→ Blasius
→ Prandtl
→ Ekman

Lecture VIII — Laminar Boundary Layers

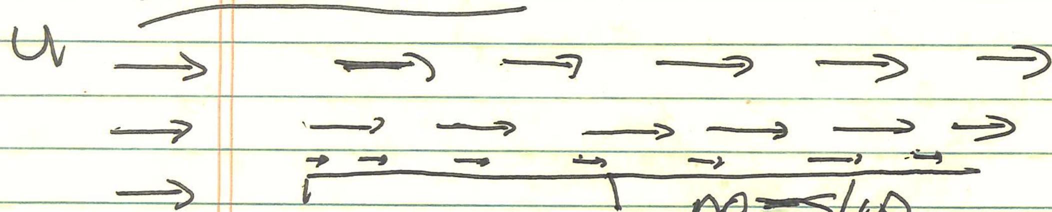
{ Acheson
{ Landau/Lifshitz

viscous fluid

①

- Consider a flat plate immersed in flow

Blasius B.L.

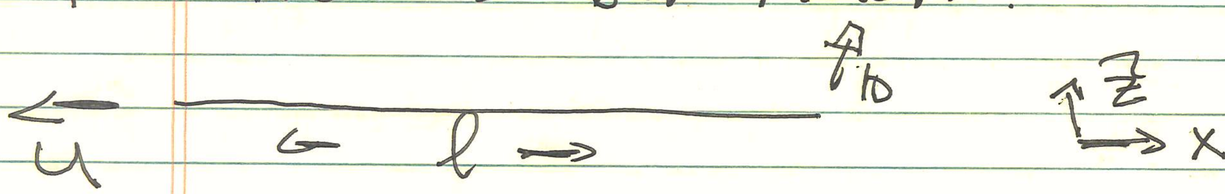


$Re > 1$
but
not
turbulent

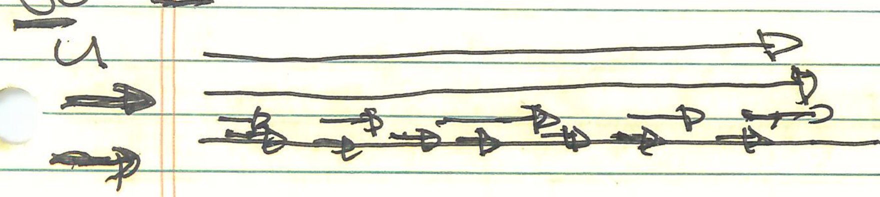
~ away from plate, flow is as imposed

~ near plate, no-slip B.C.
$V_x(0) = 0$ forces boundary layer
which connects:
- near plate → viscous
- mean flow U
(i.e. potential)

for practical question: calculate drag on plate of width b, length ℓ for $Re > 1$ but laminar.



so =



then as before:

$$\underline{V}_T = U\hat{x} + \underline{V}$$

$$\partial_t \underline{V} + \underline{V}\cdot\underline{\nabla}\,\underline{V} - \nu\nabla^2\underline{V} = -\frac{\underline{\nabla} p}{\rho}$$

$$U\partial_x \underline{V} - \nu(\partial_x^2 + \partial_z^2)\underline{V} = -\frac{\underline{\nabla} p}{\rho}$$

$$\partial_x^2 \sim \frac{1}{l_x^2} \sim \frac{1}{l^2} \qquad \text{(anisotropy)}$$

$$\partial_z^2 \sim \frac{1}{w^2} \qquad w^2 \ll l^2$$

so, noting P irrelevant,

$$U\partial_x \underline{V}_x - \nu\partial_z^2\underline{V}_x \simeq 0$$

$$\Rightarrow \boxed{w \sim (\nu x/U)^{1/2}} \rightarrow \text{thickness B.L. at location } x \text{ in B.L. (and wake)}$$

width thickens downstream.

Now, for drag

$$F_d = -\eta b\int_0^l \frac{\partial V_x}{\partial z}\,dx$$

$$\simeq -\eta b\int_0^l \frac{\Delta V_x}{w(x)}\,dx$$

$$\therefore \quad F_d \sim -\eta b U \int_0^{l} \frac{1}{\left(\frac{\nu x}{U}\right)^{1/2}}\, dx$$

$$= -\eta b \frac{U^{3/2}}{\nu^{1/2}} l^{1/2}$$

$$= -\rho \nu^{1/2} U^{3/2} b l^{1/2}$$

$$= -\rho \frac{U^2 l b}{(Re)^{1/2}} \qquad Re \sim \frac{U l}{\nu}$$

so

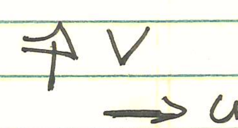
$$\boxed{\begin{aligned} F_d &\sim \rho U^2 A / \sqrt{Re} \\ &\sim \rho \nu^{1/2} U^{3/2} b \underline{l}^{1/2} \end{aligned}}$$

compare Stokes:

$$F_d \sim 6\pi \eta l$$

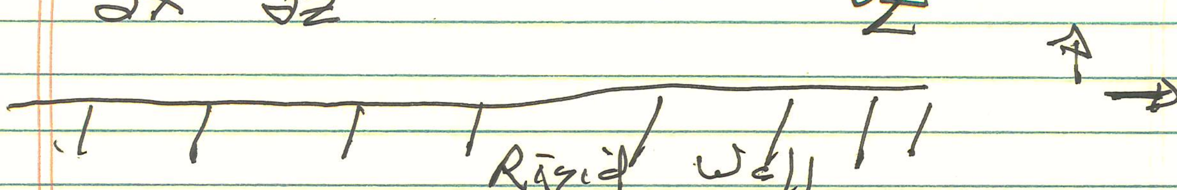
⇒ Blasius B.L. thickens with length along plate

→ Now, Prandtl Boundary Layer Analysis



$$u \partial_x u + v \partial_z u = -\frac{1}{\rho}\frac{dp}{dx} + \nu \frac{\partial^2 u}{\partial z^2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial z} = 0$$



Key ideas:

→ variation in z more rapid than in x

→ variation of u with z enough so ν non-negligible.

Now, need require:

$u \to U(x)$ as $z/w \to \infty$

so, for plate, ($p + \frac{1}{2}U^2\rho \sim$ const at edge BL

$U \to$ const. here

$p \to$ const.

$$u\,\partial_x u + v\,\partial_z u = \nu \frac{\partial^2 u}{\partial z^2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial z} = 0$$

→ z dependence is strong fcn

know $w(x)$ → scale

$(\nu x/U)^{1/2}$

$x, z \to \eta = \frac{z}{w(x)}$

Seek similarity solution

$u = u(\eta)$, $\eta = \cancel{z}/g(x)$

i.e. - reduce pde to ode

- guess $g(x) \to w = (\nu x / U)^{1/2}$

can show.

stretched coord.

2D, $\nabla \cdot v = 0$

$v = \nabla\phi \times \hat{y}$ stream function ψ

$$\Rightarrow \boxed{u = \partial\psi/\partial z\,, \quad v = -\partial\psi/\partial x}$$

$$u = U\,\underbrace{h(\eta)}_{\text{some fctn}}$$

then $$\boxed{\frac{\partial\psi}{\partial z} = U h(\eta)}$$

so

$$\psi = U g(x)\int_0^{\eta} h(s)\,ds + k(x)$$

want $\psi = 0$ at $\eta = 0$
(plate a streamline)

$\Rightarrow k(x) = 0.$

re-write:

$$\boxed{\psi = U g(x) f(\eta)}\,, \qquad f(\eta) = 0.$$

so $' \Rightarrow df/d\eta$

$$u = U g f'(\eta)$$

$$v = -U\left(g'(x) f + g f'(\eta)\frac{\partial\eta}{\partial x}\right)$$

so

$$v = -\frac{\partial \psi}{\partial x} = u\,(\eta f'(\eta) - f)\, g'(x)$$

so now plug into basic flow eqn

$$u\,\partial_x u + v\,\partial_z u = \nu\,\partial_z^2 u$$

where

$$u = u g f'$$

$$\partial_x u = u g' f' + u g f''\left(\frac{-z}{g^2}\right) g'$$

$$v = u(\eta f' - f) g'$$

$$\partial_z v = u\left(\frac{1}{g} f' + \eta f'' \frac{1}{g} - \frac{f'}{g}\right) g'$$

$$\nu\,\partial_z^2 u = \nu u g \left(f'' \frac{1}{g}\right)'$$

$$= \nu u g \left(\frac{f'''}{g^2}\right)$$

assembling:

$$\boxed{-u^2 f' f'' \frac{z g'}{g^2} + u^2(\eta f' - f) g' \frac{f''}{g} = \nu u \frac{f'''}{g^2}}$$

$$-u f' f'' z g' + u(\eta f' - f) g g' f'' = \nu f'''$$

$$-u g' f f'' z + u \frac{z}{\cancel{g}} f f'' \cancel{g} g' - u f f'' g g' = \nu f'''$$

Finally:

$$\boxed{f''' + \frac{u g g'}{\nu} f f'' = 0}$$

here: $f = f(\eta)$

$\eta = z / g(x)$

Now, idea of similarity solution is to reduce pde to ode in η.

So, as g to be specified:

$$g g' = \nu / u \qquad g = g(x)$$

$$\left(\frac{g^2}{2}\right)' = \nu / u$$

$$g^2 = \frac{2 \nu x}{u} + \cancel{c}$$

thickness must vanish at leading edge.

so $g(x) = \left(\frac{2\nu x}{U}\right)^{1/2}$

no surprise!

$$\Rightarrow \boxed{\begin{aligned} \psi &= (2\nu U x)^{1/2} f(\eta) \\ \eta &= z \Big/ \left(\frac{2\nu x}{U}\right)^{1/2} \end{aligned}}$$

and $\boxed{f''' + f f'' = 0}$ requires numerics

with b.c.'s:

$f'(\infty) = 1$ $\quad u \to U$ at $z \to \infty$

$f(0) = f'(0) = 0$ $\quad \Rightarrow$ no slip

and

$$\nu \frac{\partial u}{\partial z} = \nu U \left(\frac{U}{2\nu x}\right)^{1/2} f''(0)$$

N.B. $\partial_x U + \partial_z V = 0$

$$\frac{U}{l} \sim \frac{V}{\left(\frac{\nu l}{U}\right)^{1/2}} \qquad \underbrace{\frac{V}{U} \sim 1/Re}_{V \text{ small.}}$$

and, define drag coefficient C_d via:

defn.

$$C_d = \left[\frac{F}{L} \Big/ \tfrac{1}{2}\rho U^2\right] 2\ell$$

2D defn.

↳ for 2D flow.

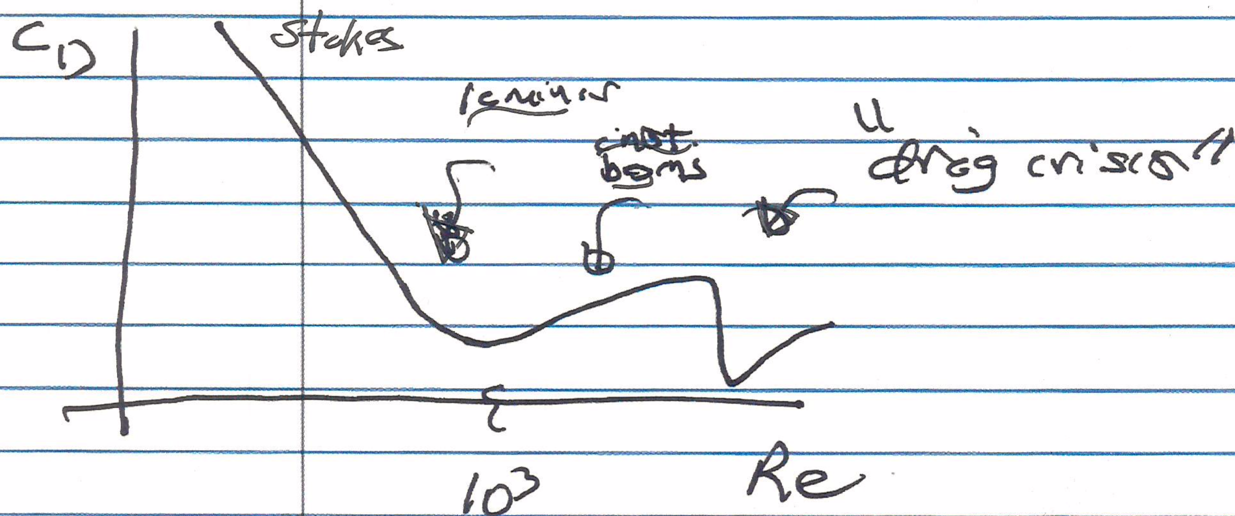
$$\sim \# / \sqrt{Re}$$

in general

$$C_d = F / \tfrac{1}{2}\rho U^2 S$$

S: surface area.

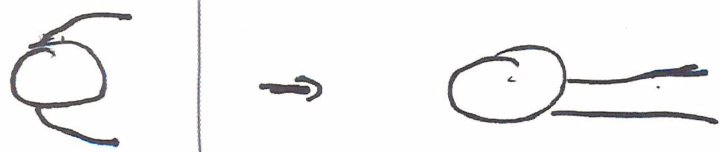
Comments on Drag:



→ $Re > 1$ ⇒ laminar BL

→ $Re > 10^3$ ⇒ instability, separation begin, reach body.

C_D rises



8b.

→ turbulence onset ⇒ C_D drops

'drag crisis'

~ B.L. energized

Why golf balls have dimples

B.L. flow with turb. can beat viscous losses.

~ C_D indep. Re ⇒ turbulence physics

→ Wakes

For laminar wake — see Lecture 7.

Point of wake: Viscosity changes global structure of flow.

C.f.: "Finnegans Wake"

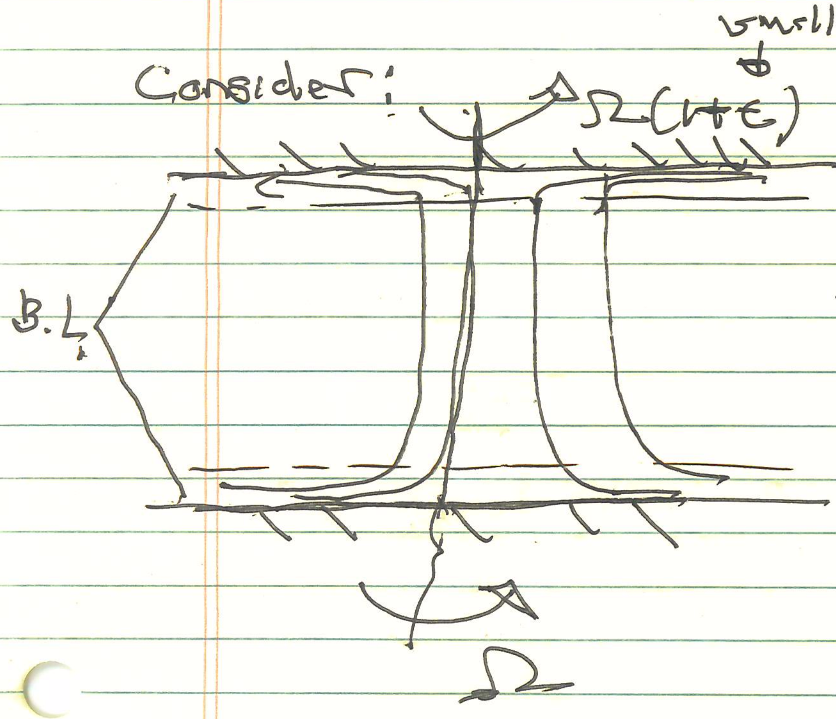
by James Joyce.

⇒ classical Joyce novel
→ incomprehensible
→ origin of "quark" (word):
("3 quarks for Muster Mark")

⇒ not related to fluid dynamics except possibly beer consumption....

(b) Ekman Layer – Rotating Fluid

Consider:



- two rotating boundaries, slight difference
- "suction" $\Rightarrow w$!

Now, recall for rotating fluid:

$$\frac{\partial \underline{V}}{\partial t} + 2\underline{\Omega} \times \underline{V} = -\frac{\underline{\nabla} P^*}{\rho} + \nu \nabla^2 \underline{V}$$

P^* absorbs centrifugal

$$\underline{\nabla} \cdot \underline{V} = 0$$

For steady flow, $\nu = 0$, I = inviscid

$$-2\Omega v_I = -\frac{1}{\rho}\frac{\partial P_I}{\partial x}$$

$$2\Omega u_I = -\frac{1}{\rho}\frac{\partial P_I}{\partial y}$$

$$0 = -\frac{1}{\rho}\frac{\partial P_I}{\partial z}$$

$\left\{ \begin{matrix} u \\ v \\ w \end{matrix} \right\}$ uniform

$(0, 0, \Omega)$

$$\partial_x u_I + \partial_y v_I + \partial_z w_Z = 0$$

$\Rightarrow$

P_I indep z, i.e. $P_I = P(x, y)$

so u_I, v_I indep z

and using u_I, v_I in $\underline{\nabla} \cdot \underline{v} = 0$

$\Rightarrow$

$$\frac{\partial w_I}{\partial z} = 0$$

so $\underline{v}$ indep z $\to$ T-P Thm. !

$\rightsquigarrow$ Consider B. L. at $\underline{z = 0}$:

Now viscous effects sensitive to z $\Rightarrow$ i.e. Boundary Layer depends z, so

for $z \sim 0$ BL:

$$-2\Omega v = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial z^2}$$

$$2\Omega u = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \frac{\partial^2 v}{\partial z^2}$$

$$0 = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \frac{\partial^2 w}{\partial z^2}$$

$$\partial_x u + \partial_y v + \partial_z w = 0$$

Now $\frac{\partial w}{\partial z} + \underline{\nabla}_\perp \cdot \underline{v}_\perp = 0$

so

$|w| \ll |\underline{v}_\perp|$ as $\Delta z \ll r_{typical}$

$\Rightarrow |\nabla_\perp P| \gg |\nabla_z P|$

$\Rightarrow P \cong P(x, y)$, only

{ view flow eqns as for P }

so

$$\frac{\partial P}{\partial x} \cong \frac{\partial P_I}{\partial x} \Rightarrow -\frac{1}{\rho}\frac{\partial P}{\partial x} \cong -2\Omega V_I$$

$$\frac{\partial P}{\partial y} \cong \frac{\partial P_I}{\partial y} \Rightarrow -\frac{1}{\rho}\frac{\partial P}{\partial y} \cong +2\Omega U_I$$

so

$$-2\Omega(V - V_I) = \nu \frac{\partial^2 U}{\partial z^2} \qquad (1)$$

$$+2\Omega(U - U_I) = \nu \frac{\partial^2 V}{\partial z^2} \qquad (2)$$

(2) * i

$$\nu \frac{\partial^2 f}{\partial z^2} = \left[(U - U_I(x,y)) + i\,(V - V_I(x,y))\right] 2\Omega i$$
$$= f(2\Omega i)$$

$f = (U - U_I) + i(V - V_I)$ $\quad$ U_I, V_I no z dependence!

so

$$\nu \frac{\partial^2 f}{\partial z^2} = 2\Omega i f$$

$\Rightarrow$

$$f = A e^{-(1+i) z_*} + \cancel{B e^{(1+i) z_*}}$$

$$z_* = z \Big/ \left(\frac{\nu}{\Omega}\right)^{1/2}$$

Ekman B.L has thickness $d \sim (\nu/\Omega)^{1/2}$

Now, to match:

$f \to 0$ as $z_* \to \infty$
$\left(\begin{array}{l} u \to u_I \\ v \to v_I \end{array}\right)$

$\Rightarrow B = 0$ unphysical

And since $u = v = 0$ at $z = 0$
(boundary at rest in rotating frame) $\Rightarrow$ <u>no slip</u>

$$f = -(u_I + i v_I)\, e^{-(1+i) z_*}$$

$\Rightarrow$ finally,

$$\boxed{\begin{aligned} u &= u_I - e^{-z_*}\left(u_I \cos z_* + v_I \sin z_*\right) \\ v &= v_I - e^{-z_*}\left(v_I \cos z_* - u_I \sin z_*\right) \end{aligned}}$$

Now, further can note:

$$\frac{\partial w}{\partial z} = \left(\frac{\Omega}{\nu}\right)^{1/2} \frac{\partial w}{\partial z_*} = -\left(\partial_x u + \partial_y v\right)$$

upon plug in $\quad \underline{u}_I = \nabla p \times \hat{z}$

$$= \left(\frac{\partial v_I}{\partial x} - \frac{\partial u_I}{\partial y}\right) e^{-z_*} \sin z_* - \cancel{\left(\frac{\partial u_I}{\partial x} + \frac{\partial u_I}{\partial y}\right)\left(1 + e^{-z_*}\cos z_*\right)}$$

$$W_E(x,y) = \int_0^\infty dz_* \left(\frac{\Omega}{\nu}\right)^{1/2} \frac{\partial w}{\partial z_*}$$

Ekman velocity (vert)

$$= \frac{1}{2}\left(\frac{\nu}{\Omega}\right)^{1/2}\left(\frac{\partial v_I}{\partial x} - \frac{\partial u_I}{\partial y}\right)$$

$$\boxed{W_E(x,y) = \frac{1}{2}\frac{\nu}{\Omega}\,\omega_I}$$

[z component vorticity]

Now easy to show from B.C. that if lower boundary rotates at Ω_B relative to rotating frame:

$$W_E(x,y) = (\nu/\Omega)^{1/2}\left(\frac{1}{2}\omega_I - \Omega_B\right)$$

Similarly, for top at Ω_T:

$$w_E(x,y) = (\nu/\Omega)^{1/2}\left(\Omega_T - \tfrac{1}{2}\omega_I\right)$$

Now, for rapid rotation T-P Thm. says vertical uniformity $\Rightarrow$

$$w_{E_B} = w_{E_T}$$

$$\boxed{\omega_I/2 - \Omega_B = \Omega_T - \tfrac{1}{2}\omega_I}$$ global match

so $$\boxed{\omega_I = \Omega_B + \Omega_T}$$

so $$\Omega_B = 0$$

$$\Omega_T = \epsilon\Omega$$

$$\therefore \boxed{\omega_I = \epsilon\Omega}$$

and trivially,

$$\frac{1}{r}\frac{d}{dr}\left(r u_{\theta_I}\right) = \epsilon\Omega$$

$$\boxed{u_{\theta_I} = \tfrac{1}{2}\epsilon\Omega r}$$ $\simeq$ {avg. top btm rot freq.

and $w_E \Rightarrow$ $$\boxed{\tfrac{1}{2}(\nu\Omega)^{1/2}\epsilon = W_{Z_I}}$$

no radial flow.