapunov exponents, chaos

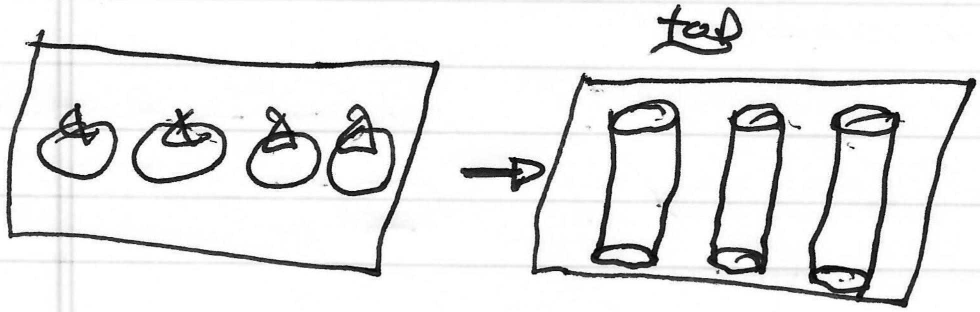
- Patterns I - Phase Dynamics → Time: (CGL)
  - single oscillator synch, noise
  - local coupling → phase diffusion (KPZ) model → domains of synch. Pdf(ϕ)
  - global coupling (vs. dispersion, noise) → K transition → global synch.
    (n.b.: range of coupling is key)
  - phase turbulence, repulsive coupling → K-S eqn, etc.

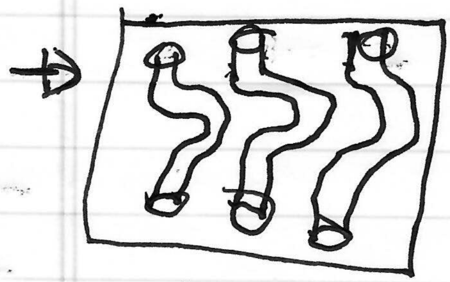
now:

→ Patterns II - Convection near marginal → Space

Focus: Secondary instability in ensemble of convection cells/rolls near marginality (i.e. 'weakly nonlinear')

i.e.





zig-zag

i.e. development

- Eckhaus → modulate array
- zig-zag → bending

- subject is classic, as easily amenable to experiment, tractable as 'near equilibrium'.
- approach:

  - basic model of marginal roll ⇒ Swift-Hohenberg model.

  - modulations in ensemble of rolls (some similarity to WKE).
  i.e. pattern dynamics on scales larger than that of individual roll...

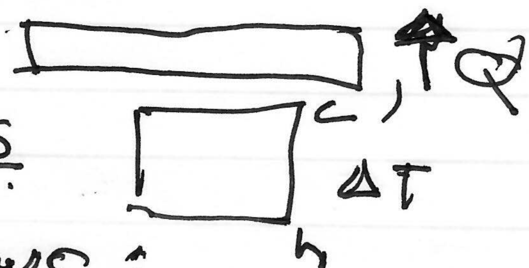
⇒ envelope equations (Landau Theory)

- implications of envelope formalism for patterns.

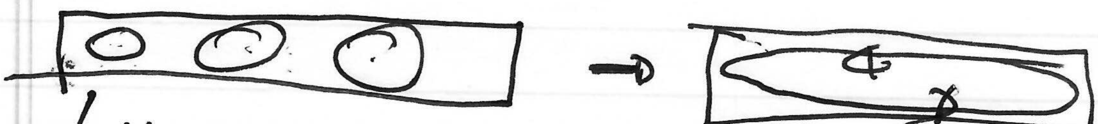
Comment:

- more elegant and 'classical' than useful. A <u>must</u> for a basic course. . . .

- related, more relevant models:
  - → fixed flux convection: (Chapman – Proctar) <u>vs</u>



  - → convection driving flows:



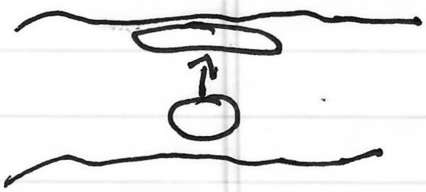
(Howard – Krishnamurti)

Refs:

- → several posts → Cross + Hohenberg is encyclopedic.
- → books: { Cross + Greenside, P. Manneville, Rebecca Hoyle
- → later: { Chapman – Proctar, Howard, Krishnamurti } Key papers.

→ Physics of Convection (Rayleigh – Benard)

(See Lectures VI, Phys. 216, W 2017 or any book).



stratified fluid/atmosphere

$\underline{\nabla}\cdot\underline{V} = 0$ as $\tau > \ell/C_s$;

→ parcel will rise by buoyancy if:

$$dS/dz < 0 \Rightarrow \frac{1}{\bar{T}}\frac{d\bar{T}}{dz} < \frac{(\gamma-1)}{\rho}\frac{d\rho}{dz} \quad (g<0)$$

$$1/\tau_b^2 \cong g/\gamma \; \partial S_0/\partial z$$

→ if dissipation:

$$\left.\begin{aligned} \partial_t \tilde{T} &\to \partial_t \tilde{T} - \chi \nabla^2 \tilde{T} \\ \partial_t \underline{\tilde{V}} &\to \partial_t \underline{\tilde{V}} - \nu \nabla^2 \underline{\tilde{V}} \end{aligned}\right\}$$

viscosity, heat diffusion can damp convection

so, natural to require for instability:

$$\frac{\tau_\nu \tau_\chi}{\tau_b^2} > 1 \Rightarrow g \frac{\partial \langle s \rangle}{\partial z} \ell^4 / \nu\chi \equiv Ra$$

$$Ra > Ra_{crit.}$$

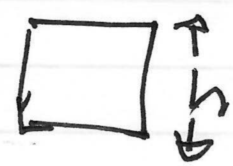
in box:

$$\boxed{Ra \equiv \frac{g \Delta T \alpha h^3}{\nu \chi}}$$

Rayleigh #

$$\delta\rho = -\alpha \, \delta T$$

↓ coeff of thermal expansion



basic equations:

(2D)

$$\partial_t \nabla_\perp^2 \phi - \nu \nabla_\perp^2 \nabla^2 \phi = -g \frac{\partial}{\partial x}\left(\hat{T}/T_0\right)$$

$$\partial_t \left(\gamma \frac{\hat{T}}{T_0}\right) = -\hat{v}_z \frac{d\langle s \rangle}{dz} + \chi \nabla_\perp^2 T$$

$$\left(\underline{V} = \underline{\nabla}\phi \times \hat{y}\right)$$

or (with $w = v_z$)

$$\partial_t \nabla^2 W = g\alpha\left(\frac{\partial^2\Theta}{\partial x^2}+\frac{\partial^2\Theta}{\partial y^2}\right) + \nu\nabla^2\nabla^2 W \quad \left(\nabla_\perp\right)$$

$$\frac{\partial\Theta}{\partial t} = \beta W + \chi \nabla^2\Theta$$

What sets (behavior of) critical Rayleigh # $\rightarrow$ dissipation, and boundary conditions ?

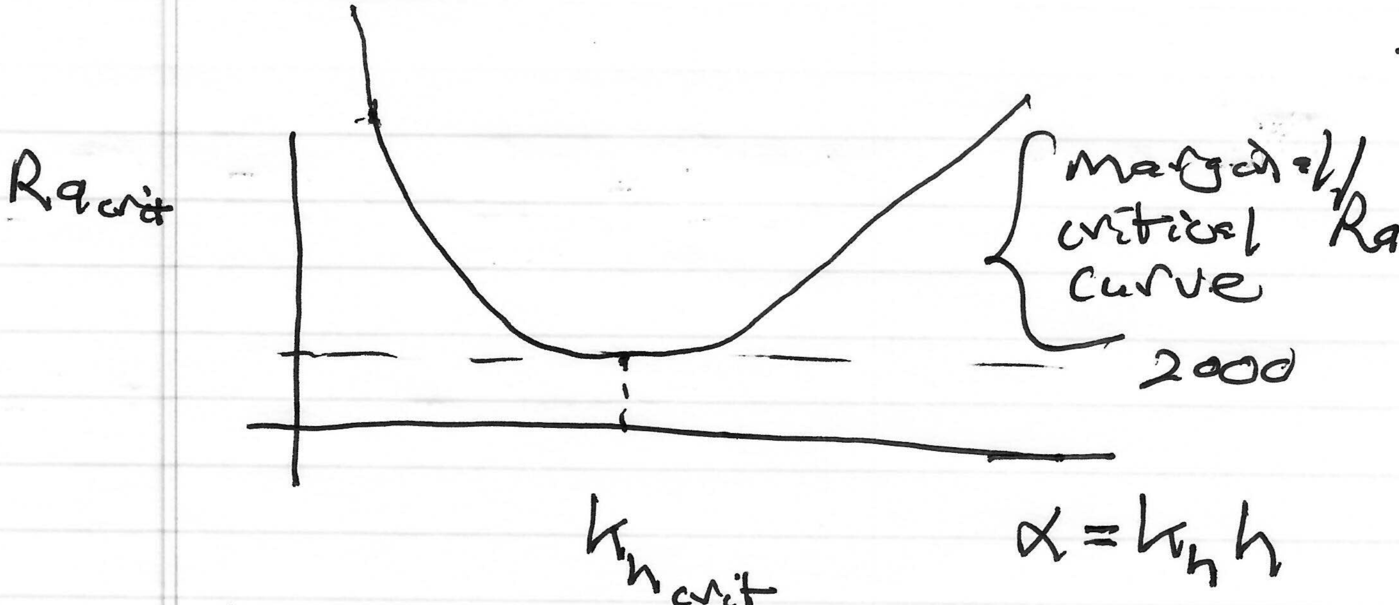
i.e. for no slip:

$$\tilde{V}_z\Big|_{0,h} = 0 \; ; \qquad \tilde{V}_h\Big|_{0,h} = 0$$

n.b.: $\nabla_h\cdot\tilde{V}_h + \nabla_z \tilde{V}_z = 0$

no slip $\Rightarrow$ $\nabla_h\cdot\tilde{V}_h = 0$

so $\partial_z W\Big|_{0,h} = 0$



→ high k_h → minimum Ra_{crit}.
$\sim \nu k^2$ etc

→ low k_h 

~ no slip ($V_h = 0$) damping.

N.B.: In stress free, numbers change, but similar structure.

→ Now, how describe convection for

$$Ra = Ra_{crit} + \underbrace{\epsilon}_{\delta Ra}$$

? i.e. small excursion into super critical

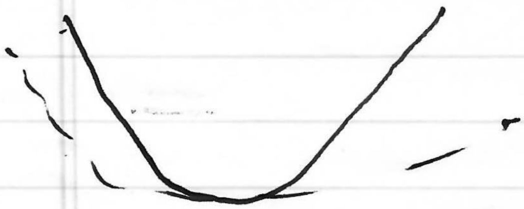
1D Key elements:

- Ra_{crit}
- k_{crit}

→  curve

→ saturation

~ system approaches steady state.

so  fit with parabola.

so

$$\boxed{\gamma \tau_0 = (Ra - Ra_{crit}) - \tau_0 \xi_0^2 (k - k_{crit})^2}$$

model of growth near marginal.

so, schematically:

$$\boxed{\begin{aligned} \tau_0 \frac{\partial W}{\partial t} = {} & (Ra - Ra_{crit}) W \\ & - \tau_0 \xi_0^2 \left(\sqrt{-\partial_x^2} - k_{crit}\right)^2 W \\ & - |W|^2 W \end{aligned}}$$

has form of Landau Eqn.

How simplify?

This is a step toward $\boxed{\text{Swift - Hohenberg Model}}$ a reduced model of convection near onset. See Swift, Hohenberg 1977 for derivation.

Now, beyond 1D; consider: (de - 1D)

- uniform base state
- rotationally invariant in 2D plane ($\perp$ rolls).

so $\gamma_{\underline{q}}$ can depend only on $|\underline{q}| = q$, not $\underline{q}$.

$\Rightarrow$

$$\gamma_{\underline{q}} = \underbrace{\rho}_{\text{control parameter } (Ra - Ra_{crit})} - c\,(\underbrace{q - q_c}_{\text{min. } q\text{, growth}})^2$$

$\Rightarrow$

$$\partial_t W = \left(\rho - c\left(\sqrt{-\underline{\nabla}^2} - q_c\right)^2\right) W + \ldots$$

Now, near onset:

$q + q_c \cong 2q_c$, so can write 1 in a "creative way":

$$\gamma = \rho - c(q - q_c)^2$$
$$\cong \rho - c\frac{(q+q_c)^2}{(2q_c)^2}(q - q_c)^2$$
$$\cong \rho - \frac{c}{4q_c^2}(q^2 - q_c^2)^2$$

$\Rightarrow$ re-scale:

$$\partial_t W = rW - (\nabla^2 + \underbrace{q_c^2}_{\text{frequently} \to 1})^2 W$$

Finally to saturate, need restrict growth at finite amplitude and:

- respect inversion symmetry $w \to -w$ (cf. basic eqns)
- respect phase symmetry.

$\Rightarrow -u^3, \quad -|W|^2 W$

So

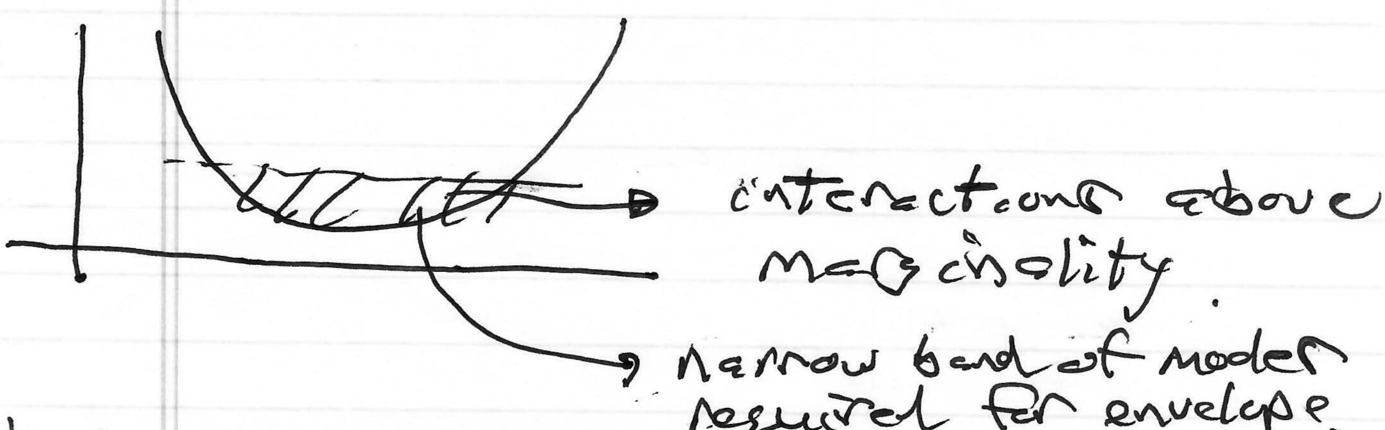
$$\boxed{\partial_t W = rW - (\nabla^2 + k_0^2)^2 W - |W|^2 W}$$

$\Rightarrow$ Swift - Hohenberg Model

$\rightsquigarrow$ describes convection near onset

$\rightsquigarrow$ can't quantify its own breakdown.

i.e. describes:



Now:

- S-H can be derived from basic eqns, 'systematically'. Tedious, not instructive.

- S-H is derivable from variational principle — i.e. Lyapunov Function

Now, for w^3 form:

$$V[w] = \int dx \int dy \left[-\frac{1}{2} r w^2 + \frac{1}{4} w^4 + \frac{1}{2} \left[(\nabla^2 + 1) w \right]^2 \right]$$

and

$$\frac{dV}{dt} = - \int dx \int dy \, (\partial_t w)^2$$

i.e. any evolution of w tends to decrease V. V is minimal at stationarity of w.

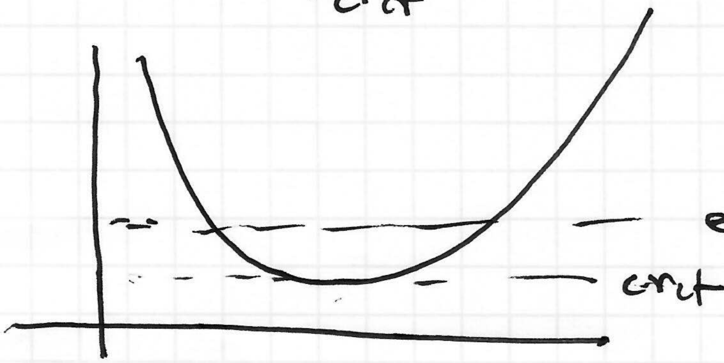
→ easily shown.

→ can write: $\partial_t w = -\frac{\delta V}{\delta w}$

→ J-H Model is a reduced model, applicable to band of modes near Ra_{crit}.

→ $Ra = Ra_{crit} + \delta Ra$

⇒



excited → ensemble/array of convection rolls.

what happens? → 13a.

With models:

Key ⇒ how does pattern of excited cell evolve? what configuration does it adapt.

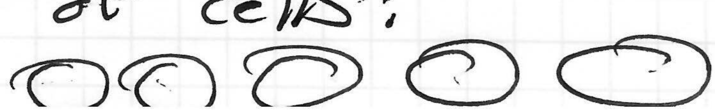
Classic:

⇒ explore stability of band of modes with:

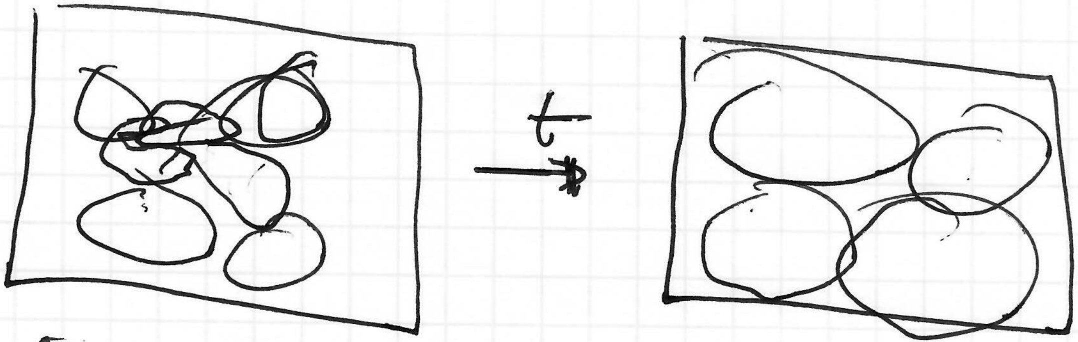
$$\underline{k} = k_0 \hat{x} + \underline{k}$$

breaks symmetry.

— $k_0 \hat{x}$ → base state is array of cells:



→ See 184–185 of Cross, Greenside



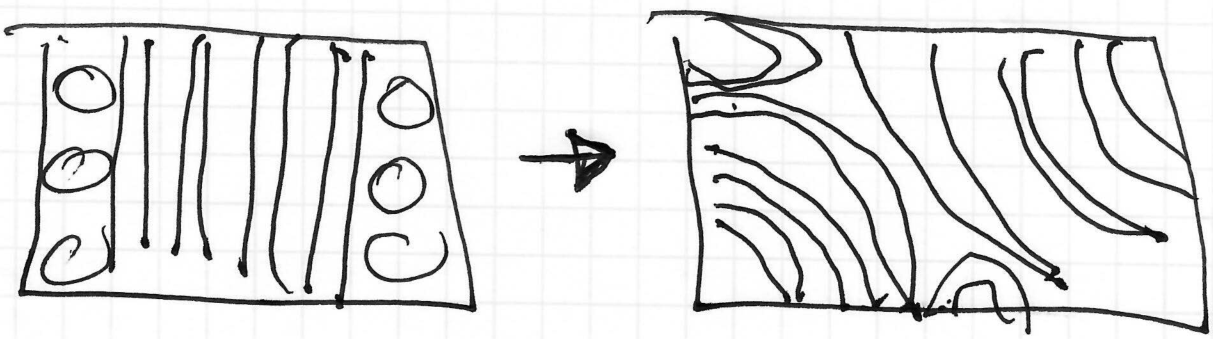
domains, periodic box

larger domains

→ coarsening

⇒ Eckhaus process.

or



$- |k|/|q_0| \ll 1$, so

$$W \cong \Big[W_0 \underbrace{A(x,y,t)}_{\text{amplitude}} \underbrace{e^{i q_0 \cdot x}}_{\text{carrier}} + c.c. \Big] + O(\epsilon)$$

Now, backing up:

$$\gamma T_0 = \underbrace{\epsilon}_{Ra-R_{crit}} - \xi_0^2 (q - q_0)^2$$

Now, $\underline{q} = \underbrace{q_0 \hat{x}}_{\text{1D base}} + \underbrace{\underline{k}}_{\text{2D.}}$

so

$$\gamma T_0 = \epsilon - \xi_0^2 \left(|q_0 \hat{x} + \underline{k}| - q_0 \right)^2$$

$$= \epsilon - \xi_0^2 \left(q_0 \left(\left(1 + \frac{k_x}{q_0}\right)^2 + \frac{k_y^2}{q_0^2} \right)^{1/2} - q_0 \right)^2$$

expanding, etc.

$$\gamma\,\gamma_0 = \epsilon - \xi_0^2\left(k_x + \frac{k_y^2}{2q_0}\right)^2$$

envelope $\underline{k}$ dependence

h.o. in k_x incl.

Note:

- k_x, k_y asymmetry due to symmetry breaking / direction set of the base state

- envelope eqn!

$$\partial_t W = \epsilon - \xi_0^2(q - q_0)^2$$

$$= \epsilon W - \xi_0^2\underbrace{\left(|[\underline{q}_0 + \underline{k}]| - |q_0|\right)^2}_{}W$$

→ op.

$$W = W_0 A e^{i q_0 x}$$

$$\underline{k} \to \partial_{\underline{x}} A$$

$$\gamma \to \partial_t A$$

So $\gamma \to \partial_t$

$k_x \to -i\partial_x$

$k_y \to -i\partial_y$

and: $w \to -w$ $\Rightarrow$ w^3 saturation

$w \to we^{i\varphi}$ $\Rightarrow$ spatial shift of pattern irrelevant (boundaries ?)

$\Rightarrow$ Newell - Whitehead Eqn. (Amplitude)

$$\boxed{\partial_T A = A + \left(\partial_X - \frac{1}{2} i \partial_Y^2\right)^2 A - |A|^2 A}$$

where: re-scale

$$\left\{\begin{aligned} X &= |\epsilon|^{1/2} x/\xi_0 \\ Y &= |\epsilon|^{1/4} y \,(q_0/\xi_0)^{1/2} \\ T &= \epsilon t/\tau_0 \\ A &= (g_0/|\epsilon|)^{1/2} A. \end{aligned}\right.$$

→ NW has Lyapunov Function

→ Difference of $\begin{cases} \underset{\sim}{SH} \to \text{maintains rotational symmetry base state} \\ \underset{\sim}{NW} \to \text{breaks symmetry by } \underline{\Sigma} = \varepsilon_0 \hat{x} \text{ assumption.} \end{cases}$

Now, have N-W eqn:

$$\tau_0 \partial_t A = rA + \varepsilon_0^2 \left(\partial_x + \frac{1}{2ik_c} \partial_y^2 \right)^2 A - g|A|^2 A$$

(akin CGL)

and useful to re-write as:

$$A = |A| e^{i\phi} \to a e^{i\phi}$$

as complex structure guarantees phase dynamics relevant;

ignoring y dep,

$$\tau_0 \partial_t a = \left(r - \varepsilon_0^2 (\partial_x \phi)^2 \right) a + \varepsilon_0^2 \partial_x^2 a - g \underbrace{|a|^2 a}_{a^3}$$

<u>amplitude</u>.

$$\partial_t \phi = \frac{\varepsilon_0^3}{\tau_0}\left(\partial_x^2 \phi + 2\frac{\partial_x a}{a}\partial_x \phi\right)$$

phase

Observe:

- amplitude evolves via r and q^3, favoring long wavelengths
- phase diffusive. Can $2\frac{\partial_x a}{a}\partial_x\phi$ (effectively) flip sign of effective diffusivity?

Time independent solutions:

- phase winding

$$\partial_x a = 0$$

$$\phi = \delta k x + \phi_0$$

$$\Rightarrow \quad 0 = (r - \varepsilon_0^2 \delta k^2) a - g a^3$$

$$a = \left[(r - \varepsilon_0^2 \delta k^2)/g\right]^{1/2}.$$

$$\Rightarrow \quad A = e^{i\delta k x + \phi_0} a.$$