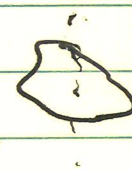
Vorticity, Vortices, Vortex Physics (3D)
$\rightsquigarrow$ A Selection

Recall $\underline{\omega} = \underline{\nabla} \times \underline{V}$

$\underset{=}{I}\,\underline{\omega} = \underline{L}$ [see Lighthill]

moment



leads to:

diffusion due viscosity

$$\partial_t \underline{\omega} = \underline{\nabla} \times (\underline{V} \times \underline{\omega}) + \nu \nabla^2 \underline{\omega}$$

induction equation

$\underline{\omega}$ frozen-in

$$\partial_t \underline{\omega} + \underline{V} \cdot \underline{\nabla}\,\underline{\omega} = \underline{\omega} \cdot \underline{\nabla}\,\underline{V} + \nu \nabla^2 \underline{\omega}$$

$$= \underline{\omega} \cdot \underset{=}{S}_R + \nu \nabla^2 \underline{\omega}$$

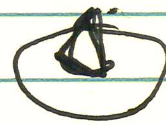
$\left\{ \begin{array}{l} \underline{\nabla} \cdot \underline{V} = 0 \\ P = P(\rho) \\ ds = 0 \end{array} \right.$

$\left\{ \underset{=}{S}_R = \underline{\nabla}\,\underline{V} \right.$ rate of strain

Conservation Theorems:

Key Thm:

$\nu \to 0$



$$\frac{d}{dt}\left(\oint \underline{v}\cdot d\underline{\ell}\right) = 0$$

Kelvin

and equivalently:

$$\frac{d}{dt}\int \underline{\omega}\cdot d\underline{a} = 0$$

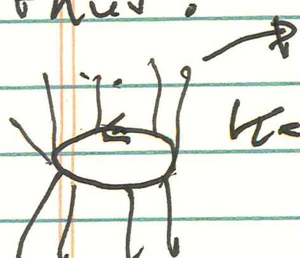
i.e. latter form, for small surface areas (diff'nt)

$$\boxed{\frac{d}{dt}\left[\omega S\right]} \to \boxed{\frac{d}{dt}\left[\omega\, dS\right] = 0}$$

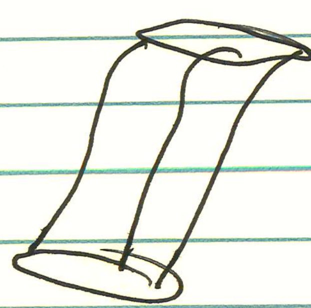
i.e. conservation of vortices

Structures

Further: lines $\underline{\omega}$



Kelvin loop →



Vortex tube

"Vortex line": $\frac{dx}{\omega_x} = \frac{dy}{\omega_y} = \frac{dz}{\omega_z}$

"Tube": – bundle of lines piercing loop.

– "bundle" defines surface + contents = tube.

Now, heuristically, for incompressible flow

$\bar{V} = \sigma \ell = \text{const.}$

N.B.: Volume must either:

– be a ring → close on self.

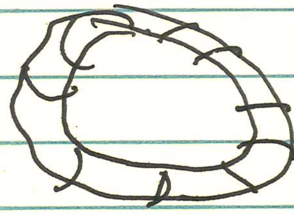
⇒ vortex ring

(a.k.a. smoke ring)

– connect to boundary.

reason: $\underline{\nabla} \cdot \underline{\omega} = 0$

So, for ring, $\ell = R$



$s R \cong \text{const}$ $\qquad -ds\,R = dR\,s$

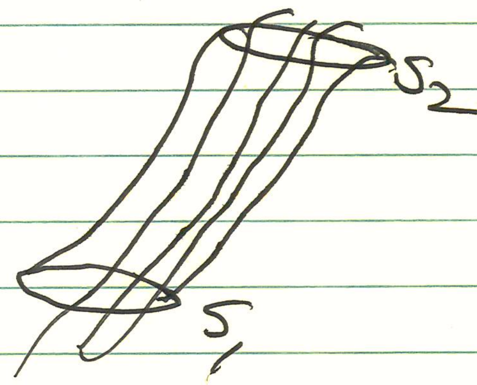
$$\frac{d}{dt}\left[\omega \,\text{[redacted]}\, s\right] = \frac{d}{dt}\left[\omega / R\right] \cong 0$$

$$\boxed{\omega / R \cong \text{const}, \text{ up to } \nu \text{ for vortex ring.}}$$

For vortex tube, can show:

$\int \underline{\omega} \cdot \hat{n}\, ds = \text{const}$ for all surfaces spanning a given vortex tube.

Proof:



$$\underline{\nabla} \cdot \underline{\omega} = 0$$

$$\int d^3x \ \underline{\nabla} \cdot \underline{\omega} = 0$$

$\Rightarrow$

$$\int \underline{\omega} \cdot d\underline{S} = 0$$

in particular, as $\underline{\omega} \perp$ sides,

$$\int dS_1 \ \underline{\omega} \cdot \hat{n}_1 + \int dS_2 \ \underline{\omega} \cdot \hat{n}_2 = 0$$

so

$$\int dS_1 \ \underline{\omega} \cdot \hat{n}_1 = -\int dS_2 \ \underline{\omega} \cdot \hat{n}_2$$

but $\hat{n}_2$ opposite $\hat{n}_1$. If reverse direction,

$$\int dS_1 \, \underline{\omega} \cdot \hat{n}_1 = \int dS_2 \, \underline{\omega} \cdot \hat{n}_2 .$$ ✓

Of course, <u>lines</u> frozen into flow.

→ Flow due Vorticity.

Follow Biot-Savart :

$$\underline{\nabla} \times \underline{v} = \underline{\omega}$$

Given $\underline{\omega}(x)$ – localized vortex patch

→ Flow ?

Let $\underline{v} = \underline{\nabla} \times \underline{A}$

↑ vector potential

$$\cancel{\underline{\nabla}(\underline{\nabla} \cdot \underline{A})} - \nabla^2 \underline{A} = \underline{\omega}$$

Gauge: $\underline{\nabla} \cdot \underline{A} = 0$

$$\nabla^2 \underline{A} = -\underline{\omega}$$

$$\underline{A} = \int d\underline{x}' \frac{\underline{\omega}(\underline{x}')}{4\pi |\underline{x} - \underline{x}'|}$$

so

$$\underline{\nabla} \times \underline{A} = \underline{v} = \int d\underline{x}' \frac{\nabla \times \omega(\underline{x}')}{4\pi |x - x'|}$$

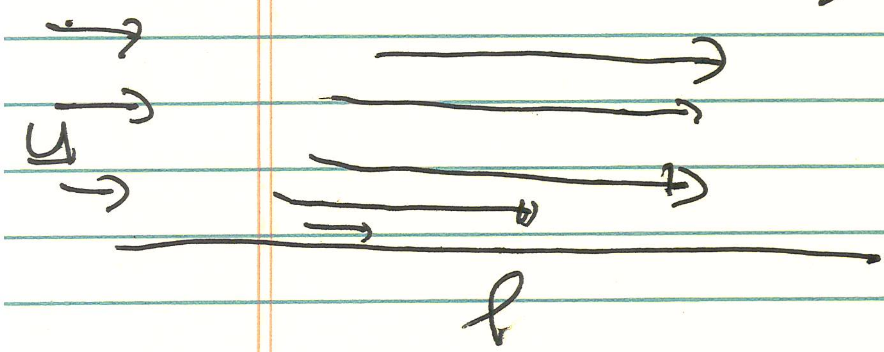
$$= \underline{\nabla} \times \int d\underline{x}' \frac{\underline{\nabla} \times \underline{\omega}(x')}{4\pi |x - x'|}$$

etc.

Point: $\underline{v} \sim 1/r^2$

→ Boundary + Vorticity

Vorticity is often associated with boundaries, and so with viscosity. Why?



- flow over boundary:
  - forms boundary layer near surface, due no slip.
  - why? → ultimately must match to potential flow far away, and $\underline{u} = 0$ at wall
  - typically:

$(\nu T)^{1/2}$

$$W \sim \left(\frac{\nu l}{u}\right)^{1/2} \sim l\left(\nu / u l\right)^{1/2}$$

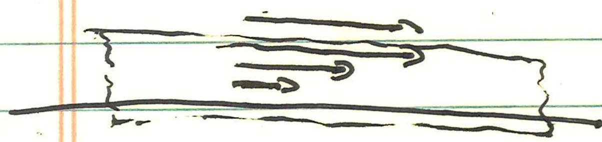
Blasius

$w \sim \ell / Re^{1/2}$

i.e. Layer is thin

$w \ll \ell$, for $Re \gg 1$

— so:



"Stokesian pill-box":

$U\ell - 0 = \Gamma$

so $\omega \sim \Gamma / A \sim U\ell / w\ell$

$\sim U/w$

can replace boundary layer
equate

with <u>vortex</u> <u>sheet</u>

x = out board

x x x x x x

$\underline{\omega} = u/w$, $\downarrow$ $\sim\nu$

$\omega\, dl_{\perp} \sim \Delta u \sim U$.

$\downarrow$ jump in velocity in layer.

$\Rightarrow$ Natural event that vortex sheets form near boundaries

$\rightarrow$ no slip condition.

$\rightarrow$ Vorticity Evolution

Recall:

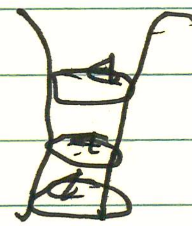
$$\frac{d\underline{\omega}}{dt} = \partial_t \underline{\omega} + \underline{v}\cdot\underline{\nabla}\,\underline{\omega} = \underline{\omega}\cdot\underline{\nabla}\,\underline{v} + \nu\, \underline{\nabla}^2 \underline{\omega}$$

ignoring viscosity,

$$\frac{d\underline{\omega}}{dt} = \underline{\omega}\cdot\underline{\nabla}\,\underline{V}$$

$\downarrow$ stretching

for $\underline{\omega} = \omega\hat{z}$



$$\frac{d\omega_z}{dt} = \omega_z\,\underbrace{\partial_z V_z}$$

$\downarrow$ vertical rate of strain stretches vortex tube

"vortex tube stretching"

(central to turbulence).

obviously $\partial_z V_z > 0$ (for $\omega_z > 0$)
amplifies vorticity of tornado

$\Rightarrow$ high wind, Toto blown away from Dorothy, etc.

$\partial_z V_z > 0$? $\Rightarrow$ thermal updrafts!

$\rightarrow$ heating of surface

N.B. Vortex structures:

- line
- tube
- ring
- sheet

⇒ use to describe geometry and structure of flow.