(1) Consider the Boltzmann equation for quasiparticle transport

$$I[n_{k,\sigma}] = \frac{\partial n_{k,\sigma}(\boldsymbol{r})}{\partial t} + \frac{1}{\hbar}\frac{\partial \widetilde{\varepsilon}_{k,\sigma}}{\partial \boldsymbol{k}} \cdot \frac{\partial n_{k,\sigma}(\boldsymbol{r})}{\partial \boldsymbol{r}} - \frac{1}{\hbar}\frac{\partial \widetilde{\varepsilon}_{k,\sigma}}{\partial \boldsymbol{r}} \cdot \frac{\partial n_{k,\sigma}(\boldsymbol{r})}{\partial \boldsymbol{k}} \quad .$$

(a) Linearize to obtain an equation valid to order $\delta n_{k,\sigma}$. You may abbreviate the linearized collision integral simply as $I[\delta n_{k,\sigma}]$.

(b) Multiplying the *nonlinear* Boltzmann equation by $1, \sigma, \boldsymbol{k}$, and $\widetilde{\varepsilon}_{k,\sigma}$, derive the hydrodynamic relations for the local number density ρ and particle current $\boldsymbol{j}$,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{j} = 0 \quad ,$$

the local momentum density $\boldsymbol{g}$ and the momentum flux tensor $\Pi_{\alpha\beta}$,

$$\frac{\partial g_\alpha}{\partial t} + \frac{\partial \Pi_{\alpha\beta}}{\partial r^\beta} = 0 \, ,$$

the local energy density $\mathcal{E}$ and the energy current $\boldsymbol{Q}$,

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot \boldsymbol{Q} = 0 \quad ,$$

and the local spin density σ^z and the associated spin current $\boldsymbol{j}_{\sigma^z}$,

$$\frac{\partial \sigma^z}{\partial t} + \nabla \cdot \boldsymbol{j}_{\sigma^z} = 0 \quad .$$

[Note that these are collisional invariants and are therefore preserved by the collision integral.] Derive expressions for all these quantities, and then linearize your formulae to order δn.

Hint: If $\mathcal{E}(\boldsymbol{r})$ is the local energy density, then

$$\frac{\partial \mathcal{E}}{\partial \boldsymbol{r}} = \frac{\partial}{\partial \boldsymbol{r}}\left(\sum_{k,\sigma} \frac{\delta \mathcal{E}}{\delta n_{k,\sigma}(\boldsymbol{r})} \delta n_{k,\sigma}(\boldsymbol{r})\right) = \frac{\partial}{\partial \boldsymbol{r}}\left(\sum_{k,\sigma} \widetilde{\varepsilon}_{k,\sigma}\, \delta n_{k,\sigma}(\boldsymbol{r})\right) \quad .$$

This is helpful in considering the momentum and energy hydrodynamics.

(2) Landau's collisionless Boltzmann equation is

$$(\cos\theta - \lambda)\, \delta k_{\mathrm{F}}(\hat{\boldsymbol{k}}) + \cos\theta \int \frac{d\hat{\boldsymbol{k}}'}{4\pi}\, F(\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{k}}')\, \delta k_{\mathrm{F}}(\hat{\boldsymbol{k}}') = 0 \, ,$$

where the channel label (symmetric or antisymmetric) has been suppressed. The interaction function $F(\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{k}}')$ can be expanded in spherical harmonics:

$$F(\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{k}}') = \sum_{l,m} \frac{4\pi F_l}{2l+1}\, Y_{lm}(\hat{\boldsymbol{k}})\, Y_{lm}^*(\hat{\boldsymbol{k}}') \quad .$$

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(a) By similarly expanding the deviation $\delta k_{\mathrm{F}}(\hat{\boldsymbol{k}})$,

$$\delta k_{\mathrm{F}}(\hat{\boldsymbol{k}}) = \sum_{l,m} \beta_{lm}\, Y_{lm}(\hat{\boldsymbol{k}}) \ ,$$

convert the integral form of the collisionless Boltzmann equation to the algebraic form

$$\beta_{lm} = -\sum_{l',m'} D^{lm}_{l'm'}(\lambda)\, \frac{F_{l'm'}\,\beta_{l'm'}}{2l'+1}$$

where

$$D^{lm}_{l'm'}(\lambda) \equiv \int d\hat{\boldsymbol{k}}\, \frac{\cos\theta}{\cos\theta - \lambda}\, Y^{*}_{lm}(\hat{\boldsymbol{k}})\, Y_{l'm'}(\hat{\boldsymbol{k}}) \quad .$$

(b) Assuming an azimuthally symmetric solution, and retaining only the Landau parameters F_0 and F_1, show that zero sound propagates when $\lambda = \omega/q v_{\mathrm{F}}$ satisfies

$$\frac{\lambda}{2} \ln\left(\frac{\lambda+1}{\lambda-1}\right) - 1 = \frac{1 + \frac{1}{3}F_1}{F_0(1 + \frac{1}{3}F_1) + \lambda^2 F_1} \quad .$$

(c) Solve this equation numerically for $^3\mathrm{HeN}$ at $p = 0.28$ atm, where $F^{\mathrm{s}}_0 = 10.77$ and $F^{\mathrm{s}}_1 = 6.25$ from Wheatley's measurements of specific heat and first sound (Wheatley, 1966). Experiments at $p = 0.32$ atm find that $(c_0 - c_1)/c_1 = 0.035 \pm 0.003$ (Abel, Anderson, and Wheatley, 1966). How self-consistent is Fermi-Liquid theory if one uses only the first multipole F_0 to model zero sound? What is one uses the first two multipoles F_0 and F_1? [*Note:* All Landau parameters for this problem are in the symmetric channel. You may also assume that the Landau parameters change sufficiently slowly with pressure that you are justified in ignoring the difference between 0.28 atm and 0.32 atm.]

(3) (a) In an anisotropic system, the Landau interaction function $F(\hat{\boldsymbol{k}}, \hat{\boldsymbol{k}}')$ is not a function of the dot-product $\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{k}}'$ alone. Suppose, rather, that F is *separable, i.e.* $F(\hat{\boldsymbol{k}}, \hat{\boldsymbol{k}}') = F_0\, \Delta(\hat{\boldsymbol{k}})\Delta(\hat{\boldsymbol{k}}')$, where $\Delta(\hat{\boldsymbol{k}})$ is some prescribed function. Solve the collisionless Landau-Boltzmann equation for the Fermi surface oscillation $\delta k_{\mathrm{F}}(\hat{\boldsymbol{k}})$. What equation must the parameter $\lambda = \omega/v_{\mathrm{F}} q$ satisfy? Note that $\lambda = \lambda(F_0, \hat{\boldsymbol{q}})$.

(b) Take $\Delta(\hat{\boldsymbol{k}}) = 1 - \eta\cos^2\theta$, where θ is the polar angle for $\hat{\boldsymbol{k}}$. When $\eta = 0$, we recover the model zero sound treatment discussed in class. For $0 < \eta \le 1$, the interaction function is anisotropic in that quasiparticles near the poles ($\theta = 0, \pi$) of the Fermi sphere interact more weakly than those along the equator ($\theta = \frac{1}{2}\pi$). Obtain an algebraic equation for $\lambda(F_0, \hat{\boldsymbol{z}})$ in terms of the constants F_0 and η.