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Chapter 9

Landau Fermi Liquid Theory

9.1 Normal ^{3}He Liquid

^{3}He is a neutral atom consisting of two protons, one neutron, and two electrons. A composite of five fermions, it behaves as a hard-sphere (radius $a \approx 1.35$Å) fermion of (nuclear) spin $I = \frac{1}{2}$ at energies below the scale of electronic transitions[1]. It exhibits a fairly rich phase diagram, depicted in the left hand panel of Fig. 9.1. ^{3}He A and ^{3}He B are superfluid phases which differ in the symmetry of their respective order parameters. ^{3}He N is a normal fluid which behaves much like a free Fermi gas, but in which interaction effects play an essential role in its physical properties. It is known as a *Fermi liquid*[2] In a Fermi liquid, as in the noninteracting Fermi gas, the low-temperature specific heat $c_V(T)$ is linear in T and the magnetic susceptibility $\chi(T)$ is Pauli-like ($\chi \propto T^0$), as shown in Fig. 9.2. An important distinction between ^{3}He N and most metals is that the mass of the ^{3}He atom is about 6,000 times greater than that of the electron. Thus at a typical density $n = 1.64 \times 10^{22}\,\text{cm}^{-3}$ and $m_3 = 5.01 \times 10^{-24}\,\text{g}$ one obtains a Fermi temperature

$$T_{\mathsf{F}} = \frac{\hbar^2}{2mk_{\text{B}}}\,(3\pi^2 n)^{2/3} = 4.97\,\text{K} \quad , \tag{9.1}$$

which is much much smaller than $T_{\mathsf{F}}(\text{Cu}) \approx 81,000\,\text{K}$ and $T_{\mathsf{F}}(\text{Al}) \approx 135,000\,\text{K}$. This explains why one begins to see Curie-like behavior in the magnetic susceptibility, *i.e.* $\chi(T) \simeq n\mu_0^2/k_{\text{B}}T$, at temperatures $T \gtrsim 1\,\text{K}$. Here $\mu_0 = -10.746 \times 10^{-27}\,\text{J/T} = -1.1574\,\mu_{\text{B}}$ is the ^{3}He nuclear magnetic moment, and $\mu_{\text{B}} = e\hbar/2m_{\text{e}}c$ is the Bohr magneton, with m_{e} the electron mass. Recall these basic

[1] $E_1 - E_0 \approx 20\,\text{eV}$, and the first ionization energy is $24.6\,\text{eV}$.
[2] The general theory of Fermi liquids was developed principally by the Russian physicist Lev Landau in the 1950s.

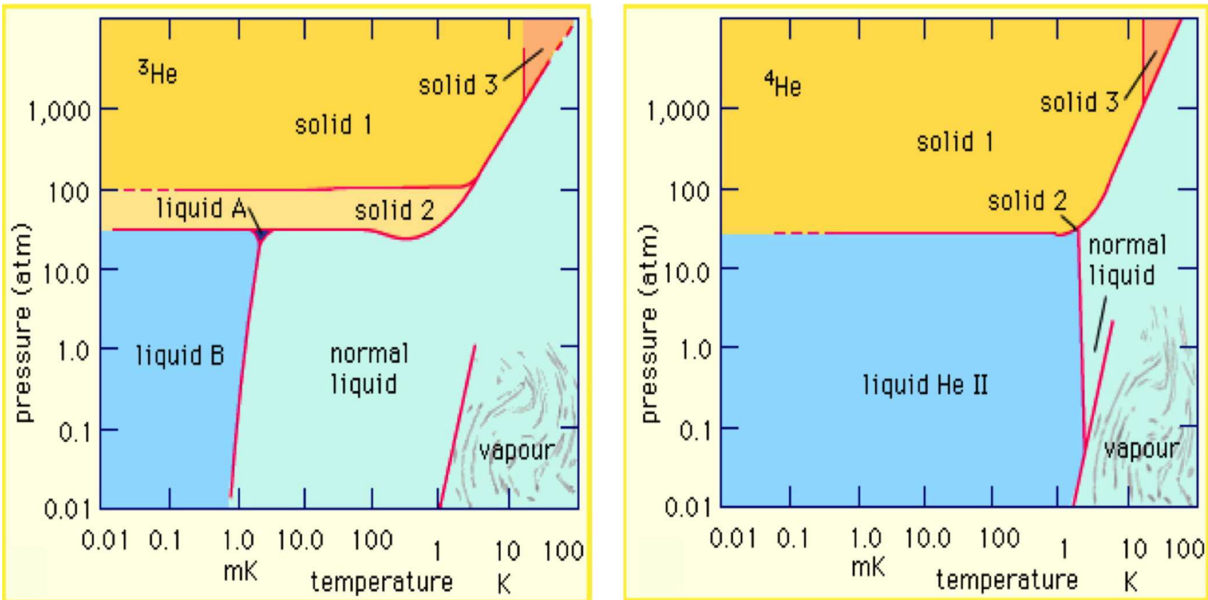

Figure 9.1: Phase diagrams of ^{3}He (left) and ^{4}He (right).

results for the free spin-$\frac{1}{2}$ Fermi gas with ballistic dispersion $\varepsilon(\boldsymbol{k}) = \hbar^2\boldsymbol{k}^2/2m$:

$$\begin{aligned}
\text{Fermi wavevector} &: \; k_{\rm F} = (3\pi^2 n)^{1/3} \\
\text{density of states} &: \; g(\varepsilon_{\rm F}) = \frac{m k_{\rm F}}{\pi^2\hbar^2} \\
\text{occupancy} &: \; f(\varepsilon) = \left[\exp\!\left(\frac{\varepsilon-\mu}{k_{\rm B}T}\right)+1\right]^{-1} \\
\text{specific heat} &: \; c_V = \frac{1}{V}\left(\frac{\partial E}{\partial T}\right)_{N,V} = \frac{\pi^2}{3}\, g(\varepsilon_{\rm F})\, k_{\rm B}^2 T + \mathcal{O}(T^3) \\
\text{magnetic susceptibility} &: \; \chi = \left(\frac{\partial M}{\partial H}\right)_{N,V} = \mu_0^2\, g(\varepsilon_{\rm F}) + \mathcal{O}(T^2) \\
\text{compressibility} &: \; \kappa = n^{-2}\left(\frac{\partial n}{\partial \mu}\right)_T = n^{-2} g(\varepsilon_{\rm F}) + \mathcal{O}(T^2) \quad .
\end{aligned} \tag{9.2}$$

Experimental data for $c_V(T)$ and $\chi(T)$ in ^{3}He are shown in Fig. 9.2. Note that $c_V(T)/T$ and $\chi(T)$ are each pressure-dependent constants as $T \to 0$. The same is true for the compressibility $\kappa(T)$, which is obtained from measurements of the velocity of thermodynamic sound, $s = (m_3 n\kappa)^{-1/2}$. In a noninteracting Fermi gas, all these quantities are proportional to the density of states $g(\varepsilon_{\rm F})$, up to constant factors. We can define $c_V^0(T,n)$, $\chi^0(T,n)$, and $\kappa^0(T,n)$ to be the corresponding free Fermi gas expressions for a system of spin-$\frac{1}{2}$ fermions of mass m_3 and density n. One finds that the ratios c_V/c_V^0, χ/χ^0, and κ/κ^0 all tend to different constants as $T \to 0$. Thus, it is impossible to reconcile the data by positing a phenomenological effective

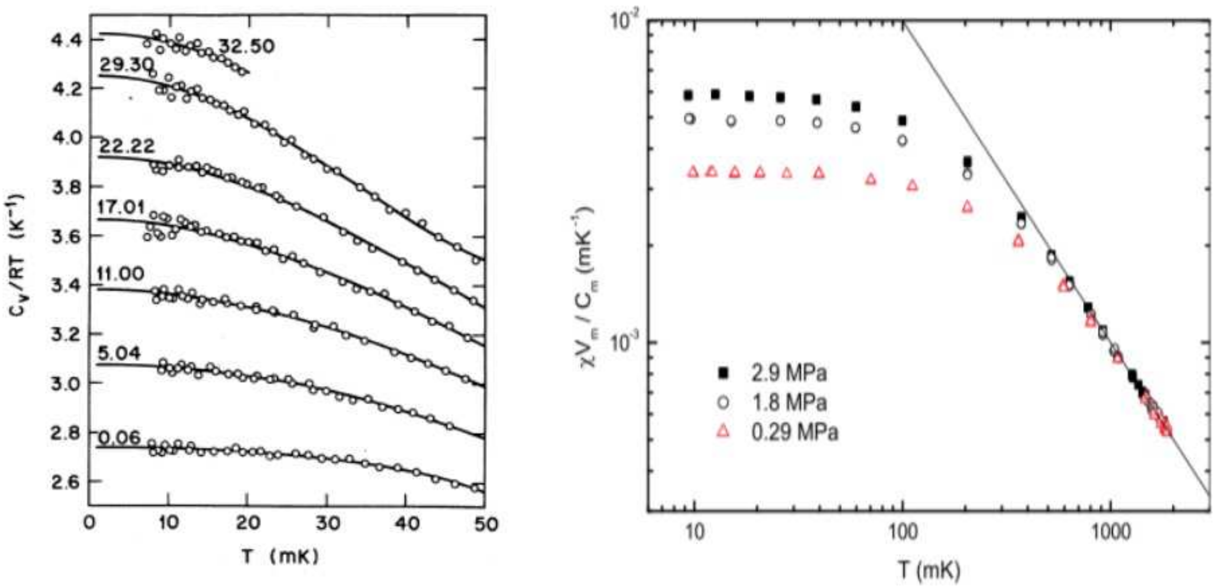

Figure 9.2: Left: $c_V(T)/RT$ for normal ^{3}He. From D. S. Greywall, *Phys. Rev. B* **27**, 2747 (1983). Numbers give the sample pressures in bars at $T = 0.1\,\mathrm{K}$. Right: Normalized magnetic susceptibility $\chi(T)\,v_0/C_\mathrm{m}$ of normal ^{3}He, where v_0 is the molar volume and $C_\mathrm{m} \equiv \lim_{T\to\infty} T\chi(T)$ is the Curie constant. From V. Goudon *et al.*, *J. Phys.: Conf. Ser.* **150**, 032024 (2009).

mass m^*, since that would require that these ratios all tend to the same value. Furthermore, the $T \to 0$ limits of these ratios are all pressure-dependent. Another issue is that the first correction to the low temperature linear specific heat in a Fermi gas go as T^3, whereas experiments yield a co.rrection on the order of $T^3 \ln T$ We shall see below how Landau's theory is capable of reproducing the observed temperature dependences, and moreover introduces additional interaction parameters which allow us to describe all these behaviors in a consistent way. We shall largely follow here the treatments by Nozieres and Pines, and by Baym and Pethick[3].

9.2 Fermi Liquid Theory : Statics and Thermodynamics

9.2.1 Adiabatic continuity

The idea behind Fermi liquid theory is that the many-body eigenstates of the free Fermi gas with Hamiltonian $\hat{H}_0$, which are Slater determinants, each evolve adiabatically into eigenstates of the interacting Hamiltonian $\hat{H} = \hat{H}_0 + \hat{H}_1$, where $\hat{H}_1$ is the interaction part. Typically we will

[3]P. Nozieres and D. Pines, *Theory of Quantum Liquids* (Avalon, 1999); G. Baym and C. Pethick, *Landau Fermi-Liquid Theory : Concepts and Applications* (Wiley, 1991).

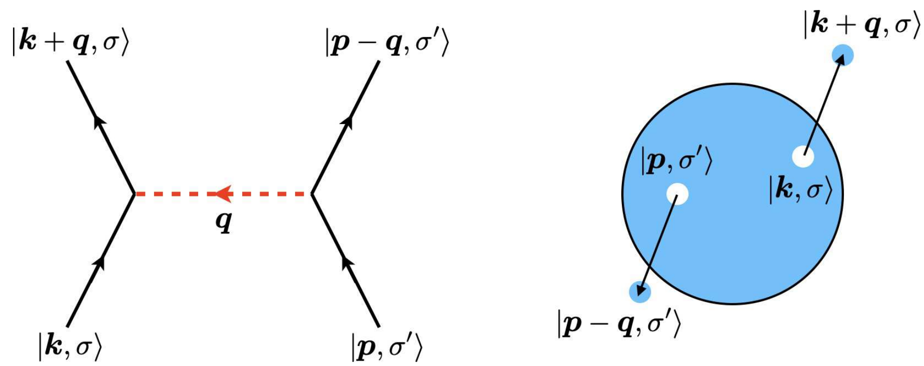

Figure 9.3: Two particle, two hole excitation of the state $|\mathsf{F}\rangle$ obtained via first order perturbation theory in the interaction Hamiltonian $\hat{H}_1$.

consider

$$\hat{H}_0 = \sum_{\boldsymbol{k},\sigma} \varepsilon^0_{\boldsymbol{k}}\, c^\dagger_{\boldsymbol{k},\sigma} c_{\boldsymbol{k},\sigma} \quad , \tag{9.3}$$

with $\varepsilon^0_{\boldsymbol{k}} = \hbar^2\boldsymbol{k}^2/2m$. The general form of interactions in a translationally invariant system is

$$\hat{H}_1 = \frac{1}{2} \sum_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}} \sum_{\alpha,\beta} \sum_{\alpha',\beta'} \hat{u}_{\alpha\beta\alpha'\beta'}(\boldsymbol{q})\, c^\dagger_{\boldsymbol{k}+\boldsymbol{q},\alpha}\, c^\dagger_{\boldsymbol{p}-\boldsymbol{q},\alpha'}\, c_{\boldsymbol{p},\beta'}\, c_{\boldsymbol{k},\beta} \quad . \tag{9.4}$$

In systems with spin isotropy, we can write

$$\hat{u}_{\alpha\beta\alpha'\beta'}(\boldsymbol{q}) = \hat{u}^{\mathsf{S}}(\boldsymbol{q})\, \delta_{\alpha\beta}\, \delta_{\alpha'\beta'} + \hat{u}^{\mathsf{H}}(\boldsymbol{q})\, \boldsymbol{\tau}_{\alpha\beta} \cdot \boldsymbol{\tau}_{\alpha'\beta'} \quad , \tag{9.5}$$

where $\hat{u}^{\mathsf{S,H}}(\boldsymbol{q})$ are the scalar and Heisenberg exchange parts of the interaction, respectively.

We will focus here on the case where $\hat{u}^{\mathsf{H}} = 0$, in which case we may write

$$\hat{H}_1 = \frac{1}{2} \sum_{\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}} \sum_{\sigma,\sigma'} \hat{u}(\boldsymbol{q})\, c^\dagger_{\boldsymbol{k}+\boldsymbol{q},\sigma}\, c^\dagger_{\boldsymbol{p}-\boldsymbol{q},\sigma'}\, c_{\boldsymbol{p},\sigma'}\, c_{\boldsymbol{k},\sigma} \quad . \tag{9.6}$$

When $\hat{H}_1 = 0$, the N-particle ground state is the filled Fermi sphere, $|\,\mathrm{F}\,\rangle = \prod'_{\boldsymbol{k},\sigma} c^\dagger_{\boldsymbol{k},\sigma}\,|\,0\,\rangle$, where the prime denotes the restriction $|\boldsymbol{k}| \le k_{\mathrm{F}}$. Treating the interaction in first order perturbation theory, we have the perturbed ground state $|\,\mathrm{F}'\,\rangle$ is given by

$$|\,\mathrm{F}'\,\rangle = |\,\mathrm{F}\,\rangle + \sum_\alpha \frac{|\,\alpha\,\rangle\,\langle\,\alpha\,|\,\hat{H}_1\,|\,\mathrm{F}\,\rangle}{E^0_{\mathrm{F}} - E^0_\alpha} + \mathcal{O}(\hat{H}_1^2) \quad . \tag{9.7}$$

This results in contributions such as that depicted in Fig. 9.3. Proceeding to still higher orders of perturbation theory, the perturbed ground state appears as a seething, bubbling 'soup' of particle-hole pairs.

We can associate interacting and noninteracting eigenstates, however, through the process of adiabatic evolution. Define

$$\hat{H}(\lambda) = \hat{H}_0 + \lambda \hat{H}_1 \quad , \tag{9.8}$$

so $\hat{H}(0) = \hat{H}_0$ and $\hat{H}(1) = \hat{H}_0 + \hat{H}_1 = \hat{H}$. Suppose $\lambda(t)$ is a monotonically increasing function of t for $t < 0$, with $\lambda(-\infty) = 0$ and $\lambda(0) = 1$. The unitary evolution operator is then

$$\begin{aligned} \hat{U}(0,-\infty) &= \mathcal{T} \exp\left\{ -\frac{i}{\hbar} \int\limits_{-\infty}^{0} \! dt\, \hat{H}(t) \right\} \\ &= \mathcal{T} \exp\left\{ -\frac{i}{\hbar} \int\limits_{0}^{1} \! \frac{d\lambda}{\dot{\lambda}}\, \hat{H}(\lambda) \right\} = \mathcal{T} \exp\left\{ -\frac{i}{\hbar\epsilon} \int\limits_{0}^{1} \! \frac{d\lambda}{\lambda}\, \hat{H}(\lambda) \right\} \equiv \hat{U}_\epsilon \quad , \end{aligned} \tag{9.9}$$

where in the final expression we take $\lambda(t) = \exp(-\epsilon|t|)$. Thus, we can consider the adiabatic map,

$$\hat{U}_\epsilon \;:\; |\,\mathrm{F}\,\rangle \rightarrow |\,\mathrm{F}'_\epsilon\,\rangle = \hat{U}_\epsilon\,|\,\mathrm{F}\,\rangle \tag{9.10}$$

where $\hat{H}\,|\,\mathrm{F}'\,\rangle = E'\,|\,\mathrm{F}'\,\rangle$. We then consider the limit as $\epsilon \to 0$. One wrinkle here is that the phase of $|\,\mathrm{F}'_\epsilon\rangle$ in the limit $\epsilon \to 0$ is generally divergent, and to cancel it out we could instead define the state

$$|\,\widetilde{\mathrm{F}}'\,\rangle \equiv \lim_{\epsilon\to 0} \left\{ \left(\frac{\langle\,\mathrm{F}\,|\,U^\dagger_\epsilon\,|\,\mathrm{F}\,\rangle}{\langle\,\mathrm{F}\,|\,U_\epsilon\,|\,\mathrm{F}\,\rangle} \right)^{1/2} \hat{U}_\epsilon\,|\,\mathrm{F}\,\rangle \right\} \quad , \tag{9.11}$$

in which the phase cancels.

Suppose that rather starting with the N-particle state $|\,\mathrm{F}\,\rangle$, we start with the state $c^\dagger_{\boldsymbol{k},\sigma}\,|\,\mathrm{F}\,\rangle$, where $|\boldsymbol{k}| > k_{\mathrm{F}}$. We then adiabatically evolve with $\hat{U}_\epsilon$ as described above (including our nifty phase divergence cancellation protocol). We then obtain a state $|\,\Psi_{\boldsymbol{k},\sigma}\,\rangle$, about which we know three things: (i) its total particle number is $N+1$, (ii) its total momentum is $\hbar\boldsymbol{k}$, and (iii) its total spin polarization is σ. We may write

$$|\,\Psi'_{\boldsymbol{k},\sigma}\,\rangle = q^\dagger_{\boldsymbol{k},\sigma}\,|\,\mathrm{F}'\,\rangle \quad , \tag{9.12}$$

where

$$\begin{aligned} q^\dagger_{\boldsymbol{k},\sigma} &= \lim_{\epsilon\to 0} \left\{ U_\epsilon\, c^\dagger_{\boldsymbol{k},\sigma}\, U^\dagger_\epsilon \right\} \\ &= Z_{\boldsymbol{k},\sigma}\, c^\dagger_{\boldsymbol{k},\sigma} + \sum_{\boldsymbol{k}_1,\boldsymbol{k}_2} \sum_{\sigma_1,\sigma_2} A^{\sigma_1,\sigma_2}_{\boldsymbol{k}_1,\boldsymbol{k}_2}\, c^\dagger_{\boldsymbol{k}_1,\sigma_1} c^\dagger_{\boldsymbol{k}_2,\sigma_2} c_{\boldsymbol{k}_1+\boldsymbol{k}_2-\boldsymbol{k},\sigma_1+\sigma_2-\sigma} + \ldots \quad . \end{aligned} \tag{9.13}$$

Thus, the operator which *when acting on the interacting ground state* $|\,\mathrm{F}'\,\rangle$ creates the excited state $|\,\Psi_{\boldsymbol{k},\sigma}\,\rangle$ is a complicated linear combination of products of creation and annihilation operators where each term has fixed total particle number, momentum, and spin polarization. We say that $q^\dagger_{\boldsymbol{k},\sigma}$ creates a *quasiparticle* of momentum $\hbar\boldsymbol{k}$ and spin polarization σ. The factor $Z_{\boldsymbol{k},\sigma}$ is called the *quasiparticle weight* (typically independent of σ in unmagnetized systems) and tells

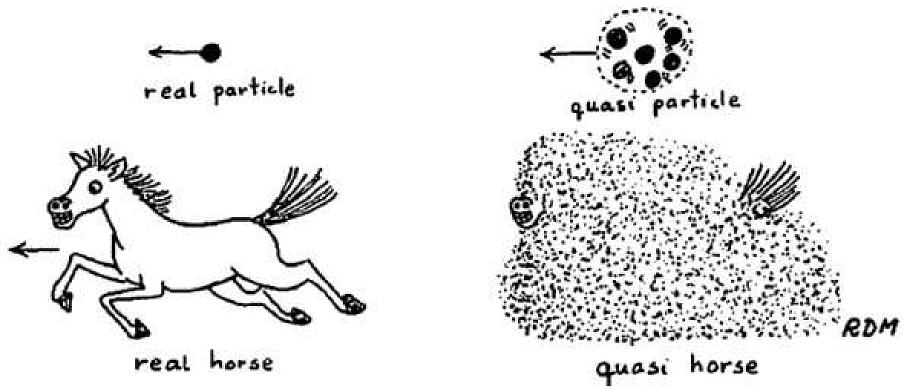

Figure 9.4: A quasi-particle is to a real particle as a quasi-horse is to a real horse. From R. D. Mattuck, *A Guide to Feynman Diagrams in the Many-Body Problem* (Dover, 1992).

us what fraction of the quasiparticle content is the single bare fermion $c^\dagger_{\boldsymbol{k},\sigma}$. The rest is what we in the many-body biz call *dressing*. The bare particle, or what's left of it, is surrounded by a cloud of particle-hole pairs in various combinations. See Fig. 9.4 for a vivid analogy.

Now imagine starting with a general Fock basis state,

$$\big|\,\Psi_0\big[\{N_{\boldsymbol{k},\sigma}\}\big]\,\big\rangle = \prod_{\boldsymbol{k},\sigma}\big(c^\dagger_{\boldsymbol{k},\sigma}\big)^{N_{\boldsymbol{k},\sigma}}\,|\,0\,\rangle \quad , \tag{9.14}$$

which is an eigenstate of $\hat{H}_0$ with eigenvalue $E^0\big[\{N_{\boldsymbol{k},\sigma}\}\big] = \sum_{\boldsymbol{k},\sigma} N_{\boldsymbol{k},\sigma}\,\varepsilon^0_{\boldsymbol{k},\sigma}$. We then perform our adiabatic evolution, which generates the interacting eigenstate $\big|\Psi\big[\{N_{\boldsymbol{k},\sigma}\}\big]\big\rangle$, which must be an eigenstate of $\hat{H} = \hat{H}_0 + \hat{H}_1$. Its associated eigenvalue E must then be a function, however complicated, of the set $\{N_{\boldsymbol{k},\sigma}\}$, *i.e.* $E = E\big[\{N_{\boldsymbol{k},\sigma}\}\big]$. Since we can adiabatically evolve any many-body eigenstate of $\hat{H}_0$, we can also evolve a *density matrix* of the form

$$\varrho_0\big[\{n_{\boldsymbol{k},\sigma}\}\big] = \bigotimes_{\boldsymbol{k},\sigma}\Big[\big(1-n_{\boldsymbol{k},\sigma}\big)\,|\,0\,\rangle\langle\,0\,| + n_{\boldsymbol{k},\sigma}\,c^\dagger_{\boldsymbol{k},\sigma}\,|\,0\,\rangle\langle\,0\,|\,c_{\boldsymbol{k},\sigma}\Big] \tag{9.15}$$

Here we may take the distribution $\{n_{\boldsymbol{k},\sigma}\}$ to be smooth as a function of $\boldsymbol{k}$ for each σ, and regard the energy to be a function (or functional[4]) of the distributions $\{n_{\boldsymbol{k},\sigma}\}$.

It is important to note that the principle of adiabatic continuity can easily fail, for example when a phase boundary is crossed as λ evolves over the interval $\lambda \in [0,1]$. This is indeed the case for phases of matter such as charge and spin density waves, exciton condensates, superconductors, *etc.*

[4]If we regard $\boldsymbol{k}$ as a continuous variable, then $E[\{n_{\boldsymbol{k},\sigma}\}]$ is a functional of the functions $n_{\boldsymbol{k},\uparrow}$ and $n_{\boldsymbol{k},\downarrow}$.

9.2.2 First law of thermodynamics for Fermi liquids

We begin with the formula for the entropy of a distribution of fermions,

$$\begin{aligned} S[\{n_{\boldsymbol{k},\sigma}\}] &= -k_{\rm B}\, \mathsf{Tr}\,(\varrho_0 \ln \varrho_0) \\ &= -k_{\rm B} \sum_{\boldsymbol{k},\sigma} \Big\{ n_{\boldsymbol{k},\sigma} \ln n_{\boldsymbol{k},\sigma} + (1 - n_{\boldsymbol{k},\sigma}) \ln(1 - n_{\boldsymbol{k},\sigma}) \Big\} \quad . \end{aligned} \tag{9.16}$$

Note that the entropy does not change under adiabatic evolution of the density matrix. The first variation of the entropy is then

$$\delta S = -k_{\rm B} \sum_{\boldsymbol{k},\sigma} \ln\!\left(\frac{n_{\boldsymbol{k},\sigma}}{1 - n_{\boldsymbol{k},\sigma}} \right) \delta n_{\boldsymbol{k},\sigma} \quad . \tag{9.17}$$

The total particle number operator is $\hat{N} = \sum_{\boldsymbol{k},\sigma} \hat{n}_{\boldsymbol{k},\sigma}$, hence

$$N = \mathsf{Tr}\,\big(\varrho_0\, \hat{N}\big) = \sum_{\boldsymbol{k},\sigma} n_{\boldsymbol{k},\sigma} \qquad , \qquad \delta N = \sum_{\boldsymbol{k},\sigma} \delta n_{\boldsymbol{k},\sigma} \quad . \tag{9.18}$$

Note that the particle number, like the entropy, is preserved by adiabatic evolution.

Finally, the energy E, as discussed in the previous section, is a functional of the distribution, which means that we may write

$$\delta E = \sum_{\boldsymbol{k},\sigma} \widetilde{\varepsilon}_{\boldsymbol{k},\sigma}\, \delta n_{\boldsymbol{k},\sigma} \qquad , \qquad \widetilde{\varepsilon}_{\boldsymbol{k},\sigma} = \frac{\delta E}{\delta n_{\boldsymbol{k},\sigma}} \tag{9.19}$$

is the first functional variation of E. The energy is *not* an adiabatic invariant. It is crucial to note that $\widetilde{\varepsilon}_{\boldsymbol{k},\sigma}$ is simultaneously a function of $\boldsymbol{k}$ and σ *and* a functional of the distribution. Indeed, we shall write

$$\frac{\delta^2 E}{\delta n_{\boldsymbol{k},\sigma}\, \delta n_{\boldsymbol{k}',\sigma'}} = \frac{\delta \widetilde{\varepsilon}_{\boldsymbol{k},\sigma}}{\delta n_{\boldsymbol{k}',\sigma'}} \equiv \frac{1}{V}\, \widetilde{f}_{\boldsymbol{k}\sigma,\boldsymbol{k}'\sigma'} \quad , \tag{9.20}$$

where $\widetilde{f}_{\boldsymbol{k}\sigma,\boldsymbol{k}'\sigma'}$ has dimensions of energy $\times$ volume and is itself, in principle, a functional of the distribution.

Writing the First Law as

$$T\, \delta S = \delta E - \mu\, \delta N \quad , \tag{9.21}$$

and using the fact that the $\delta n_{\boldsymbol{k},\sigma}$ are all independent variations, we have

$$-\, k_{\rm B} T\, \ln\!\left(\frac{n_{\boldsymbol{k},\sigma}}{1 - n_{\boldsymbol{k},\sigma}} \right) = \widetilde{\varepsilon}_{\boldsymbol{k},\sigma} - \mu \quad , \tag{9.22}$$

for each $(\boldsymbol{k},\sigma)$, which is equivalent to

$$n_{\boldsymbol{k},\sigma} = \frac{1}{\exp\!\Big(\frac{\tilde{\varepsilon}_{\boldsymbol{k},\sigma}-\mu}{k_{\rm B}T}\Big)+1} \quad . \tag{9.23}$$

This has the innocent appearance of the Fermi distribution familiar from elementary quantum statistical physics, but it must be emphasized again that $\tilde{\varepsilon}_{\boldsymbol{k},\sigma}$ is a functional of the distribution, hence Eqn. 9.23 is in fact a complicated implicit, nonlinear equation for the individual occupations $n_{\boldsymbol{k},\sigma}$.

At $T=0$, however, we have

$$n_{\boldsymbol{k},\sigma}(T=0) = \Theta(\mu - \tilde{\varepsilon}_{\boldsymbol{k},\sigma}) \equiv n^0_{\boldsymbol{k},\sigma} \quad . \tag{9.24}$$

It is now convenient to define the deviation

$$\delta n_{\boldsymbol{k},\sigma} \equiv n_{\boldsymbol{k},\sigma} - n^0_{\boldsymbol{k},\sigma} \quad , \tag{9.25}$$

where $n^0_{\boldsymbol{k},\sigma}$ is the ground state distribution at $T=0$. In an isotropic system with no external magnetic field, we have $n^0_{\boldsymbol{k},\sigma} = \Theta(k_{\rm F}-k)$. We may now write the energy E as a functional of the $\delta n_{\boldsymbol{k},\sigma}$, *viz.*

$$E = E_0 + \sum_{\boldsymbol{k},\sigma} \varepsilon_{\boldsymbol{k},\sigma}\, \delta n_{\boldsymbol{k},\sigma} + \frac{1}{2V} \sum_{\boldsymbol{k},\sigma} \sum_{\boldsymbol{k}',\sigma'} f_{\boldsymbol{k}\sigma,\boldsymbol{k}'\sigma'}\, \delta n_{\boldsymbol{k},\sigma}\, \delta n_{\boldsymbol{k}',\sigma'} + \ldots \quad . \tag{9.26}$$

Though it may not be obvious at this stage, it turns out that this is as far as we need to go in the expansion of the energy as a functional Taylor series in the $\delta n_{\boldsymbol{k},\sigma}$. Note that

$$\tilde{\varepsilon}_{\boldsymbol{k},\sigma} = \frac{\delta E}{\delta n_{\boldsymbol{k},\sigma}} = \varepsilon_{\boldsymbol{k},\sigma} + \frac{1}{V} \sum_{\boldsymbol{k}',\sigma'} f_{\boldsymbol{k}\sigma,\boldsymbol{k}'\sigma'}\, \delta n_{\boldsymbol{k}',\sigma'} + \ldots \tag{9.27}$$

and thus

$$\varepsilon_{\boldsymbol{k},\sigma} = \frac{\delta E}{\delta n_{\boldsymbol{k},\sigma}}\bigg|_{\delta n=0} \quad . \tag{9.28}$$

Similarly,

$$\frac{\delta^2 E}{\delta n_{\boldsymbol{k},\sigma}\, \delta n_{\boldsymbol{k}',\sigma'}}\bigg|_{\delta n=0} = \frac{\delta \tilde{\varepsilon}_{\boldsymbol{k},\sigma}}{\delta n_{\boldsymbol{k}',\sigma'}}\bigg|_{\delta n=0} \equiv \frac{1}{V}\, f_{\boldsymbol{k}\sigma,\boldsymbol{k}'\sigma'} \quad . \tag{9.29}$$

Compare with Eqn. 9.20. In isotropic systems, the Fermi velocity is given by

$$\frac{1}{\hbar}\, \frac{\partial \varepsilon_{\boldsymbol{k},\sigma}}{\partial \boldsymbol{k}}\bigg|_{k=k_{\rm F}} = v_{\rm F}\, \hat{\boldsymbol{k}} \quad , \tag{9.30}$$

and we define the *effective mass* m^* by the relation $v_{\rm F} = \hbar k_{\rm F}/m^*$. The Fermi energy is then given by $\varepsilon_{\rm F} = \varepsilon_{\boldsymbol{k},\sigma}\big|_{k=k_{\rm F}}$, and the density of states at the Fermi energy is

$$g(\varepsilon_{\rm F}) = \sum_\sigma \int \frac{d^3k}{(2\pi)^3}\,\delta(\varepsilon_{\rm F} - \varepsilon_{\boldsymbol{k},\sigma}) = \frac{m^* k_{\rm F}}{\pi^2\hbar^2} \quad , \tag{9.31}$$

where, recall, $k_{\rm F} = (3\pi^2 n)^{1/3}$.

In systems with spin isotropy, we may define the functions $f^{\rm s}_{\boldsymbol{k},\boldsymbol{k}'}$ and $f^{\rm a}_{\boldsymbol{k},\boldsymbol{k}'}$ as follows:

$$\begin{aligned} f_{\boldsymbol{k}\uparrow,\boldsymbol{k}'\uparrow} &= f_{\boldsymbol{k}\downarrow,\boldsymbol{k}'\downarrow} \equiv f^{\rm s}_{\boldsymbol{k},\boldsymbol{k}'} + f^{\rm a}_{\boldsymbol{k},\boldsymbol{k}'} \\ f_{\boldsymbol{k}\uparrow,\boldsymbol{k}'\downarrow} &= f_{\boldsymbol{k}\downarrow,\boldsymbol{k}'\uparrow} \equiv f^{\rm s}_{\boldsymbol{k},\boldsymbol{k}'} - f^{\rm a}_{\boldsymbol{k},\boldsymbol{k}'} \quad . \end{aligned} \tag{9.32}$$

Equivalently,

$$f_{\boldsymbol{k}\sigma,\boldsymbol{k}'\sigma'} = f^{\rm s}_{\boldsymbol{k},\boldsymbol{k}'} + \sigma\sigma' f^{\rm a}_{\boldsymbol{k},\boldsymbol{k}'} \quad . \tag{9.33}$$

Recall that $f_{\boldsymbol{k}\sigma,\boldsymbol{k}'\sigma'}$ has dimensions of energy $\times$ volume. Thus we may define the dimensionless function $F_{\boldsymbol{k}\sigma,\boldsymbol{k}'\sigma'}$ by multiplying $f_{\boldsymbol{k}\sigma,\boldsymbol{k}'\sigma'}$ by the density of states $g(\varepsilon_{\rm F})$:

$$F_{\boldsymbol{k}\sigma,\boldsymbol{k}'\sigma'} \equiv g(\varepsilon_{\rm F})\, f_{\boldsymbol{k}\sigma,\boldsymbol{k}'\sigma'} \qquad , \qquad F^{\rm s,a}_{\boldsymbol{k},\boldsymbol{k}'} \equiv g(\varepsilon_{\rm F})\, f^{\rm s,a}_{\boldsymbol{k},\boldsymbol{k}'} \quad , \tag{9.34}$$

with $F_{\boldsymbol{k}\sigma,\boldsymbol{k}'\sigma'} = F^{\rm s}_{\boldsymbol{k},\boldsymbol{k}'} + \sigma\sigma' F^{\rm a}_{\boldsymbol{k},\boldsymbol{k}'}$. When $\boldsymbol{k}$ and $\boldsymbol{k}'$ both lie on the Fermi surface, we may write

$$F^{\rm s,a}_{k_{\rm F}\hat{\boldsymbol{k}},k_{\rm F}\hat{\boldsymbol{k}}'} \equiv F^{\rm s,a}(\vartheta_{\hat{\boldsymbol{k}},\hat{\boldsymbol{k}}'}) \quad , \tag{9.35}$$

where $\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{k}}' = \cos\vartheta_{\hat{\boldsymbol{k}},\hat{\boldsymbol{k}}'}$. Furthermore, we may expand $F^{\rm s,a}(\vartheta)$ in terms of the Legendre polynomials, *viz.*

$$F^{\rm s,a}(\vartheta) = \sum_{n=0}^\infty F^{\rm s,a}_n\, P_n(\cos\vartheta) \quad . \tag{9.36}$$

Recall the generating function for the Legendre polynomials,

$$(1 - 2xt + t^2)^{-1/2} = \sum_{n=0}^\infty t^n\, P_n(x) \quad , \tag{9.37}$$

as well as the recurrence relation

$$P_{n+1}(x) = \frac{2n+1}{n+1}\, x\, P_n(x) - \frac{n}{n+1}\, P_{n-1}(x) \quad , \tag{9.38}$$

and the orthogonality relation

$$\int\limits_{-1}^{1} dx\, P_m(x)\, P_n(x) = \frac{2}{2n+1}\,\delta_{mn} \quad . \tag{9.39}$$

Therefore if $F(\vartheta) = \sum_\ell F_\ell\, P_\ell(\vartheta)$ then

$$\int \frac{d\Omega}{4\pi}\, F(\vartheta)\, P_n(\cos\vartheta) = \frac{F_n}{2n+1} \quad , \tag{9.40}$$

where $d\Omega$ is the differential solid angle.

parameter	$p = 0$ bar	$p = 27$ bar
m^*/m	2.80	5.17
F_0^{s}	9.28	68.17
F_1^{s}	5.39	12.79
F_0^{a}	-0.696	-0.760
$(F_1^{\mathrm{a}})^*$	-0.54	-1.00
$(F_1^{\mathrm{a}})^*$	-0.46	-0.27
v_{F} (cm/sec)	5.90×10^3	3.57×10^3
c_1 (cm/sec)	1.829×10^4	3.893×10^4

Table 9.1: Fermi liquid parameters for $^3\mathrm{He}\,\mathsf{N}$ (from Baym and Pethick, p. 117). Two estimates for the parameter F_1^{a} are given, based on two different methods.

9.2.3 Low temperature equilibrium properties

Entropy and specific heat

From the first law, we have

$$\begin{aligned} T\,\delta S &= \sum_{\boldsymbol{k},\sigma} (\widetilde{\varepsilon}_{\boldsymbol{k},\sigma} - \mu)\,\delta n_{\boldsymbol{k},\sigma} \\ &= \sum_{\boldsymbol{k},\sigma} (\widetilde{\varepsilon}_{\boldsymbol{k},\sigma} - \mu) \left\{ \frac{\partial n_{\boldsymbol{k},\sigma}}{\partial \widetilde{\varepsilon}_{\boldsymbol{k},\sigma}}\,\delta\widetilde{\varepsilon}_{\boldsymbol{k},\sigma} + \frac{\partial n_{\boldsymbol{k},\sigma}}{\partial\mu}\,\delta\mu + \frac{\partial n_{\boldsymbol{k},\sigma}}{\partial T}\,\delta T \right\} \\ &= \sum_{\boldsymbol{k},\sigma} (\widetilde{\varepsilon}_{\boldsymbol{k},\sigma} - \mu) \left(\frac{\partial n_{\boldsymbol{k},\sigma}}{\partial \widetilde{\varepsilon}_{\boldsymbol{k},\sigma}} \right) \left\{ (\delta\widetilde{\varepsilon}_{\boldsymbol{k},\sigma} - \delta\mu) - \left(\frac{\widetilde{\varepsilon}_{\boldsymbol{k},\sigma} - \mu}{T} \right) \delta T \right\} \quad . \end{aligned} \tag{9.41}$$

It turns out that the contribution of the $(\delta\widetilde{\varepsilon}_{\boldsymbol{k},\sigma} - \delta\mu)$ term inside the curly brackets results in a contribution of order $T^3 \ln T$, which we shall accept on faith for the time being[5]. Thus, we are left with

$$\begin{aligned} \delta S &= -\sum_{\boldsymbol{k},\sigma} \left(\frac{\partial n_{\boldsymbol{k},\sigma}}{\partial \widetilde{\varepsilon}_{\boldsymbol{k},\sigma}} \right) (\widetilde{\varepsilon}_{\boldsymbol{k},\sigma} - \mu)^2\, \frac{\delta T}{T^2} = -V g(\varepsilon_{\mathrm{F}})\, \frac{\delta T}{T^2} \int\limits_0^\infty \! d\varepsilon\, \frac{\partial n}{\partial\varepsilon}\,(\varepsilon - \mu)^2 \\ &= -V g(\varepsilon_{\mathrm{F}})\, k_{\mathrm{B}}^2\, \delta T \int\limits_{-\infty}^{\infty} \! dx\, \frac{\partial}{\partial x} \left(\frac{1}{\exp(x) + 1} \right) x^2 = \frac{\pi^2}{3}\, V\, g(\varepsilon_{\mathrm{F}})\, k_{\mathrm{B}}^2\, \delta T \quad . \end{aligned} \tag{9.42}$$

We conclude

$$S(T, V, N) = V\, \frac{\pi^2}{3}\, g(\varepsilon_{\mathrm{F}})\, k_{\mathrm{B}}^2\, T \tag{9.43}$$

[5]For a justification, see §1.4 of Baym and Pethick.

and

$$c_V(T,n) = \frac{T}{V}\left(\frac{\partial S}{\partial T}\right)_{V,N} = \frac{\pi^2}{3}\, g(\varepsilon_{\rm F})\, k_{\rm B}^2\, T \quad . \tag{9.44}$$

The difference between this result and that of the free fermi gas is the appearance of the effective mass m^* in the density of states $g(\varepsilon_{\rm F})$. If $c_V^0(T)$ is defined to be the low-temperature specific heat in a free Fermi gas of particles of mass m at the same density n, then

$$\frac{c_V(T)}{c_V^0(T)} = \frac{m^*}{m} \quad . \tag{9.45}$$

From $\delta F\big|_{V,N} = -S\,\delta T$, we integrate and obtain the temperature dependence of the ltz free energy,

$$F(T,V,N) = E_0(V,N) + V\,\frac{\pi^2}{6}\, g(\varepsilon_{\rm F})\,(k_{\rm B}T)^2 \quad . \tag{9.46}$$

Thus the chemical potential is

$$\begin{aligned}\mu(n,T) &= -\frac{\partial F}{\partial N}\bigg|_{T,V} = -\left(\frac{\partial(F/V)}{\partial(N/V)}\right)_T \\ &= \mu(n,T=0) + \frac{\pi^2}{6}\,(k_{\rm B}T)^2\,\frac{\partial g(\varepsilon_{\rm F})}{\partial n} \\ &= \mu(n,0) - \frac{\pi^2}{4}\,k_{\rm B}\left(\frac{1}{3} + \frac{\partial \ln m^*}{\partial \ln n}\right)\frac{T^2}{T_{\rm F}} \quad ,\end{aligned} \tag{9.47}$$

where $k_{\rm B}T_{\rm F} \equiv \hbar^2 k_{\rm F}^2/2m^*$.

Compressibility and sound velocity

Consider a swollen Fermi surface of radius $k_{\rm F} + dk_{\rm F}$, as depicted in Fig. 9.5. The change in the chemical potential is then given by

$$d\mu = \tilde{\varepsilon}_{k_{\rm F}+dk_{\rm F}} - \tilde{\varepsilon}_{k_{\rm F}} = d\tilde{\varepsilon}_{k_{\rm F}} \quad , \tag{9.48}$$

where we assume no spin dependence in the dispersion. Thus,

$$\begin{aligned}d\mu &= d\varepsilon_{k_{\rm F}} + \frac{1}{V}\sum_{\boldsymbol{k}',\sigma'} f_{\boldsymbol{k}_{\rm F}\sigma,\boldsymbol{k}'\sigma'}\,\delta n_{\boldsymbol{k}',\sigma'} = \hbar v_{\rm F}\, dk_{\rm F}\left\{1 + \int\!\frac{d^3k'}{(2\pi)^3}\sum_{\sigma'} f_{\boldsymbol{k}_{\rm F}\sigma.\boldsymbol{k}'\sigma'}\,\delta(\varepsilon_{\boldsymbol{k}'}-\mu)\right\} \\ &= \hbar v_{\rm F}\, dk_{\rm F}\left\{1 + 2\int\!\frac{d\Omega}{4\pi}\, f^{\rm s}(\vartheta)\int\!\frac{d^3k'}{(2\pi)^3}\,\delta(\varepsilon_{\boldsymbol{k}'}-\mu)\right\} = \hbar v_{\rm F}\, dk_{\rm F}\,\big\{1 + F_0^{\rm s}\big\} \quad .\end{aligned} \tag{9.49}$$

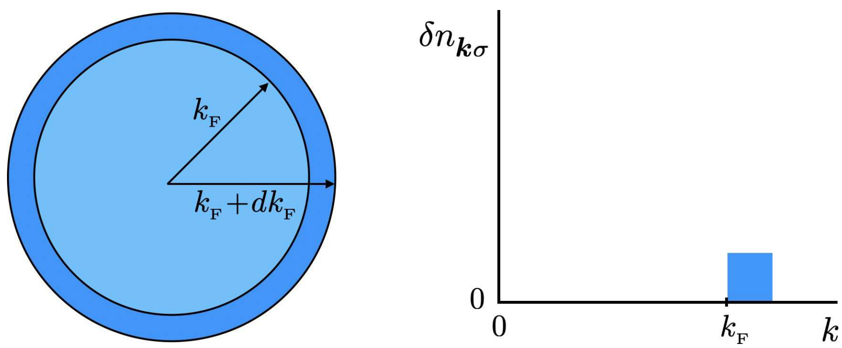

Figure 9.5: $\delta n_{\boldsymbol{k}\sigma}$ for a swollen Fermi surface.

We can now write

$$\begin{aligned}\kappa &= n^{-2}\,\frac{\partial n}{\partial \mu} = n^{-2}\,\frac{\partial n}{\partial k_F}\,\frac{\partial k_F}{\partial \mu} \\ &= n^{-2}\,\frac{k_F^2}{\pi^2}\,\frac{1}{\hbar v_F\,(1+F_0^s)} = \frac{n^{-2}\,g(\varepsilon_F)}{1+F_0^s} = \frac{9\pi^2 m^*}{\hbar^2 k_F^5\,(1+F_0^s)} \quad .\end{aligned} \tag{9.50}$$

Thus, if $\kappa^0 = n^{-2}\,g_0(\varepsilon_F)$ is the compressibility of the free Fermi gas with mass m at the same density n, we have

$$\frac{\kappa}{\kappa^0} = \frac{m^*/m}{1+F_0^s} \quad . \tag{9.51}$$

To derive the connection with sound propagation, we examine the inviscid, weak flow limit of the Navier-Stokes equations, yielding $\partial_t(\varrho \boldsymbol{u}) = -\boldsymbol{\nabla} p$, where $\varrho = mn$ is the density, with m the *bare* mass and n the number density, and p the pressure. Local thermodynamics then gives $\boldsymbol{\nabla} p = (\partial p/\partial \varrho)\,\boldsymbol{\nabla}\varrho = (1/\varrho\kappa)\boldsymbol{\nabla}\varrho$. Taking the divergence,

$$-\frac{1}{\kappa}\boldsymbol{\nabla}\cdot\bigg(\frac{1}{\varrho}\boldsymbol{\nabla}\varrho\bigg) = \frac{\partial}{\partial t}\,\boldsymbol{\nabla}\cdot(\varrho\boldsymbol{u}) = -\frac{\partial^2\varrho}{\partial t^2} \quad , \tag{9.52}$$

where in the last equality we have invoked the continuity equation $\partial_t\varrho + \boldsymbol{\nabla}\cdot(\varrho\boldsymbol{u}) = 0$. Since $\boldsymbol{\nabla}\varrho$ is presumed to be small, we arrive at the Helmholtz equation,

$$\frac{1}{\bar{\varrho}\kappa}\,\nabla^2\varrho = \frac{\partial^2\varrho}{\partial t^2} \quad , \tag{9.53}$$

with wave propagation speed $s = 1/\sqrt{\bar{\varrho}\kappa}$, where $\bar{\varrho}$ is the average density.

Uniform magnetic susceptibility

In the presence of an external magnetic field B, there is an additional Zeeman contribution to the Hamiltonian, $\hat{H}_Z = -\mu_0 B \sum_{\boldsymbol{k},\sigma} \sigma n_{\boldsymbol{k},\sigma}$. This causes the $\uparrow$ Fermi surface to expand and the $\downarrow$

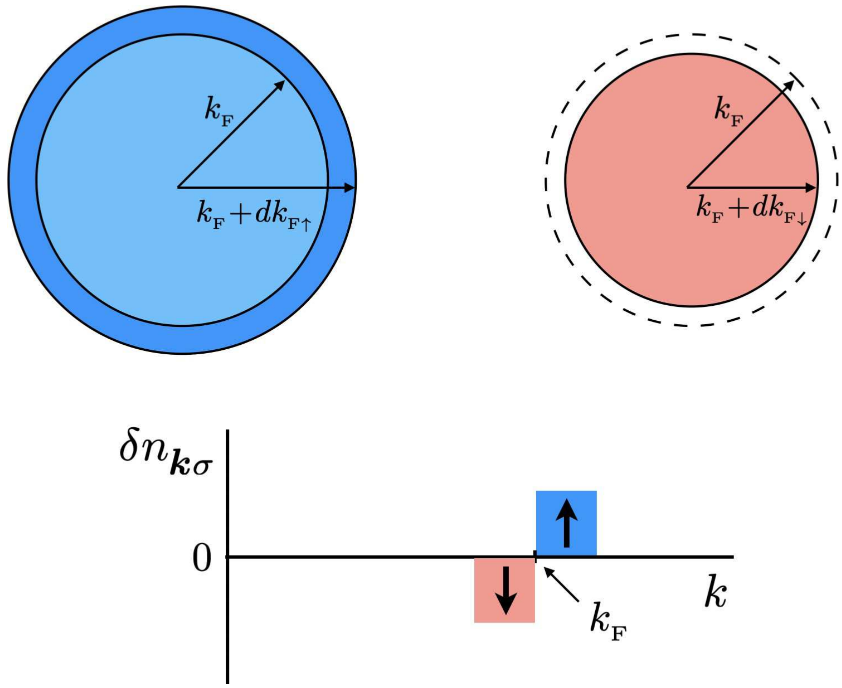

Figure 9.6: $\delta n_{\boldsymbol{k}\sigma}$ in the presence of a magnetic field.

Fermi surface to contract. Thus $dk_{F\uparrow} = -dk_{F\downarrow} \equiv dk_F$ and $\delta n_{\boldsymbol{k},\sigma} = \sigma\,\delta(k_F - k)\,dk_F$. The situation is depicted in Fig. 9.6. If particle number is conserved, then the chemical potential, which is the same for each spin species, is unchanged to lowest order in B. Thus,

$$\begin{aligned} 0 = d\tilde{\varepsilon}_{k_F,\sigma} &= -\sigma\mu_0\,dB + d\varepsilon_{k_F,\sigma} + \frac{1}{V}\sum_{\boldsymbol{k}',\sigma'} f_{\boldsymbol{k}_F\sigma,\boldsymbol{k}'\sigma'}\,\delta n_{\boldsymbol{k}',\sigma'} \\ &= -\sigma\mu_0\,dB + \hbar v_F\,dk_F\left\{\sigma + \int\frac{d^3k'}{(2\pi)^3}\,f_{\boldsymbol{k}_F\sigma,\boldsymbol{k}'\sigma'}\,\sigma'\,\delta(\varepsilon_{\boldsymbol{k}'}-\mu)\right\} \\ &= -\sigma\mu_0\,dB + \sigma\hbar v_F\,dk_F\left\{1 + g(\varepsilon_F)\int\frac{d\Omega}{4\pi}\,f^{a}(\vartheta)\right\} \\ &= -\sigma\mu_0\,dB + \sigma\hbar v_F\,(1+F_0^{a})\,dk_F \quad . \end{aligned} \tag{9.54}$$

Note that we have invoked the fact that $\sum_{\sigma'}\sigma' f_{\boldsymbol{k}\sigma,\boldsymbol{k}'\sigma'} = 2\sigma f^{a}_{\boldsymbol{k},\boldsymbol{k}'}$. We conclude that

$$\frac{\partial k_F}{\partial B} = \frac{\mu_0}{\hbar v_F\,(1+F_0^{a})} \quad . \tag{9.55}$$

The magnetic susceptibility is then

$$\chi = \frac{1}{V}\left(\frac{\partial M}{\partial B}\right)_{N,V,B=0} = \mu_0\left(\frac{\partial n_\uparrow}{\partial B} - \frac{\partial n_\downarrow}{\partial B}\right) = \mu_0\left(\frac{\partial n_\uparrow}{\partial k_{F\uparrow}} + \frac{\partial n_\downarrow}{\partial k_{F\downarrow}}\right)\left(\frac{\partial k_F}{\partial B}\right)_{B=0} = \frac{\mu_0^2\,g(\varepsilon_F)}{1+F_0^{a}} \quad , \tag{9.56}$$

and therefore

$$\frac{\chi}{\chi^0} = \frac{m^*/m}{1+F_0^{\mathrm{a}}} \quad , \tag{9.57}$$

where $\chi^0 = \mu_0^2\, g_0(\varepsilon_{\mathrm{F}})$

Galilean invariance

Consider now a Galilean transformation to an inertial primed frame of reference moving at constant velocity $\boldsymbol{u}$ with respect to our unprimed inertial laboratory frame. The Hamiltonian in the primed frame is

$$\begin{aligned} \hat{H}' &= \sum_{i=1}^{N} \frac{(\boldsymbol{p}_i - m\boldsymbol{u})^2}{2m} + \hat{H}_1 \\ &= \hat{H} - \boldsymbol{u}\cdot\boldsymbol{P} + \tfrac{1}{2}M\boldsymbol{u}^2 \quad , \end{aligned} \tag{9.58}$$

where $\boldsymbol{P} = \sum_i \boldsymbol{p}_i$ is the total momentum and $M = Nm$ is the total mass. Let's now add a particle of momentum $\boldsymbol{p} = \hbar\boldsymbol{k}$ and spin polarization σ in the lab frame at $T=0$, where its energy is then $\varepsilon_{\boldsymbol{k},\sigma}$. In the primed frame, however, the added particle has momentum $\hbar\boldsymbol{k} - m\boldsymbol{u}$ and energy $\widetilde{\varepsilon}_{\boldsymbol{k},\sigma} = \varepsilon_{\boldsymbol{k},\sigma} - \hbar\boldsymbol{k}\cdot\boldsymbol{u} + \frac{1}{2}m\boldsymbol{u}^2$. Thus, $\tilde{\varepsilon}'_{\boldsymbol{k}-\hbar^{-1}m\boldsymbol{u},\sigma} = \varepsilon_{\boldsymbol{k},\sigma} - \hbar\boldsymbol{k}\cdot\boldsymbol{u} + \frac{1}{2}m\boldsymbol{u}^2$, or, equivalently,

$$\tilde{\varepsilon}'_{\boldsymbol{k},\sigma} = \varepsilon_{\boldsymbol{k}+\hbar^{-1}m\boldsymbol{u},\sigma} - \hbar\boldsymbol{k}\cdot\boldsymbol{u} - \tfrac{1}{2}m\boldsymbol{u}^2 \quad . \tag{9.59}$$

Note though that $\tilde{\varepsilon}'_{\boldsymbol{k},\sigma} = \tilde{\varepsilon}'_{\boldsymbol{k},\sigma}[\{n'_{\boldsymbol{k},\sigma}\}]$, with

$$\begin{aligned} n'_{\boldsymbol{k},\sigma} = n^0_{\boldsymbol{k}+\hbar^{-1}m\boldsymbol{u},\sigma} &= n^0_{\boldsymbol{k},\sigma} + \frac{m\boldsymbol{u}}{\hbar}\cdot\boldsymbol{\nabla}_{\boldsymbol{k}}\, n^0_{\boldsymbol{k},\sigma} \\ &= n^0_{\boldsymbol{k}} - mv_{\mathrm{F}}\,\boldsymbol{u}\cdot\hat{\boldsymbol{k}}\,\delta(\varepsilon_{\boldsymbol{k},\sigma}-\mu) \quad . \end{aligned} \tag{9.60}$$

This relation is illustrated in Fig. 9.7. Thus, we have

$$\begin{aligned} \tilde{\varepsilon}'_{\boldsymbol{k},\sigma} &= \varepsilon_{\boldsymbol{k},\sigma} + \frac{1}{V}\sum_{\boldsymbol{k}',\sigma'} f_{\boldsymbol{k}\sigma,\boldsymbol{k}'\sigma'}\,\delta n'_{\boldsymbol{k}'\sigma'} \\ &= \varepsilon_{\boldsymbol{k},\sigma} - mv_{\mathrm{F}}\sum_{\sigma'}\int\!\frac{d^3k'}{(2\pi)^3}\, f_{\boldsymbol{k}\sigma,\boldsymbol{k}'\sigma'}\,\boldsymbol{u}\cdot\hat{\boldsymbol{k}}'\,\delta(\varepsilon_{\boldsymbol{k}',\sigma'}-\mu) \\ &= \varepsilon_{\boldsymbol{k},\sigma} - mv_{\mathrm{F}}\, g(\varepsilon_{\mathrm{F}})\,\boldsymbol{u}\cdot\!\int\!\frac{d\hat{\boldsymbol{k}}'}{4\pi}\,\hat{\boldsymbol{k}}'\, f^{\mathrm{s}}_{\boldsymbol{k},\boldsymbol{k}'_{\mathrm{F}}} \end{aligned} \tag{9.61}$$

We are only interested in the case where $|\boldsymbol{k}| \approx k_{\mathrm{F}}$, and thus we may write

$$\begin{aligned} \tilde{\varepsilon}'_{\boldsymbol{k}_{\mathrm{F}},\sigma} &= \varepsilon_{\boldsymbol{k}_{\mathrm{F}},\sigma} - mv_{\mathrm{F}}\,\boldsymbol{u}\cdot\!\int\!\frac{d\hat{\boldsymbol{k}}'}{4\pi}\,\hat{\boldsymbol{k}}'\,F^{\mathrm{s}}_{\boldsymbol{k}_{\mathrm{F}},\boldsymbol{k}'_{\mathrm{F}}} \\ &= \varepsilon_{\boldsymbol{k}_{\mathrm{F}},\sigma} - mv_{\mathrm{F}}\,\boldsymbol{u}\cdot\hat{\boldsymbol{k}}\!\int\!\frac{d\hat{\boldsymbol{k}}'}{4\pi}\,\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{k}}'\,F^{\mathrm{s}}_{\boldsymbol{k}_{\mathrm{F}},\boldsymbol{k}'_{\mathrm{F}}} \\ &= \varepsilon_{\boldsymbol{k}_{\mathrm{F}},\sigma} - \tfrac{1}{3}\,F^{\mathrm{s}}_1\,mv_{\mathrm{F}}\,\boldsymbol{u}\cdot\hat{\boldsymbol{k}} \quad . \end{aligned} \tag{9.62}$$

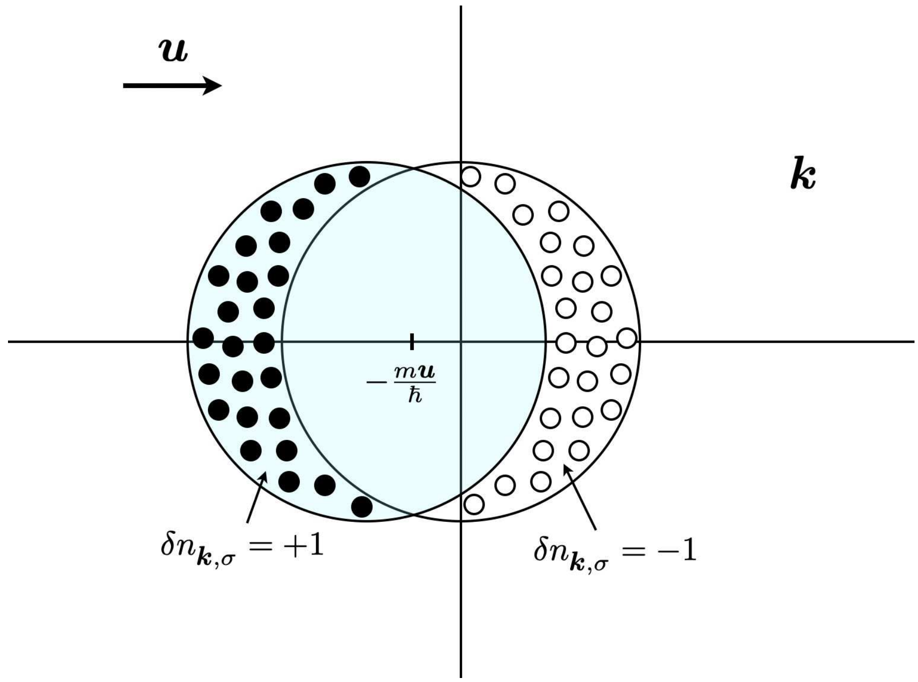

Figure 9.7: Distribution of quasiparticle occupancies in a frame moving with velocity $\boldsymbol{u}$.

Note that we have used above the fact that the integral

$$\int \frac{d\hat{\boldsymbol{k}}'}{4\pi}\,\hat{\boldsymbol{k}}'\,F^{\rm s}_{\boldsymbol{k}_{\rm F},\boldsymbol{k}'_{\rm F}} = C\hat{\boldsymbol{k}} \tag{9.63}$$

must by rotational isotropy lie along $\hat{\boldsymbol{k}}$. Taking the dot product with $\hat{\boldsymbol{k}}$ then gives

$$C = \int \frac{d\hat{\boldsymbol{k}}'}{4\pi}\,\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{k}}'\,F^{\rm s}(\vartheta_{\hat{\boldsymbol{k}},\hat{\boldsymbol{k}}'}) = \tfrac{1}{3}F^{\rm s}_1 \quad . \tag{9.64}$$

Putting this all together, we have

$$\begin{aligned}
\tilde{\varepsilon}'_{\boldsymbol{k}_{\rm F},\sigma} &= \varepsilon_{\boldsymbol{k}_{\rm F},\sigma} - \tfrac{1}{3}\,F^{\rm s}_1\,mv_{\rm F}\,\boldsymbol{u}\cdot\hat{\boldsymbol{k}} \\
&= \varepsilon_{\boldsymbol{k}_{\rm F}+\hbar^{-1}m\boldsymbol{u},\sigma} - \hbar\boldsymbol{k}\cdot\boldsymbol{u} - \tfrac{1}{2}m\boldsymbol{u}^2 \\
&= \varepsilon_{\boldsymbol{k}_{\rm F},\sigma} + \frac{m\boldsymbol{u}}{\hbar}\cdot\boldsymbol{\nabla}_{\boldsymbol{k}}\,\varepsilon_{\boldsymbol{k},\sigma}\big|_{\boldsymbol{k}=\boldsymbol{k}_{\rm F}} - \hbar k_{\rm F}\,\boldsymbol{u}\cdot\hat{\boldsymbol{k}} - \tfrac{1}{2}m\boldsymbol{u}^2 \\
&= \varepsilon_{\boldsymbol{k}_{\rm F},\sigma} + (m-m^*)v_{\rm F}\,\boldsymbol{u}\cdot\hat{\boldsymbol{k}} - \tfrac{1}{2}m\boldsymbol{u}^2 \quad ,
\end{aligned} \tag{9.65}$$

Thus, to lowest order in $\boldsymbol{u}$, we have

$$(m-m^*) = -\tfrac{1}{3}F^{\rm s}_1\,m \quad \Rightarrow \quad \frac{m^*}{m} = 1 + \tfrac{1}{3}F^{\rm s}_1 \quad . \tag{9.66}$$

This result is connected with the following point. The total particle current is given by

$$\boldsymbol{J} = \sum_{\boldsymbol{k},\sigma} \frac{1}{\hbar} \frac{\partial \widetilde{\varepsilon}_{\boldsymbol{k},\sigma}}{\partial \boldsymbol{k}} \, n_{\boldsymbol{k},\sigma} \quad , \tag{9.67}$$

where it is $\widetilde{\varepsilon}_{\boldsymbol{k},\sigma}$ and not $\varepsilon_{\boldsymbol{k},\sigma}$ which appears.

We again stress that this relationship between m^*/m and $F_1^{\rm s}$ is valid only in Galilean invariant systems, such as liquid ^{3}He N. The imposition of a crystalline lattice potential breaks the Galilean symmetry and invalidates the above result.

9.2.4 Thermodynamic stability at $T = 0$

Consider a $T = 0$ distortion of the Fermi surface. The Landau free energy $\Omega = E - TS + \mu N$ must be a minimum with respect to all possible such distortions. We adopt the parameterization

$$\begin{aligned} n_{\boldsymbol{k},\sigma} &= \Theta(k_{\rm F}(\hat{\boldsymbol{k}},\sigma) - k) = \Theta(k_{\rm F} + \delta k_{\rm F}(\hat{\boldsymbol{k}},\sigma) - k) \\ &= \Theta(k_{\rm F} - k) + \delta(k_{\rm F} - k)\, \delta k_{\rm F}(\hat{\boldsymbol{k}},\sigma) + \tfrac{1}{2}\delta'(k_{\rm F} - k)\, \big[\delta k_{\rm F}(\hat{\boldsymbol{k}},\sigma)\big]^2 + \ldots \quad , \end{aligned} \tag{9.68}$$

where $\delta k_{\rm F}(\hat{\boldsymbol{k}},\sigma)$ is the local FS distortion in the direction $\hat{\boldsymbol{k}}$ for spin polarization σ. We now evaluate $\Omega(T = 0) = E - \mu N$ to second order in $\delta k_{\rm F}$:

$$\begin{aligned} \Omega &= \Omega_0 + \sum_{\boldsymbol{k},\sigma} (\varepsilon_{\boldsymbol{k},\sigma} - \mu)\, \delta n_{\boldsymbol{k},\sigma} + \frac{1}{2V} \sum_{\boldsymbol{k},\sigma} \sum_{\boldsymbol{k}',\sigma'} f_{\boldsymbol{k}\sigma,\boldsymbol{k}'\sigma'}\, \delta n_{\boldsymbol{k},\sigma}\, \delta n_{\boldsymbol{k}',\sigma'} \\ &= \Omega_0 + \sum_{\boldsymbol{k},\sigma} (\varepsilon_{\boldsymbol{k},\sigma} - \mu) \Big\{ \delta(k_{\rm F} - k)\, \delta k_{\rm F}(\hat{\boldsymbol{k}},\sigma) + \tfrac{1}{2}\delta'(k_{\rm F} - k)\, \big[\delta k_{\rm F}(\hat{\boldsymbol{k}},\sigma)\big]^2 \Big\} \\ &\qquad + \frac{1}{2V} \sum_{\boldsymbol{k},\sigma} \sum_{\boldsymbol{k}',\sigma'} f_{\boldsymbol{k}\sigma,\boldsymbol{k}'\sigma'}\, \delta(k_{\rm F} - k)\, \delta(k_{\rm F} - k')\, \delta k_{\rm F}(\hat{\boldsymbol{k}},\sigma)\, \delta k_{\rm F}(\hat{\boldsymbol{k}}',\sigma') \quad , \end{aligned} \tag{9.69}$$

which entails

$$\begin{aligned} \frac{\Omega - \Omega_0}{V} &= \sum_\sigma \int \frac{d^3k}{(2\pi)^3} \left\{ -\frac{\partial}{\partial k}\, \delta(k_{\rm F} - k) \right\} \big[\delta k_{\rm F}(\hat{\boldsymbol{k}},\sigma)\big]^2 \\ &\qquad + \frac{k_{\rm F}^4}{8\pi^4} \sum_{\sigma,\sigma'} \int \frac{d\hat{\boldsymbol{k}}}{4\pi} \int \frac{d\hat{\boldsymbol{k}}'}{4\pi}\, f_{\sigma,\sigma'}(\vartheta_{\hat{\boldsymbol{k}},\hat{\boldsymbol{k}}'})\, \delta k_{\rm F}(\hat{\boldsymbol{k}},\sigma)\, \delta k_{\rm F}(\hat{\boldsymbol{k}}',\sigma') \\ &= \frac{\hbar^2 k_{\rm F}^3}{4\pi^2 m^*} \Bigg\{ \sum_\sigma \int \frac{d\hat{\boldsymbol{k}}}{4\pi} \big[\delta k_{\rm F}(\hat{\boldsymbol{k}},\sigma)\big]^2 \\ &\qquad + \frac{1}{2} \sum_{\sigma,\sigma'} \int \frac{d\hat{\boldsymbol{k}}}{4\pi} \int \frac{d\hat{\boldsymbol{k}}'}{4\pi}\, F_{\sigma,\sigma'}(\vartheta_{\hat{\boldsymbol{k}},\hat{\boldsymbol{k}}'})\, \delta k_{\rm F}(\hat{\boldsymbol{k}},\sigma)\, \delta k_{\rm F}(\hat{\boldsymbol{k}}',\sigma') \Bigg\} \quad . \end{aligned} \tag{9.70}$$

Recall now that $F_{\boldsymbol{k}\sigma,\boldsymbol{k}'\sigma'} = F^{\rm s}_{\boldsymbol{k},\boldsymbol{k}'} + \sigma\sigma' F^{\rm a}_{\boldsymbol{k},\boldsymbol{k}'}$, so if we define the symmetric and antisymmetric components of the FS distortion

$$\delta k^{\rm s}_{\rm F}(\hat{\boldsymbol{k}}) \equiv \sum_\sigma \delta k_{\rm F}(\hat{\boldsymbol{k}},\sigma) \qquad , \qquad \delta k^{\rm a}_{\rm F}(\hat{\boldsymbol{k}}) \equiv \sum_\sigma \sigma\, \delta k_{\rm F}(\hat{\boldsymbol{k}},\sigma) \quad , \tag{9.71}$$

then

$$\frac{\Omega-\Omega_0}{V} = \frac{\hbar^2 k_{\rm F}^3}{8\pi^2 m^*} \sum_{\nu={\rm s,a}} \Bigg\{ \int\!\frac{d\hat{\boldsymbol{k}}}{4\pi} \big[\delta k^\nu_{\rm F}(\hat{\boldsymbol{k}})\big]^2 + \int\!\frac{d\hat{\boldsymbol{k}}}{4\pi}\int\!\frac{d\hat{\boldsymbol{k}}'}{4\pi}\, F^\nu(\vartheta_{\hat{\boldsymbol{k}},\hat{\boldsymbol{k}}'})\, \delta k^\nu_{\rm F}(\hat{\boldsymbol{k}})\, \delta k^\nu_{\rm F}(\hat{\boldsymbol{k}}')\Bigg\} \quad . \tag{9.72}$$

Having resolved the free energy into contributions from the spin symmetric and antisymmetric distortions of the FS, we now further resolve it into angular momentum channels, writing

$$\delta k^\nu_{\rm F}(\hat{\boldsymbol{k}}) = \sum_{\ell=0}^\infty \sum_{m=-\ell}^{\ell} A^\nu_{\ell,m}\, Y_{\ell,m}(\hat{\boldsymbol{k}}) \quad , \tag{9.73}$$

where $A^\nu_{\ell,-m} = A^{\nu *}_{\ell,m}$ since $\delta k^\nu_{\rm F}(\hat{\boldsymbol{k}})$ is real. We also have

$$F^\nu(\vartheta_{\hat{\boldsymbol{k}},\hat{\boldsymbol{k}}'}) = \sum_{\ell=0}^\infty F^\nu_\ell\, P_\ell(\vartheta_{\hat{\boldsymbol{k}},\hat{\boldsymbol{k}}'}) = \sum_{\ell=0}^\infty \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1}\, F^\nu_\ell\, Y^*_{\ell,m}(\hat{\boldsymbol{k}})\, Y_{\ell,m}(\hat{\boldsymbol{k}}') \quad , \tag{9.74}$$

and invoking the orthonormality of the spherical harmonics,

$$\int\! d\hat{\boldsymbol{k}}\, Y^*_{\ell,m}(\hat{\boldsymbol{k}})\, Y_{\ell' m'}(\hat{\boldsymbol{k}}) = \delta_{\ell\ell'}\,\delta_{mm'} \quad , \tag{9.75}$$

we obtain the pleasingly compact expression

$$\frac{\Omega-\Omega_0}{V} = \frac{\hbar^2 k_{\rm F}^3}{32\,\pi^3 m^*} \sum_{\nu={\rm s,a}} \bigg(1+\frac{F^\nu_\ell}{2\ell+1}\bigg)\, |A^\nu_{\ell,m}|^2 \quad . \tag{9.76}$$

The stability criterion in each angular momentum channel is then

$$F^\nu_\ell > -(2\ell+1) \quad , \tag{9.77}$$

where $\nu\in\{{\rm s,a}\}$.

What happens when these stability criteria are violated? According to Eqn. 9.76, the free energy can be made arbitrarily negative by increasing the amplitude(s) $A^\nu_{\ell,m}$ of any FS distortion for which $F^\nu_\ell < -(2\ell+1)$. This is unphysical, and an artifact of going only to order $(\delta k^\nu_{\rm F})^2$ in the expansion of the Landau free energy. Suppose though we add a fourth order correction to Ω of the form

$$\frac{\Delta\Omega}{V} = \frac{\hbar^2 k_{\rm F}^3}{4\pi m^*} \sum_{\nu={\rm s,a}} \lambda_\nu \bigg(\int\!\frac{d\hat{\boldsymbol{k}}}{4\pi}\big[\delta k^\nu_{\rm F}(\hat{\boldsymbol{k}})\big]^2\bigg)^2 = \frac{\hbar^2 k_{\rm F}^3}{64\pi^3 m^*} \sum_{\nu={\rm s,a}} \lambda_\nu \Bigg(\sum_{\ell,m} |A^\nu_{\ell,m}|^2\Bigg)^2 \tag{9.78}$$

so that

$$\frac{\Omega+\Delta\Omega-\Omega_0}{V}=\frac{\hbar^2 k_F^3}{32\,\pi^3 m^*}\sum_{\nu=\mathrm{s,a}}\Bigg\{\sum_{\ell,m}\bigg(1+\frac{F^\nu_\ell}{2\ell+1}\bigg)\,|A^\nu_{\ell,m}|^2+\tfrac{1}{2}\,\lambda_\nu\bigg(\sum_{\ell,m}|A^\nu_{\ell,m}|^2\bigg)^2\Bigg\} \quad . \tag{9.79}$$

Such a term lies beyond the expansion for the internal energy of a Fermi liquid that we have considered thus far. To minimize the free energy, we set the variation with respect to each $A^{\nu *}_{\ell,m}$ to zero. For stable channels where $F^\nu_\ell > -(2\ell+1)$, we then find $A^\nu_{\ell,m}=0$. But for unstable channels, we obtain

$$\sum_{m=-\ell}^{\ell}|A^\nu_{\ell,m}|^2=-\frac{1}{\lambda_\nu}\bigg(1+\frac{F^\nu_\ell}{2\ell+1}\bigg)>0 \quad . \tag{9.80}$$

Thus, the weight of the distortion in each unstable (ν,ℓ) sector is distributed over all $(2\ell+1)$ of the coefficients $A^\nu_{\ell,m}$ such that the sum of their squares is fixed as specified above. Thus, an $\ell=1$ instability results in a dipolar distortion of the FS, while an $\ell=2$ instability results in a quadrupolar distortion of the FS, *etc.*

9.3 Collective Dynamics of the Fermi Surface

9.3.1 Landau-Boltzmann equation

We first review some basic features of the Boltzmann equation, which was discussed earlier in §5.6. Consider the classical dynamical system governing flow on an N-dimensional phase space Γ, where $\boldsymbol{X}=(X^1,\ldots,X^N)\in\Gamma$ is a point in phase space. The dynamical system is

$$\frac{d\boldsymbol{X}}{dt}=\boldsymbol{V}(\boldsymbol{X}) \tag{9.81}$$

where each $V^\mu=V^\mu(X^1,\ldots,X^N)$[6]. Now consider a distribution function $f(\boldsymbol{X},t)$. The continuity equation says

$$\frac{\partial f}{\partial t}+\boldsymbol{\nabla}\cdot(\boldsymbol{V}f)=0 \quad , \tag{9.82}$$

where $\boldsymbol{\nabla}=\big(\frac{\partial}{\partial X^1},\ldots,\frac{\partial}{\partial X^N}\big)$. Assuming phase flow is *incompressible*, $\boldsymbol{\nabla}\cdot\boldsymbol{V}=0$ and the continuity equation takes the form

$$\frac{Df}{Dt}=\frac{\partial f}{\partial t}+\boldsymbol{V}\cdot\boldsymbol{\nabla}f=0 \quad , \tag{9.83}$$

where $\frac{Df}{Dt}=\frac{d}{dt}f(\boldsymbol{X}(t),t)$, called the *convective derivative*, is the total derivative of the distribution in the frame comoving with the flow.

[6]This autonomous system can be extended to a time-dependent one, *i.e.* $\dot{\boldsymbol{X}}=\boldsymbol{V}(\boldsymbol{X},t)$, which is a dynamical system in one higher $(N+1)$ dimensions, taking $X^{N+1}=t$ and $V^{N+1}=1$.

For our application, phase space has dimension $N = 6$, with $\boldsymbol{X} = (\boldsymbol{r}, \boldsymbol{k})$. We also add to the RHS a source/sink term corresponding to *collisions* between particles. Typically these are local in position $\boldsymbol{r}$ but nonlocal in the wavevector $\boldsymbol{k}$. An example is shown in Fig. 9.3, where a collision results in an instantaneous wavevector $\boldsymbol{q}$ transfer between two interacting particles. We also must account for spin, and the most straightforward way to do this is to specify independent distributions for each spin polarization. Writing $f(\boldsymbol{r},\boldsymbol{k},\sigma,t) = n_{\boldsymbol{k},\sigma}(\boldsymbol{r},t)$, our Boltzmann equation takes the form

$$\frac{\partial n_{\boldsymbol{k},\sigma}(\boldsymbol{r},t)}{\partial t} + \langle \dot{\boldsymbol{r}} \rangle_\sigma \cdot \frac{\partial n_{\boldsymbol{k},\sigma}(\boldsymbol{r},t)}{\partial \boldsymbol{r}} + \langle \dot{\boldsymbol{k}} \rangle_\sigma \cdot \frac{\partial n_{\boldsymbol{k},\sigma}(\boldsymbol{r},t)}{\partial \boldsymbol{k}} = I[n] \quad , \tag{9.84}$$

where $I[n]$ is the collision term. We now invoke Landau's Fermi liquid theory, but on a local scale, and write the energy density $\mathcal{E}(\boldsymbol{r},t)$ as a functional of the distribution $\delta n_{\boldsymbol{k},\sigma}(\boldsymbol{r},t)$, *viz.*

$$\mathcal{E}(\boldsymbol{r},t) = \mathcal{E}_0 + \sum_\sigma \int \frac{d^3k}{(2\pi)^3}\, \varepsilon_{\boldsymbol{k},\sigma}\, \delta n_{\boldsymbol{k},\sigma}(\boldsymbol{r},t) + \frac{1}{2} \sum_{\sigma,\sigma'} \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3}\, f_{\boldsymbol{k}\sigma,\boldsymbol{k}'\sigma'}\, \delta n_{\boldsymbol{k},\sigma}(\boldsymbol{r},t)\, \delta n_{\boldsymbol{k}',\sigma'}(\boldsymbol{r},t) \quad , \tag{9.85}$$

where $\delta n_{\boldsymbol{k},\sigma}(\boldsymbol{r},t)$ is dimensionless and indicates the local number density of fermions of wavevector $\boldsymbol{k}$ and spin polarization σ in units of the bulk number density n. Note that the above expression is local in position space. We then have the Landau-Boltzmann equation[7],

$$\frac{\partial n_{\boldsymbol{k},\sigma}(\boldsymbol{r},t)}{\partial t} + \overbrace{\frac{1}{\hbar}\frac{\partial \widetilde{\varepsilon}_{\boldsymbol{k},\sigma}(\boldsymbol{r},t)}{\partial \boldsymbol{k}}}^{\langle \dot{\boldsymbol{r}} \rangle_\sigma} \cdot \frac{\partial n_{\boldsymbol{k},\sigma}(\boldsymbol{r},t)}{\partial \boldsymbol{r}} - \overbrace{\frac{1}{\hbar}\frac{\partial \widetilde{\varepsilon}_{\boldsymbol{k},\sigma}(\boldsymbol{r},t)}{\partial \boldsymbol{r}}}^{-\langle \dot{\boldsymbol{k}} \rangle_\sigma} \cdot \frac{\partial n_{\boldsymbol{k},\sigma}(\boldsymbol{r},t)}{\partial \boldsymbol{k}} = I[n] \quad , \tag{9.86}$$

where

$$\widetilde{\varepsilon}_{\boldsymbol{k},\sigma}(\boldsymbol{r},t) = V_\sigma(\boldsymbol{r},t) + \varepsilon_{\boldsymbol{k},\sigma}(\boldsymbol{r},t) + \sum_{\sigma'} \int \frac{d^3k'}{(2\pi)^3}\, f_{\boldsymbol{k}\sigma,\boldsymbol{k}'\sigma'}\, \delta n_{\boldsymbol{k}',\sigma'}(\boldsymbol{r},t) \quad . \tag{9.87}$$

Here we have included $V_\sigma(\boldsymbol{r},t)$, the external local potential for particles at position $\boldsymbol{r}$ at time t. Note that

$$\frac{\partial}{\partial \boldsymbol{r}}\, \widetilde{\varepsilon}_{\boldsymbol{k},\sigma}(\boldsymbol{r},t) = \frac{\partial}{\partial \boldsymbol{r}}\, V_\sigma(\boldsymbol{r},t) + \sum_{\sigma'} \int \frac{d^3k'}{(2\pi)^3}\, f_{\boldsymbol{k}\sigma,\boldsymbol{k}'\sigma'}\, \frac{\partial}{\partial \boldsymbol{r}}\, \delta n_{\boldsymbol{k}',\sigma'}(\boldsymbol{r},t) \tag{9.88}$$

Now we write linearize, writing $n = n^0 + \delta n$, obtaining

$$\frac{\partial\, \delta n_{\boldsymbol{k},\sigma}}{\partial t} + \frac{1}{\hbar}\frac{\partial \varepsilon_{\boldsymbol{k},\sigma}}{\partial \boldsymbol{k}} \cdot \frac{\partial\, \delta n_{\boldsymbol{k},\sigma}}{\partial \boldsymbol{r}} - \frac{1}{\hbar}\frac{\partial n^0_{\boldsymbol{k},\sigma}}{\partial \boldsymbol{k}} \cdot \frac{\partial \widetilde{\varepsilon}_{\boldsymbol{k},\sigma}}{\partial \boldsymbol{r}} = I[n^0 + \delta n] \quad . \tag{9.89}$$

If $V_\sigma(\boldsymbol{r},t) = \delta\hat{V}_\sigma\, e^{i(\boldsymbol{q}\cdot\boldsymbol{r} - \omega t)}$, then the solution for the distribution in the linearized theory will be $\delta n_{\boldsymbol{k},\sigma}(\boldsymbol{r},t) = \delta\hat{n}_{\boldsymbol{k},\sigma}\, e^{i(\boldsymbol{q}\cdot\boldsymbol{r} - \omega t)}$, with

$$\omega\, \delta\hat{n}_{\boldsymbol{k},\sigma} - \boldsymbol{q}\cdot\boldsymbol{v}_{\boldsymbol{k},\sigma}\, \delta\hat{n}_{\boldsymbol{k},\sigma} + \left(\frac{\partial n^0_{\boldsymbol{k},\sigma}}{\partial \varepsilon_{\boldsymbol{k},\sigma}}\right) \boldsymbol{q}\cdot\boldsymbol{v}_{\boldsymbol{k},\sigma} \left[\delta\hat{V}_\sigma + \sum_{\sigma'} \int \frac{d^3k'}{(2\pi)^3}\, f_{\boldsymbol{k}\sigma,\boldsymbol{k}'\sigma'}\, \delta\hat{n}_{\boldsymbol{k}',\sigma'}\right] = -\big[\mathcal{L}\, \delta\hat{n}\big]_{\boldsymbol{k},\sigma} \quad , \tag{9.90}$$

[7]We assume no curvature $\boldsymbol{\Omega}(\boldsymbol{k})$ contributing to the velocity $\dot{\boldsymbol{r}}$.

where $\mathcal{L}$ is the *linearized collision operator*. Note that this is a linear integral (or integrodifferential, depending on the form of $\mathcal{L}$) equation for $\delta\hat{n}_{\boldsymbol{k},\sigma}$ in terms of $\delta\hat{V}_\sigma$.

9.3.2 Zero sound : free FS oscillations in the collisionless limit

We now consider the case of free oscillations of the Fermi surface, *i.e.* the case $V_\sigma(\boldsymbol{r},t)=0$, in the collisionless limit ($\mathcal{L}=0$). We are left with

$$(\omega-\boldsymbol{q}\cdot\boldsymbol{v}_{\boldsymbol{k},\sigma})\,\delta\hat{n}_{\boldsymbol{k},\sigma}+\boldsymbol{q}\cdot\boldsymbol{v}_{\boldsymbol{k},\sigma}\bigg(\frac{\partial n^0_{\boldsymbol{k},\sigma}}{\partial\varepsilon_{\boldsymbol{k},\sigma}}\bigg)\sum_{\sigma'}\int\!\frac{d^3k'}{(2\pi)^3}\,f_{\boldsymbol{k}\sigma,\boldsymbol{k}'\sigma'}\,\delta\hat{n}_{\boldsymbol{k}',\sigma'}=0 \quad . \tag{9.91}$$

This is an *eigenvalue equation* for $\omega(\boldsymbol{q})$, where the eigenvector is the distribution $\delta\hat{n}_{\boldsymbol{k},\sigma}$. If we write

$$\delta n_{\boldsymbol{k},\sigma}(\boldsymbol{r},t)=\hbar v_{\rm F}\,\delta(\varepsilon_{\rm F}-\varepsilon_{\boldsymbol{k},\sigma})\,\delta k_{\rm F}(\hat{\boldsymbol{k}},\sigma)\,e^{i(\boldsymbol{q}\cdot\boldsymbol{r}-\omega t)} \quad , \tag{9.92}$$

then we arrive at

$$(\omega-\boldsymbol{q}\cdot\boldsymbol{v}_{\boldsymbol{k}_{\rm F},\sigma})\,\delta k_{\rm F}(\hat{\boldsymbol{k}},\sigma)-\boldsymbol{q}\cdot\boldsymbol{v}_{\boldsymbol{k}_{\rm F},\sigma}\sum_{\sigma'}\int\!\frac{d^3k'}{(2\pi)^3}\,\delta(\varepsilon_{\rm F}-\varepsilon_{\boldsymbol{k}',\sigma'})\,f_{\boldsymbol{k}_{\rm F}\sigma,\boldsymbol{k}'_{\rm F}\sigma'}\,\delta k_{\rm F}(\hat{\boldsymbol{k}}',\sigma')=0 \quad . \tag{9.93}$$

We now take $\boldsymbol{v}_{\boldsymbol{k},\sigma}=v_{\rm F}\hat{\boldsymbol{k}}$, independent of σ. Thus,

$$(\lambda-\hat{\boldsymbol{q}}\cdot\hat{\boldsymbol{k}})\,\delta k_{\rm F}(\hat{\boldsymbol{k}},\sigma)-\tfrac{1}{2}\,\hat{\boldsymbol{q}}\cdot\hat{\boldsymbol{k}}\int\!\frac{d\hat{\boldsymbol{k}}'}{4\pi}\,F_{\sigma,\sigma'}(\vartheta_{\hat{\boldsymbol{k}},\hat{\boldsymbol{k}}'})\,\delta k_{\rm F}(\hat{\boldsymbol{k}}',\sigma')=0 \quad , \tag{9.94}$$

where $\lambda\equiv\omega/v_{\rm F}q$. This is immediately resolved into symmetric and antisymmetric channels $\nu\in\{{\rm s},{\rm a}\}$, *viz.*

$$(\hat{\boldsymbol{q}}\cdot\hat{\boldsymbol{k}}-\lambda)\,\delta k^\nu_{\rm F}(\hat{\boldsymbol{k}})+\hat{\boldsymbol{q}}\cdot\hat{\boldsymbol{k}}\int\!\frac{d\hat{\boldsymbol{k}}'}{4\pi}\,F^\nu(\vartheta_{\hat{\boldsymbol{k}},\hat{\boldsymbol{k}}'})\,\delta k^\nu_{\rm F}(\hat{\boldsymbol{k}}')=0 \tag{9.95}$$

Thus,

$$\delta k^\nu_{\rm F}(\hat{\boldsymbol{k}})=\frac{\hat{\boldsymbol{q}}\cdot\hat{\boldsymbol{k}}}{\lambda-\hat{\boldsymbol{q}}\cdot\hat{\boldsymbol{k}}}\int\!\frac{d\hat{\boldsymbol{k}}'}{4\pi}\,F^\nu(\vartheta_{\hat{\boldsymbol{k}},\hat{\boldsymbol{k}}'})\,\delta k^\nu_{\rm F}(\hat{\boldsymbol{k}}') \quad , \tag{9.96}$$

and resolving into angular momentum channels as before, writing

$$F^\nu(\vartheta_{\hat{\boldsymbol{k}},\hat{\boldsymbol{k}}'})=\sum_{\ell,m}\frac{4\pi\,F^\nu_\ell}{2\ell+1}\,Y_{\ell,m}(\hat{\boldsymbol{k}})\,Y^*_{\ell,m}(\hat{\boldsymbol{k}}') \qquad , \qquad \delta k^\nu_{\rm F}(\hat{\boldsymbol{k}})=\sum_{\ell=0}^\infty\sum_{m=-\ell}^{\ell}A^\nu_{\ell,m}\,Y_{\ell,m}(\hat{\boldsymbol{k}}) \quad , \tag{9.97}$$

multiplying the above equation by $Y^*_{\ell,m}(\hat{\boldsymbol{k}})$ and then integrating over the unit $\hat{\boldsymbol{k}}$ sphere, we obtain

$$A^\nu_{\ell,m}=\sum_{\ell',m'}\frac{F^\nu_{\ell'}}{2\ell'+1}\left[\int\!d\hat{\boldsymbol{k}}\,\frac{\hat{\boldsymbol{q}}\cdot\hat{\boldsymbol{k}}}{\lambda-\hat{\boldsymbol{q}}\cdot\hat{\boldsymbol{k}}}\,Y^*_{\ell,m}(\hat{\boldsymbol{k}})\,Y_{\ell',m'}(\hat{\boldsymbol{k}})\right]A^\nu_{\ell',m'} \tag{9.98}$$

The oscillations of the FS are called *zero sound*.

Simple model for zero sound

Eqn. 9.98 defines an eigenvalue equation for the infinite length vector $\boldsymbol{A} = \{A_{0,0}, A_{1,-1}, A_{1,0}, \ldots\}$. So simplify matters, consider the case where $F^\nu_\ell = F^\nu_0\,\delta_{\ell,0}$. We drop the ν superscript for clarity. Eqn. 9.98 then reduces to

$$1 = F_0 \int \frac{d\hat{\boldsymbol{k}}}{4\pi}\, \frac{\hat{\boldsymbol{q}}\cdot\hat{\boldsymbol{k}}}{\lambda - \hat{\boldsymbol{q}}\cdot\hat{\boldsymbol{k}}} = F_0 \left[\frac{\lambda}{2}\ln\!\left(\frac{\lambda+1}{\lambda-1}\right) - 1\right] \quad , \tag{9.99}$$

which is equivalent to

$$\left(1 + \frac{1}{F_0}\right)\lambda^{-1} = \tanh^{-1}(\lambda^{-1}) \quad . \tag{9.100}$$

This is a transcendental equation for $\lambda(F_0)$. It may be solved graphically by plotting the LHS and RHS *versus* the quantity $u \equiv \lambda^{-1}$. One finds that a nontrivial solution with real λ exists provided $F_0 > 0$. For $F_0 \in [-1, 0]$, a complex solution exists, corresponding to a damped oscillation. We may also solve explicitly in two limits:

$$\begin{aligned}
&F_0 \to 0 \quad \Rightarrow \quad \lambda \to 1 \quad \Rightarrow \quad \frac{\lambda}{2}\ln\left(\frac{\lambda+1}{\lambda-1}\right) = \frac{1}{2}\ln\left(\frac{2}{\lambda-1}\right) + \ldots \quad \Rightarrow \quad \lambda \simeq 1 + 2\,e^{-2/F_0}\\
&F_0 \to \infty \quad \Rightarrow \quad \lambda \to \infty \quad \Rightarrow \quad \frac{\lambda}{2}\ln\left(\frac{\lambda+1}{\lambda-1}\right) = 1 + \frac{1}{3\lambda^2} + \ldots \quad \Rightarrow \quad \lambda \simeq \sqrt{\frac{F_0}{3}}
\end{aligned} \tag{9.101}$$

The ratio of zero sound to first sound velocities is thus

$$\frac{c_0}{c_1} = \frac{\sqrt{3}\,\lambda(F^{\rm s}_0)}{\sqrt{(1+F^{\rm s}_0)(1+\frac{1}{3}F^{\rm s}_1)}} \quad . \tag{9.102}$$

Another zero sound mode

Consider next the truncated Landau interaction function

$$\begin{aligned}
F(\vartheta_{\hat{\boldsymbol{k}},\hat{\boldsymbol{k}}'}) &= F_0 + F_1\,\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{k}}' \\
&= F_0 + F_1\,\cos\theta\cos\theta' + \tfrac{1}{2}F_1\,\sin\theta\sin\theta'\left(e^{i\phi}\,e^{-i\phi'} + e^{-i\phi}e^{i\phi'}\right) \quad .
\end{aligned} \tag{9.103}$$

We posit a Fermi surface distortion of the form $\delta k_{\rm F}(\hat{\boldsymbol{k}}) = u(\theta)\,e^{i\phi}$, resulting in the eigenvalue equation

$$u(\theta) = \frac{F_1}{4}\,\frac{\sin\theta\cos\theta}{\lambda - \cos\theta}\int\limits_0^\pi d\theta'\,\sin^2\!\theta'\,u(\theta') \quad . \tag{9.104}$$

Multiply by $\sin\theta$ and integrate to obtain

$$\frac{4}{F_1} = \int\limits_{-1}^{1} dx\, \frac{x-x^3}{\lambda - x} = -\lambda(\lambda^2-1)\ln\!\left(\frac{\lambda+1}{\lambda-1}\right) + 2\lambda^2 - \tfrac{4}{3} \quad , \tag{9.105}$$

where $x = \cos\theta$. Note that at the limiting value $\lambda = 0$ the integral returns a value of $\frac{2}{3}$, corresponding to $F_1 = 6$. In the opposite limit $\lambda \to \infty$, the RHS takes the value $2/3\lambda^2$. Thus, there should be a solution for $F_1 \in [6, \infty]$. According to Tab. 9.1, in ^{3}He N at high pressure one indeed has $F_1^s > 6$, yet so far as I am aware this mode has yet to be observed.

Separable kernel

Finally, consider the case of the *separable kernel*,

$$F(\hat{\boldsymbol{k}}, \hat{\boldsymbol{k}}') = L\, w(\hat{\boldsymbol{k}})\, w(\hat{\boldsymbol{k}}') \quad , \tag{9.106}$$

resulting in the eigenvalue equation

$$\delta k_{\rm F}(\hat{\boldsymbol{k}}) = \frac{L\, \hat{\boldsymbol{q}}\cdot\hat{\boldsymbol{k}}\, w(\hat{\boldsymbol{k}})}{\lambda - \hat{\boldsymbol{q}}\cdot\hat{\boldsymbol{k}}} \int \frac{d\hat{\boldsymbol{k}}'}{4\pi}\, w(\hat{\boldsymbol{k}}')\, \delta k_{\rm F}(\hat{\boldsymbol{k}}') \quad . \tag{9.107}$$

Multiplying by $w(\hat{\boldsymbol{k}})$ and integrating, we obtain

$$\int \frac{d\hat{\boldsymbol{k}}}{4\pi} \left(\frac{\hat{\boldsymbol{q}}\cdot\hat{\boldsymbol{k}}}{\lambda - \hat{\boldsymbol{q}}\cdot\hat{\boldsymbol{k}}}\right) w^2(\hat{\boldsymbol{k}}) = L^{-1} \quad . \tag{9.108}$$

Note that $\lambda = \lambda(\hat{\boldsymbol{q}})$ will in general be a function of direction if the function $w(\hat{\boldsymbol{k}})$ is not isotropic.

9.4 Dynamic Response of the Fermi Liquid

We now restore the driving term $V(\boldsymbol{r},t) = \delta\hat{V}(\boldsymbol{q},\omega)\, e^{i(\boldsymbol{q}\cdot\boldsymbol{r}-\omega t)}$, taken to be spin-independent, and solve the inhomogeneous linear equation Eqn. 9.89 at $T = 0$ for $\delta\hat{n}_{\boldsymbol{k},\sigma}(\boldsymbol{q},\omega)$ in the collisionless limit. The Fourier components of the bulk density are given by

$$\delta\hat{n}(\boldsymbol{q},\omega) = \int\frac{d^3k}{(2\pi)^3}\, \delta\hat{n}_{\boldsymbol{k},\sigma}(\boldsymbol{q},\omega) \equiv -\chi(\boldsymbol{q},\omega)\, \delta\hat{V}(\boldsymbol{q},\omega) \quad , \tag{9.109}$$

where $\chi(\boldsymbol{q},\omega)$ is the dynamical density response function, which we first met in chapter 9. We work in the symmetric channel and suppress the symmetry index $\nu = {\rm s}$. The linearized collisionless Landau-Boltzmann equation then takes the form

$$\delta k_{\rm F}(\hat{\boldsymbol{k}}) = \frac{\hat{\boldsymbol{q}}\cdot\hat{\boldsymbol{k}}}{\lambda - \hat{\boldsymbol{q}}\cdot\hat{\boldsymbol{k}}} \left\{ \int \frac{d\hat{\boldsymbol{k}}'}{4\pi}\, F(\vartheta_{\hat{\boldsymbol{k}},\hat{\boldsymbol{k}}'})\, \delta k_{\rm F}(\hat{\boldsymbol{k}}') + \frac{\delta\hat{V}(\hat{\boldsymbol{q}},\omega)}{\hbar v_{\rm F}} \right\} \quad , \tag{9.110}$$

with $\lambda = \omega/qv_{\rm F}$ as before. The density response is related to the Fermi surface distortion according to

$$\delta\hat{n}(\boldsymbol{q},\omega) = \frac{k_{\rm F}^2}{\pi^2}\int\frac{d\hat{\boldsymbol{k}}}{4\pi}\,\delta k_{\rm F}(\hat{\boldsymbol{k}}) \quad . \tag{9.111}$$

Note that $\delta k_{\rm F}(\hat{\boldsymbol{k}})$ is implicity a function of $\boldsymbol{q}$ and ω.

The difficulty in solving the above equation is that the different angular momentum channels don't decouple. However, in the simplified model where the interaction function $F(\vartheta) = F_0$ is isotropic, we can make progress. We then have

$$\delta\hat{n}(\boldsymbol{q},\omega) = \overbrace{\int\frac{d\hat{\boldsymbol{k}}}{4\pi}\left(\frac{\hat{\boldsymbol{q}}\cdot\hat{\boldsymbol{k}}}{\lambda-\hat{\boldsymbol{q}}\cdot\hat{\boldsymbol{k}}}\right)}^{\equiv\,-G(\lambda)}\left\{F_0\,\delta\hat{n}(\hat{\boldsymbol{q}},\omega) + \frac{k_{\rm F}^2}{\pi^2}\,\frac{\delta\hat{V}(\hat{\boldsymbol{q}},\omega)}{\hbar v_{\rm F}}\right\} \tag{9.112}$$

where

$$G(\lambda) = -\int\frac{d\hat{\boldsymbol{k}}}{4\pi}\left(\frac{\hat{\boldsymbol{q}}\cdot\hat{\boldsymbol{k}}}{\lambda-\hat{\boldsymbol{q}}\cdot\hat{\boldsymbol{k}}}\right) = 1 - \frac{\lambda}{2}\,\ln\!\left(\frac{\lambda+1}{\lambda-1}\right) \quad . \tag{9.113}$$

Thus we find

$$\chi(\boldsymbol{q},\omega) = \frac{g(\varepsilon_{\rm F})\,G(\omega/v_{\rm F}|\boldsymbol{q}|)}{1+F_0\,G(\omega/v_{\rm F}|\boldsymbol{q}|)} \quad . \tag{9.114}$$

Note that the pole of the response function lies at the natural frequency of the FL oscillations, *i.e.* when $1 + F_0\,G(\omega/qv_{\rm F}) = 0$.