PHYSICS 211C : CONDENSED MATTER PHYSICS
HW ASSIGNMENT #2

(1) Show that

$$\sum_{\sigma,\sigma'} c^\dagger_{i\sigma}\, c_{i\sigma'}\, c^\dagger_{j\sigma'}\, c_{j\sigma} = \tfrac{1}{2}\, n_i\, n_j + \tfrac{1}{2}\, \boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j \quad .$$

(2) Consider the paramagnetic phase of the Hubbard model within the Stoner approximation. Compute the charge susceptibility, $\chi_{\rm c} = \partial n/\partial\mu$, at $T=0$. This is related to the isothermal compressibility via $\kappa_T = n^{-2}\chi_{\rm c}$. Show that the bare (*i.e.* $U=0$) value is $\chi_{\rm c} = g(\varepsilon_{\rm F})$, *i.e.* the density of states at the Fermi level. Show within Stoner theory how this result is modified when $U>0$.

(3) Investigate within the Stoner approximation how the Curie temperature $T_{\rm c}(U)$ varies as a function of U for $U \gtrsim U_{\rm c}$.

(4) Consider the antiferromagnetic Heisenberg model on a bipartite lattice:

$$\hat{H} = J \sum_{\langle ij \rangle} \boldsymbol{S}_i \cdot \boldsymbol{S}_j$$

where $J>0$ and the sum is over the bonds of the lattice.

(a) Break up the lattice into a dimer covering. There are exponentially many such dimer coverings, *i.e.* the number grows as $e^{\alpha N}$ where N is the number of lattice sites. Think about tiling a chessboard with with dominoes. The analysis of this problem was performed by H. N. V. Temperley and M. E. Fisher, *Phil. Mag.* **6**, 1061 (1961). Denote one sublattice as A and the other as B. You are to develop a mean field theory of interacting dimers in the presence of a self-consistent staggered field

$$\langle \boldsymbol{S}_{\rm A} \rangle = -\langle \boldsymbol{S}_{\rm B} \rangle \equiv m\hat{\boldsymbol{z}} \ .$$

The mean field Hamiltonian then breks up into a sum over dimer Hamiltonians

$$\begin{aligned}\hat{H}_{\rm dimer} &= J\boldsymbol{S}_{\rm A}\cdot\boldsymbol{S}_{\rm B} + (z-1)J\,\langle \boldsymbol{S}_{\rm B}\rangle \cdot \boldsymbol{S}_{\rm A} + (z-1)J\,\langle \boldsymbol{S}_{\rm A}\rangle\cdot\boldsymbol{S}_{\rm B} \\ &= J\boldsymbol{S}_{\rm A}\cdot\boldsymbol{S}_{\rm B} - H_{\rm s}\,(S^z_{\rm A} - S^z_{\rm B})\end{aligned}$$

where the effective staggered field is $H_{\rm s} = (z-1)Jm$, and z is the lattice coordination number. Find the eigenvalues of $\hat{H}_{\rm dimer}$ when $S=\frac{1}{2}$.

(b) Define $x = 2h/J$. What is the self-consistent equation for x when $T=0$? Under what conditions is there a nontrivial solution? What then is the self-consistent staggered magnetization? How does it compare with the result of spin-wave theory?