PHYSICS 211C : CONDENSED MATTER PHYSICS
HW ASSIGNMENT #4

(1) For two particles i and j each with spin $S=1$, find the projection operators $P^J_{S=1}(i,j)$ for $J \in \{0,1,2\}$ in terms of $\boldsymbol{S}_i \cdot \boldsymbol{S}_j$. Show that $\sum_{J=0}^{2} P^J_{S=1}(i,j) = 1$.

(2) For $S=2$ particles on the square lattice, find the nearest neighbor projection operator Hamiltonian which renders the AKLT valence bond solid state an exact zero energy ground state.

(3) Consider the two Majumdar-Ghosh states $|\,\mathsf{A}\,\rangle$ and $|\,\mathsf{B}\,\rangle$ on a ring of N sites, where $N=2K$ is even. Compute the overlap between $\langle\,\mathsf{A}\,|\,\mathsf{B}\,\rangle$. Note that you may write these states as

$$
\begin{aligned}
\big|\,\mathsf{A}\,\big\rangle &= 2^{-K/2}\,\varepsilon_{\sigma_1\sigma_2}\,\varepsilon_{\sigma_3\sigma_4}\cdots\varepsilon_{\sigma_{2K-1}\sigma_{2K}}\,\big|\,\sigma_1\,,\,\sigma_2\,,\,\ldots\,,\,\sigma_{2K}\,\big\rangle \\
\big|\,\mathsf{B}\,\big\rangle &= 2^{-K/2}\,\varepsilon_{\sigma_2\sigma_3}\,\varepsilon_{\sigma_4\sigma_5}\cdots\varepsilon_{\sigma_{2K}\sigma_1}\,\big|\,\sigma_1\,,\,\sigma_2\,,\,\ldots\,,\,\sigma_{2K}\,\big\rangle \quad ,
\end{aligned}
$$

where $\varepsilon_{+-}=+1$, $\varepsilon_{-+}=-1$, and $\varepsilon_{++}=\varepsilon_{--}=0$.

(4) Show, for two spin-$\frac{1}{2}$ particles, that the operator

$$
X(i,j) = P^{J=1}_{S=\frac{1}{2}}(i,j) - P^{J=0}_{S=\frac{1}{2}}(i,j) = \tfrac{1}{2} + 2\boldsymbol{S}_i \cdot \boldsymbol{S}_j
$$

is the exchange operator, such that

$$
X(i,j)\,|\,\sigma_i,\sigma_j\,\rangle = |\,\sigma_j,\sigma_i\,\rangle \quad .
$$

I.e. $X(i,j)$ exchanges the two spin polarizations.

(5) The unnormalized wavefunction for the $S=1$ ALKT valence bond solid chain may be written in the matrix product form

$$
\big|\,\Psi\,\big\rangle = \mathsf{Tr}\,\big[M(1)\,M(2)\cdots M(N)\big]
$$

where

$$
M(n) = \begin{pmatrix} -|\,0\,\rangle_n & \sqrt{2}\,|\,1\,\rangle_n \\ -\sqrt{2}\,|\,\bar{1}\,\rangle_n & |\,0\,\rangle_n \end{pmatrix} \quad .
$$

Here, $|\,1\,\rangle$, $|\,0\,\rangle$, and $|\,\bar{1}\,\rangle$ are the three polarization states for $S=1$, corresponding to $S^z=+1$, 0, and -1, respectively. Compute the normalization $\langle\,\Psi\,|\,\Psi\,\rangle$ for an N-site chain.