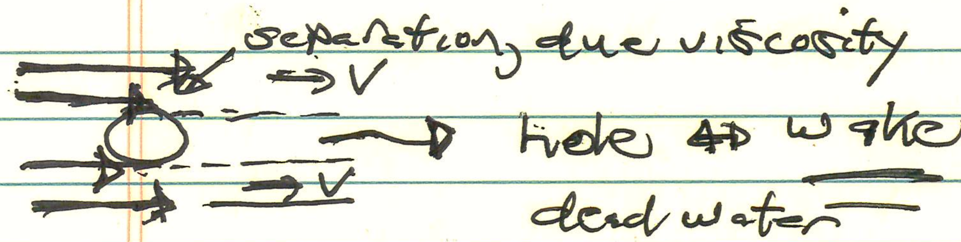
Notes 4a: Kelvin-Helmholtz

Consider higher Re # flow over sphere

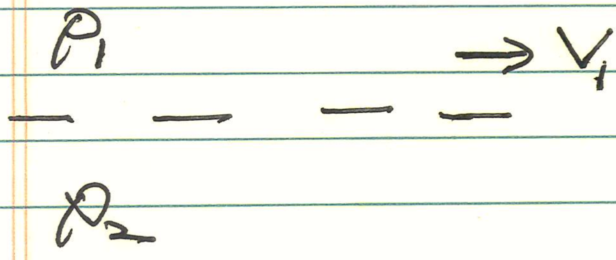


no slip

but hole fills in! — how

- viscous diffusion
- stability

→ motivates KH instability — interface in flow

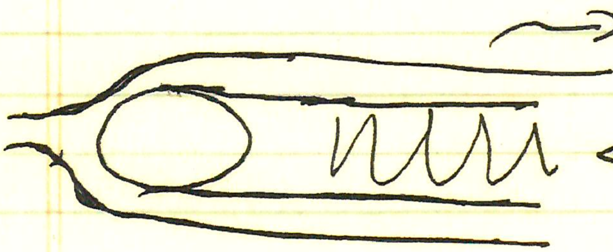


tangential discontinuity (limit of shear layer)

→ Kelvin - Helmholtz

stability of discontinuity ?

Recall OV of main story:



- with no-slip B.C.'s, $v_n|_{surf} = v_T|_{surf} = 0$

how does wake fill in?

separation happens, wake forms

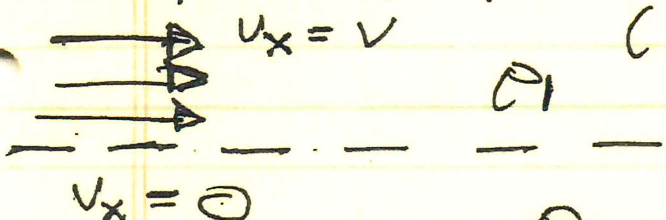
- separation → instability → turbulence.

How?

Instability ⇒ Kelvin-Helmholtz

⇒ free energy → ΔV — flow shear

⇒ simplification: shear layer (interface) idealization



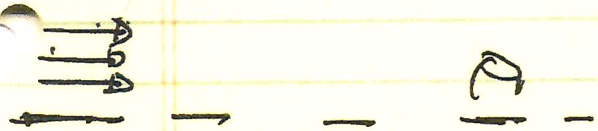
Note: { classic example of interfacial instability.

- $\nabla V = 0$, except interface

- vorticity $\partial V_x / \partial z$ localized to interface] ★

⇒ can play game w/ R–T, now with $\underline{V}$, η.

Physical ideas: (1)



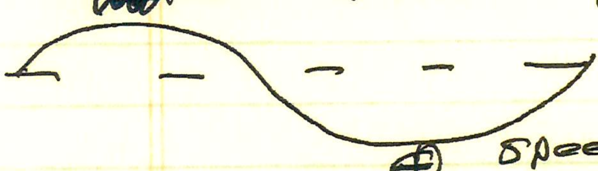
⊕ high V, low p

low, zero V, high p ⊕

$p + \rho V^2/2 = \text{const}$

(2) δV perturbation → ripple interface $\delta V < 0$

⊖ bring in slow fluid −slow δV



⊕ speeds up → fast draught in $\delta V > 0$

so Bernoulli ⇒

$\delta p > 0$ δp $\delta p > 0$

$\delta p < 0$

$\delta p < 0$

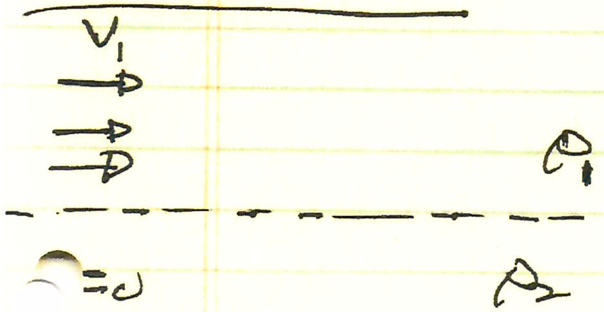
but

$\delta P > 0 \Rightarrow \delta V < 0$, further → i.e. re-enforces critical perturbation

→ unstable

⇒ KH instability drives viscous mixing via turbulence, mixing, billows, etc.

To calculate:



$p + \frac{V^2}{2} = \text{const}$

assumes eqbrm.

so, as before:

$\underline{\nabla} \cdot \underline{V} = 0$

$\underline{V} = \underline{\nabla}\phi$ $\qquad \underline{\omega} = 0$, except interface

$\nabla^2\phi = 0$

⇒

wave along interface

$$\phi = \sum_k \phi_k \, e^{ikx} \, e^{-k|z|} \, e^{-i\omega_k t}$$

decays away from interface.

as before:

$$\tilde{p}_1(0_+) = \tilde{p}_2(0_-)$$

$\eta \rightarrow$ interface ripple displacement

and
$$\left.\frac{\partial \phi}{\partial z}\right|_1 = \left.\frac{\partial \phi}{\partial z}\right|_2$$

$$\frac{d\eta}{dt} = V_z\Big|_0$$

$$\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\underline{\nabla}\,\underline{v} = -\frac{\nabla p}{\rho}$$

$$\partial_t \tilde{V}_{z_1} + V_1 \partial_x \tilde{V}_{z_1} = -\frac{\partial_z p_1}{\rho_1} = \frac{k_z \tilde{p}_1}{\rho_1}$$

$$\partial_t \tilde{V}_{z_2} + 0 = -\frac{\partial_z p_2}{\rho} = -\frac{k_z \tilde{p}_2}{\rho_2}$$

$$-i(\Omega - kV_1)\tilde{V}_1 = \frac{k_z}{\rho_1}\tilde{p}_{(1)}$$

$$-i\Omega \tilde{V}_2 = -\frac{k_z \tilde{p}_2}{\rho_2}$$

$$\left(\frac{\rho_1}{k_z}\right) - i(\Omega - kV_1)\tilde{V}_1 = i\Omega \frac{\rho_2}{k_z}\tilde{V}_2$$

but $\tilde{V}_{z_1}(0) = \partial_t \eta + V_1 \partial_x \eta$

$\tilde{V}_{z_2}(0) = \partial_t \eta$

$$\left(\frac{\rho_1}{k_z}\right)\left[-i(\Omega - kV_1)\right]\left[-i(\Omega - kV_1)\right]\eta$$

$$= i\Omega \frac{\rho_2}{k_z}\left(-i\Omega\eta\right)$$

$$\frac{\rho_1}{k_1} - (kV - \Omega)^2 \cancel{\eta} = \Omega^2 \frac{\rho_2}{k_z} \cancel{\eta}$$

$$\boxed{-\frac{\rho_1}{k_1}(k_1 V - \Omega)^2 = \frac{\rho_2}{k_1}\Omega^2}$$

$$\boxed{-\frac{\rho_1}{k}(kV-\omega)^2 = \frac{\rho_2}{k}\omega^2}$$

$$\Rightarrow \boxed{\omega = kV\left(\frac{\rho_1 + i(\rho_1\rho_2)^{1/2}}{\rho_1+\rho_2}\right)}$$

$\Rightarrow$

$\rightarrow \quad \gamma \sim kV \dfrac{\sqrt{\rho_1\rho_2}}{\rho_1+\rho_2}$ $\rightarrow$ KH growth

note $\quad \omega_r \sim kV \left(\rho_1 / (\rho_1+\rho_2)\right)$

no "exchange of stabilities" here!,

— $\rho_1=\rho_2 \qquad \gamma = \dfrac{kV}{2}$

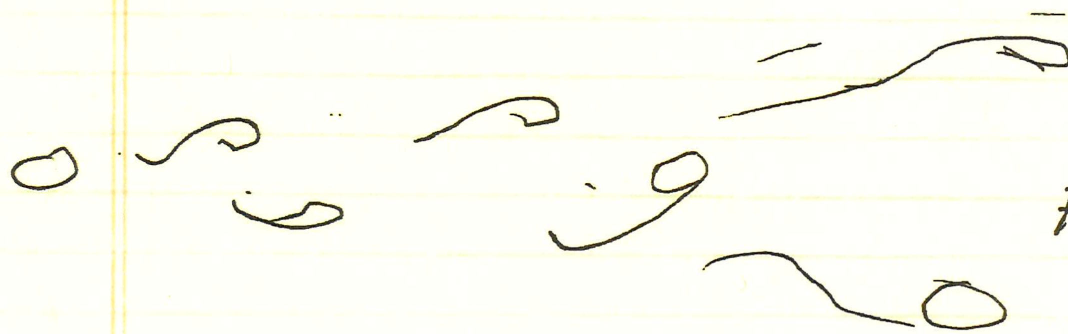
generally $\boxed{\gamma \sim k(\Delta V)}$ ..

→ what happens?

→ vortex roll-up, billows

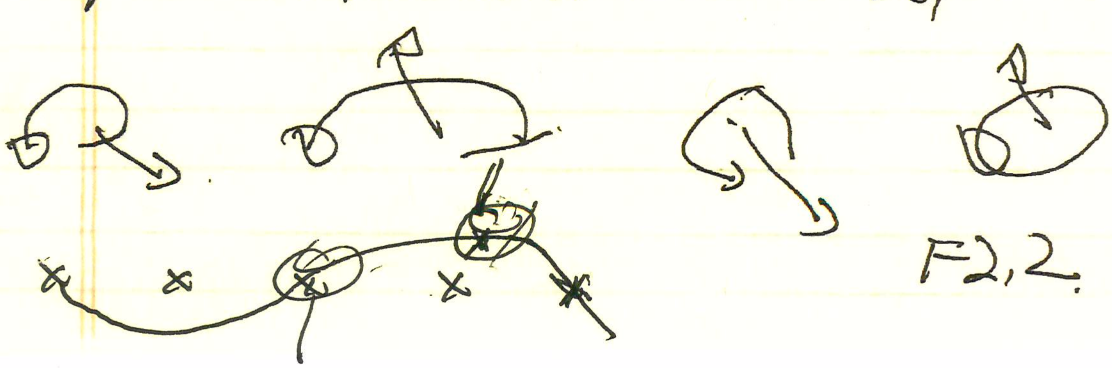
F2.3, F2.4

→ Van-Karman vortex street



Fl.15

→ n.b. array vortex lines unstable w/r displacements as shown



F2.2

Array vortices unstable if vert- maximum.

<u>Refs</u>:

→ R-T, K-H:

· S. Chandrasekhar, "Hydrodynamic and Hydromagnetic Stability"

→ K-H: Falkovich

→ Surface Waves, Surface Tension (Laplace Law), KH: Landau/Lifshitz.

· see also: G. K. Batchelor, "An Introduction to Fluid Mechanics"