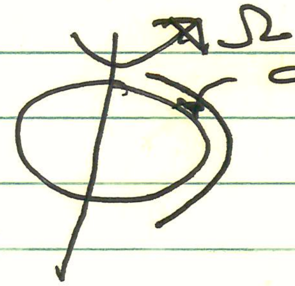
→ Vorticity, cont'd

→ 2D

~ 2D vorticity dynamics of interest in geophysical context

i.e.



– constrained dynamics $\begin{cases} \text{rotation} \\ \text{stratification} \\ \underline{B}_0 \to \underline{\Omega}_0 \end{cases}$

– treat fluid as effectively 2-dimensional

Now,

$$\partial_t \underline{\omega} + \underline{v} \cdot \underline{\nabla}\, \underline{\omega} = \underline{\omega} \cdot \underline{\nabla}\, \underline{v}$$

$$\underline{\omega} = \omega\, \hat{z}$$
$$\underline{v} = v_x\, \hat{x} + v_y\, \hat{y}$$

$$\underline{\omega} \cdot \underline{\nabla}\, \underline{v} = 0$$

→ no vortex tube stretching

$$\partial_t \underline{\omega} + \underline{v}\cdot\underline{\nabla}\,\underline{\omega} = 0$$

$$\partial_t \underline{\omega} + \underline{v}\cdot\underline{\nabla}\,\underline{\omega} = \nu \nabla^2 \underline{\omega}$$

$$\underline{\omega} = \omega(\underline{x})\,\hat{z}$$

$\underline{\nabla}\cdot\underline{v} = 0$ $\qquad\qquad$ $\underline{v} = \underline{\nabla}\phi \times \hat{z}$

$$\underline{\omega} = -\nabla^2\phi\,\hat{z}$$

$$\partial_t \nabla^2\phi + \underline{\nabla}\phi\times\hat{z}\cdot\underline{\nabla}\,\nabla^2\phi - \nu\nabla^2\nabla^2\phi = 0$$

$$\partial_t \rho + \underline{\nabla}\phi\times\hat{z}\cdot\underline{\nabla}\,\rho - \nu\nabla^2\rho = 0$$

i.e. - vorticity conserved (to ν) along particle trajectories.

- system is Hamiltonian

i.e.
$$H = \phi(x, y, t)$$
$$\dot{x} = \partial H/\partial y$$
$$\dot{y} = -\partial H/\partial x \qquad \begin{pmatrix} p \\ x \end{pmatrix}$$

- ρ is Liouvillian.

Eqn. → conservation phase space density

→ Observe:

$$\rho_0 = 1$$

$$E = E_K = \int \frac{v^2}{2} d^3x = \int \frac{(\nabla\phi)^2}{2} d^3x$$

$$\partial_t \int \phi \nabla^2\phi = -\int \phi \, \nabla\phi \times \hat{z} \cdot \nabla \nabla^2\phi$$

$$= +\int [\nabla\phi \cdot \nabla\phi \times \hat{z}] \, \nabla^2\phi$$

$$= 0$$

energy conserved (to ν)

equivalent to:

$$\frac{\partial \underline{v}}{\partial t} - \underline{v} \times \underline{\omega} = -\underline{\nabla}\left(W + \frac{v^2}{2}\right)$$

$\underline{v} \cdot$

$$\partial_t \int \frac{v^2}{2} - 0 = -\int d^3x \, \underline{v} \cdot \underline{\nabla}\left(W + \frac{v^2}{2}\right)$$

$$\partial_t \int \frac{v^2}{2} = -\int d^3x\, \underline{\nabla} \cdot \left[\underline{v} \left(w + \frac{v^2}{2} \right) \right]$$

$$= 0 \,, \quad \text{for} \quad v_n = 0$$

Now, however,

$$\frac{d}{dt} \underline{\omega} = 0$$

trivial to show:

$$\frac{d}{dt} \omega^2 = 0 \quad \rightarrow \quad \partial_t \int \omega^2 = \partial_t \int \frac{(\nabla \phi)^2}{2} d^2x = 0$$

i.e. $\underline{\underline{2}}$ inviscid quadratic conserved quantities

$$E = \int d^2x \frac{(\underline{\nabla}\phi)^2}{2} \quad \rightarrow \text{energy}$$

$$\Omega = \int d^2x \frac{(\nabla^2\phi)^2}{2} \quad \rightarrow \text{enstrophy}$$

mean square vorticity

→ Dual conservation laws makes for profound difference from 3D.

→ Key is absence of vortex tube stretching.

→ Relaxation in 2D — Vorticity Homogenization

- Prandtl - Bachelor Theorem.

- Non-Ideal Process

messages

⇒ Vorticity tends toward homogeneous distribution ⟺ homogenization.

⇒ A little viscosity makes a global difference.

→ Flux Expulsion and $\boxed{\text{Homogenization}}$ — Non-Ideal, cont'd

— so far, have encountered:

→ S-P reconnection ⇒ weak dissipation ($R_m \gg 1$) has strong effect at singularity — BOUNDARY LAYER

→ Taylor Hypothesis ⇒ small flux tubes destroyed by stochasticity, leaving $\int d^3x \; \underline{A}\cdot\underline{B}$ as robust invariant

∴ $\boxed{\text{diffusive dissipation most effective at breaking freezing-in on small scales}}$

Another examples: { singular behavior in 2D, closed-streamline flow

→ $\boxed{\text{Homogenization Theory}}$ → { Prandtl, Batchelor; Weiss; Rhines, Young

recall ω evolution for $\underline{\nabla}\cdot\underline{V} = 0$

$$\boxed{\frac{\partial}{\partial t}\underline{\omega} + \underline{v}\cdot\underline{\nabla}\,\underline{\omega} = \underline{\omega}\cdot\underline{\nabla}\,\underline{v} + \nu\nabla^2\underline{\omega}}$$

$\underline{2D}$ ⇒ $\underline{\omega}\cdot\nabla\underline{v} = 0$ $\qquad \underline{\omega} = \omega(\hat{z})$

$\qquad \underline{v} = \nabla\phi\times\hat{z}$

c.f. P.A. Davidson 2004 — c.f.

A little viscosity ~~goes a long way~~ makes a global difference.

then $\partial_t \omega + \underline{\nabla}\phi \times \hat{z} \cdot \underline{\nabla}\omega = \nu \nabla^2 \omega$

more generally scalar ξ: $\left\{ \begin{array}{l} \text{active} \\ \text{or} \\ \text{passive} \end{array} \right.$

$$\partial_t \xi + \underline{\nabla}\phi \times \hat{z} \cdot \nabla \xi = \nu \nabla^2 \xi$$

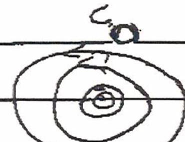
Now: $t \to \infty$, $\partial_t \xi \to 0$

$$\underline{\nabla}\phi \times \hat{z} \cdot \nabla \xi = \nu \nabla^2 \xi$$

$\nu \to 0$ $\quad \underline{\nabla}\phi \times \hat{z} \cdot \nabla \xi = 0$

$Re \sim Pe \sim \frac{VL}{\nu} \to \infty$ $\sim Re$ $\quad \xi = \xi(\phi)$

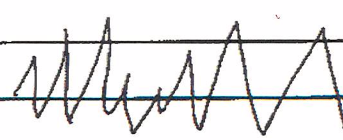
i.e. bounded domain, closed streamline solution



$\to \xi = \xi(\phi(r))$ is arbitrary solution

can develop arbitrarily fine scale $\xi(\phi)$ $\to$ closed streamlines perfect memory $\Rightarrow$

$\to$ $\therefore$ fine scale structure develops, no inter-streamline communication, & persists

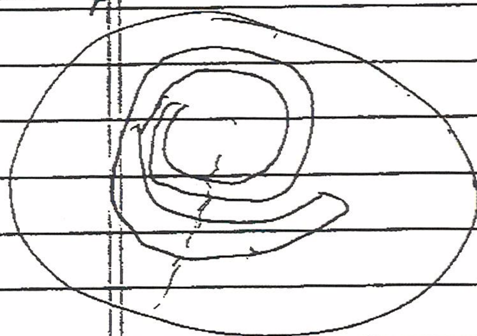
i.e.  $\to$  $\left\{ \begin{array}{l} \text{i.e. tag each} \\ \text{streamline} \\ \text{arbitrarily} \end{array} \right.$

$\sim$ no smoothing of sharp gradients

"Not all solutions of the Navier-Stokes equations are realized in nature."
Landau & Lifshitz

→ i.e. of ω, T → blob in concentric shear flow

blow-up



→ non-diffusive stretching produces arbitrarily fine scale structure!

now, point is that for $\nu \neq 0$
$Re, Pe \gg 1$,
instead of arbitrarily fine scale structure
must have: $\omega(\phi) \to \text{const}$ as $t \to \infty$ — i.e. small ν ⇒ global behavior

⇒ i.e. finite ν at large Re ⇒ vorticity homogenization! — $\omega \to \text{const}$ within C_0

⇒ highly singular behavior!
$\nu = 0$ → Euler Eqn. (2D) → $\omega = \omega(\phi)$ solutions

$\nu \neq 0$ → large Re 2D Navier-Stokes Eqn. ⇒ $\omega = \text{const}$ soln.

note contrast!!

Issues:

→ how long to homogenization? (what means asymptotic)

→ where is $\nabla\omega \neq 0$ ⇒ boundary layer thickness?

→ analogy in MHD? — Flux Expulsion

$$\underline{E} + \frac{\underline{v}\times\underline{B}}{c} = \eta \underline{J} \qquad \begin{aligned} \underline{v} &= \nabla\phi\times\hat{z} \\ \underline{B} &= \nabla A\times\hat{z} \end{aligned}$$

$$-\frac{1}{c}\partial_t \underline{A} - \nabla\phi + \frac{(\nabla\phi\times\hat{z})}{c}\times(\nabla A\times\hat{z}) = \eta \underline{J}$$

$\hat{z}\cdot(\quad)$

$$\Rightarrow -\frac{1}{c}\partial_t A_z - \cancel{\nabla\phi\cdot\hat{z}} + \hat{z}\cdot\left[\cancel{(\nabla\phi\times\hat{z})\cdot\hat{z}}\right]\nabla A$$

$$- \frac{(\nabla\phi\times\hat{z})}{c}\cdot\underline{\nabla A} = \eta \underline{J}$$

$$\therefore \quad \partial_t \underline{A} + \underline{\nabla\phi\times\hat{z}}\cdot\underline{\nabla A} = \eta\nabla^2 A$$

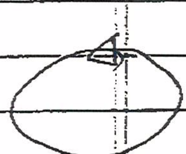
$$\Rightarrow \text{2D convection} \begin{cases} \nabla\cdot\underline{v} = 0 \\ \eta \neq 0 \end{cases}$$

⇒ expect$_{t\to\infty}$ $\nabla A = 0$, except boundaries

→ analogy is "flux expulsion"

E. Prandtl - Batchelor Theorem

* G. Batchelor, JFM 1 177 (1956) (posted)
P.B. Rhines and W.R. Young, JFM 122, 347 '82 (posted)
JFM 133, 133 '83
J. Pedlosky, "Ocean Circulation Theory"
see also Springer 1996, esp. 3.8.



Prandtl - Batchelor Theorem

Thm: Consider a region of 2D incompressible flow (i.e. vorticity advection) enclosed by closed streamline C_0. Then, if diffusive dissipation,

i.e. $\partial_t \omega + \nabla\phi \times \hat{z} \cdot \nabla \omega = \nabla \cdot (\nu \nabla \omega)$

then vorticity → uniform (homogenization), as $t \to \infty$, within C_0.

Proof

N.B.: ① finite ν ⇒ radically different final state

② no comment on "how long"?

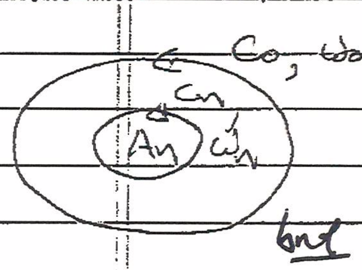
$f \to \infty$ before $\nu \to 0$

$$- \nabla\phi \times \hat{z} \cdot \nabla \omega = \nabla \cdot \nu \nabla \omega$$

for stationarity

[note $f \to \infty$ before $\nu \to 0$]

- choose arbitrary closed C_n within C_0. Here C_n a streamline



n.b. - assume simply connected region, i.e. no holes
- stationarity $\Rightarrow$ ω constant along streamlines
- $C_0 \to$ specified on 1 bndry.

$\therefore$ $\omega \to \omega_0$ on C_0 (ultimately C_0 sets b.c.)

$\omega \to \omega_n$ on C_n

if A_n is area enclosed by C_n;

$$\int_{A_n} d^2x \, \underline{v} \cdot \nabla \omega = \int_{A_n} d^2x \, \nabla \cdot (\nu \nabla \omega)$$

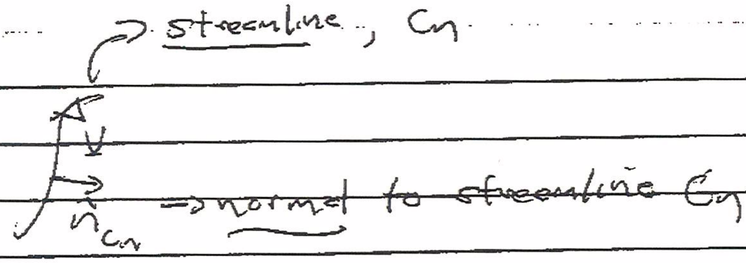
but

$$\int_{A_n} d^2x \, \underline{v} \cdot \nabla \omega = \int d^2x \, \nabla \cdot [\underline{v}\,\omega]$$

$$= \int_{C_n} d\ell \, \hat{n}_{C_n} \cdot (\underline{v}\,\omega)$$

$\hat{n}_{C_n}$: normal

now



so $$\int_{C_n} dl \left(\hat{n}_{C_n} \cdot V\right) \omega = 0$$

as V is along streamline

∴

$$0 = \int_{A_n} d^3x \; \nabla \cdot \left(\nu \nabla \omega\right)$$

$$= \nu \int_{C_n} dl \; \hat{n}_{C_n} \cdot \nabla \omega \quad \Leftarrow$$

now in stationary state must have $\omega \to$ const along streamline

∴ $\omega = \omega(\phi)$

so $\omega_{C_n} = \omega(\phi_n)$

$$0 = \nu \int_{C_n} dl \; \hat{n}_{C_n} \cdot \nabla \phi_n \frac{d\omega}{d\phi_n}$$

$$0 = \nu \frac{d\omega}{d\phi_n} \int_{C_n} dl \; \hat{n}_{C_n} \cdot \nabla \phi_n$$

but

$$\Gamma = \int d\underline{\ell} \cdot \underline{v}$$

$$= \int d\underline{\ell} \cdot (\underline{\nabla}\phi \times \underline{\hat{z}})$$

$$= \int (\hat{z} \times \vec{n}) \cdot (\underline{\nabla}\phi \times \hat{z})$$

$$= -\int d\ell\, (\underline{\nabla}\phi \cdot \vec{n}) = -\int d\ell\, (\underline{\nabla}\phi \cdot \vec{n})$$

$$\boxed{0 = \nu \frac{\partial \omega}{\partial \phi_n} \Gamma_n}$$

$$\partial\omega / \partial\phi_n = 0$$

but ϕ_n arbitrary $\Rightarrow$ $\boxed{\partial\omega/\partial\phi = 0, \text{ all } \phi}$

arbitrary $\Rightarrow$ no variation from line to line

$\Rightarrow$ $\boxed{\omega \text{ homogenized}}$!

C bnd bounding stuff

so, expect $\nabla\omega$ larger at bounding contour C_0.

$\nabla\omega \to 0$, within $\Rightarrow$ $\nabla\omega$ held at boundary !!

Some Comments:

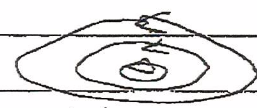
⇒ Homogenization theory looks 'magical' → caveat emptor!

i.e.

*(1.) note assumptions of:

$t \to \infty$ ⇒ time asymptopia

$q = q(\phi)$ ⇒ concentric streamlines

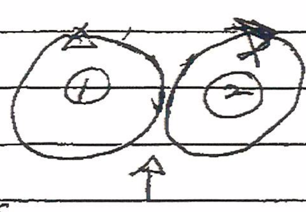


how long to achieve configuration?!

2.) simply connected domain ⇒ annulus? (2 F ?)

3.) single structure → expulsion from neighbors and possible interaction not addressed

i.e. what happens if? → 'interference of boundary layers'?



or

⇒ straining interaction

⇒ ② 'strips' ① etc.

4.) Key Assumptions:

→ { closed, bounding streamline
{ viscous dissipation

i.e. can envision:

→ [exact streamline, molecular viscosity
or
→ [coarse-grained streamline, eddy viscosity

⇒ correspond to homogenization of
→ total vorticity
→ mean/coarse-grained vorticity

⇒ time scales different

→ $\boxed{\dfrac{\tau_{circulation}}{\tau_{diffusion}} \ll 1 \rightarrow Re \gg 1 \quad \leftarrow L \rightarrow}$

– to establish concentric circulation lines
then
– diffuse across to homogenize → but slow ??!

$$\frac{\tau_c}{\tau_d} = \frac{1}{(V/L)} \frac{D}{L^2} \ll 1 \Rightarrow \frac{D}{VL} \ll 1$$

i.e. $Re \gg 1$

or equivalently $\frac{VL}{D} \gg 1$

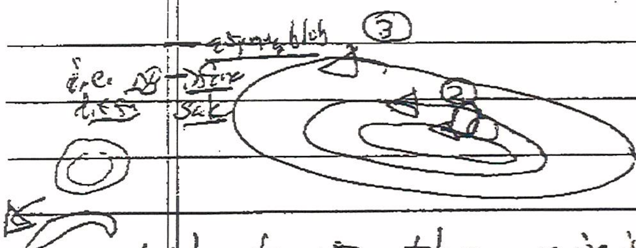
i.e. $(Re)_{\substack{eff \\ cell}} \gg 1$ $Pe \gg 1$

related: – essential idea is that q constant along streamlines established on fast ($\sim \tau_c$) scale
few

– dissipation homogenizes on slower ($\sim \tau_D$) time scale
(but this is slow...)

→ What are the time scales? – {how resolve slow time scale problem

– useful to consider differentially rotating sheared flow within closed pattern



$V_1 \neq V_2 \neq V_3$

need flows with finite l_y

What is the mixing time scale?

① finer, finer scale ....

② key: synergism between {shear, diffusion

$\boxed{\text{shear dispersion}}$!

c.f. {H. Biglari, P.H. Diamond, P.W. Terry, Phys Fluids (B2), 1, 1990
(first noted by G.I. Taylor)

→ 3D structures

- Recall vorticity structures

  - tubes
  - rings
  - sheets
  - lines

  — dynamics → Kelvin

  - 'spot' – 2D (phase element)

  - helicity

- In 3D, can identify quadratic inviscid invariant

$$H = \int \underline{v} \cdot \underline{\omega} \, d^3\underline{x}$$

→ pseudoscalar

$\underline{x} \to -\underline{x}$

to show conserved:

(1) $\dfrac{\partial \underline{v}}{\partial t} - \underline{v} \times \underline{\omega} = -\underline{\nabla}\left(W + \dfrac{v^2}{2}\right) + \nu \nabla^2 \underline{v}$

(2) $\dfrac{\partial \underline{\omega}}{\partial t} = \underline{\nabla} \times \underline{v} \times \underline{\omega} + \nu \nabla^2 \underline{\omega}$

$\underline{\omega} \cdot 1 + \underline{V} \cdot 2 \Rightarrow$

$$\partial_t \int d^3x \, \underline{V} \cdot \underline{\omega} = \overset{②}{\int \underline{V} \cdot \underline{\nabla} \times [\underline{V} \times \underline{\omega}] \, d^3x}$$

$$\overset{①}{- \int \underline{V} \cdot \underline{\nabla} \left(w + \frac{V^2}{2} \right) d^3x}$$

$$\overset{③}{+ \int d^3x \left[\underline{\omega} \cdot \nu \nabla^2 \underline{V} + \underline{V} \cdot \nu \nabla^2 \underline{\omega} \right]}$$

$$① = - \int d^3x \, \underline{\nabla} \cdot \left[\underline{V} \left(w + \frac{V^2}{2} \right) \right]$$

for $V_n \big|_{surf} = 0$

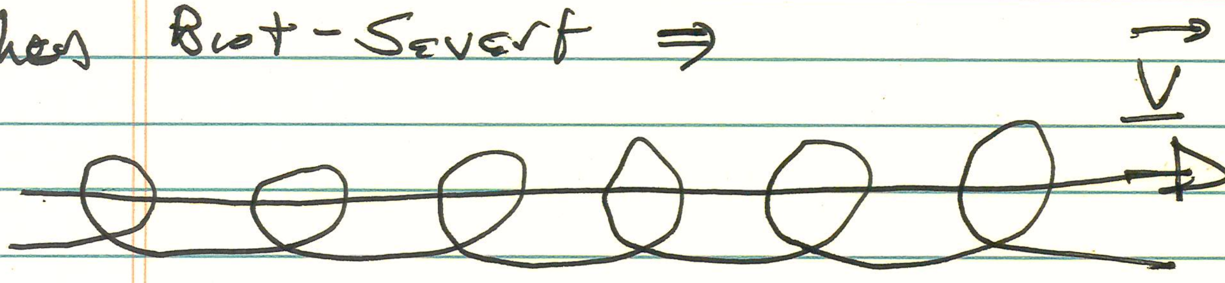
$$① = 0$$

$$② = \int \underline{\nabla} \cdot \left[\underline{V} \times (\underline{V} \times \underline{\omega}) \right] - \int d^3x \, (\cancel{\underline{\nabla} \times \underline{V}}) \cdot (\underline{V} \times \underline{\omega})$$

surf

$= 0.$

then Biot-Savart $\Rightarrow$

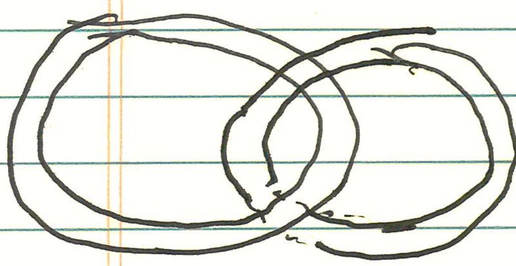


solenoid $\Longleftrightarrow \langle \underline{v} \cdot \underline{\omega} \rangle \neq 0$

Helicity $\rightarrow$ - helical symmetry in flow

- breaking of reflection symmetry

And/or: Linkage — topological



2 linked vortex rings

$\rightarrow \Phi_\omega$

$$\int d^3x \, \underline{v} \cdot \underline{\omega} = \left(\int d^2x \, \underline{\omega} \cdot d\underline{a} \right) * \left(\oint \underline{v} \cdot d\underline{l} \right)$$

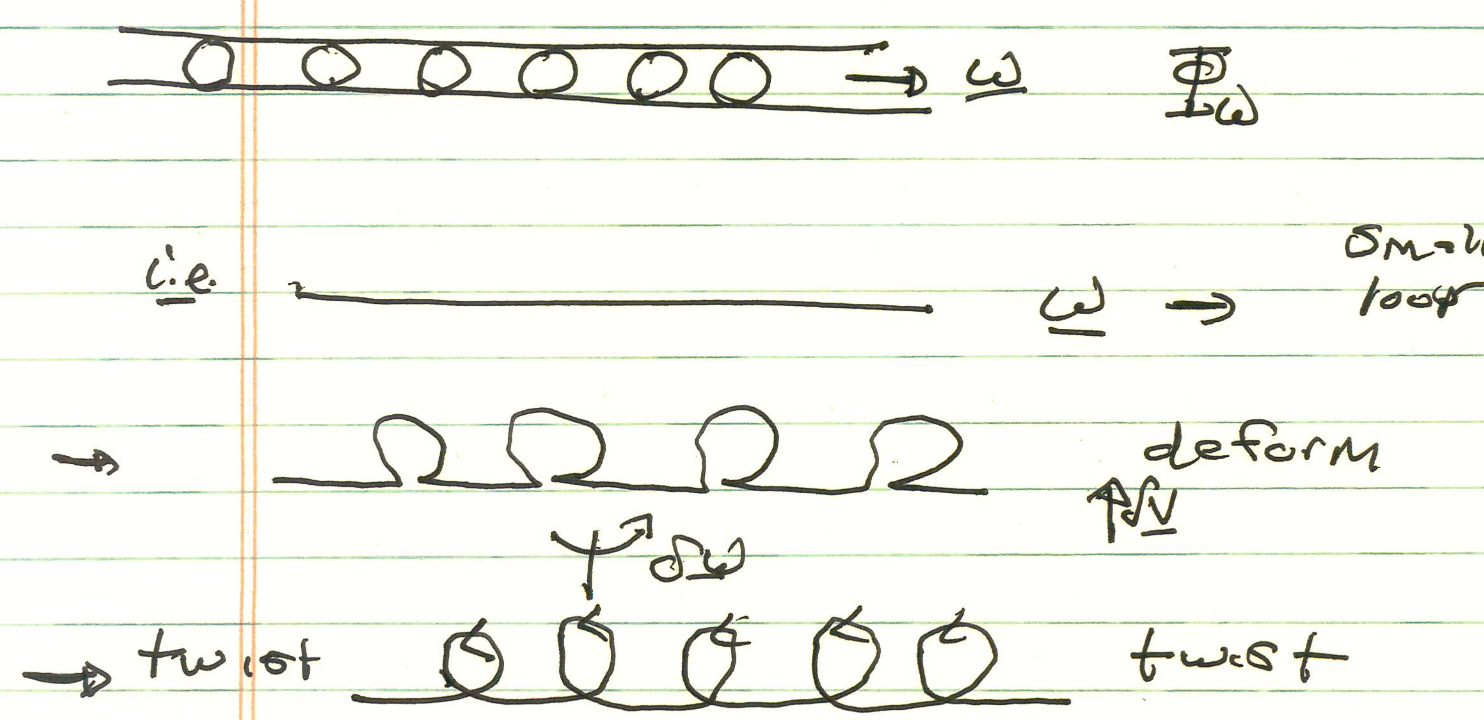
$$\partial_t \int d^3x \left[(\underline{V}\cdot\underline{\omega}) \right] = 0 \quad + O(\nu)$$

Fluid Helicity conserved.

Invariant invariant.

What does it mean ?

– consider a vortex tube



and can show;

$$\int \underline{V} \cdot \underline{\omega} = 2 \, \Phi_{\omega_1} \Phi_{\omega_2}$$