→ MHD and MRI

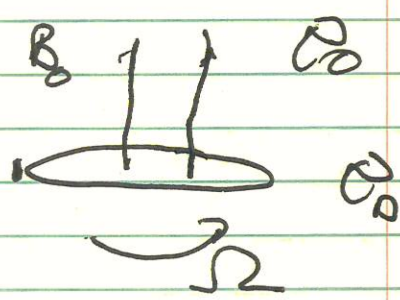
→ Seek basic theory MRI (c.f. RMP, Balbus & Hawley)

→ First: Crash course on MHD ($\underline{\nabla} \cdot \underline{v} = 0$)

- Eqns. { J×B force — k; Pressure, Tension }
- Induction Eqn.
- $\underline{\nabla} \cdot \underline{v} = 0$ MHD → 2 inter-penetrating fluids
  - ideal
  - resistive
  - [Rm, Re]
  - Magnetic stresses
- Alfven's Thm. (vs. Kelvin).
- comments on Reconnection
  - $\tau \sim v_A / \sqrt{R_m}$.
- $\underline{B}_0 = B\hat{z}$ → Waves
  - shear Alfven
  - $T \sim B/4\pi$
  - $\mu \sim \rho/B$

see 218 B W/2018

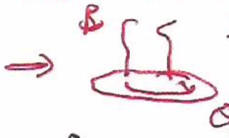
Problem



Time to slow down?

Openers:

→ what is $(\nabla \cdot \underline{v} = 0)$ MHD?

→  disk, rotating

When slow down.

1 fluid model

Basics of MHD

{N.B.: Read Kulsrud, Chapt. 3, 4}

{1 fluid, Large scale, slow}

→ MHD Equations → Eulerian Fluid

① $$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{v}) = 0}$$

(Continuity)

→ Lorentz, $\underline{E}$?

② $$\boxed{\rho\left(\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\nabla \underline{v}\right) = -\nabla P + \frac{\underline{J}\times\underline{B}}{c} + \underline{f}_{body}}$$

(momentum balance)

[frequently $\underline{f}_{body} = \rho \underline{g}$]

③ $$\frac{d}{dt} S = \frac{\partial S}{\partial t} + \underline{v}\cdot\nabla S = 0$$

$$S = C_v \ln(P/\rho^\gamma)$$

entropy

{eqn of state more general}

(isentropic fluid)

[frequent form of equation of state]

④ $$\boxed{\underline{E} + \frac{\underline{v}\times\underline{B}}{c} = \left(\eta \underline{J}, \frac{\underline{J}}{\sigma}\right)}$$

(Ohms Law)

[resistivity η is usually most significant dissipation]

Ideal MHD

$$\underline{E} + \frac{\underline{v}\times\underline{B}}{c} = 0$$

and

$$\boxed{\begin{aligned} &⑤ \quad \underline{\nabla} \cdot \underline{B} = 0 \\ &⑥ \quad \underline{\nabla} \times \underline{E} = -\frac{1}{c}\frac{\partial \underline{B}}{\partial t} \\ &⑦ \quad \underline{\nabla} \times \underline{B} = \frac{4\pi}{c}\underline{J} \end{aligned}}$$

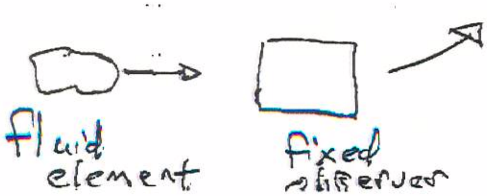
} from Maxwell's Eqns. neglecting displacement current

→ Meaning, Restriction, Validity

– MHD is simplest, closed, self-consistent plasma model, and the most heavily exploited for dynamical modelling.

– variants : reduced MHD → strong $\underline{B}_0$ (tokamaks)
2D MHD
E MHD → stationary ions (ICF)
FLR MHD } → MHD + additional
Reduced Braginsky } effects (MFE, space)
hybrid → { bulk – MHD
{ hot species – kinetic (i.e. α's energetics)
⋮

– MHD – Eulerian



"fluid element" ⇒ "glue"

here "glue" → collisions

applies $L > \lambda_{mfp}$

collisions on $\tau_{phen} < \tau_{fall\ apart}$

$(\omega > \nu)$

Why MHD ~~works~~ often works at low collisionality

→ MHD is:
- 1 fluid - electrons and ions
- strongly collisional
- low frequency
- large scale

i.e. frequencies relevant:

$$\omega \ll \Omega_{e,i};\ \omega_{p_{e,i}};\ \nu_{ee,ei};\ \nu_{ii};\ \omega_{T_{e,i}}$$

etc.

→ scales relevant:

$$L \gg \lambda_{D_{e,i}},\ \rho_{e,i},\ c/\omega_{p_{e,i}},\ l_{mfp_{e,i}}$$

$l_{mfp} < L$

and

collisions isotropise, equilibrate $\underline{\underline{P}}$.

$$\left(\text{i.e. } \underline{\underline{P}} \sim \int d^3v\ \tilde{v}_i \tilde{v}_j\ f(\underline{x}, \underline{v}, t) \right)$$

→ Some Specific Points:

- re: continuity ①;

$$\rho = m_i n_i + m_e n_e$$

i.e. (ions control fluid inertia)

$\left\{ \begin{array}{l} \text{total density} \\ \text{ion dominated} \end{array} \right.$

- re: momentum balance ② ;

→ $\underline{V} = \left(\int d^3v_i \, m_i \underline{v}_i f_i + \int d^3v_e \, m_e \underline{v}_e f_e\right) / \rho$

i.e. (ions control flow − $\rho \frac{d\underline{V}}{dt}$)

→ $\boxed{\text{where has } \underline{E} \text{ gone?}}$ → $L \gg \lambda_D$ → quasi-neutrality

↑ fluid QN

is it consistent?

$$\rho_i \frac{d\underline{v}_i}{dt} = n_i q_i \underline{E} + n_i q_i \frac{\underline{v}_i \times \underline{B}}{c} + \cdots$$

$$\rho_e \frac{d\underline{v}_e}{dt} = -n_e q_i \underline{E} - n_e q_i \frac{\underline{v}_e \times \underline{B}}{c} + \cdots$$

if add: → 0 cancel! $n_i = n_e$ (quasi-neutrality) → $\frac{\underline{J} \times \underline{B}}{c}$ (Lorentz force term in momentum balance)

note also: $\rho_i, \rho_e \to \rho$

→ re-writing the $\underline{J} \times \underline{B}$ force:

$$\frac{\underline{J} \times \underline{B}}{c} = \frac{(\nabla \times \underline{B}) \times \underline{B}}{4\pi} = -\nabla\left(\frac{B^2}{8\pi}\right) + \frac{\underline{B} \cdot \nabla \underline{B}}{4\pi}$$

so can write:

$$\rho \frac{d\underline{v}}{dt} = -\underline{\nabla}\Big(\underbrace{P}_{P_{tot}} + \underbrace{\frac{B^2}{8\pi}}_{\text{magnetic pressure (field energy density)}}\Big) + \underbrace{\frac{\underline{B}\cdot\underline{\nabla}\,\underline{B}}{4\pi}}_{\text{magnetic tension}}$$

tension

a) what / why "Magnetic Tension"?

$\underline{B} = B\,\hat{b}$ $\qquad B = |\underline{B}|\,,\quad \hat{b} = \underline{B}/B$

$$\underline{B}\cdot\underline{\nabla}\,\underline{B} = B\,\hat{b}\cdot\underline{\nabla}\,(B\hat{b})$$

$$= B^2\,\underbrace{\hat{b}\cdot\underline{\nabla}\hat{b}}_{①} + \underbrace{\hat{b}\,\hat{b}\cdot\underline{\nabla}}_{②}\left(\frac{B^2}{2}\right)$$

$\hat{b}\cdot\underline{\nabla}\,\hat{b}$ → curvature of $\hat{b}$
(i.e. rate of change of $\hat{b}$ along itself)

$\equiv d\hat{b}/ds$

n.b. in general: $\quad$ curve: $\underline{x}(t)$

tangent: $\underline{T} = d\underline{x}/ds$

$(ds^2 = d\underline{x}\cdot d\underline{x})$ $\qquad s \equiv$ distance along curve curvature $\underline{\kappa} = \frac{d\underline{T}}{ds} = \frac{d\underline{T}/dt}{ds/dt} = \frac{\dot{\underline{T}}}{|\underline{V}|}$

Now: $\underline{\kappa} = \hat{b} \cdot \underline{\nabla} \hat{b}$ → points in direction of turning of $\hat{b}$, orthogonal to $\hat{b}$

i.e.

$\underline{\kappa} = d\underline{T}/ds$

∴ $\underline{\kappa} = +\frac{\hat{n}}{R_c}$ $R_c \equiv$ radius of curvature

as curved field line suggests "tension" → "magnetic tension".

b.) What about ②?

But $\underline{J} \times \underline{B} \perp \underline{B}$ yet $\underline{\nabla}\left(\frac{B^2}{8\pi}\right)$ can have component along $\underline{B}$??

→ recombining total $\underline{J} \times \underline{B}$ gives:

$$-\underline{\nabla}\left(\frac{B^2}{8\pi}\right) + B^2\frac{\hat{b}\cdot\underline{\nabla}\hat{b}}{4\pi} + \hat{b}\,\hat{b}\cdot\underline{\nabla}\left(\frac{B^2}{8\pi}\right)$$

$$= -\nabla_\perp\left(\frac{B^2}{8\pi}\right) \cancel{-\hat{b}\hat{b}\cdot\nabla\left(\frac{B^2}{8\pi}\right)} + \cancel{\hat{b}\,\hat{b}\cdot\underline{\nabla}\left(\frac{B^2}{8\pi}\right)} + B^2\frac{\hat{b}\cdot\nabla\hat{b}}{4\pi}$$

$$\Rightarrow \boxed{\frac{\underline{J}\times\underline{B}}{c} = -\underline{\nabla}_\perp\left(\frac{B^2}{8\pi}\right) + B^2\frac{\hat{b}\cdot\underline{\nabla}\hat{b}}{4\pi}}$$

③ $dE = đQ - p\,dV$ (Thermo)

$C_v\,dT = T\,dS - p\,dV$

$\left\{\begin{array}{l} đQ = T\,dS \\ dE = C_v\,dT \\ \quad\text{(normalized)} \end{array}\right.$

$V = 1/\rho \qquad dV = -\,d\rho/\rho^2$

$$C_v\frac{dT}{T} = dS + \frac{d\rho}{\rho}$$

$$\Rightarrow \ln T = \frac{S}{C_v} + \ln\rho^{1/C_v}$$

$$\therefore \boxed{S' = C_v\ln\left(T/\rho^{1/C_v}\right)}$$

$p = \rho T$

$$\Rightarrow \quad S = c_v \ln\left(P/\rho^{(c_v+1)/c_v}\right)$$
$$= c_v \ln\left(P/\rho^{\gamma}\right)$$

$\gamma = 5/3$, ideal gas

$\left(c_v = 3/2 \text{ (normalized)}\right)$

$$\frac{dS}{dt} = 0 \quad \Rightarrow \quad \frac{d}{dt}\left(P/\rho^{\gamma}\right) = 0$$

i.e.

$$\boxed{\frac{\partial}{\partial t}\left(P/\rho^{\gamma}\right) + \underline{v}\cdot\underline{\nabla}\left(P/\rho^{\gamma}\right) = 0}$$

eqn. of state

perfect $\begin{cases}\text{homogeneity} \\ \text{stationarity}\end{cases}$ $\left(P/\rho^{\gamma} = \text{const}\right)$

– "adiabatic equation of state"

(4) Ohm's Law – most sensitive part of MHD (since controlled by electrons)

MHD variants differ primarily in Ohm's Law

- Hall MHD → Hall term
- EMHD → electron inertia
- Braginsky / drift MHD → ∇P terms

etc., etc.

Ohm's Law ⇒ subtract moments on electron equations → electrons $\left(\underline{J} = n\,e(\underline{V}_i - \underline{V}_e) \right)$

Simple resistive MHD:

$$\underline{E} + \frac{\underline{v} \times \underline{B}}{c} = \frac{\eta}{\varsigma}\,\underline{J}$$

$\sim \nu_{e,i}$ → momentum transfer to ions ...

ideal MHD:

$$\underline{E} + \frac{\underline{v} \times \underline{B}}{c} = 0$$

→ field "frozen into" fluid

⑤, ⑥, ⑦: Only 1 approximation:

$$\nabla \times \underline{B} = \frac{4\pi}{c}\,\underline{J} + \frac{1}{c}\frac{\partial \underline{E}}{\partial t}$$

Why drop displacement

$|\partial \underline{E}/\partial t| << |\underline{J}|$ → condition on ω !?

$$\rightarrow \quad \omega \frac{v B}{c} \overset{?}{<<} \frac{k}{c^{-1}} B$$

$\Rightarrow |\underline{v}|(\omega/k)/c^2 << 1$ is condition on ω.

→ Skeptic: "Does it Hang Together"?

i.e. is electric force negligible? Consistency

$$\rho \frac{d\underline{v}}{dt} = n\underline{q}\,\underline{E} + \cdots\cdots$$

and $q \neq 0$, as

$$nq = \frac{\nabla\cdot E}{4\pi}$$

$$\underline{E} = -\frac{\underline{v}\times\underline{B}}{c}$$

so

$$nq\underline{E} = +\frac{(\underline{v}\times\underline{B})}{c}\,\nabla\cdot\left(\frac{\underline{v}\times\underline{B}}{c}\right) \neq 0 \quad !$$

but

$$\sim \frac{v^2}{c^2}\,B^2 k$$

$$\sim \frac{v^2}{c^2}\,(\underline{J}\times\underline{B}) \longrightarrow \text{negligible, if } v^2/c^2 \ll 1.$$

Thus, yes indeed it does!

→ Putting it together:

$$\underline{E} + \frac{\underline{v}\times\underline{B}}{c} = \eta\,\underline{J} \quad , \quad \nabla\times\underline{E} = -\frac{1}{c}\frac{\partial\underline{B}}{\partial t}$$

⇒ the induction equation for $\underline{B}$ evolution ...

→ $$\boxed{\frac{\partial \underline{B}}{\partial t} = \underline{\nabla} \times (\underline{V} \times \underline{B}) + \eta \nabla^2 \underline{B}}$$ Induction eqn.

- with momentum equation, defines MHD as problem of 2 coupled fluid fields (vector) – $\underline{V}(\underline{X},t)$, $\underline{B}(\underline{X},t)$ evolving simultaneously

↓

- useful and instructive to re-write induction equation

$$\nabla \times \underline{V} \times \underline{B} = -\underline{V}\cdot\underline{\nabla}\,\underline{B} + \underline{B}\cdot\underline{\nabla}\,\underline{V} - \underline{B}\,\underline{\nabla}\cdot\underline{V}$$

so $$\frac{\partial \underline{B}}{\partial t} + \underline{V}\cdot\underline{\nabla}\,\underline{B} - \eta\nabla^2\underline{B} = \underline{B}\cdot\underline{\nabla}\,\underline{V} - \underline{B}\,\underline{\nabla}\cdot\underline{V}.$$

This brings us to ......

→ What Does "MHD", as a system, really mean ....?

this is answered most clearly for the case of
incompressible MHD ---

$\underline{\nabla} \cdot \underline{V} = 0$ → defines equation of state
→ sets P_{total} field

$(\omega/k \ll \underbrace{C_s}_{\text{sound}}, \underbrace{V_{MS}}_{\text{magnetosonic}})$

$$\nabla \cdot \left\{ \frac{\partial \underline{V}}{\partial t} + \underline{V} \cdot \underline{\nabla} \underline{V} = -\frac{\underline{\nabla}}{\rho}\left(P + \frac{B^2}{8\pi}\right) + \frac{\underline{B} \cdot \underline{\nabla} \underline{B}}{4\pi\rho} \right\}$$

$$\frac{d\rho}{dt} = -\rho \, \underline{\nabla} \cdot \underline{V} = 0$$

so ρ → constant ρ_0 (can relax to slow variation)

$$\nabla^2 \left[\underbrace{\left(P + \frac{B^2}{8\pi}\right)}_{\text{total pressure}} \Big/ \rho_0 \right] = \underline{\nabla} \cdot \left(\frac{\underline{B} \cdot \underline{\nabla} \underline{B}}{4\pi\rho_0} - \underline{V} \cdot \underline{\nabla} \underline{V} \right)$$

a la' Poisson's equation:

$$\frac{P + B^2}{8\pi} = -\int \frac{d^3x'}{4\pi |x - x'|} \left\{ \underline{\nabla} \cdot \left(\frac{\underline{B} \cdot \underline{\nabla}' \underline{B}}{4\pi\rho_0} - \underline{V} \cdot \underline{\nabla}' \underline{V} \right) \right\}$$

solves for: P_{tot} field → eliminates eqn. state.

Basic MHD $\nabla \cdot \underline{V} = 0$ $\qquad p^* = p_{tot}$

$$\boxed{\begin{aligned} &\frac{\partial \underline{V}}{\partial t} + \underline{V} \cdot \underline{\nabla}\, \underline{V} = -\underline{\nabla}\left(\frac{p^*}{\rho_0}\right) + \frac{\underline{B} \cdot \underline{\nabla}\, \underline{B}}{4\pi\rho_0} \\ &\frac{\partial \underline{B}}{\partial t} + \underline{V} \cdot \underline{\nabla}\, \underline{B} - \eta \nabla^2 \underline{B} = \underline{B} \cdot \underline{\nabla}\, \underline{V} \end{aligned}}$$

with $\nabla \cdot \underline{V} = 0$, constitute equations of incompressible MHD.

→ Rather clearly, this system is one of <u>two</u>, dynamically coupled, evolving vector fields $\underline{V}(\underline{x},t)$, $\underline{B}(\underline{x},t)$.

→ <u>Compressible MHD</u> is really a problem in <u>3</u> fields, two of which are vectors

<u>i.e.</u>
- $\underline{V}(\underline{x},t)$ → fluid velocity
- $\underline{B}(\underline{x},t)$ → magnetic field
- $S(\underline{x},t)$ → entropy ⇒ energy density

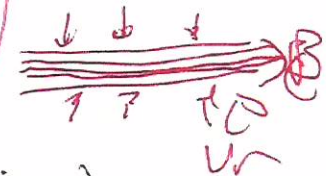
<u>i.e.</u> scalar equation of state provides 3rd field.

→ Key Question: How closely coupled are $\underline{V}$, $\underline{B}$?

⇒ the key physics element in MHD ......

⇒ $\boxed{\text{Frozen-in Law, Flux Freezing}}$

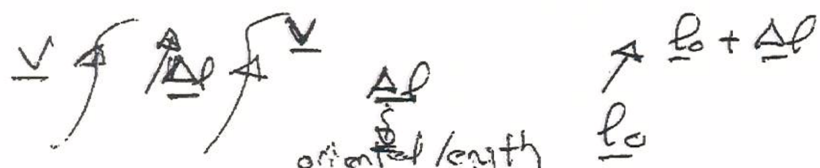
2 versions
→ local
→ integrated



B scale

① $\boxed{\text{Frozen-in Law}}$

- consider a (for the moment, passive) vector field:
  - frozen into flow $\underline{V}(\underline{x}, t)$
  - consisting of oriented, flexible strands



{ i.e. massless rubber strands in flow }

How does $\underline{\Delta l}$ evolve ?

in dt, $d(\underline{\Delta l}) = (\underline{V}(\underline{l_0} + \underline{\Delta l}) - \underline{V}(\underline{l_0}))\,dt$

$$= \underline{\Delta l} \cdot \nabla \underline{V} \quad dt$$

$$\therefore \quad \frac{d(\underline{\Delta l})}{dt} = \underline{\Delta l} \cdot \nabla \underline{V}$$

$$\frac{d}{dt}\underline{\Delta l} = \underline{\Delta l}\cdot\underline{\nabla V}$$

i.e. $\frac{d}{dt}\underline{\Delta l} = \underline{\Delta l}\cdot\underline{\underline{S}}$

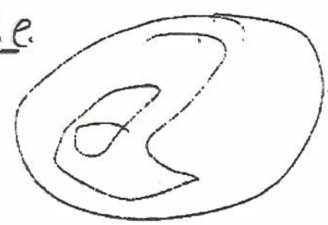
$$\left\{\begin{aligned} &\frac{d}{dt}(\Delta l)_i = \underline{\Delta l}_j \cdot S_{ij} \\ &S_{ij} = \frac{1}{2}\left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i}\right) \end{aligned}\right.$$

$\to$ strain rate tensor

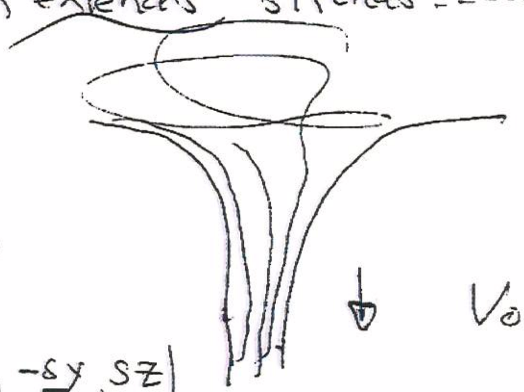
says that $\to$ $\underline{\Delta l}$ strands orient along strain

$\to$ strain extends strands ....

i.e.



$\to$ siphon flow



$\underline{V} = V_0\left(-\frac{sx}{2}, -\frac{sy}{2}, sz\right)$

plausible to say that Δl "frozen onto" the flow.

Now, if $\eta \to 0$, ... in MHD

$$\frac{\partial \underline{B}}{\partial t} + \underline{V} \cdot \underline{\nabla}\, \underline{B} = \underline{B} \cdot \underline{\nabla}\, \underline{V} - \underline{B}\, \underline{\nabla} \cdot \underline{V}$$

$$-\underline{\nabla} \cdot \underline{V} = +\frac{1}{\rho} \frac{d\rho}{dt}$$

$$\frac{\partial}{\partial t} \underline{B} + \underline{V} \cdot \underline{\nabla}\, \underline{B} = \frac{d\underline{B}}{dt} = \underline{B} \cdot \underline{\nabla}\, \underline{V} + \frac{\underline{B}}{\rho} \frac{d\rho}{dt}$$

$$\frac{1}{\rho} \frac{d\underline{B}}{dt} - \frac{\underline{B}}{\rho^2} \frac{d\rho}{dt} = \frac{\underline{B}}{\rho} \cdot \underline{\nabla} V$$

$$\therefore \quad \boxed{\frac{d}{dt}\left(\frac{\underline{B}}{\rho}\right) = \frac{\underline{B}}{\rho} \cdot \underline{\nabla}\, \underline{V}}$$

→ $\underline{B}/\rho$ obeys same equation as $\underline{\Delta l}$!

→ $\underline{B}/\rho$ is frozen into flow field $\underline{V}(\underline{x}, t)$

Note: → $\underline{B}/\rho$ is not passive → due $\underline{J} \times \underline{B}$ force

→ $\underline{B}$ determines flow, while frozen into it!

→ (essence of coupling problem)

- For $\underline{\nabla} \cdot \underline{V} = 0$, $\underline{B}$ frozen in

⇒ if $\eta \neq 0$, freezing in is broken ----

i.e. $\frac{d}{dt}\left(\frac{\underline{B}}{\rho}\right) - \frac{\eta}{\rho} \nabla^2 \underline{B} = \frac{\underline{B}}{\rho} \cdot \underline{\nabla} \underline{V}$

form of frozen evolution broken

Vorticity connection

⇒ Observe: → this motivates attention to resistivity in MHD above other dissipations ν, χ, etc..

→ η ⇒ $\underline{B}$ diffusion $\sim \eta \nabla^2$

∴ decoupling of $\underline{V}$, $\underline{B}$ occurring on small scales

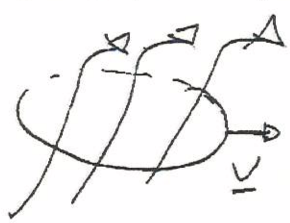
⇒ motivates 'magnetic reconnection' as study of singularity dynamics in MHD.

→ A Word to the Wise: In modelling, describing complex dynamics in MHD (i.e. MHD turbulence, dynamos, etc.) always think carefully about frozen-in law ...

What is frozen in for other systems?

→ Closely Related: Flux Freezing

- consider flux thru surface in flow



i.e. imaginary loop drawn in flow field ...

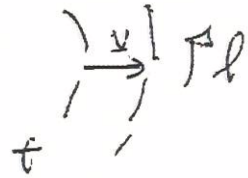
$$\frac{\partial \underline{B}}{\partial t} = \underline{\nabla} \times \underline{V} \times \underline{B}$$

$$\Phi = \int \underline{B} \cdot d\underline{s}$$

① change in $\underline{B}$ ② change in $d\underline{s}$

$$\frac{d\Phi}{dt} = \int d\underline{s} \cdot \frac{\partial \underline{B}}{\partial t} + \int \frac{d\underline{s}}{dt} \cdot \underline{B}$$

change in $\underline{B}$ motion of loop ...

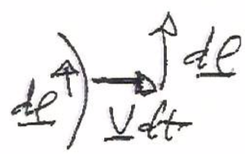


$$① = \int d\underline{s} \cdot \underline{\nabla} \times (\underline{V} \times \underline{B})$$

$$= \oint d\underline{l} \cdot (\underline{V} \times \underline{B})$$

$$ds = \underline{V}\,\Delta t \times d\underline{l}$$

For ②



so $d\underline{s} = \underline{V} dt \times d\underline{l}$

↳ {change in $\underline{s}$ in dt.

$dt\,(d\Phi_{②}/dt)$

$$② dt = \int (\underline{V} dt \times d\underline{l}) \cdot \underline{B} = d\Phi$$

$$\left.\frac{d\Phi}{dt}\right)_{②} = \int (\underline{V} \times d\underline{l}) \cdot B = -\int d\underline{l} \cdot (\underline{V} \times \underline{B})$$

so $\boxed{\dfrac{d\Phi}{dt} = ① + ② \Rightarrow 0}$ ✓

so $\Rightarrow$ magnetic flux invariant $\Leftrightarrow$ cancellation

$\Rightarrow$ in absence of resistivity, flux thru surface in flow is invariant, or frozen in

$\Rightarrow$ no surprise: $\underline{B}$ frozen in $\Rightarrow$ Φ frozen in

$\Rightarrow$ analogue in hydro: Circulation (Kelvin's Thm.)

$$\Gamma_c = \oint \underline{v} \cdot d\underline{l} = \int d\underline{a} \cdot \underline{\omega} \qquad \omega = \nabla \times \underline{v}$$

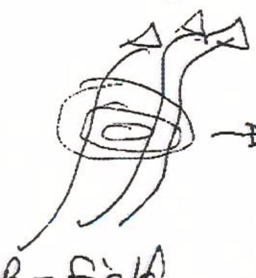
In inviscid hydro, $(\nu \to 0)$ circulation Γ_c is conserved.

[Exercise: Prove this!
Note relation between $\underline{\omega}$ equation and $\underline{B}$ eqn. Assume $\rho = $ const., $\underline{g} = 0$.]

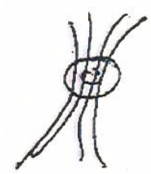
[Extra Credit: ① Discuss the extension to the case where $\rho \neq$ const.]
② what is 'frozen in' for Vlasov plasma?

→ What Does "Freezing" Mean?

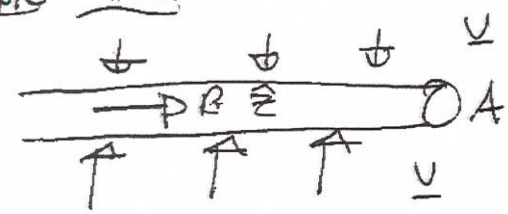
⇒ can relate field evolution in a flow to density evolution, since $\underline{B}/\rho$ is "frozen in".

Application:  → proto-stellar cloud

Star formation

$\underline{B}$-field →  (?)

Simple Cases → How does $\underline{B}$ change in a flow?

(i) 

radial compression

Cylinder of cross-sectional area A squeezed in radial flow. $\underline{B} \equiv B\,\hat{z}$

2 ways:

$$\frac{d}{dt}\left(B\hat{z}/\rho\right) = \frac{\underline{B}\hat{z}\cdot\underline{\nabla}\,\underline{V}}{\rho}$$

$\underline{V} = v\,\hat{r}$

$\Rightarrow \underline{V} \perp \underline{B}$

$$= 0$$

so $\underline{B}/\rho = \text{const}$

Now: $\rho A L = \text{const}$ so $\underline{B \sim A^{-1}}$
$\rho \sim A^{-1}$
(L const.)

or

Flux Frozen: $BA = \Phi = \text{const.}$
$\rho AL = \text{const} = M$
L const.

$BA \sim \Phi_0$, $B \sim A^{-1}$
$\rho A \sim M_0$, $\rho \sim A^{-1}$
so $B \sim \rho^{(1)} \Rightarrow B/\rho \sim \text{const}!$

ii.) $V = V(z)\hat{z}$ — compressible!
$\longrightarrow B\hat{z}$ i.e. stretch, 1D

here $\dfrac{\underline{B}}{\rho} \cdot \underline{\nabla}\, \underline{V} \neq 0$, but easier to work with $\underline{B}$ than $\underline{B}/\rho$

$$\frac{\partial \underline{B}}{\partial t} + \underline{v} \cdot \underline{\nabla}\, \underline{B} = \underline{B} \cdot \underline{\nabla}\, \underline{v} - \underline{B}\, \underline{\nabla} \cdot \underline{v}$$

$$= B \frac{\partial}{\partial z} V(z) - B \frac{\partial V(z)}{\partial z}$$

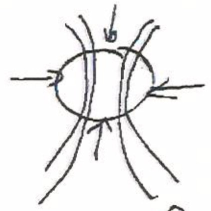
$$= 0 \; !$$

For ρ, $\quad \frac{d\rho}{dt} = -\rho \, \underline{\nabla} \cdot \underline{V} = -\rho \frac{\partial V_z}{\partial z}$

here $\underline{B}$ invariant, ρ changes $\qquad \frac{d}{dt}\frac{\underline{B}}{\rho} = \frac{\underline{B}}{\rho} \cdot \nabla \underline{V}$

i.e. $\quad B \sim \rho^{(0)}$ $\qquad$ Freezing in $\not\Rightarrow$ const

iii.)



collapsing sphere: $\underline{V} = V \hat{r}$

(i.e. $\Phi = 0$, total sphere)

consider hemispherical surface (i.e. mushroom cap)

$$\Phi \sim B R^2 \sim \text{const}$$

$$M \sim \rho R^3 \sim \text{const.}$$

$\Rightarrow \quad B \sim r^{-2}, \quad \rho \sim r^{-3} \quad \Rightarrow \quad B/\rho^{2/3} \sim \text{const.}$

why the scaling ? $\Leftrightarrow$ why of interest ?

→ { "implosion", gravitational collapse } problems sensitive to equation of state of material collapsing

If: $p \to p_{tot} = p + \frac{B^2}{8\pi}$

$p = p_0 (\rho/\rho_0)^{\gamma}$

collapse threaded by magnetic field

[then natural to ask: Can one write $B^2 = B^2(\rho)$ and thus extend equation of state to encompass magnetic pressure contribution ?

⇒ Proceed via flux-freezing !

$B \sim \rho^{2/3} \Rightarrow B^2 \sim \rho^{4/3}$

so [p_{B^2} has "γ_{eff}" $= 4/3$. This resembles equation of state for degenerate gas (see Handouts I).

⇒ More on this in discussion of flux freezing and virial theorems ....

→ Pragmatic Question: Is flux 'frozen' during star formation ? ↔ Does resistivity matter ?

$$\eta \sim \frac{4 \times 10^6}{T_{eV}^{3/2}} \text{ cm}^2/\text{sec.} \qquad \text{(Spitzer)}$$

start → collapse → protostar

$n \sim 1 \text{ atom}/\text{cm}^3 \longrightarrow$ $\rho \sim 1 \text{ g}/\text{cm}^3$
$n \sim 10^{24}$ atm$/\text{cm}^3$
(related N_A)

but

$B/\rho^{2/3} \sim$ const

$\Rightarrow \; B/B_0 \sim \left(10^{24}\right)^{2/3} \sim 10^{16}$! huge amplification

so $B_0 \sim 10^{-6}$ G, characteristic of ISM

$\Rightarrow B \sim 10^{10}$ G in protostar

$\therefore \; P_{B^2} \sim 10^{19}$ erg/cm^3 $\qquad (P_{B^2} \sim B^2/8\pi)$

but P_{Th} for normal star $\sim 10^{14}$ erg/cm^3

$P_{B^2} \gg P_{Th}$?? $\Rightarrow$ clearly flux-freezing is bad assumption.

→ In terms of time scales:

$$\underset{①}{\frac{\partial \underline{B}}{\partial t}} = \underset{②}{\underline{\nabla} \times (\underline{v} \times \underline{B})} + \underset{③}{\eta \nabla^2 \underline{B}}$$

$$\frac{1}{T_{collapse}} \sim \frac{1}{T_{dynamic}} + \underbrace{\frac{\eta}{L^2}}_{1/T_{diffn}}$$

3 scales
2 balance
i.e. ① ~ ② ③ negligible
① & ③ ② negligible.

if $T_{collapse} \ll T_{diffn}$ → {flux frozen, OK

$T_{collapse} \gg T_{diffn}$ → {must consider diffusion, freezing invalid

N.B.: In star formation, $T_{coll.} \ll T_{diffn}$

but [ISM has large neutral component.

Plasma-neutral drag sets dissipation
→ Ambipolar diffusion.

Linear Waves, Instabilities and Energy Principle

→ Contents

- this unit presents the linear structure, response theory and energetics for MHD

- proceed by: a.) linear waves
  b.) Least Action and Energy Principle
  c.) simple linear instabilities

- Later discuss nonlinear evolution, i.e.:
  i.e. a.) MHD shocks
  b.) collisionless shocks
  c.) MHD turbulence (later).

A.) Linear Waves in MHD

i.) Simple Cases

- before proceeding with full crank useful to discuss some limiting cases in depth
- always have $\left.\begin{array}{l}\underline{B}_0 = B_0\hat{z} \\ \rho = \rho_0,\ P = P_0\end{array}\right\}$ uniform

- consider

	$\nabla\cdot\underline{V} = 0$	$\nabla\cdot\underline{V} \neq 0$	
$\underline{k} = k\hat{z}$	Shear Alfven	Acoustic	– parallel propagation
$\underline{k} = k\hat{x}$	X	Magnetosonic	– perpendicular propagation

$\rightarrow \underline{k} = k\hat{z}, \quad \nabla \cdot \underline{V} = 0$ Shear Alfven Wave

$$\left.\begin{aligned} \rho_0 \frac{\partial \tilde{\underline{V}}}{\partial t} &= -\nabla\left(\tilde{p} + \frac{\tilde{B}^2}{8\pi}\right) + \frac{\underline{B_0}\cdot\underline{\nabla}\,\tilde{\underline{B}}}{4\pi} \\ \frac{\partial \tilde{\underline{B}}}{\partial t} &= \underline{B_0}\cdot\underline{\nabla}\,\tilde{\underline{V}} \end{aligned}\right\} \text{linearized eqns.}$$

Now, $\underline{\nabla} \cdot \tilde{\underline{V}} = 0 \Rightarrow$

$$-\nabla^2\left(\tilde{p} + \frac{\underline{B_0}\cdot\tilde{\underline{B}}}{8\pi}\right) + B_0\cdot\nabla\cancel{(\nabla\cdot\tilde{\underline{B}})}^{\,0} = 0$$

$\left\{\begin{array}{l}\rho_0, B_0 \\ \text{uniform}\end{array}\right.$

$$\therefore \quad \tilde{p} + \frac{\underline{B_0}\cdot\tilde{\underline{B}}}{8\pi} = 0$$

$\rightarrow$ "perturbed pressure balance"

$\rightarrow$ holds for incompressible (and weakly compressible) modes

$$\Rightarrow \quad \rho_0 \frac{\partial \tilde{\underline{V}}}{\partial t} = \frac{B_0}{4\pi}\frac{\partial}{\partial z}\tilde{\underline{B}}$$

$$\frac{\partial \tilde{\underline{B}}}{\partial t} = B_0 \frac{\partial}{\partial z}\tilde{\underline{V}}$$

$$\therefore \quad \boxed{\frac{\partial^2 \tilde{\underline{V}}}{\partial t^2} = \frac{B_0^2}{4\pi\rho_0}\frac{\partial^2}{\partial z^2}\tilde{\underline{V}}}$$

$B_0^2/4\pi\rho_0 \equiv V_A^2$ Alfven velocity

$\Rightarrow \begin{cases} \omega^2 = k_\parallel^2 V_A^2 & \rightarrow \text{dispersion relation for } \underline{\text{shear}}\ \underline{\text{Alfven}}\ \underline{\text{wave}} \\ V_{ph} = V_{gr} = V_A \vec{z} & \rightarrow \text{speed} \begin{cases} \text{phase} \\ \text{group} \end{cases} \end{cases}$

wave propagates along $\vec{z}$ at Alfven speed

$\Rightarrow$ wave is consequence of <u>magnetic</u> <u>tension</u>

$\frac{T}{\mu} \rightarrow \frac{B/4\pi}{\rho_0/B}$ ~ <u>tension</u> - <u>in</u> <u>line</u> $\Rightarrow V_A^2$

$\hookrightarrow$ <u>mass</u> - <u>per</u> - <u>line</u>

$\Rightarrow$ tension $\Leftrightarrow$ plucking $\Rightarrow \tilde{\underline{V}} \perp B_0$

$\left(\begin{matrix} \nabla \cdot \underline{V} = 0 \\ \text{parallel variation} \end{matrix} \right)$

<u>i.e.</u> $\begin{cases} \tilde{\underline{V}} = \tilde{V}_x\, \hat{x} \\ \tilde{\underline{B}} = \frac{\partial}{\partial z}(\tilde{V}_x B_0) = \tilde{B}_x\, \hat{x} \end{cases}$

in shear Alfven wave:

$\begin{cases} \tilde{\underline{V}},\ \tilde{\underline{B}} \perp \underline{B_0} \\ \tilde{\underline{V}} \parallel \tilde{\underline{B}}, \text{ but out of phase} \end{cases}$

→ energetics → construct "Poynting theorem"

$$\rho_0 \frac{\partial \tilde{\underline{V}}}{\partial t} = \frac{B_0}{4\pi} \frac{\partial}{\partial z} \tilde{\underline{B}} \qquad (1)$$

$$\frac{\partial \tilde{\underline{B}}}{\partial t} = B_0 \frac{\partial}{\partial z} \tilde{\underline{V}} \qquad (2)$$

∴ construct energy evolution

$$\mathcal{E} = \frac{\rho \tilde{V}^2}{2} + \frac{\tilde{B}^2}{8\pi} \quad \rightarrow \text{energy density}$$

∴ $(1)\cdot \underline{V}$ and $(2)\cdot \underline{B}$ ⇒

$$\frac{\partial}{\partial t}\left(\rho_0 \frac{\tilde{\mathbf{V}}^2}{2} + \frac{\tilde{B}^2}{8\pi}\right) = \frac{B_0}{4\pi}\left(\underline{\tilde{V}} \cdot \frac{\partial \tilde{\underline{B}}}{\partial z} + \tilde{\underline{B}} \cdot \frac{\partial \underline{V}}{\partial z}\right)$$

∴

$$\frac{\partial}{\partial t}\left(\rho_0 \frac{\tilde{V}^2}{2} + \frac{\tilde{B}^2}{8\pi}\right) = \frac{B_0}{4\pi} \frac{\partial}{\partial z} (\tilde{\underline{V}} \cdot \tilde{\underline{B}})$$

and have Poynting form: $\frac{\partial \mathcal{E}}{\partial t} + \underline{\nabla} \cdot \underline{S} = 0$

$$\underline{S} = -\frac{B_0}{4\pi} (\underline{V} \cdot \underline{B}) \quad \rightarrow \text{wave energy density flux}$$

$\int d^3x\, \underline{V} \cdot \underline{B}$ → cross helicity

N.B. $\underline{S} = \frac{c}{4\pi} \underline{E} \times \underline{B}$, $\underline{P} = \underline{S}/c^2$

wave energy density flux ↳ wave momentum density

$\underline{E} = -\frac{\underline{V} \times \underline{B}_0}{c}$

$$\underline{S} = -\frac{1}{4\pi} (\underline{\tilde{V}} \times \underline{B}_0) \times \underline{\tilde{B}} = \frac{1}{4\pi}\left[\cancel{(\underline{\tilde{B}} \cdot \underline{B}_0)} \underline{\tilde{V}} - (\underline{\tilde{V}} \cdot \underline{\tilde{B}}) \underline{B}_0\right]$$

$$= -\frac{\underline{B}_0}{4\pi} (\underline{\tilde{V}} \cdot \underline{\tilde{B}})$$

$\underline{S} = -\frac{\underline{B}_0}{4\pi} \underline{V} \cdot \underline{B}$

i.e. – energy flows along field

– $\underline{S} \sim \underline{V} \cdot \underline{B}$

$H_c = \int d^3x \, \underline{\tilde{V}} \cdot \underline{\tilde{B}}$ → cross helicity
→ conserved in ideal MHD

Ex.: Show H_c conserved.

→ another way to formulate shear Alfven wave

since $\underline{\tilde{V}} \perp \underline{B}_0$ write $\underline{\tilde{V}} = \nabla\phi \times \hat{z}$ ↳ velocity potential
$\underline{\tilde{B}} \perp \underline{B}_0$ $\underline{B} = \nabla A \times \hat{z}$
↳ magnetic potential

i.e. $\underline{E} = \underline{E}_\perp$ so $\underline{\tilde{V}} = \frac{c}{B_0^2} \underline{E} \times \underline{B}_0$ in shear Alfven now, $\quad \frac{\partial \underline{v}}{\partial t} = -\frac{1}{\rho_0} \underline{\nabla} \left(p + \frac{\tilde{B}^2}{8\pi} \right) + \frac{\underline{B}_0 \cdot \underline{\nabla}\, \underline{B}}{4\pi \rho_0}$

as $\tilde{\underline{v}}, \tilde{\underline{B}} \perp B_0$, take $\hat{z} \cdot \underline{\nabla} \times \quad \Rightarrow$

$$\hat{z} \cdot \frac{\partial \underline{\omega}}{\partial t} = 0 + \frac{B_0}{4\pi\rho_0} \frac{\partial}{\partial z} \hat{z} \cdot (\underline{\nabla} \times \underline{B})$$

Now, $\quad \underline{v} = \underline{\nabla}\phi \times \hat{z} \qquad\qquad \hat{z} \cdot \underline{\nabla} \times \tilde{\underline{B}} = \frac{4\pi}{c} \tilde{J}_z$

$\qquad = (\partial_y \phi, -\partial_x \phi, 0)$

$\qquad\qquad \underline{\nabla}(\underline{\nabla} \cdot \underline{A}) - \nabla^2 A = +\frac{4\pi}{c} \tilde{J}_z$

$\qquad \underline{\omega}_z = \hat{z} \cdot \omega = -\nabla_\perp^2 \phi \rightarrow \hat{z}$ component vorticity

$\Rightarrow \qquad\qquad\qquad\qquad \rightarrow$ magnetic torque

$$\frac{\partial}{\partial t} \nabla_\perp^2 \phi = \frac{B_0}{4\pi\rho_0} \frac{\partial}{\partial z} \nabla_\perp^2 A$$

vorticity evolution $\qquad \underline{\nabla} \times (\underline{\nabla} \times \underline{A})$

and $\quad \frac{\partial \tilde{\underline{B}}}{\partial t} = B_0 \frac{\partial}{\partial z} \underline{v} \qquad\qquad$ and $\hat{z} \cdot \underline{\nabla} \times \quad \Rightarrow$

$$\frac{\partial}{\partial t} \nabla_\perp^2 A = B_0 \frac{\partial}{\partial z} \nabla_\perp^2 \phi$$

current evolution $\qquad$ $\parallel$ vorticity gradient observe if "un-$\nabla_\perp^2$", have:

$$\frac{\partial A}{\partial t} - B_0 \frac{\partial}{\partial z} \phi = 0$$

$\Rightarrow$ basically means $E_{\parallel} = 0$ for Alfven waves.

$$\underline{E} = -\frac{\underline{V} \times \underline{B}_0}{c} \quad \therefore \quad \hat{z} \cdot \frac{\tilde{\underline{V}} \times B_0 \hat{z}}{c} = 0 \quad \checkmark$$

$\therefore$ can write shear Alfven wave equations as

$$\left.\begin{aligned} E_{\parallel} &= 0 = \frac{\partial A}{\partial t} - B_0 \frac{\partial \phi}{\partial z} = 0 \\ \frac{\partial}{\partial t} \nabla_\perp^2 \phi &= \frac{B_0}{4\pi \rho_0} \frac{\partial}{\partial z} \nabla_\perp^2 A \end{aligned}\right\}$$

$\Rightarrow$ example of 'reduced equations'.

Now, need also consider:

$\rightarrow$ $\underline{k} = k\hat{z}$, $\quad \underline{\nabla} \cdot \underline{V} \neq 0$

What happens?

MRI – Linear Theory

$$\underline{B}_0 = \begin{pmatrix} \hat{z} \\ \hat{\phi} \end{pmatrix}$$

→ MHD Eqns. $(\underline{\nabla} \cdot \underline{v} = 0)$

[diff'ntl rot + Alfven wave]

$$\frac{d}{dt} \tilde{B}_r = (\underline{B}_0 \cdot \underline{\nabla}) \tilde{v}_r + \eta \nabla^2 \tilde{B}_r$$

→ diff'ntl rotation

$$\frac{d}{dt} \tilde{B}_\phi = \tilde{B}_r \frac{d\Omega}{dr} + (\underline{B}_0 \cdot \underline{\nabla}) \tilde{v}_\phi + \eta \nabla^2 \tilde{B}_\phi$$

$$\frac{d}{dt} \tilde{B}_z = \underline{B}_0 \cdot \nabla \tilde{v}_z + \eta \nabla^2 \tilde{B}_z$$

and

$$\kappa^2 = \frac{2\Omega}{r} \frac{d}{dr} (r^2 \Omega)$$

$$\rho \left(\frac{d\tilde{v}_r}{dt} - 2\Omega \tilde{v}_\phi \right) = -\frac{\partial}{\partial r} P_* + \left(\frac{\underline{B}_0 \cdot \nabla}{4\pi} \right) \tilde{B}_r$$

$$\rho \left(\frac{d}{dt} \tilde{v}_\phi + \frac{\kappa^2}{2\Omega} \tilde{v}_r \right) = -\frac{1}{r} \frac{\partial}{\partial \phi} P_*$$

$$\kappa^2 = \frac{2\Omega}{r} \partial_r (r^2 \Omega) \qquad + \left(\frac{\underline{B}_0 \cdot \underline{\nabla}}{4\pi} \right) \tilde{B}_\phi + \eta \nabla^2 \tilde{v}_\phi$$

$$\rho \left(\frac{d\tilde{v}_z}{dt} \right) = -\frac{\partial}{\partial z} \left(p + \frac{B^2}{4\pi} \right) - \rho \frac{\partial \Phi}{\partial z}$$

$$+ \left(\frac{\underline{B}_0 \cdot \nabla}{4\pi} \right) \tilde{B}_z + \eta \nabla^2 \tilde{v}_z$$

→ Energetics

energy density

$$\frac{d}{dt} \langle \rho (\tilde{V}^2 + \tilde{V}_A^2 + \Phi) \rangle +$$

$$+ \nabla \cdot \langle \underline{v} \left(\frac{1}{2}\rho V^2 + \rho \Phi + P\right) + \frac{\underline{B}}{4\pi} \times (\underline{v} \times \underline{B}) \rangle$$

Flux

$$= - \underbrace{\tilde{T}_{r\phi} \frac{d\Omega}{d\ln r}} + \text{Dissp}^{n}$$

gain from Ω'

T → torque density

$$T_{R,\phi} = \langle \rho (\tilde{u}_r \tilde{u}_\phi - \tilde{U}_{Ar} \tilde{U}_{A\phi}) \rangle$$

$\tilde{B}_r \tilde{B}_\phi$

stresses   magnetic.

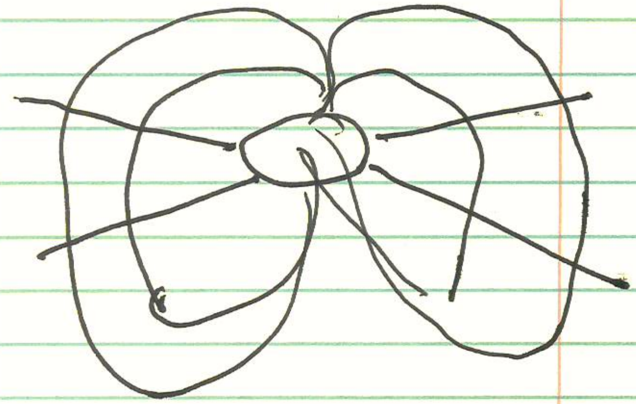
these two count.

deviation from Alfvenic balance

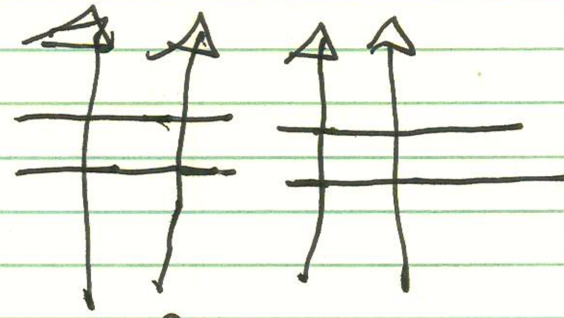
→ energy gain from Ω' ⟷ agrees with spring intuition.

→ begs for formulation QL problem.

→ Simple MRI → Uniform B_z.

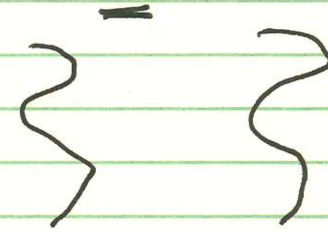


→



$\underline{B_0}$

$\underline{\nabla}\cdot\underline{V} = 0$



i.e. indep ϕ

$\underline{V}_r \to \underset{\substack{\uparrow \\ \text{radial}}}{\xi_r}\, e^{ikz}\hat{r}$

σ

$\delta B_r = ikB_0\,\xi_r$

$\tilde{\xi}_\phi$

$\delta\tilde{B}_\phi$

$\delta B_z = \xi_z = 0$

$\dfrac{\underline{B}\cdot\nabla B}{4\pi} \to \dfrac{ikB_0}{4\pi}\,\delta\tilde{\underline{B}} = -(k\cdot V_A)^2\,\tilde{\xi}$

$$\Rightarrow \boxed{\begin{aligned} \ddot{\tilde{\xi}}_r - 2\Omega\,\dot{\tilde{\xi}}_\phi &= -\left(\frac{d\Omega^2}{d\ln R} + (k\cdot V_A)^2\right)\tilde{\xi}_r \\ \ddot{\tilde{\xi}}_\phi + 2\Omega\,\dot{\tilde{\xi}}_r &= -(k\cdot V_A)^2\,\xi_\phi \end{aligned}}$$

Crank ⇒

$$\omega^4 - \omega^2\left[k^2 + 2(k\cdot V_A)^2\right] + (k\cdot V_A)^2\left[(k\cdot V_A)^2 + \frac{d\Omega^2}{d\ln R}\right] = 0$$

$\hookrightarrow$ need $\underline{\frac{d\Omega}{dr} < 0}$.

note:

- if $(k\cdot V_A)^2 > -\frac{d\Omega^2}{d\ln R}$ → stable

i.e. weak field case, for MRI.
B facilitates, not drives!

- $\gamma_{max} \sim \frac{1}{2}\left(\frac{d\Omega}{d\ln r}\right)$ → YUGE
→ k? → scale

for $(k\cdot V_A)^2_{max} = -\left(\frac{1}{4} + \frac{k^2}{16\Omega^2}\right)\frac{d\Omega^2}{d\ln R}$

- Keplerian: $\omega_{max} = \frac{3}{4}\Omega$

$$(k\cdot V_A)_{max} = \frac{\sqrt{15}}{4}\Omega$$

→ relevance, linear theory?

what is the message ?

- nonlinear theory, simulations essential

- role dissp'n. → higher h more unstable.

→ Transport ? ! → Reynolds vs Maxwell.

systematics ? ? → not addressed.

→ Two key issues:

- Tr'p → governing physics !
  → what can be said in general.
- dynamo.
  → $\langle \underline{B} \rangle$ vs $\langle \tilde{B}^2 \rangle$

→ Is it sufficient → simulations ! ?

QL estn. ? ! → profile variation

→ Transport to saturate ? !