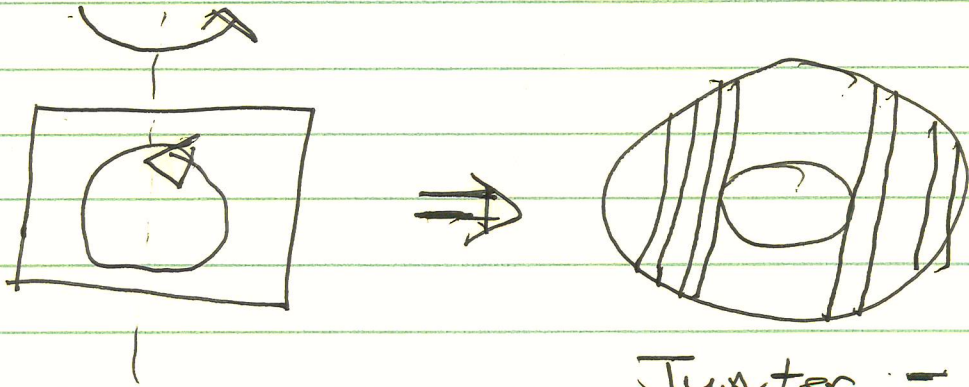


Lecture VI

Rotating Convection

Now,



Jupiter - bands
 - day - nights

how do convection and rotation interact?

Key point: rotation changes the freezing-in law for vorticity,

Recall:

$$\partial_t \underline{\omega} - \nu \nabla^2 \underline{\omega} = \underline{\nabla} \times (\underline{v} \times \underline{\omega})$$

induction equation with vorticity.

$$\underline{\nabla} \cdot \underline{v} = 0 \Rightarrow \begin{cases} \partial_t \underline{\omega} + \underline{v} \cdot \underline{\nabla} \underline{\omega} - \nu \nabla^2 \underline{\omega} = \underline{\omega} \cdot \underline{\nabla} \underline{v} \\ \frac{d \underline{\omega}}{dt} - \nu \nabla^2 \underline{\omega} = \underline{\omega} \cdot \underline{\nabla} \underline{v} \end{cases}$$

Why "freezing-in"?

Consider two test particles at
 $\underline{r}_1, \underline{r}_2$, "frozen into" flow, i.e.

• 2

$$\frac{d\underline{y}}{dt} = -\gamma (\underline{y} - \underline{v}(\underline{x}, t)) + \underline{f}$$

$\gamma \rightarrow \infty$ (freezing in):

$$\frac{d\underline{r}_1}{dt} = \underline{v}(\underline{r}_1, t)$$

$$\frac{d\underline{r}_2}{dt} = \underline{v}(\underline{r}_2, t)$$

$$\text{let } d\underline{l} = \underline{r}_2 - \underline{r}_1$$

think of as a filament

$$\frac{d d\underline{l}}{dt} = \underline{v}(\underline{r}_1 + \frac{d\underline{l}}{2}, t) - \underline{v}(\underline{r}_1 - \frac{d\underline{l}}{2}, t)$$

$$= d\underline{l} \cdot \underline{\nabla} \underline{v}$$

$\Rightarrow d\underline{l}$ follows the flow and is stretched by d .

$$\text{Now, } -\frac{d\underline{\omega}}{dt} = \underline{\omega} \cdot \underline{\nabla} \underline{v}$$

$$\frac{d d\underline{l}}{dt} = d\underline{l} \cdot \underline{\nabla} \underline{v}$$

$\underline{\omega}, d\underline{l}$
obey same equation

- so vorticity is frozen - into the flow (up to v).

- vorticity filaments follow and are stretched by the flow.

- obviously, freezing-in related to Kelvin's Theorem.

Now, in rotating fluid:

$$\underline{\underline{\Omega}} = \Omega \hat{z}$$

$$\underline{v}_{\text{lab}} = \underline{v}_{\text{fluid}} + \underline{\Omega} \times \underline{r}$$

$$\underline{v} \rightarrow \underline{v} + \underbrace{\Omega \times r}_{\text{centrifugal}}$$

so, for $\nabla \cdot \underline{v} = 0$, eqn. in frame: $\nabla \cdot \underline{v} = 0$ (centrifugal force)

$$\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} = -\nabla \left(\frac{p}{\rho} - \frac{\Omega^2 r^2}{2} \right) \quad (\text{absorb into pressure})$$

$$+ \underline{v} \times 2\underline{\Omega}$$

$$\underline{v} \cdot \nabla \underline{v} = \nabla \left(\frac{v^2}{2} \right) - \underline{v} \times \underline{\omega}$$

Coriolis force

\Rightarrow

$$\frac{\partial \underline{v}}{\partial t} = -\nabla \left(\frac{p}{\rho} + \frac{v^2}{2} - \Omega^2 r^2 \right)$$

$$+ \underline{v} \times \underline{\omega} + \underline{v} \times 2\underline{\Omega}$$

curl \Rightarrow

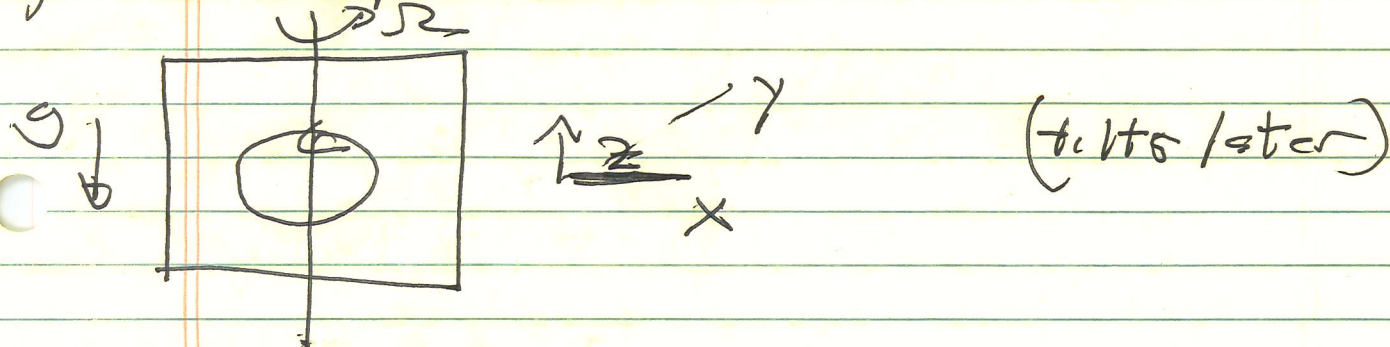
$$\frac{\partial \underline{\omega}}{\partial t} (\underline{\omega} + 2\underline{\Omega}) = \underline{\nabla} \times \underline{v} \times (\underline{\omega} + 2\underline{\Omega}) + r \nabla^2 \underline{\omega}$$

relative (fluid) vorticity
 planetary vorticity

- $\underline{\omega} + 2\underline{\Omega}$ frozen-in vorticity

- $\int d\underline{a} \cdot (\underline{\omega} + 2\underline{\Omega})$ conserved.

or, for simple case:



$$\frac{\partial \underline{\omega}}{\partial t} + \underline{v} \cdot \underline{\nabla} (\underline{\omega} + 2\underline{\Omega}) - r \nabla^2 \underline{\omega} = (2\underline{\Omega} + \underline{\omega}) \cdot \underline{\nabla} \underline{v}$$

$$\frac{d\underline{\omega}}{dt} - r \nabla^2 \underline{\omega} = (2\underline{\Omega} + \underline{\omega}) \cdot \underline{\nabla} \underline{v} = 2\underline{\Omega} \cdot \underline{\nabla} \underline{v} + \underline{\omega} \cdot \underline{\nabla} \underline{v}$$

For: $\Omega \gg |\underline{\omega}|, |\underline{\nabla} \underline{v}|$ etc.

\downarrow
 mean vorticity

or system rotation strong compared to relative vorticity.

- motions can't vary in direction of Ω
- all ^{slow} steady motions of rotating inviscid fluid are necessarily 2D,

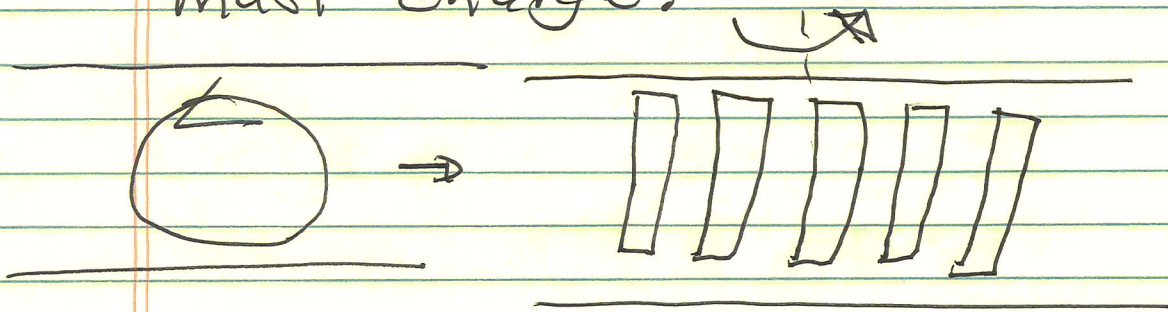
i.e.

$$2\Omega \partial_z \underline{v} \approx 0$$

→ Taylor - Proudman Theorem (1921, 1916)

→ Rapid rotation "two-dimensionalizes" the flow.

→ convection cells/modes organize into "Taylor Columns", "Proudman Pillars" → cartoon must change:



- cells must be tall, thin

- viscous BL at top, bottom (Ekman layer)

i.e. $-r \partial_z^2 \underline{w} = 2\Omega \partial_z \underline{v} = 0$

$l \sim (\nu/\Omega)^{1/2}$ → thickness.

→ Physics of Taylor - Proudman Theorem

↳ Vortex lines extract energy penalty when bent.

↳ Vortex lines don't like being bent.

→ Inertial Waves

→ Radiated Buoyancy waves in rotating fluid.

Recall HW Set 1:

$$\omega^2 = k_z^2 (4\Omega^2) / (k_z^2 + kr^2)$$

and radial boundary condition sets eigenvalue.

Derivation:

From vorticity:

$$\frac{\partial \tilde{\omega}}{\partial t} + \underline{\tilde{v}} \cdot \nabla (\underline{\tilde{\omega}} + 2\underline{\Omega}) + \underline{\omega} \cdot \nabla \underline{\tilde{v}} = 2\Omega \partial_z \underline{\tilde{v}}$$

$$\partial_t \tilde{\omega}_z = 2\Omega \partial_z \tilde{v}_z$$

And from EOM \Rightarrow

$$\nabla \times \nabla \times \underline{\tilde{v}} \Rightarrow$$

$$\partial_t \underline{\nabla} (\underline{\nabla} \cdot \underline{\tilde{v}}) - \partial_t \nabla^2 \underline{\tilde{v}}$$

$$= \underline{\nabla} \times \underline{\nabla} \times [\underline{v} \times 2\underline{\Omega} \hat{z}]$$

$$-\partial_t \nabla^2 \underline{\tilde{v}} = \underline{\nabla} \times (2\underline{\Omega} \partial_z \underline{\tilde{v}}) - \underline{\tilde{v}} \cdot \underline{\nabla} / 2\underline{\Omega} \hat{z}$$

so \hat{z} :

$$-\partial_t \nabla^2 \tilde{v}_z = 2\underline{\Omega} \partial_z \tilde{\omega}_z$$

$$\partial_t \tilde{\omega}_z = 2\underline{\Omega} \partial_z \tilde{v}_z$$

axis gradient
in $\omega \rightarrow$
axis acceleration

stretching vortex \rightarrow
vorticity fluctuation

$$\Rightarrow \omega^2 = k_z^2 4\underline{\Omega}^2 / (kr^2 + k_z^2)$$

- physics is rotating (vortex) flow
lines don't like being 'bent'
($k_z \neq 0$) \Rightarrow impedance on easy density
for finite k_z motions.
 $\Rightarrow \oplus$ definite $\partial \omega$.

- rough picture is one of gyroscopic
restoring force (conservation L_z)



- analogous Alfvén wave in MHD.

- N.B. inertial wave

- $u_{gr} < 0$

- $\omega = 0$ finite for shear layers

The point:

→ Convection requires cellular overturning (i.e. $k_z > 1/ht_{box}$)
 → minimal k_z

→ $\gamma^2 \sim k_x^2 / (k_z^2 + k_x^2)$

⇒ low k_z motions favored.

⇒ [Rotation is (strong) stabilizing effect on convection, as energy coupled to inertial wave.]

Another dimensionless number:

Ra, Pr

~~Ta~~ $Ta = 4 \Omega d^2 \nu / r^2$ $d \sim \text{box scale}$

$\tau = 4 (\Omega d^2 / \nu)^2$

Taylor #

~ Taylor number captures natural competition between rotation and viscous diffusion.

~ Ta joins Ra , Pr as key parameters in convective stability theory

~ $Ra_{crit} = Ra_{crit}(Pr, Ta)$ is now

stability threshold problem.

N.B. Ra , Ta both involve γ but are distinct - $\alpha g \Delta T / d$ vs Ω^2 .

Can combine stationary convection and inertial wave calculations to obtain basic equations:

$$\partial_t \theta = \theta W + K \nabla^2 \theta$$

$$\partial_t \nabla^2 W = g \alpha \nabla_h^2 \theta + \gamma \nabla^2 \nabla^2 W - 2\Omega \partial_z \omega_z$$

$$\partial_t \omega_z = -2\Omega \partial_z W$$

(derive)

notation as before

For ideal stability:

$$\omega^2 = [g \alpha \beta k_h^2 + (4\Omega^2) k_z^2] / k^2$$

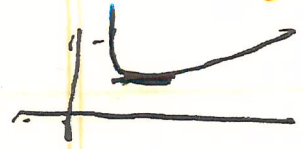
- favor high k_h , low k_z cells
- ⇒
- Taylor columns (thin) and Proudman pillars, as shown in movies.
- stabilizing effect of rotation evident ⇒ cell anisotropy.

For viscous, conductive stability (rotational) with rotation, then:

$$Ra_{crit} = Ra_{crit}(Ta, \alpha), \text{ for } Pr \gg 1,$$

↓
 $k_h h$

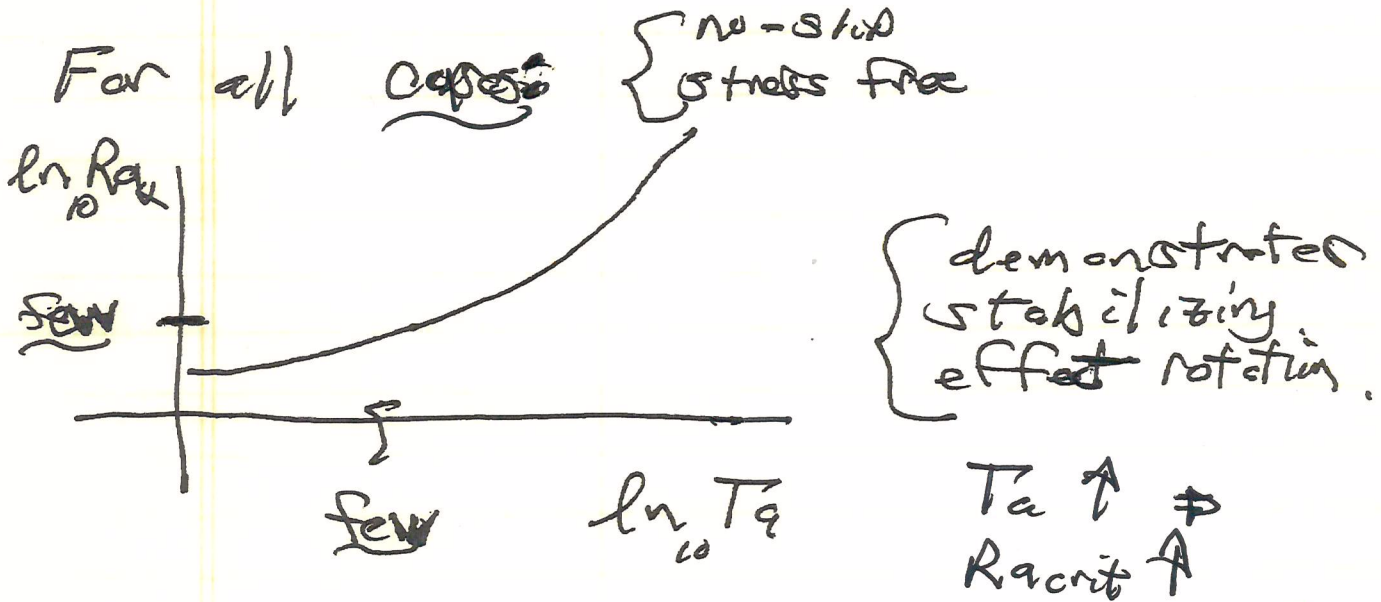
One can further specify Ra_{crit} as Ra_{crit} minimum (vs $\alpha = k_h h$)



i.e. smallest $Ra_{crit}(Ta, \alpha)$ scanned over α .

$$Ra_{crit} = Ra_{crit}(Ta)$$

↓
Taylor # dependence

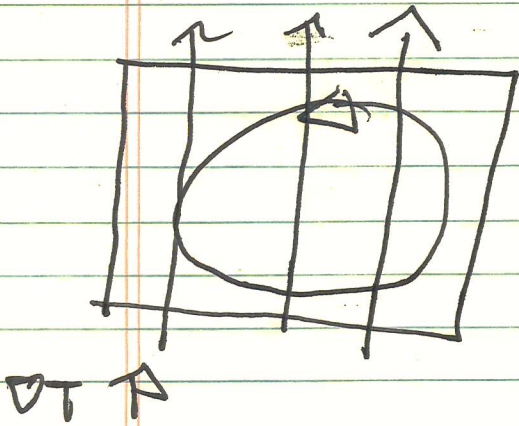


→ can develop variational principle for aside exchange-of-stabilities case. Need also treat over-stable limit.

→ Cultural Aside : Magnetoconvection

Dynamo-generated magnetic fields can feed back on convection. A particular example is sunspots (i.e. dark - lower T \downarrow - convection weakened), which are

associated with strong B fields.



$B_0 \downarrow \underline{g}$

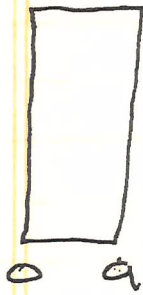
- similar to rotation problem with 'bending' due B_0
- energy penalty.
- Alfvén wave replaced inertial wave

Aside: What of tall, thin box?

~~12~~

tall, thin box

Now:

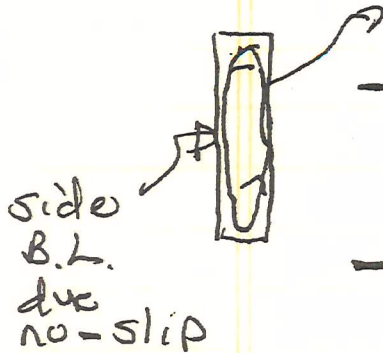


→ ignore top/bottom boundary

→ no-slip on side walls is key → and only relevant case

i.e. $v_z \Rightarrow w$; $w(0) = w(a) = 0$

side B.L.



- long thin cell must fight no-slip d.c.'s on side wall

- higher k_h will introduce high k_x damping.

∴ - expect high Ra due side-wall no slip, even at $k_h \rightarrow k_{h, min}$.

- curvature of Ra_{crit} vs k_h curve TBD. Speculate rather weak curvature,

- as side surface area \gg top surface area, expect top no-slip vs stress free comparison not significant.