6-2 (a) Normalization requires

$$1=\int_{-\infty}^{\infty}|\psi|^2dx=A^2\int_{-\frac{L}{4}}^{\frac{L}{4}}\cos^2\left(\frac{2\pi x}{L}\right)dx=\left(\frac{A^2}{2}\right)\int_{-\frac{L}{4}}^{\frac{L}{4}}\left(1+\cos\left(\frac{4\pi x}{L}\right)\right)dx$$

so $A=\frac{2}{\sqrt{L}}$.

(b)

$$P=\int_0^{\frac{L}{8}}|\psi|^2dx=A^2\int_0^{\frac{L}{8}}\cos^2\left(\frac{2\pi x}{L}\right)dx=\left(\frac{4}{L}\right)\left(\frac{1}{2}\right)\int_0^{\frac{L}{8}}\left(1+\cos\left(\frac{4\pi x}{L}\right)dx\right)$$

$$=\left(\frac{2}{L}\right)\left(\frac{L}{8}\right)+\left(\frac{2}{L}\right)\left(\frac{L}{4\pi}\right)\sin\left(\frac{4\pi x}{L}\right)\Bigg|_0^{\frac{L}{8}}=\frac{1}{4}+\frac{1}{2\pi}=0.409$$

6-4 The time development of Ψ is given by Equation 6.8 or

$$\Psi(x,\ t) = \int a(k) e^{i\{kx-\omega(k)t\}} dk = \left(\frac{C\alpha}{\sqrt{\pi}}\right)\int_{-\infty}^{\infty} e^{\{ikx - i\omega(k)t - \alpha^2 k^2\}} dk\ ,$$

with $\omega(k) = \dfrac{\hbar k^2}{2m}$ for a free particle of mass m. As in Example 6.3, the integral may be reduced to a recognizable form by completing the square in the exponent. Since $\omega(k)t = \left(\dfrac{\hbar t}{2m}\right)k^2$, we group this term together with $\alpha^2 k^2$ by introducing $\beta^2 = \alpha^2 + \dfrac{i\hbar t}{2m}$ to get

$$ikx - \omega(k)t - \alpha^2 k^2 = -\left(\beta k - \frac{ix}{2\beta}\right)^2 - \frac{x^2}{4\beta^2}\ .$$

Then, changing variables to $z = \beta k - \dfrac{ix}{2\beta}$ gives

$$\Psi(x,\ t) = \left(\frac{C\alpha}{\beta\sqrt{\pi}}\right) e^{-x^2/4\beta^2} \int_{-\infty}^{\infty} e^{-z^2} = \left(\frac{C\alpha}{\beta}\right) e^{-x^2/4\beta^2}\ .$$

To interpret this result, we must recognize that β is complex and separate real and imaginary parts. Thus, $\left|\beta^2\right|^2 = \left|\alpha^2 + \dfrac{i\hbar t}{2m}\right|^2 = \alpha^4 + \left(\dfrac{\hbar t}{2m}\right)^2$ and the exponent for Ψ is

$$\frac{x^2}{4\beta^2} = \frac{x^2\left(\alpha^2 - \frac{i\hbar t}{2m}\right)}{4\left|\beta^2\right|^2} = \frac{x^2}{4\left[\alpha^2 + \left(\frac{\hbar t}{2m\alpha}\right)^2\right]} + (\text{imaginary terms})$$

then

$$|\Psi(x,\ t)| = \frac{C\alpha}{\left(\alpha^4 + \left(\frac{\hbar t}{2m}\right)^2\right)^{1/4}} e^{\left\{-x^2/\left[4\left\{\alpha^2 + (\hbar t/2m\alpha)^2\right\}\right]\right\}}\ .$$

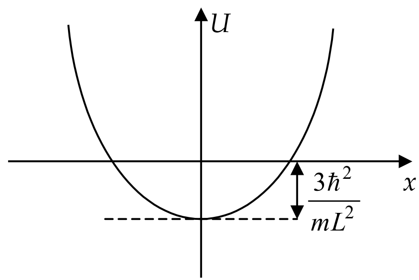
We see that apart from a phase factor, $\Psi(x, t)$ is still a gaussian but with amplitude diminished by $\frac{\alpha}{\left(\alpha^4+\left(\frac{\hbar t}{2m}\right)^2\right)^{1/4}}$ and a width $\Delta x(t)=\left(\alpha^2+\left(\frac{\hbar t}{2m\alpha}\right)^2\right)^{1/2}$ where $\alpha=\Delta x(0)$ is the initial width.

6-5 (a) Solving the Schrödinger equation for U with $E=0$ gives

$$U=\left(\frac{\hbar^2}{2m}\right)\frac{\left(\frac{d^2\psi}{dx^2}\right)}{\psi}.$$

If $\psi = Ae^{-x^2/L^2}$ then $\frac{d^2\psi}{dx^2}=\left(4Ax^3-6AxL^2\right)\left(\frac{1}{L^4}\right)e^{-x^2/L^2}$, $U=\left(\frac{\hbar^2}{2mL^2}\right)\left(\frac{4x^2}{L^2}-6\right)$.

(b) $U(x)$ is a parabola centered at $x=0$ with $U(0)=\frac{-3\hbar^2}{mL^2}<0$:



6-7 Since the particle is confined to the box, Δx can be no larger than L, the box length. With $n=0$, the particle energy $E_n=\frac{n^2h^2}{8mL^2}$ is also zero. Since the energy is all kinetic, this implies $\langle p_x^2\rangle=0$. But $\langle p_x\rangle=0$ is expected for a particle that spends equal time moving left as right, giving $\Delta p_x=\sqrt{\langle p_x^2\rangle-\langle p_x\rangle^2}=0$. Thus, for this case $\Delta p_x\Delta x=0$, in violation of the uncertainty principle.

[redacted]

6-9 $E_n=\frac{n^2h^2}{8mL^2}$, so $\Delta E=E_2-E_1=\frac{3h^2}{8mL^2}$

$$\Delta E=(3)\frac{(1240\text{ eV nm}/c)^2}{8\left(938.28\times10^6\text{ eV}/c^2\right)\left(10^{-5}\text{ nm}\right)^2}=6.14\text{ MeV}$$

$$\lambda=\frac{hc}{\Delta E}=\frac{1240\text{ eV nm}}{6.14\times10^6\text{ eV}}=2.02\times10^{-4}\text{ nm}$$

This is the gamma ray region of the electromagnetic spectrum.

6-10 $E_n=\frac{n^2h^2}{8mL^2}$

$$\frac{h^2}{8mL^2}=\frac{\left(6.63\times10^{-34}\text{ Js}\right)^2}{8\left(9.11\times10^{-31}\text{ kg}\right)\left(10^{-10}\text{ m}\right)^2}=6.03\times10^{-18}\text{ J}=37.7\text{ eV}$$

(a) $E_1=37.7\text{ eV}$

$E_2=37.7\times2^2=151\text{ eV}$

$E_3=37.7\times3^2=339\text{ eV}$

$E_4=37.7\times4^2=603\text{ eV}$

(b) $hf=\frac{hc}{\lambda}=E_{n_i}-E_{n_f}$

$$\lambda=\frac{hc}{E_{n_i}-E_{n_f}}=\frac{1\,240\text{ eV}\cdot\text{nm}}{E_{n_i}-E_{n_f}}$$

For $n_i=4$, $n_f=1$, $E_{n_i}-E_{n_f}=603\text{ eV}-37.7\text{ eV}=565\text{ eV}$, $\lambda=2.19\text{ nm}$

$n_i=4$, $n_f=2$, $\lambda=2.75\text{ nm}$

$n_i=4$, $n_f=3$, $\lambda=4.70\text{ nm}$

$n_i=3$, $n_f=1$, $\lambda=4.12\text{ nm}$

$n_{\mathrm{i}} = 3$, $n_{\mathrm{f}} = 2$, $\lambda = 6.59$ nm

$n_{\mathrm{i}} = 2$, $n_{\mathrm{f}} = 1$, $\lambda = 10.9$ nm

[redacted]

6-12 $\Delta E = \dfrac{hc}{\lambda} = \left(\dfrac{h^2}{8mL^2}\right)\left[2^2 - 1^2\right]$ and $L = \left[\dfrac{(3/8)h\lambda}{mc}\right]^{1/2} = 7.93\times10^{-10}\ \text{m} = 7.93\ \text{Å}$.

6-13 (a) Proton in a box of width $L = 0.200\ \text{nm} = 2\times10^{-10}\ \text{m}$

$$E_1 = \frac{h^2}{8m_pL^2} = \frac{\left(6.626\times10^{-34}\ \text{J}\cdot\text{s}\right)^2}{8\left(1.67\times10^{-27}\ \text{kg}\right)\left(2\times10^{-10}\ \text{m}\right)^2} = 8.22\times10^{-22}\ \text{J}$$

$$= \frac{8.22\times10^{-22}\ \text{J}}{1.60\times10^{-19}\ \text{J/eV}} = 5.13\times10^{-3}\ \text{eV}$$

(b) Electron in the same box:

$$E_1 = \frac{h^2}{8m_eL^2} = \frac{\left(6.626\times10^{-34}\ \text{J}\cdot\text{s}\right)^2}{8\left(9.11\times10^{-31}\ \text{kg}\right)\left(2\times10^{-10}\ \text{m}\right)^2} = 1.506\times10^{-18}\ \text{J} = 9.40\ \text{eV}.$$

(c) The electron has a much higher energy because it is much less massive.

6-15 (a) $$U=\left(\frac{e^2}{4\pi\,\varepsilon_0 d}\right)\left[-1+\frac{1}{2}-\frac{1}{3}+\left(-1+\frac{1}{2}\right)+(-1)\right]=\frac{(-7/3)e^2}{4\pi\,\varepsilon_0 d}=\frac{(-7/3)ke^2}{d}$$

(b) $$K=2E_1=\frac{2h^2}{8m\times 9d^2}=\frac{h^2}{36md^2}$$

(c) $E=U+K$ and $\frac{dE}{dd}=0$ for a minimum $\left[\frac{(+7/3)e^2k}{d^2}\right]-\frac{h^2}{18md^3}=0$

$$d=\frac{3h^2}{(7)\left(18ke^2m\right)} \text{ or } d=\frac{h^2}{42mke^2}$$

$$d=\frac{\left(6.63\times10^{-34}\ \text{J}\cdot\text{s}\right)^2}{(42)\left(9.11\times10^{-31}\ \text{kg}\right)\left(9\times10^9\ \text{N}\cdot\text{m}^2\cdot\text{C}^{-2}\right)\left(1.6\times10^{-19}\ \text{C}\right)^2}=0.5\times10^{-10}\ \text{m}=0.050\ \text{nm}$$

(d) Since the lithium spacing is a, where $Na^3=V$ and the density is $\frac{Nm}{V}$ where m is the mass of one atom, we get

$$a=\left(\frac{Vm}{Nm}\right)^{1/3}=\left(\frac{m}{\text{density}}\right)^{1/3}=\left(1.66\times10^{-27}\ \text{kg}\times\frac{7}{530\ \text{kg}/\text{m}^3}\right)^{1/3}\ \text{m}=2.8\times10^{-10}\ \text{m}$$
$$=0.28\ \text{nm}$$

(2.8 times larger than $2d$)

6-16 (a) $\psi(x) = A\sin\left(\frac{\pi x}{L}\right)$, $L = 3$ Å. Normalization requires

$$1 = \int_0^L |\psi|^2\, dx = \int_0^L A^2 \sin^2\left(\frac{\pi x}{L}\right) dx = \frac{LA^2}{2}$$

so $A = \left(\frac{2}{L}\right)^{1/2}$

$$P = \int_0^{L/3} |\psi|^2\, dx = \left(\frac{2}{L}\right)\int_0^{L/3} \sin^2\left(\frac{\pi x}{L}\right) dx = \frac{2}{\pi}\int_0^{\pi/3} \sin^2 \phi\, d\phi = \frac{2}{\pi}\left[\frac{\pi}{6} - \frac{(3)^{1/2}}{8}\right] = 0.195\,5\,.$$

(b) $\psi = A\sin\left(\frac{100\pi x}{L}\right)$, $A = \left(\frac{2}{L}\right)^{1/2}$

$$P = \frac{2}{L}\int_0^{L/3} \sin^2\left(\frac{100\pi x}{L}\right) dx = \frac{2}{L}\left(\frac{L}{100\pi}\right)\int_0^{100\pi/3} \sin^2 \phi\, d\phi = \frac{1}{50\pi}\left[\frac{100\pi}{6} - \frac{1}{4}\sin\left(\frac{200\pi}{3}\right)\right]$$
$$= \frac{1}{3} - \left[\frac{1}{200\pi}\right]\sin\left(\frac{2\pi}{3}\right) = \frac{1}{3} - \frac{\sqrt{3}}{400\pi} = 0.331\,9$$

(c) Yes: For large quantum numbers the probability approaches $\frac{1}{3}$.

6-17 (a) The wavefunctions and probability densities are the same as those shown in the two lower curves in Figure 6.16 of the text.

(b) $P_1 = \int_{1.5\text{ Å}}^{3.5\text{ Å}} |\psi|^2\, dx = \frac{2}{10\text{ Å}}\int_{1.5\text{ Å}}^{3.5\text{ Å}} \sin^2\left(\frac{\pi x}{10}\right) dx$

$$\frac{1}{5}\left[\frac{x}{2} - \frac{10}{4\pi}\sin\left(\frac{\pi x}{5}\right)\right]_{1.5}^{3.5}$$

In the above result we used $\int \sin^2 ax dx = \frac{x}{2} - \frac{1}{4a}\sin(2ax)$. Therefore,

$$P_1 = \frac{1}{10}\left[x - \frac{5}{\pi}\sin\left(\frac{\pi x}{5}\right)\right]_{1.5}^{3.5} = \frac{1}{10}\left\{3.5 - \frac{5}{\pi}\sin\left[\frac{\pi(3.5)}{5}\right] - 1.5 + \frac{5}{\pi}\sin\left[\frac{\pi(1.5)}{5}\right]\right\}$$
$$= \frac{1}{10}\left[2.0 + \frac{5}{\pi}(\sin 0.3\pi - \sin 0.7\pi)\right] = \frac{1}{10}[2.00 + 0.0] = 0.200$$

(c) $P_2 = \frac{1}{5}\int_{1.5}^{3.5} \sin^2\left(\frac{\pi x}{5}\right) dx = \frac{1}{5}\left[\frac{x}{2} - \frac{5}{4\pi}\sin(0.4\pi x)\right]_{1.5}^{3.5} = \frac{1}{10}\left[x - \frac{5}{2\pi}\sin(0.4\pi x)\right]_{1.5}^{3.5}$

$$= \frac{1}{10}\{2.0 + (0.798)\{\sin[0.4\pi(1.5)] - \sin[0.4\pi(3.5)]\}\} = 0.351$$

(d) Using $E = \frac{n^2 h^2}{8mL^2}$ we find $E_1 = 0.377$ eV and $E_2 = 1.51$ eV .

6-28 A particle within the well is subject to no forces and, hence, moves with uniform speed, spending equal time in all parts of the well. Thus, for such a particle the probability density is *uniform*. That is, $P_c(x) = \text{constant}$. The constant is fixed by requiring the integrated probability to be unity, that is, $1 = \int_0^L P_c(x)\,dx = CL$ or $C = \frac{1}{L}$. To find $\langle x \rangle$ we weight the possible particle positions according to the probability density P_c to get $\langle x \rangle = \int_0^L x P_c(x)\,dx = \frac{1}{L}\left(\frac{x^2}{2}\right)\Big|_0^L = \frac{L}{2}$. Similarly, $\langle x^2 \rangle$ is found by weighting the possible values of x^2 with P_c:

$$\langle x^2 \rangle = \int_0^L x^2 P_c(x)\,dx = \frac{1}{L}\left(\frac{x^3}{3}\right)\Big|_0^L = \frac{L^2}{3}.$$

The classical and quantum results for $\langle x \rangle$ agree exactly; for $\langle x^2 \rangle$ the quantum prediction is smaller by an amount $\frac{L^2}{2(n\pi)^2}$ which, however, goes to zero in the limit of large quantum numbers *n*, where classical and quantum results must coincide (correspondence principle).

6-29 (a) Normalization requires

$1 = \int_{-\infty}^{\infty} |\psi|^2\,dx = C^2 \int_0^{\infty} e^{-2x}\left(1 - e^{-x}\right)^2 dx = C^2 \int_0^{\infty}\left(e^{-2x} - 2e^{-3x} + e^{-4x}\right)dx$. The integrals are elementary and give $1 = C^2\left\{\frac{1}{2} - 2\left(\frac{1}{3}\right) + \frac{1}{4}\right\} = \frac{C^2}{12}$. The proper units for *C* are those of $(\text{length})^{-1/2}$ thus, normalization requires $C = (12)^{1/2}\ \text{nm}^{-1/2}$.

(b) The most likely place for the electron is where the probability $|\psi|^2$ is largest. This is also where ψ itself is largest, and is found by setting the derivative $\frac{d\psi}{dx}$ equal zero:

$$0 = \frac{d\psi}{dx} = C\left\{-e^{-x} + 2e^{-2x}\right\} = Ce^{-x}\left\{2e^{-x} - 1\right\} .$$

The RHS vanishes when $x = \infty$ (a minimum), and when $2e^{-x} = 1$, or $x = \ln 2 \text{ nm}$. Thus, the most likely position is at $x_p = \ln 2 \text{ nm} = 0.693 \text{ nm}$.

(c) The average position is calculated from

$$\langle x\rangle = \int_{-\infty}^{\infty} x|\psi|^2\,dx = C^2\int_0^{\infty} xe^{-2x}\left(1-e^{-x}\right)^2 dx = C^2\int_0^{\infty} x\left(e^{-2x} - 2e^{-3x} + e^{-4x}\right)dx .$$

The integrals are readily evaluated with the help of the formula $\int_0^{\infty} xe^{-ax}dx = \frac{1}{a^2}$ to get $\langle x\rangle = C^2\left\{\frac{1}{4} - 2\left(\frac{1}{9}\right) + \frac{1}{16}\right\} = C^2\left\{\frac{13}{144}\right\}$. Substituting $C^2 = 12 \text{ nm}^{-1}$ gives

$$\langle x\rangle = \frac{13}{12}\text{ nm} = 1.083 \text{ nm} .$$

We see that $\langle x\rangle$ is somewhat greater than the most probable position, since the probability density is skewed in such a way that values of x larger than x_p are weighted more heavily in the calculation of the average.

6-30 The possible particle positions within the box are weighted according to the probability density $|\psi|^2 = \frac{2}{L}\sin^2\left(\frac{n\pi x}{L}\right)$ The position is calculated as $\langle x\rangle = \int_0^L x|\psi|^2\,dx = \frac{2}{L}\int_0^L x\sin^2\left(\frac{n\pi x}{L}\right)dx$. Making the change of variable $\theta = \frac{nx}{L}$ (so that $d\theta = \frac{\pi\,dx}{L}$) gives $\langle x\rangle = \frac{2L}{\pi^2}\int_0^{\pi}\theta\sin^2 n\theta\,d\theta$. Using the trigonometric identity $2\sin^2\theta = 1 - \cos 2\theta$, we get $\langle x\rangle = \frac{L}{\pi^2}\left\{\int_0^{\pi}\theta\,d\theta - \int_0^{\pi}\theta\cos 2n\theta\,d\theta\right\}$. An integration by parts shows that the second integral vanishes, while the first integrates to $\frac{\pi^2}{2}$. Thus, $\langle x\rangle = \frac{L}{2}$, independent of n. For the computation of $\left\langle x^2\right\rangle$, there is an extra factor of x in the integrand. After changing variables to $\theta = \frac{\pi x}{L}$ we get $\left\langle x^2\right\rangle = \frac{L^2}{\pi^3}\left\{\int_0^{\pi}\theta^2 d\theta - \int_0^{\pi}\theta^2\cos 2n\theta\,d\theta\right\}$. The first integral evaluates to $\frac{\pi^3}{3}$, the second may be integrated by parts twice to get

$$\int_0^{\pi}\theta^2\cos 2n\theta\,d\theta = -\frac{1}{n}\int_0^{\pi}\theta\sin 2n\theta\,d\theta = \left(\frac{1}{2n^2}\right)\theta\cos 2n\theta\Big|_0^{\pi} = \frac{\pi}{2n^2} .$$

Then $\langle x^2 \rangle = \frac{L^2}{\pi^3}\left\{\frac{\pi^3}{3} - \frac{\pi}{2n^2}\right\} = \frac{L^2}{3} - \frac{L^2}{2(n\pi)^2}$.

6-31 The symmetry of $|\psi(x)|^2$ about $x = 0$ can be exploited effectively in the calculation of average values. To find $\langle x \rangle$

$$\langle x \rangle = \int_{-\infty}^{\infty} x|\psi(x)|^2\, dx$$

We notice that the integrand is antisymmetric about $x = 0$ due to the extra factor of x (an odd function). Thus, the contribution from the two half-axes $x > 0$ and $x < 0$ cancel exactly, leaving $\langle x \rangle = 0$. For the calculation of $\langle x^2 \rangle$, however, the integrand is symmetric and the half-axes contribute equally to the value of the integral, giving

$$\langle x \rangle = \int_0^{\infty} x^2 |\psi|^2\, dx = 2C^2 \int_0^{\infty} x^2 e^{-2x/x_0}\, dx.$$

Two integrations by parts show the value of the integral to be $2\left(\frac{x_0}{2}\right)^3$. Upon substituting for C^2, we get $\langle x^2 \rangle = 2\left(\frac{1}{x_0}\right)(2)\left(\frac{x_0}{2}\right)^3 = \frac{x_0^2}{2}$ and $\Delta x = \left(\langle x^2 \rangle - \langle x \rangle^2\right)^{1/2} = \left(\frac{x_0^2}{2}\right)^{1/2} = \frac{x_0}{\sqrt{2}}$. In calculating the probability for the interval $-\Delta x$ to $+\Delta x$ we appeal to symmetry once again to write

$$P = \int_{-\Delta x}^{+\Delta x} |\psi|^2\, dx = 2C^2 \int_0^{\Delta x} e^{-2x/x_0}\, dx = -2C^2\left(\frac{x_0}{2}\right)e^{-2x/x_0}\Big|_0^{\Delta x} = 1 - e^{-\sqrt{2}} = 0.757$$

or about 75.7% independent of x_0.