Interfacial Instabilities – Supplemental

a.) Instability – why?

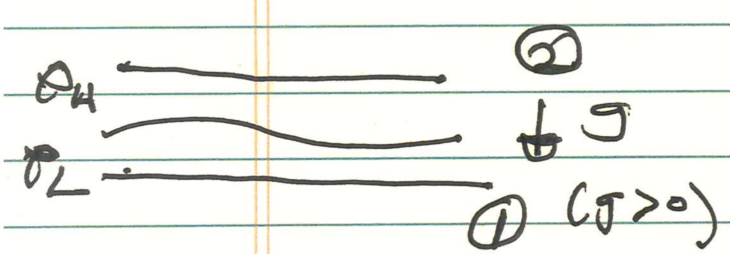
- origin of fluid motion
- relaxation

$$\nabla\rho,\ \nabla T,\ \text{etc} \rightarrow \rho\langle v^2\rangle$$

- onset of chaos/turbulence; symmetry breaking

Look for exponential growth of eigenmodes.

b.) Rayleigh–Taylor (with Interface)



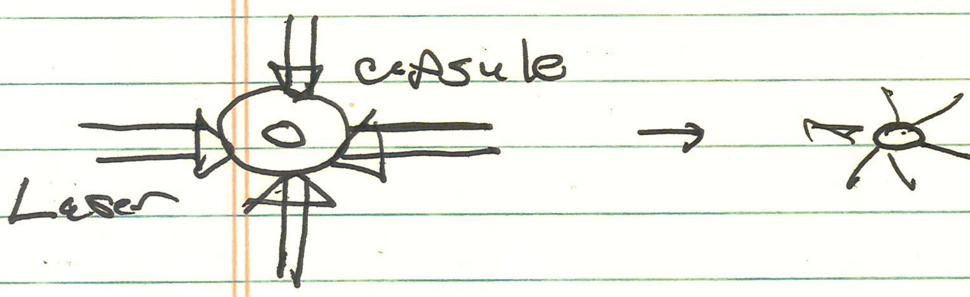
$$\nabla_z \rho > 0$$

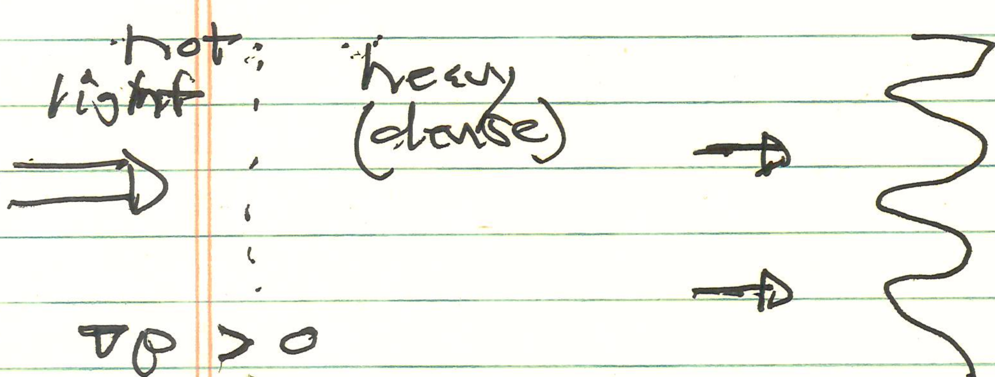
$$\nabla_z P = -\rho g$$

$$-\omega^2 = \gamma^2 = kg\left(\frac{\rho_H-\rho_L}{\rho_H+\rho_L}\right)$$

$$(\nabla P)(\nabla\rho) < 0$$

Why – ICF / implosions





$\nabla \rho > 0$

$\nabla P < 0$

rippled interface
→ breaks symmetry
of implosion

Problem: Continuous gradient case ?!

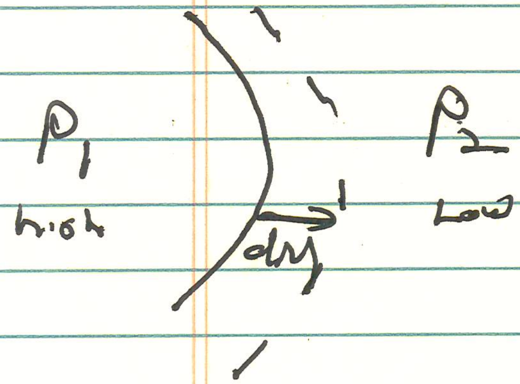
How: – $\nabla^2 \phi = 0$
for fluid, with Bernoulli.

– $\tilde{P}_1 = \tilde{P}_2$ matching
$\tilde{V}_{z_1} = \tilde{V}_{z_2}$

– interface condition,

$\tilde{V}_z = \frac{d\tilde{\eta}}{dt}$ $\eta = \eta(x,t)$ displacement of interface.

(a) Surface Tension



$$dF = -(P_1 - P_2)\,dV + \sigma\, dA$$
1/2

$$dV = dA\, d\eta$$

and can show Laplace's Law

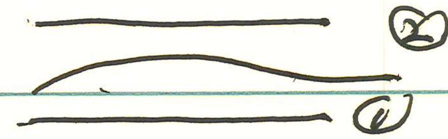
$$\boxed{P_1 - P_2 = \sigma\left(1/R_1 + 1/R_2\right)}$$

$R_1, R_2 \equiv$ radii of curvature of interface.

For small ripple of interface,
(how small is small ?)

$$P_2 - P_1 = \sigma \nabla_\perp^2 \eta$$

$$\boxed{\tilde{P}_1 - \tilde{P}_2 = -\sigma \nabla_\perp^2 \tilde{\eta}}$$



B/

$$\tilde{p}_1 \Big|_0 = \tilde{p}_2 \Big|_0 \quad \rightarrow \quad \tilde{p}_1 - \tilde{p}_2 = +\sigma \nabla_\perp^2 \tilde{\eta}$$

(2 'eqs' into 1, #'s)

$$\tilde{\phi}_1 \Big|_0 = -\tilde{\phi}_2 \Big|_0$$

$$\tilde{p}_2 = \rho_2 \frac{\partial \tilde{\phi}_2}{\partial t} + g \rho_2 \tilde{\eta}$$

$$\tilde{p}_1 = \rho_1 \frac{\partial \tilde{\phi}_1}{\partial t} + g \rho_1 \tilde{\eta}$$

$$\tilde{p}_1 - \tilde{p}_2 = +\sigma \nabla_\perp^2 \tilde{\eta}$$

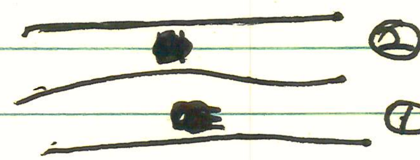
$$\left(\rho_1 \frac{\partial \tilde{\phi}_1}{\partial t} + g \rho_1 \tilde{\eta}\right) - \left(\rho_2 \frac{\partial \tilde{\phi}_2}{\partial t} + g \rho_2 \tilde{\eta}\right) = +\sigma \nabla_\perp^2 \eta$$

$$\tilde{\phi}_1 = -\tilde{\phi}_2$$

$$\boxed{(\rho_1 + \rho_2) \frac{\partial \tilde{\phi}}{\partial t} + g(\rho_1 - \rho_2) \tilde{\eta} = +\sigma \nabla_\perp^2 \tilde{\eta}}$$

and

$$\frac{\partial_t \tilde{\eta}}{\partial t} = \frac{\partial \tilde{\phi}_1}{\partial z}$$



$$(\rho_1+\rho_2)\frac{\partial^2 \tilde{\phi}}{\partial t^2} + g\,(\rho_1-\rho_2)\frac{\partial \tilde{\phi}_1}{\partial z}$$

$$= +\sigma \nabla_\perp^2 \frac{\partial \tilde{\phi}_1}{\partial z}$$

$$\frac{\partial \tilde{\phi}_1}{\partial z} = k\tilde{\phi}_1 \qquad (e^{kz})$$

$$(\rho_1+\rho_2)(-\omega^2\tilde{\phi}) + g(\rho_1-\rho_2)k\,\tilde{\phi}$$

$$= +\sigma k_\perp^2 k\,\tilde{\phi}$$

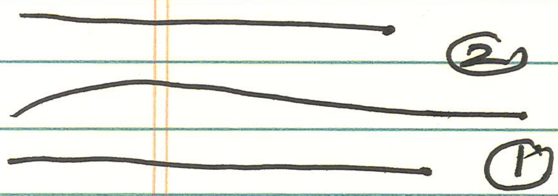
$$k = (k_x^2+k_y^2)^{1/2}$$

$$\partial_z^2\phi - k_\perp^2\phi = 0$$

$$k_\perp^2 = k_x^2+k_y^2$$

thus, have general dispersion relation:

$$\boxed{-\omega^2 = -g\frac{(\rho_1-\rho_2)}{(\rho_1+\rho_2)}k + \frac{T k_\perp^2 k}{(\rho_1+\rho_2)}}$$



② → ρ_H

① → ρ_L → R-T unstable

$$\boxed{\gamma^2 = \frac{g(\rho_H-\rho_L)k}{(\rho_H+\rho_L)} - \frac{T k_\perp^2 k}{(\rho_H+\rho_L)}}$$

→

$\gamma^2 \to 0$ at $\boxed{k_\perp^2 = \frac{g(\rho_H-\rho_L)}{T}}$

i.e. cut-off scale.

If box size L:

$$1/L^2 \sim \frac{g(\rho_H - \rho_L)}{\sigma}$$

specifies maximal density contrast ($\nabla\rho$) that system can hold, based on stability.

Stability limit.

Now, if $\rho_1 \gg \rho_2$

$$\boxed{\omega^2 = g k + \frac{\sigma}{\rho} k_\perp^2 k}$$

stable oscillation.

Gravity - capillary wave

$$gk \gg \frac{\sigma}{\rho} k_\perp^2 k$$

→ gravity wave

$$\omega^2 = gk$$

$$gk \ll \frac{\sigma}{\rho} k_\perp^2 k$$

→ capillary wave / ripple

$$\omega^2 = \frac{\sigma k_\perp^2 k}{\rho}$$

d.

Now can address several phenomena in surface tension.

a

water droplet oscillation? (Kelvin)

(capillary!)

Dimensional analysis:

$$\omega^2 \sim \frac{\sigma/\rho}{R^3}$$

What is # ?

→ think multipoles! $\left\{ \begin{array}{l} Y_{l,m} + \\ \text{symmetry} \end{array} \right.$

– monopole excluded $\omega^2 = 0$ (incompressible!)

$\# \sim l$

– diple excluded $\omega^2 = 0$

$\# \sim l\,(l-1)$

– quadrupole is lowest...

$\# \sim \underbrace{l\,(l-1)}\,(l+2)$

How calculate?

Could consider change in surface area of droplet

$$A = \int_0^{2\pi}\int_0^{\pi} \left[r^2 + \left(\frac{\partial r}{\partial\theta}\right)^2 + \frac{1}{\sin^2\theta}\left(\frac{\partial r}{\partial\phi}\right)^2 \right]^{1/2} r\sin\theta\, d\theta\, d\phi$$

$r = R + \gamma$

then, as before (φ small):

$$A = \int_0^{2\pi}\int_0^{\pi} \left\{ (R+\varphi)^2 + \left\{ \frac{1}{2}\left(\frac{\partial \varphi}{\partial \theta}\right)^2 + \frac{1}{\sin^2\theta}\left(\frac{\partial \varphi}{\partial \phi}\right)^2 \right\} \right\}$$

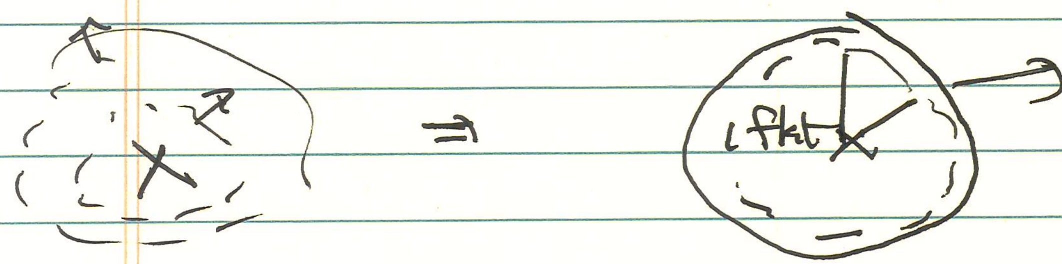
$$\cancel{*} \sin\theta \, d\theta \, d\phi$$

--- etc

and expand in $Y_{\ell m}$, etc.

(b.c. precludes m)

(b) Pond Puddle



- rock in pond → quiescent disk grows as ripples propagates outward.

- within quiescent disk - no waves.

→ rate of expansion ?

$$\omega^2 = gk + \frac{T}{\rho} k k^2 , \quad \omega = \left(gk + \frac{T}{\rho} k^3\right)^{1/2}$$

→ group velocity has minimum

i.e. $V_{gr} = \frac{d\omega}{dk}$

$\frac{dV_{gr}}{dk} \to 0$ at $k^* = \left[\left(\frac{2}{\sqrt{3}} - 1\right)\frac{g\rho}{\sigma}\right]^{1/2}$

$V_{gr}\Big|_{min} = 1.09\left(\frac{g\sigma}{\rho}\right)^{1/4}$

↓

expansion speed.