Refs: L&L: 66 and 73
Jackson: 14.1-2
Lechner: Chap 7

Chapter 4: Fields of Moving Charges

Liénard-Wiechert potential & field

To determine the field due to a point charge in arbitrary motion given by a specified trajectory $y^\mu(\lambda)$ (we use y^μ rather than x^μ, so as to not get confused with the argument of $A_\mu(x)$) we use the retarded Green's function $G_{ret}(x,x')$. Recall

$$G_{ret}(x,x') = G_{ret}(x-x')$$

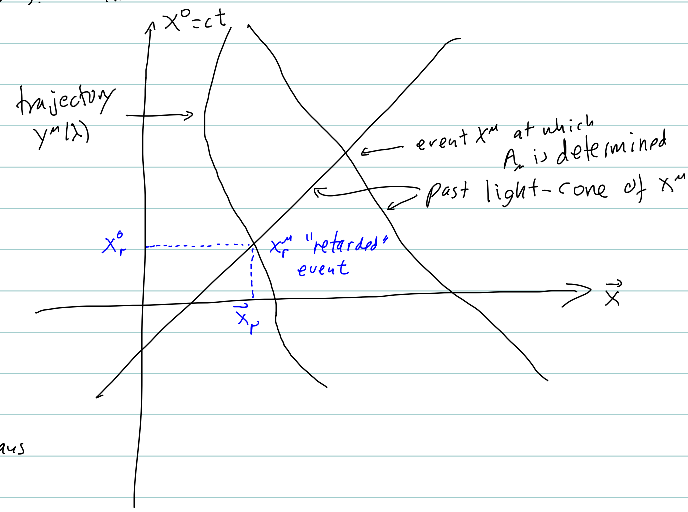
with $\partial_x^2 G_{ret}(x) = \delta^{(4)}(x)$

so that $G_{ret}(x) = \frac{1}{4\pi|\vec{x}|}\delta(x^0 - |\vec{x}|)$

and

$$A_\mu(x) = A_\mu^{(in)}(x) + \int d^4_x x' \, G_{ret}(x,x')\left(\frac{4\pi}{c} J_\mu(x')\right)$$

(recall $\partial^2 A_\mu = \frac{4\pi}{c} J_\mu$ in Lorentz gauge)



Notice that the δ-function in G_{ret} means that the field at $(x^0, \vec{x})$ is determined by the charge at the retarded time $x^0_r = ct_r$, given by $x^0 - x^0_r - |\vec{x} - \vec{x}_r| = 0 \Rightarrow x^0_r = x^0 - |\vec{x} - \vec{x}_r|$. That is $t_r = t - \frac{1}{c}\vec{R}$

where $|\vec{R}|$ is the distance from the charge to $\vec{x}$ at time t_r.

$A^{in}_\mu(x)$ has $\partial^2 A^{in}_\mu = 0$ and has a simple interpretation: if the charge is infinitely far away as $t \to -\infty$, then the only contribution to $A_\mu(x)$ is from $A^{in}_\mu(x) \Rightarrow$ it's the "initial" value of $A_\mu(x)$, specified at $t = -\infty$. We set it to zero (can add it back at no cost).

The 4-current of the point charge q is $J^\mu(x) = cq \frac{dy^\mu}{dy^0}\delta^{(3)}(\vec{x} - \vec{y}(y^0))$

Trick: multiply by $\int d\lambda \, \delta(x^0 - y^0(\lambda)) \frac{dy^0}{d\lambda} = 1$ to obtain a covariant expression and, more usefully, a $\delta^4(x)$:

$$\frac{1}{c} J^\mu(x) = \int d\lambda \, q u^\mu \delta^{(4)}(x^\mu - y^\mu(\lambda)) \qquad \text{where } u^\mu = \frac{dy^\mu}{d\lambda}$$

So we have

$$A_\mu(x) = \int d^4x' \, G_{ret}(x,x')\left[\frac{4\pi}{c}\int d\lambda \, c q u^\mu \delta^4(x' - y^\mu(\lambda))\right] = 4\pi q \int d\lambda \, u^\mu G_{ret}(x - y(\lambda))$$

There are two ways to do the integral. The elegant way involves expressing $G_{ret}(x)$ in a Lorentz invariant way: since $x^0 = |\vec{x}|$ is on the future light cone, consider

$$\delta(x^2)\theta(x^0) = \delta(x^{0\,2} - \vec{x}^2)\,\theta(x^0) = \frac{\delta(x^0 - |\vec{x}|)}{2|\vec{x}|} \quad \text{so} \quad G_{ret} = \frac{1}{2\pi}\delta((x-y)^2)\,\theta(x^0 - y^0)$$

$$\Rightarrow A_\mu(x) = 2q\int d\lambda \, u^\mu \delta((x - y(\lambda))^2)\,\theta(x^0 - y^0(\lambda)) = 2q u^\mu \frac{1}{\left|\frac{d}{d\lambda}(x - y(\lambda))^2\right|}\Bigg|_{\substack{y(\lambda_0)=x \\ y^0 < x^0}} = q\frac{u^\mu}{|(x-y)\cdot u|}\Bigg|_{\lambda_0}$$

Liénard-Wiechert potentials

Here, evaluated at λ_0 means, as anticipated, at retarded time: λ_0 is the solution to

$$(x-y(\lambda_0))^2=0 \quad \text{with } x^0>y^0(\lambda_0)$$

which, of course, is just $y^0(\lambda_0)=x^0-|\vec{x}-\vec{y}(\lambda_0)|$

The alternative way is to use $G_{ret}(x)=\frac{1}{4\pi|x|}\delta(x^0-|\vec{x}|)$ directly:

$$A_\mu(x)=q\int d\lambda\, v^\mu \frac{1}{|\vec{x}-\vec{y}|}\delta(x^0-y^0(\lambda)-|\vec{x}-\vec{y}(\lambda)|)$$

$$= q\frac{v^\mu}{|\vec{x}-\vec{y}(\lambda)|}\Big|_{\lambda_0}\frac{1}{\left|\frac{d}{d\lambda}(x^0-y^0-|\vec{x}-\vec{y}(\lambda)|)\right|}$$

where $\frac{d}{d\lambda}(y^0+|\vec{x}-\vec{y}|)=\frac{dy^0}{d\lambda}+\frac{(\vec{x}-\vec{y})\cdot(-\frac{d\vec{y}}{d\lambda})}{|\vec{x}-\vec{y}|}$

This equals the previous expression, since equals $\frac{1}{|\vec{x}-\vec{y}|}\left[(x^0-y^0)\frac{dy^0}{d\lambda}-(\vec{x}-\vec{y})\cdot\frac{d\vec{y}}{d\lambda}\right]=\frac{1}{|\vec{x}-\vec{y}|}(x-y)\cdot v$

so $A_\mu=\frac{q v_\mu}{|x-y|\cdot\frac{1}{|x-y|}[(x-y)\cdot v]}=\frac{q v_\mu}{(x-y)\cdot v}$

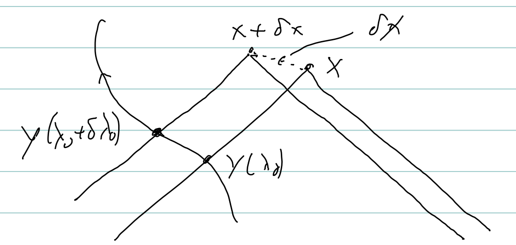
Let's write the potentials in terms of the velocity $\vec{\beta}$ of the charge and the distance $\vec{R}$ from retarded charge to $\vec{x}$. Using $\lambda=y^0$ $\frac{dy^\mu}{dy^0}=(1,\vec{\beta})$. As above,

$$A^\mu(x)=q\frac{v^\mu}{|\vec{x}-\vec{y}(\lambda)|}\Big|_{\lambda_0}\frac{1}{\frac{dy^0}{d\lambda}-\frac{(\vec{x}-\vec{y})}{|\vec{x}-\vec{y}|}\cdot\frac{d\vec{y}}{d\lambda}}=q\frac{(1,\vec{\beta})}{R}\cdot\frac{1}{1-\hat{R}\cdot\vec{\beta}}$$

where $\vec{\beta}$ is at t_r (retarded).

We can also compute $\vec{E}$ & $\vec{B}$. The complication is that in taking $\frac{\partial}{\partial x^\mu}$ we are changing not just x^μ but also λ_0 (think of $\lambda_0=\lambda_0(x)$ determined by $(x-y(\lambda_0))^2=0$).

$$(x-y(\lambda_0))^2=0 \quad \text{and} \quad (x+\delta x-y(\lambda_0+\delta\lambda_0))^2=0 \quad \text{or} \quad (x+\delta x-(y(\lambda_0)+v\,\delta\lambda_0))^2=0$$



$$(x-y)\cdot(\delta x-v\,\delta\lambda_0)\quad 0$$

$$\Rightarrow \partial_\mu\lambda_0=\frac{\partial\lambda_0}{\partial x^\mu}=\frac{(x-y)_\mu}{(x-y)\cdot v}$$

For $F_{\mu\nu}$ we need $\partial_\mu A_\nu$, and for this $\partial_\mu v_\nu=\frac{\partial\lambda_0}{\partial x^\mu}\cdot\frac{dv_\nu}{d\lambda}\Big|_{\lambda_0}=\frac{(x-y)_\mu}{(x-y)\cdot v}a_\nu$

where $a^\nu=\frac{dv^\nu}{d\lambda}=\frac{d^2y^\nu}{d\lambda^2}=\ddot{y}^\nu$. Similarly $\partial_\mu y_\nu=\frac{(x-y)_\mu v_\nu}{(x-y)\cdot v}$

Then $F_{\mu\nu} = \partial_\mu \left(\frac{q\, u_\nu}{(x-y)\cdot u} \right) - \mu \leftrightarrow \nu$

$$= \frac{q}{[(x-y)\cdot u]^2} \left\{ (x-y)_\mu a_\nu - u_\nu \left[\left(\eta^{\mu\lambda} - \frac{(x-y)^\mu u^\lambda}{(x-y)\cdot u} \right) u_\lambda + (x-y)_\lambda \frac{(x-y)_\mu a^\lambda}{(x-y)\cdot u} \right] \right\} - \mu \leftrightarrow \nu$$

This gives $\vec{E}$ and $\vec{B}$ in terms of retarded $y^\mu(\lambda_0)$, $u^\mu(\lambda_0)$ and $a^\mu(\lambda_0)$. Note that it is reparametrization invariant. It is useful to write this more explicitly in terms of $\vec{\beta}$, $\vec{R}$ and t_r. So take $\lambda = y^0$.

Let's separate this into the a-dependent part $F^{acc}_{\mu\nu}$ and the rest, $F^{vel}_{\mu\nu}$

$$F^{acc}_{\mu\nu} = \frac{q}{[(x-y)\cdot u]^2} (x-y)_\mu \left(a_\nu - \frac{u_\nu (x-y)\cdot a}{(x-y)\cdot u} \right) - \mu \leftrightarrow \nu$$

$$F^{vel}_{\mu\nu} = \frac{q\, u^2}{[(x-y)\cdot u]^3} \left[(x-y)_\mu u_\nu - (x-y)_\nu u_\mu \right]$$

Then (recall $u^\mu = (1, \vec{\beta})$, $(x-y)\cdot u = |\vec{x} - \vec{y}(\lambda_0)| - (\vec{x} - \vec{y}(\lambda_0))\cdot\vec{\beta} = R(1 - \hat{R}\cdot\vec{\beta})$)

$$F^{vel}_{\mu\nu} = \frac{q}{\gamma^2 R^3} \frac{(x-y)_\mu u_\nu - \mu \leftrightarrow \nu}{(1 - \hat{R}\cdot\vec{\beta})^3} \Rightarrow \vec{E}^{vel} = -F^{0i} = -\frac{q}{\gamma^2 R^3} \frac{1}{(1-\hat{R}\cdot\vec{\beta})^3} \left[R\beta^i - \vec{R} \right] = \frac{q}{R^2} \frac{\hat{R} - \vec{\beta}}{\gamma^2 (1 - \hat{R}\cdot\vec{\beta})^3}$$

and $$\vec{B}^{vel} = -\frac{1}{2}\epsilon^{ijk} F^{jk} = -\frac{q}{\gamma^2 R^3} \frac{\epsilon^{ijk} R^j \beta^k}{(1-\hat{R}\cdot\vec{\beta})^3} = -\frac{q}{\gamma^2 R^2} \frac{\hat{R}\times\vec{\beta}}{(1-\hat{R}\cdot\vec{\beta})^3}$$

which we recognize as the $\vec{E}$ & $\vec{B}$ fields of a moving charge with $\vec{\beta}$ = constant. But for $\dot{\vec{\beta}} \neq 0$ we have additional terms. We need

$$a^\mu = \frac{d^2 y^\mu}{dy^{0\,2}} = \frac{d}{dy^0}(1, \vec{\beta}) = (0, \vec{\alpha}) \quad \text{with } \vec{\alpha} = \frac{d\vec{\beta}}{dy^0} = \frac{1}{c^2}\vec{a}$$

$$\vec{E}^{acc} = -\frac{q}{R^2 (1-\hat{R}\cdot\vec{\beta})^2} \left[R\left(\vec{\alpha} + \frac{\vec{\beta}\, \hat{R}\cdot\vec{\alpha}}{1 - \hat{R}\cdot\vec{\beta}} \right) - \vec{R}\left(0 + \frac{\hat{R}\cdot\vec{\alpha}}{1 - \hat{R}\cdot\vec{\beta}} \right) \right] = -\frac{q}{R} \frac{1}{(1-\hat{R}\cdot\vec{\beta})^2} \left[\vec{\alpha} - \hat{R}\cdot\vec{\alpha} \frac{\hat{R} - \vec{\beta}}{1 - \hat{R}\cdot\vec{\beta}} \right]$$

and

$$\vec{B}^{acc} = -\frac{q}{R(1-\hat{R}\cdot\vec{\beta})^2} \hat{R} \times \left(\vec{\alpha} + \frac{\vec{\beta}\, \hat{R}\cdot\vec{\alpha}}{1 - \hat{R}\cdot\vec{\beta}} \right)$$

Note that $\vec{B}^{acc} = \hat{R} \times \vec{E}^{acc}$ and $|\vec{E}^{acc}| \sim \frac{1}{R}$. So $E \sim \underbrace{\frac{1}{R}}_{\text{"radiation field"}} + \frac{1}{R^2}$

Note $\hat{R} \times [(\hat{R} - \vec{\beta}) \times \vec{\alpha}] = (\hat{R} - \vec{\beta})\, \hat{R}\cdot\vec{\alpha} - \vec{\alpha}\, \hat{R}\cdot(\hat{R} - \vec{\beta}) = -(1 - \hat{R}\cdot\vec{\beta}) \left(\vec{\alpha} - \hat{R}\cdot\vec{\alpha} \frac{\hat{R} - \vec{\beta}}{1 - \hat{R}\cdot\vec{\beta}} \right)$ so one may write

$$\vec{E}^{acc} = \frac{q}{R} \frac{1}{(1 - \hat{R}\cdot\vec{\beta})^3} \hat{R} \times [(\hat{R} - \vec{\beta}) \times \vec{\alpha}]$$

At long distances only the radiation field is significant. It has $\vec{E}$ & $\vec{B}$ perpendicular to $\vec{R}$ — the retarded position vector, and are perpendicular to each other.

Non-relativistic limit: $\vec{E}^{acc} = \frac{q}{R}\hat{R}\times(\hat{R}\times\vec{\alpha}) = \frac{q}{c^2R}\hat{R}\times(\hat{R}\times\vec{a})$

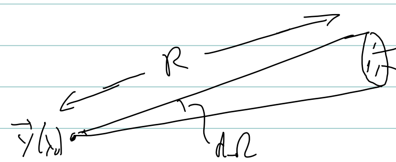
Rectilinear motion, $\vec{\beta}=\beta\hat{\alpha}$ $\Rightarrow$ $\vec{E}^{acc} = \frac{q}{c^2R}\frac{1}{(1-\beta\hat{a}\cdot\hat{R})^3}\left[\hat{R}\times(\hat{R}\times\vec{a})\right]$

Power Radiated

Poynting vector: energy flux

$$\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{B} = \frac{c}{4\pi}|E|^2\hat{R} \quad \left(= c\,u\,\hat{R} \quad \text{with} \quad u=\tfrac{1}{8\pi}(E^2+B^2)\right).$$

Consider at t a sphere centered at the retarded charge position $\vec{y}(\lambda_0)$. The energy



$d^2x = (\text{area}) = R^2\,d\Omega = R^2\sin\theta\, d\theta\, d\phi$

$dP = \text{power} = \frac{\text{energy}}{\text{time}}$ through area $= (\vec{S}\cdot\hat{R})(R^2 d\Omega)$

$$\Rightarrow \quad \frac{dP'}{d\Omega} = \vec{S}\cdot\hat{R}R^2 = \frac{c}{4\pi}\left(R|E|\right)^2 = \frac{c}{4\pi}\left|\frac{q}{(1-\hat{R}\cdot\vec{\beta})^3}\hat{R}\times\left[(\hat{R}-\vec{\beta})\times\vec{\alpha}\right]\right|^2$$

Note that as $R\to\infty$ the contribution of $\vec{E}^{vel}$ vanishes, hence we have neglected it here.

This expression has the energy per unit time in the inertial frame, that is, measured by a far-away observer. Sometimes we are interested in a time interval measured at the particle, $dP = dP'\frac{\partial x^0}{\partial y^0}$. Now, recall

$$\frac{\partial y^\mu}{\partial x^\nu} = \frac{(x-y)_\nu U^\mu}{(x-y)\cdot U} \quad \text{so that} \quad \frac{\partial y^0}{\partial x^0} = \frac{1}{1-\hat{R}\cdot\vec{\beta}}$$

and we have then

$$\frac{dP}{d\Omega} = \frac{cq^2}{4\pi}\frac{1}{(1-\hat{R}\cdot\vec{\beta})^5}\left|\hat{R}\times\left((\hat{R}-\vec{\beta})\times\vec{\alpha}\right)\right|^2$$

Note added: see discussion (below) on synchrotron radiation. The width of a pulse of radiation changes by this factor, it is the same issue, but there it is conceptually clearer.

(The reason for using time as seen by particle is (i) want total power radiated between particle at y_1^0 and y_2^0, and (ii) dP is then a Lorentz invariant: see below).

The relativistic expression presents some challenges with the integration:

$$\frac{dP}{d\Omega} = \frac{cq^2}{4\pi}\frac{1}{(1-\hat{R}\cdot\vec{\beta})^5}\left|\vec{\alpha}(1-\hat{R}\cdot\vec{\beta}) + (\vec{\beta}-\hat{R})\,\hat{R}\cdot\vec{\alpha}\right|^2$$

$$= \frac{cq^2}{4\pi}\frac{1}{(1-\hat{R}\cdot\vec{\beta})^5}\Big[\alpha^2(1-\hat{R}\cdot\vec{\beta})^2 + 2\left[\vec{\alpha}\cdot\vec{\beta}-\hat{R}\cdot\vec{\alpha}\right]\hat{R}\cdot\vec{\alpha}(1-\hat{R}\cdot\vec{\beta})$$
$$+ (\hat{R}\cdot\vec{\alpha})^2(1+\beta^2-2\vec{\beta}\cdot\hat{R})\Big]$$

$$= \frac{cq^2}{4\pi}\frac{1}{(1-\hat{R}\cdot\vec{\beta})^5}\left[\alpha^2(1-\hat{R}\cdot\vec{\beta})^2 - (\hat{R}\cdot\vec{\alpha})^2(1-\beta^2) + 2\vec{\alpha}\cdot\vec{\beta}\,\hat{R}\cdot\vec{\alpha}\,(1-\hat{R}\cdot\vec{\beta})\right]$$

To write this in terms of angles we pick a frame ($\vec{\beta}\propto\hat{z}$, $\vec{\alpha}$ in xz-plane)

$$\vec{\alpha} = \alpha(\sin\xi, 0, \cos\xi)$$
$$\vec{\beta} = \beta(0,0,1)$$
$$\hat{R} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

$$\Rightarrow \vec{\beta}\cdot\hat{R} = \beta\cos\theta\ ,\quad \vec{\alpha}\cdot\hat{R} = \alpha(\cos\theta\cos\xi + \sin\theta\sin\xi\cos\phi)$$

(and $\vec{\alpha}\cdot\vec{\beta} = \alpha\beta\cos\xi$ but these are independent of $d\Omega$).

Now $\int_0^{2\pi}d\phi$ is trivial, so, with $x = \cos\theta$

$$P = \frac{cq^2}{2}\int_{-1}^{1}dx\,\frac{1}{(1-\beta x)^5}\left[\alpha^2(1-\beta x)^2 - (1-\beta^2)\alpha^2\left(\cos^2\xi\, x^2 + \tfrac{1}{2}\sin^2\xi(1-x^2)\right) + 2\vec{\alpha}\cdot\vec{\beta}\,\alpha\cos\xi\, x(1-x\beta)\right]$$

(we used $\int_0^{2\pi}d\phi\,(1,\cos\phi,\cos^2\phi) = 2\pi(1,0,1/2)$.). The remaining integral is straightforward but tedious (and we have Mathematica):

$$P = \frac{cq^2}{2}\left[\frac{4}{3}\alpha^2\frac{1-s^2\beta^2}{(1-\beta^2)^3}\right] \qquad \text{where } s^2 = \sin^2\xi = |\hat{\alpha}\times\hat{\beta}|^2$$

or

$$\boxed{P = \frac{2}{3}\frac{q^2}{c}\gamma^6\left[\vec{a}^2 - (\vec{\beta}\times\vec{a})^2\right]} \qquad \text{Lienard (1898)}$$

In the non-relativistic limit, $\beta \ll 1$, we obtain "Larmor's formula"

$$\boxed{P = \frac{2}{3}\frac{q^2}{c}a^2} \qquad \text{(NR limit, "Larmor's formula).}$$

Comment: It is easy to obtain Larmor's formula from the non-relativistic limit of

$$\frac{dP'}{d\Omega} = \frac{cq^2}{4\pi}\frac{1}{(1-\hat{R}\cdot\vec{\beta})^6}\left|\hat{R}\times[(\hat{R}-\vec{\beta})\times\vec{\alpha}]\right|^2 \simeq \frac{cq^2}{4\pi}|\hat{R}\times\vec{\alpha}|^2 = \frac{cq^2\alpha^2}{4\pi}\sin^2\theta$$

Then $P'\simeq P$ and integrating over $d\Omega$ (recall $\vec{a}=\frac{1}{c^2}\vec{\alpha}$)

$$P \simeq \frac{2}{3}\frac{q^2a^2}{c^3}$$

Many textbooks then obtain Lienard's formula as follows: argue that P is frame invariant, then find a Lorentz scalar that reduces to Larmor's formula in the NR limit.

The 1st part of the argument is this: with $dP = \frac{\Delta E}{\Delta y^0}$, this is energy/time as measured by comoving observer.

2nd Part: substitute $\vec{a} = \frac{1}{m}\frac{d\vec{p}}{dt}$ in Larmor's:

$$a^2 = \frac{1}{m^2}\frac{d\vec{p}}{dt}\cdot\frac{d\vec{p}}{dt} = \frac{1}{m^2}\left[-c^2\frac{dp^\mu}{ds}\frac{dp_\mu}{ds}\right]$$

where the last step is valid in the NR limit (we'll verify below).

Now

$$-\frac{c^2}{m^2}\frac{dp^\mu}{ds}\frac{dp_\mu}{ds} = -\frac{c^2}{m^2}\left(\frac{dt}{ds}\right)^2\left[\left(\frac{d}{dt}(mc\gamma)\right)^2 - \left(\frac{d}{dt}(mc\gamma\vec{\beta})\right)^2\right]$$

$$= -c^2\gamma^2\left[\left(\gamma^3\vec{\beta}\cdot\dot{\vec{\beta}}\right)^2 - \left(\gamma^3\vec{\beta}\cdot\dot{\vec{\beta}}\,\vec{\beta} + \gamma\dot{\vec{\beta}}\right)^2\right]$$

$$= -c^2\gamma^2\left[\gamma^6(\vec{\beta}\cdot\dot{\vec{\beta}})^2(1-\beta^2) - 2\gamma^4(\vec{\beta}\cdot\dot{\vec{\beta}})^2 - \gamma^2(\dot{\vec{\beta}})^2\right]$$

$$= c^2\gamma^6\left[\dot{\vec{\beta}}^2(1-\beta^2) + (\vec{\beta}\cdot\dot{\vec{\beta}})^2\right]$$

$$= c^2\gamma^6\left[\dot{\vec{\beta}}^2 - \left(\beta^2\dot{\vec{\beta}}^2 - (\vec{\beta}\cdot\dot{\vec{\beta}})^2\right)\right] = c^2\gamma^6\left[\dot{\vec{\beta}}^2 - (\vec{\beta}\times\dot{\vec{\beta}})^2\right]$$

so

$$P = \frac{2}{3}\frac{q^2\gamma^6}{c^3}\left[\vec{a}^2 - (\vec{\beta}\times\vec{a})^2\right] \quad \text{as before.}$$

Ref: Jackson 14.3
L.L. 73
Lechner Chap 10

Angular Distribution

We already have the basic equation for this in $\frac{dP}{d\Omega}$ given in terms of θ & φ angles in p. 5 above.

We look at special cases:

1. Linear motion: $\vec{\beta} \parallel \vec{\alpha}$.

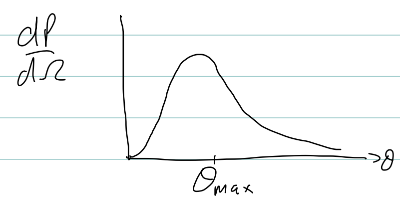
$\frac{dP}{d\Omega}$ is independent of ϕ (symmetric under rotations about axis defined by $\vec{\beta}$).

Explicitly
$$\frac{dP}{d\Omega} = \frac{cq^2}{4\pi} \frac{1}{(1-\hat{R}\cdot\vec{\beta})^5} \left| \hat{R} \times \{(\hat{R}-\vec{\beta}) \times \vec{\alpha}\} \right|^2$$
$$= \frac{cq^2}{4\pi} \frac{\alpha^2 \sin^2\theta}{(1-\beta\cos\theta)^5}$$

The direction of maximum radiation can be found analytically, but can also be quickly approximated by noting that for $\beta \sim 1$ it is at small θ and the denominator is small for

$$1-\beta\cos\theta \simeq 1-\beta(1-\tfrac{1}{2}\theta^2) \simeq 1-\beta+\tfrac{1}{2}\theta^2 = \text{small}, \text{ or with } \frac{1}{\gamma^2} = (1-\beta)(1+\beta) \simeq 2(1-\beta)$$

$$\Rightarrow \theta^2 \sim \frac{1}{\gamma^2} \Rightarrow \frac{dP}{d\Omega} \text{ is peaked at } \theta_{max} \sim \frac{1}{\gamma} \ll 1, \text{ but vanishes at } \theta = 0$$



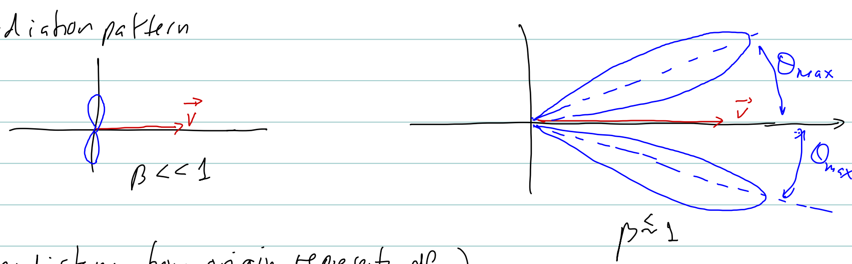
writing $1-\beta\cos\theta \simeq \frac{1}{2\gamma^2}(1+\gamma^2\theta^2)$

$$\frac{dP}{d\Omega} \simeq \frac{cq^2}{4\pi}\alpha^2 (2\gamma^2)^5 \frac{\theta^2}{(1+(\gamma\theta)^2)^5}$$

or $$\frac{dP}{d\Omega} = \frac{8q^2}{\pi} c\alpha^2 \gamma^8 \frac{(\gamma\theta)^2}{(1+(\gamma\theta)^2)^5}$$

In the opposite limit, $\beta \ll 1$ $\quad \frac{dP}{d\Omega} \simeq \frac{cq^2}{4\pi}\alpha^2\sin^2\theta$ (Larmor), so $\theta_{max} \simeq \frac{\pi}{2}$

Radiation pattern



Note: Figures of revolution (ie, the latter is a forward cone).

(The distance from origin represents $\frac{dP}{d\Omega}$)

Another case of interest is when $\vec{\beta} \perp \vec{\alpha}$, as in circular motion. Then, from previous lecture

$$\frac{dP}{d\Omega} = \frac{cq^2}{4\pi}\frac{1}{(1-\hat{R}\cdot\vec{\beta})^5}\left[\alpha^2(1-\hat{R}\cdot\vec{\beta})^2 - (\hat{R}\cdot\vec{\alpha})^2(1-\beta^2) + 2\vec{\alpha}\cdot\vec{\beta}\,\hat{R}\cdot\vec{\alpha}\,(1-\hat{R}\cdot\vec{\beta})\right]$$

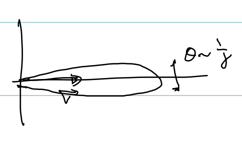
$$= \frac{cq^2}{4\pi}\frac{\alpha^2}{(1-\hat{R}\cdot\vec{\beta})^5}\left[(1-\hat{R}\cdot\vec{\beta})^2 - (\hat{R}\cdot\hat{\alpha})^2(1-\beta^2)\right]$$

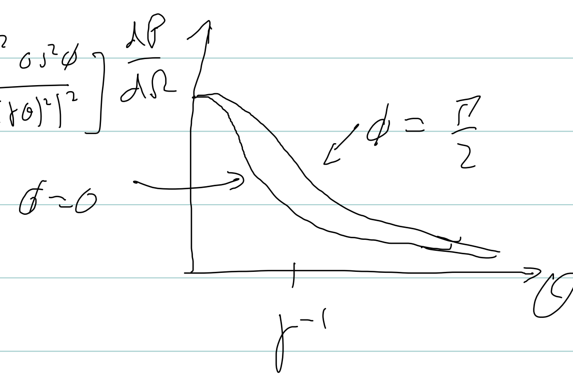
In the coordinate system we used earlier ($\hat{\beta}=\hat{z}$, $\vec{\alpha}$ in xz plane) we have now $\hat{\alpha}=\hat{x}$ and $\hat{R}\cdot\hat{\beta}=\cos\theta$, $\hat{R}\cdot\hat{\alpha}=\sin\theta\cos\phi$ (that is $\hat{R}\cdot\hat{\beta}=\hat{R}_z$ and $\hat{R}\cdot\hat{\alpha}=\hat{R}_x$)

$$\frac{dP}{d\Omega} = \frac{cq^2\alpha^2}{4\pi}\left[\frac{1}{(1-\beta\cos\theta)^3} - (1-\beta^2)\frac{\sin^2\theta\cos^2\phi}{(1-\beta\cos\theta)^5}\right]$$

Using $1-\beta\cos\theta \simeq \frac{1}{2\gamma^2}(1+(\gamma\theta)^2)$ for $\beta\simeq 1$ we have

$$\frac{dP}{d\Omega} \simeq \frac{2cq^2\alpha^2}{\pi}\gamma^6\frac{1}{(1+(\gamma\theta)^2)^3}\left[1 - \frac{4(\gamma\theta)^2\cos^2\phi}{\gamma^2(1+(\gamma\theta)^2)^2}\right]$$





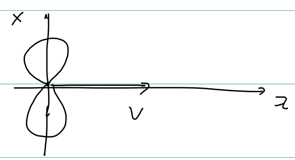
The radiation is emitted preferentially in the direction of $\vec{\beta}$, to within a cone of angular size $\theta\sim\frac{1}{\gamma}$. There is slightly more power radiated off the $\vec{\alpha}$-$\vec{\beta}$ plane (ie $\phi=\frac{\pi}{2}$) than on plane ($\phi=0$), but the difference is order $1/\gamma^2$.

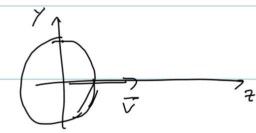
At small β, retaining lowest order in β:

$$\frac{dP}{d\Omega} = \frac{cq^2\alpha^2}{4\pi}\left[1-\sin^2\theta\cos^2\phi + \beta\cos\theta\left(3-5\sin^2\theta\cos^2\phi\right)\right]$$

$$= \frac{cq^2\alpha^2}{4\pi} \times \begin{cases} \cos^2\theta - \beta\cos\theta\left(2-5\cos^2\theta\right) & \text{for } \phi=0 \\ 1+3\beta\cos\theta & \phi=\frac{\pi}{2} \end{cases}$$

Patterns:





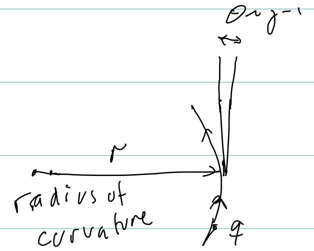
How about polarization? Recall we 1st derived (Lienord-Weichert)

$$\vec{E}_{rad} = -\frac{q}{R}\frac{1}{(1-\hat{R}\cdot\vec{\beta})^2}\left[\vec{\alpha} - \hat{R}\cdot\vec{\alpha}\,\frac{\hat{R}-\vec{\beta}}{1-\hat{R}\cdot\vec{\beta}}\right]$$

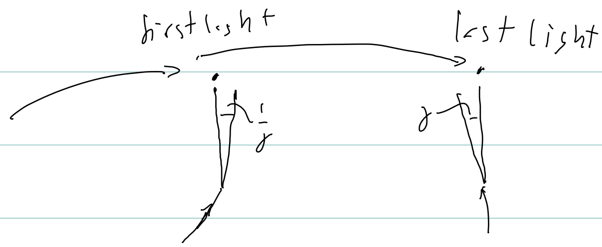
For $\gamma \gg 1$ the radiation is mostly along $\hat{R}=\hat{\beta}$, so focusing on that direction and noting that $\vec{\alpha}\cdot\vec{\beta}=0$ for circular motion, $\vec{E}_{rad} \simeq -\frac{q}{R}\frac{1}{(1-\beta)^2}\vec{\alpha}$

That is, fully in the plane of the circular motion to good approximation.

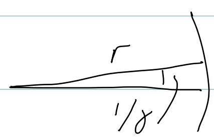
Heuristic discussion of radiation from Arbitrary motion with $\gamma >> 1$ and intro to spectral decomposition — Ref: Jackson 14.4



Arbitrary relativistic motion ($\gamma >> 1$):
Radiation in forward cone $\theta \sim \frac{1}{\gamma}$

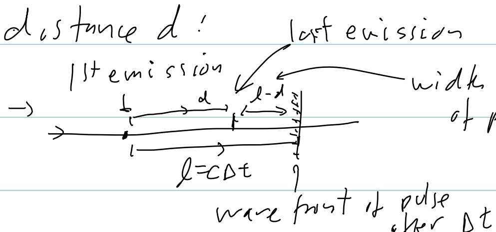


For a fixed observer as the particle transits a small section of the curved path there is radiation within the cone $\theta \sim 1/\gamma$ at observer.



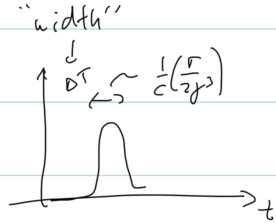
$d = r \cdot \frac{1}{\gamma}$ so burst of emission over time $\Delta t = \frac{d}{v} = \frac{r}{c\gamma\beta}$

Front edge of radiation moves a distance $l = c\Delta t = \frac{r}{\gamma\beta}$ by time the back edge of radiation "pulse" is emitted from particle that moved distance d:

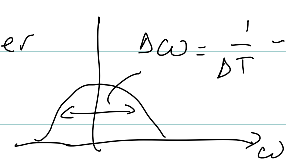


width of pulse $= l - d = \frac{r}{\gamma\beta} - \frac{r}{\gamma} = \frac{r}{\gamma\beta}(1-\beta) \simeq \frac{r}{2\gamma^3}$

Observed intensity



$\Longleftrightarrow$ Fourier



$\Delta\omega = \frac{1}{\Delta T} \sim \gamma^3 \frac{c}{r}$

For circular motion $\frac{c}{r} = \omega_q$ angular frequency and the observed frequencies $\omega \lesssim \Delta\omega \sim \gamma^3 \omega_q$

The amplification factor γ^3 is important.

For example, for $\omega_q \sim$ MHz, once can produce 10 keV X-rays ($\omega \sim 10^{19}\, s^{-1}$) with $\gamma \sim (10^{19}/10^6)^{1/3} \sim 10^4$. For electrons with $mc^2 \simeq 1$ MeV this requires $E \sim 10$ GeV → see energies of synchrotrons used as X-ray sources!

Ref: Lechner 10
Jackson 14.5

Spectral Analysis

Clearly of interest to have a more quantitative analysis.

We would like to have an expression for

$$\frac{dI}{d\Omega\, d\omega} = \frac{\text{observed intensity of radiation}}{\text{solid angle} \cdot \text{frequency}}$$

Now $\frac{dI}{d\Omega}$ is just $\frac{dP'}{d\Omega}$ integrated over time. We use P' which refers to per time of lab frame, ie, observer's time as is appropriate for this question. The expression is in these notes (p.4).

$$\frac{dI}{d\Omega} = \int_{-\infty}^{\infty} dt\, \frac{dP}{d\Omega}$$

Parseval's theorem

$$f(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \tilde{f}(k) \Leftrightarrow \int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty} \frac{dk}{2\pi} |\tilde{f}(k)|^2$$

Physicist's proof:

$$\int_{-\infty}^{\infty} dx\, |f(x)|^2 = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \frac{dk}{2\pi} \int_{-\infty}^{\infty} \frac{dk'}{2\pi}\, e^{ikx} \tilde{f}(k)\, e^{-ik'x} \tilde{f}^*(k')$$

$$= \int \frac{dk}{2\pi} \int \frac{dk'}{2\pi} \tilde{f}(k) \tilde{f}^*(k') \underbrace{\int_{-\infty}^{\infty} dx\, e^{i(k-k')x}}_{= 2\pi\, \delta(k-k')}$$

$$= \int_{-\infty}^{\infty} \frac{dk}{2\pi} |\tilde{f}(k)|^2$$

Since we had

$$\frac{dP'}{d\Omega} = \vec{S}\cdot\hat{R} R^2 = \frac{c}{4\pi} (R|E|)^2 = \frac{c}{4\pi} \left| \frac{q}{(1-\hat{R}\cdot\vec{\beta})^3} \hat{R} \times [(\hat{R}-\vec{\beta}) \times \vec{\alpha}] \right|^2$$

we write this as $\quad \dfrac{dP'}{d\Omega} = |\vec{P}(t)|^2 \quad \vec{P}(t) = \sqrt{\frac{c}{4\pi}} \frac{q}{(1-\hat{R}\cdot\vec{\beta})^3} \hat{R} \times (\hat{R} - \vec{\beta}) \times \vec{\alpha}$

Then $$\frac{dI}{d\Omega} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} |\vec{\tilde{P}}(\omega)|^2$$ (where $\vec{\tilde{P}}(\omega)$ is the Fourier transform of $\vec{P}(t)$)

$$= \int_0^{\infty} \frac{d\omega}{2\pi} \left[|\vec{\tilde{P}}(\omega)|^2 + |\vec{\tilde{P}}(-\omega)|^2 \right]$$

$$\Rightarrow \frac{dI}{d\Omega\, d\omega} = \frac{1}{2\pi} \left[|\vec{\tilde{P}}(\omega)|^2 + |\vec{\tilde{P}}(-\omega)|^2 \right] = \frac{1}{\pi} |\vec{\tilde{P}}(\omega)|^2$$

the last step if $\vec{P}(t)$ is real, then $\vec{\tilde{P}}(-\omega) = \vec{\tilde{P}}^*(\omega)$.

Useful approximations:

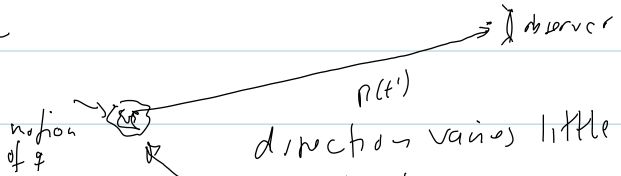
in $\int_{-\infty}^{\infty} dt\, e^{i\omega t}\,(\text{func})_{\text{retarded}}$

evaluated at $t - t' = R/c$

Change integration variable $dt \to dt'$. Then func = func(t')

$$\Rightarrow \int_{-\infty}^{\infty} dt'\,(1-\hat{n}\cdot\vec{\beta})\, e^{i\omega(t' + R(t')/c)}\,(\text{func}(t'))$$

Now



direction varies little so

(i) choose origin somewhere in blob (ie, particle trajectory)

(ii) If observer is at $\vec{r}$

$$R(t') = |\vec{r} - \vec{y}(t')| \simeq r - \hat{r}\cdot\vec{y}(t')$$

recall, trajectory of q.

So (drop primes and $e^{i\omega r}$ = constant)

$$\vec{p}(\omega) = \sqrt{\frac{c}{4\pi}}\, q \int_{-\infty}^{\infty} dt\, e^{i\omega(t - \hat{r}\cdot\vec{y}(t)/c)} \frac{\hat{r}\times[(\hat{r}-\vec{\beta})\times\vec{\alpha}]}{(1-\hat{r}\cdot\vec{\beta})^2}$$

and

$$\frac{d^2 I}{d\Omega\, d\omega} = \frac{c q^2}{4\pi^2}\left| \int_{-\infty}^{\infty} dt\, e^{i\omega(t - \hat{r}\cdot\vec{y}(t)/c)} \frac{\hat{r}\times\{(\hat{r}-\vec{\beta})\times\vec{\alpha}\}}{(1-\hat{r}\cdot\vec{\beta})^2}\right|^2$$

This integral is hard. There is a trick that simplifies significantly. Note that

$$\frac{\hat{r}\times[(\hat{r}-\vec{\beta})\times\vec{\alpha}]}{(1-\hat{r}\cdot\vec{\beta})^2} = \frac{1}{c}\frac{d}{dt}\left[\frac{\hat{r}\times(\hat{r}\times\vec{\beta})}{1-\hat{r}\cdot\vec{\beta}}\right]$$

Proof:

$$\frac{1}{c}\frac{d}{dt}\left[\frac{\hat{r}\times(\hat{r}\times\vec{\beta})}{1-\hat{r}\cdot\vec{\beta}}\right] = \frac{\hat{r}\cdot\vec{\alpha}}{(1-\hat{r}\cdot\vec{\beta})^2}\,\hat{r}\times(\hat{r}\times\vec{\beta}) + \frac{\hat{r}\times(\hat{r}\times\vec{\alpha})}{1-\hat{r}\cdot\vec{\beta}}$$

$$= \frac{1}{(1-\hat{r}\cdot\vec{\beta})^2}\left[\hat{r}\cdot\vec{\alpha}\,(\hat{r}\,\hat{r}\cdot\vec{\beta}-\vec{\beta}) + \hat{r}\times(\hat{r}\times\vec{\alpha}) - \hat{r}\cdot\vec{\beta}\,(\hat{r}\,\hat{r}\cdot\vec{\alpha}-\vec{\alpha})\right]$$

$$= \frac{1}{(1-\hat{r}\cdot\vec{\beta})^2}\left[(\hat{r}\cdot\vec{\beta})\vec{\alpha}-(\hat{r}\cdot\vec{\alpha})\vec{\beta} + \hat{r}\times(\hat{r}\times\vec{\alpha})\right]$$

$$= \frac{1}{(1-\hat{r}\cdot\vec{\beta})^2}\left[-\hat{r}\times(\vec{\beta}\times\vec{\alpha}) + \hat{r}\times(\hat{r}\times\vec{\alpha})\right]$$

Now, we willy-willy integrate by parts

$$\vec{\mathcal{P}}(\omega) = \sqrt{\frac{c}{4\pi}}\,q\int_{-\infty}^{\infty}dt\; e^{i\omega(t-\hat{r}\cdot\frac{\vec{y}(t)}{c})}\;\frac{1}{c}\frac{d}{dt}\left[\frac{\hat{r}\times(\hat{r}\times\vec{\beta})}{1-\hat{r}\cdot\vec{\beta}}\right]$$

$$\text{"="} -\sqrt{\frac{c}{4\pi}}\,\frac{q}{c}\int_{-\infty}^{\infty}dt\;\frac{d}{dt}e^{i\omega(t-\hat{r}\cdot\vec{y}/c)}\;\frac{\hat{r}\times(\hat{r}\times\vec{\beta})}{1-\hat{r}\cdot\vec{\beta}}$$

and noting that $\frac{d}{dt}(t-\hat{r}\cdot\vec{y}/c) = 1-\hat{r}\cdot\vec{\beta}$ we have

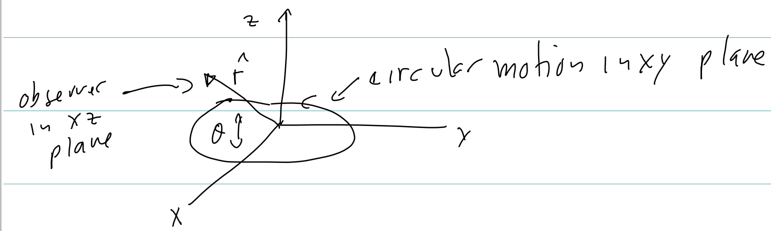
$$\vec{\mathcal{P}}(\omega) = -i\omega\sqrt{\frac{c}{4\pi}}\,\frac{q}{c}\int_{-\infty}^{\infty}dt\; e^{i\omega(t-\hat{r}\cdot\vec{y}/c)}\;\hat{r}\times(\hat{r}\times\vec{\beta})$$

This is much simpler! Now

$$\frac{d^2I}{d\Omega\,d\omega} = \frac{q^2\omega^2}{4\pi^2c}\left|\int_{-\infty}^{\infty}dt\; e^{i\omega(t-\hat{r}\cdot\vec{y}/c)}\;\hat{r}\times(\hat{r}\times\vec{\beta})\right|^2$$

Spectrum of circular motion
(Synchrotron radiation)

<u>Refs</u>: Lechner 11
Jackson 14.6



Note: I use θ for angle with x-axis (not z!).

$$\vec{y}(t) = \rho\left(\sin\left(\frac{vt}{\rho}\right), -\cos\left(\frac{vt}{\rho}\right), 0\right) \qquad \left(\text{putting } \vec{y}(0) = -\rho\,(0,1,0) \text{ and } \vec{\beta}(0) = \beta(1,0,0)\right)$$

and the argument of exp in $\tilde{\vec{P}}$ is

$$\omega\left(t - \frac{\hat{r}\cdot\vec{y}(t)}{c}\right) = \omega\left(t - \frac{\rho}{c}\sin\left(\frac{vt}{\rho}\right)\cos\theta\right)$$

$$\simeq \frac{\omega}{2}\left(\left(\frac{1}{\gamma^2} + \theta^2\right)t + \frac{c^2}{3\rho^2}t^3\right)$$

where we used our previous heuristic discussion that for $\gamma \gg 1$ the radiation is in a cone $\theta \sim 1/\gamma$ around $\hat{\beta}$, and since the observer is in the xz plane the burst of radiation that reaches him/her is from when $\hat{\beta} = \pm\hat{x}$, ie, from a small time interval around $\vec{y}(t) \simeq \rho(\pm 1, 0, 0) \Rightarrow$ expand about $\sin\left(\frac{vt}{\rho}\right) = 0$; and of course, expand about $\theta \simeq 0$.

The higher order terms are supressed by either $\theta^2 \sim 1/\gamma^2$ or by $\left(\frac{vt}{\rho}\right)^2 \sim \left(\frac{v\Delta t}{\rho}\right)^2$ with $c\Delta t \sim \frac{\rho}{\gamma\beta}$ from our previous arguments

So $\left(\frac{vt}{\rho}\right)^2 \sim \frac{1}{\gamma^2}$.

Note that this makes sense if we only integrate over one cycle of the circular motion. If we really integrate over all

times, we get infinitely many equal contributions $\to \infty$
The reason is clear: it is periodic motion, with fixed frequency = angular frequency $= \omega_q = \frac{v}{\rho}$

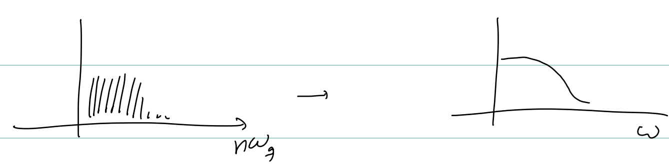
The proper mathematical analysis is then to do a Fourier series rather than an integral:

$$\vec{p}(t) = \sum_{n=-\infty}^{\infty} e^{in\omega_q t} \tilde{\vec{p}}_n$$

with

$$\tilde{\vec{p}}_n = \frac{1}{2\pi} \int_0^{\frac{2\pi}{\omega_q}} dt\, e^{-in\omega_q t} \vec{p}(t)$$

Physically we expect $n\omega_q \sim \gamma^3 \omega_q$, the number of modes that contribute is huge; the approximation above of just retaining one pulse replaces the large (but exact) number of discrete modes by a continuum:



and $\frac{1}{2\pi}\sum_n |\tilde{p}_n|^2 \to \frac{1}{2\pi}\int d\omega\, |\tilde{p}(\omega)|^2$.

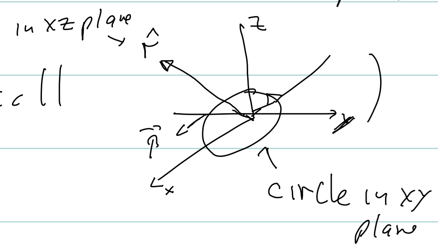
The approximation to $\vec{\rho}(t) \propto \hat{r} \times [(\hat{r}-\vec{\beta}) \times \vec{\alpha}]$

is $\perp$ to $\hat{r}$: physically the EM wave at the observer has $\vec{E}$ (and $\vec{B}$) $\perp$ to line of sight. We can decompose $\vec{E}$ into 2 polarizations in the plane $\perp$ to $\hat{r}$:

$\vec{\epsilon}_{\parallel}$: in the xy plane

$\vec{\epsilon}_{\perp}$: $\perp$ direction, $\vec{\epsilon}_{\perp} = \hat{r} \times \vec{\epsilon}_{\parallel}$

Since radiation is from $\hat{\beta} = \pm\hat{x}$ ($+\hat{x}$ if $\hat{r}\cdot\hat{x} > 0$)

(recall 

)

$\Rightarrow \vec{\epsilon}_{\parallel} = (0, 1, 0)$

(and $\vec{\beta} = \beta(\cos(\frac{vt}{\rho}), \sin(\frac{vt}{\rho}), 0)$).

Then $\vec{\epsilon}_{\perp} = \hat{r} \times \vec{\epsilon}_{\parallel} = (\cos\theta, 0, \sin\theta) \times \vec{\epsilon}_{\parallel} = (-\sin\theta, 0, \cos\theta)$

Decompose

$$\hat{r} \times (\hat{r} \times \vec{\beta}) = \rho_{\parallel}\vec{\epsilon}_{\parallel} + \rho_{\perp}\vec{\epsilon}_{\perp}$$

$$\rho_{\parallel} = \vec{\epsilon}_{\parallel}\cdot[\hat{r}\times(\hat{r}\times\vec{\beta})] = (\vec{\epsilon}_{\parallel}\times\hat{r})\cdot(\hat{r}\times\vec{\beta}) = -\vec{\epsilon}_{\perp}\cdot(\hat{r}\times\vec{\beta})$$

$$= -(\vec{\epsilon}_{\perp}\times\hat{r})\cdot\vec{\beta} = -\vec{\epsilon}_{\parallel}\cdot\vec{\beta} = -\sin\left(\frac{vt}{\rho}\right) \simeq -\frac{c\,t}{\rho}$$

$$\rho_{\perp} = \vec{\epsilon}_{\perp}\cdot[\hat{r}\times(\hat{r}\times\vec{\beta})] = (\vec{\epsilon}_{\perp}\times\hat{r})\cdot(\hat{r}\times\vec{\beta}) = \vec{\epsilon}_{\parallel}\cdot(\hat{r}\times\vec{\beta})$$

$$= (\vec{\epsilon}_{\parallel}\times\hat{r})\cdot\vec{\beta} = -\vec{\epsilon}_{\perp}\cdot\vec{\beta} = \beta\cos\left(\frac{vt}{\rho}\right)\sin\theta \simeq \theta$$

So
$$\frac{d^2I}{d\Omega\,d\omega} = \frac{q^2\omega^2}{4\pi^2 c}\left|\int_{-\infty}^{\infty} dt\; e^{i\frac{\omega}{2}\left[\left(\frac{1}{\gamma^2}+\theta^2\right)t + \frac{1}{3}\frac{c^2}{\rho^2}t^3\right]}\left(-\frac{c\,t}{\rho}\vec{\epsilon}_{\parallel} + \theta\vec{\epsilon}_{\perp}\right)\right|^2$$

The integral can be expressed in terms of the Airy function

$$Ai(x) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, e^{i(xt+\frac{1}{3}t^3)} = \frac{1}{\pi}\int_0^{\infty} dt \cos\left(xt+\frac{1}{3}t^3\right)$$

and it's derivative

$$A_i'(x) = \frac{i}{2\pi}\int_{-\infty}^{\infty} dt\, t\, e^{i(xt+\frac{1}{3}t^3)}$$

Our integral is of the form $\int_{-\infty}^{\infty} dt\, e^{i(at+\frac{1}{3}bt^3)}$

So rescale $t \to \frac{1}{b^{1/3}}t$, $\frac{1}{b^{1/3}}\int_{-\infty}^{\infty} dt\, e^{i\left(\frac{a}{b^{1/3}}t+\frac{1}{3}t^3\right)} = \frac{2\pi}{b^{1/3}} A_i\left(\frac{a}{b^{1/3}}\right)$

and $\int_{-\infty}^{\infty} dt\, t\, e^{i(at+\frac{1}{3}bt^3)} = \frac{2\pi}{b^{2/3}} A_i'\left(\frac{a}{b^{1/3}}\right)$

In our case $a = \frac{\omega}{2}\left(\frac{1}{\gamma^2}+\theta^2\right)$ and $b = \frac{\omega}{2}\frac{c^2}{\rho^2}$. Let $z = \frac{\frac{\omega}{2}\left(\frac{1}{\gamma^2}+\theta^2\right)}{\left[\frac{\omega}{2}\cdot c^2/\rho^2\right]^{1/3}}$

or $z = \left(\frac{\omega\rho}{2c}\right)^{2/3}\left(\frac{1}{\gamma^2}+\theta^2\right)$

With this

$$\frac{d^2I}{d\Omega d\omega} = \frac{q^2\omega^2}{4\pi^2 c}\left| -\frac{c}{\rho}2\pi\frac{1}{\left(\frac{\omega}{2}\frac{c^2}{\rho^2}\right)^{2/3}} A_i'(z)\,\vec{\epsilon}_{\parallel} + \theta\frac{2\pi}{\left(\frac{\omega}{2}\frac{c^2}{\rho^2}\right)^{1/3}} A_i(z)\,\vec{\epsilon}_{\perp}\right|^2$$

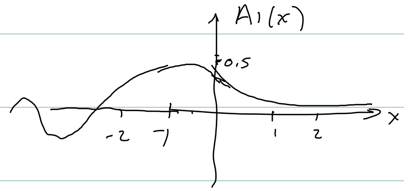
$$= \frac{q^2}{c}\left[\left(\frac{4\omega\rho}{c}\right)^{2/3}[A_i'(z)]^2 + \theta^2\left(\frac{\sqrt{2}\,\omega\rho}{c}\right)^{4/3}[A_i(z)]^2\right]$$

Note that the angular frequency of the circular motion is $\omega_0 = \frac{c}{\rho}$

so the result is given in terms of the ratio $\frac{\omega\rho}{c} = \frac{\omega}{\omega_0}$.

Garg: App. E

Now



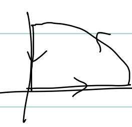
We can compute

$$Ai(0) = \frac{1}{\pi}\,\mathrm{Re}\int_0^\infty e^{i\frac{1}{3}t^3}\,dt$$

. Change variables $u = \frac{1}{3}t^3$

$\left(du = t^2 dt \Rightarrow dt = \frac{du}{(3u)^{2/3}} \right)$

Then $\displaystyle\int_0^\infty e^{i\frac{1}{3}t^3}\,dt = \int_0^\infty e^{iu}\,\frac{du}{3^{2/3}u^{2/3}}$

Then change $u = iv$ (formally consider $\oint_C dz\, e^{iz} = 0$ ).

$$= \frac{1}{3^{2/3}}\int_0^\infty e^{-v}\,\frac{i\,dv}{i^{2/3}v^{2/3}} = \frac{i^{1/3}}{3^{2/3}}\int_0^\infty e^{-v}\,v^{\frac{1}{3}-1}\,dv = \frac{e^{i\pi/6}}{3^{2/3}}\,\Gamma(1/3)$$

So $Ai(0) = \frac{1}{\pi}\,\frac{\cos(\pi/6)}{3^{2/3}}\,\Gamma(1/3)$, or using $\Gamma(\frac{1}{3})\Gamma(\frac{2}{3}) = \frac{\pi}{\cos(\pi/6)}$

$$Ai(0) = \frac{1}{3^{2/3}\,\Gamma(\frac{2}{3})} \simeq 0.355$$

Similarly one finds $Ai'(0) = -\frac{1}{3^{1/3}\,\Gamma(1/3)} \simeq -0.259$

The large x behavior can be obtained by stationary phase:

$$\frac{d}{dt}\left(xt + \frac{1}{3}t^3\right) = 0 \Rightarrow t^2 = -x \;;\; \frac{d^2}{dt^2}\left(xt + \frac{1}{3}t^3\right) = 2t$$

So $xt + \frac{1}{3}t^3 = x\sqrt{-x} + \frac{1}{3}(-x)^{3/2} + \frac{1}{2}\,2\sqrt{-x}\,(t - \sqrt{-x})^2 + \dots$

so that

$$\int_{-\infty}^{\infty} dt\, e^{i(tx + \frac{1}{3}t^3)} \simeq \int_{-\infty}^{\infty} dt\, e^{i\left(\sqrt{-x}\,\frac{2}{3}x + \sqrt{-x}\,(t-\sqrt{-x})^2\right)}$$

Use $\sqrt{-x} = i x$ (the other solution blows up (steepest "ascent").

So

$$= e^{-\frac{2}{3}x^{3/2}} \int_{-\infty}^{\infty} dt\, e^{-\sqrt{x}(t - i\sqrt{x})^2}$$

$$= e^{-\frac{2}{3}x^{3/2}} \int_{-\infty}^{\infty} dv\, e^{-\sqrt{x}\, v^2} = \frac{e^{-\frac{2}{3}x^{3/2}}}{x^{1/4}}\, 2\int_0^{\infty} du\, e^{-u^2}$$

$\xi = u^2$

$du = \dfrac{d\xi}{2\sqrt{\xi}}$

$$= \frac{2e^{-\frac{2}{3}x^{3/2}}}{x^{1/4}} \int_0^{\infty} \frac{d\xi}{2\sqrt{\xi}} e^{-\xi} = \frac{e^{-\frac{2}{3}x^{3/2}}}{x^{1/4}} \Gamma(1/2) = \sqrt{\pi}\, \frac{e^{-\frac{2}{3}x^{3/2}}}{x^{1/4}}$$

And $Ai(x) \simeq \dfrac{e^{-\frac{2}{3}x^{3/2}}}{2\sqrt{\pi}\, x^{1/4}}$ as $x \to +\infty$

For $Ai'(x)$, differentiate this

$$Ai'(x) \simeq -\frac{x^{1/4}}{2\sqrt{\pi}} e^{-\frac{2}{3}x^{3/2}}$$

For $x \to -\infty$ and relation to K_ν see Garg.

We can now analyze the behavior of $\frac{d^2 I}{d\Omega\, d\omega}$.

Recall

$$\frac{d^2 I}{d\Omega\, d\omega} = \frac{q^2}{c}\left[\left(\frac{4\omega}{\omega_0}\right)^{2/3}\left[A_i'(z)\right]^2 + \theta^2\left(\frac{\sqrt{2}\,\omega}{\omega_0}\right)^{4/3}\left[A_i(z)\right]^2\right]$$

where $z = \left(\frac{\omega}{2\omega_0}\right)^{2/3}\left(\frac{1}{\gamma^2}+\theta^2\right)$

For fixed θ^2, we have at small ω

$$\frac{d^2 I}{d\Omega\, d\omega} \simeq \frac{q^2}{c}\left[\left(\frac{4\omega}{\omega_0}\right)^{2/3}\left[A_i'(0)\right]^2 + \theta^2\left(\frac{\sqrt{2}\,\omega}{\omega_0}\right)^{4/3}\left[A_i(0)\right]^2\right] \simeq \frac{q^2}{c}\left[A_i'(0)\right]^2\left(\frac{4\omega}{\omega_0}\right)^{2/3}$$

and for large ω

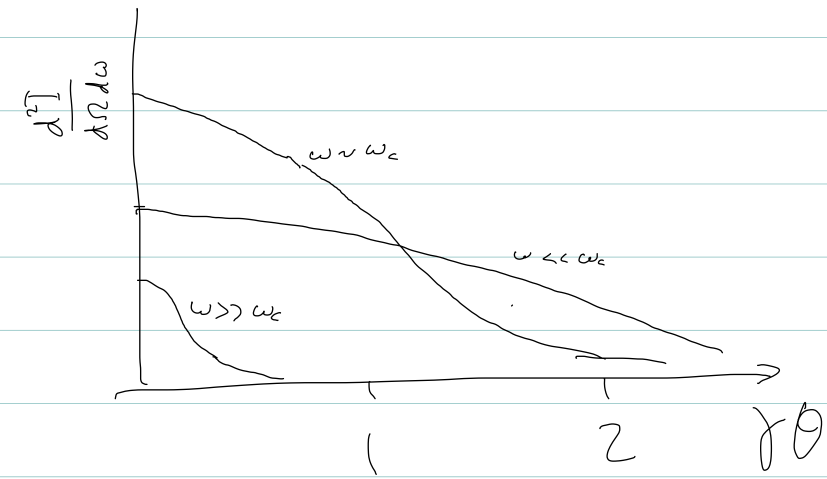
$$\frac{d^2 I}{d\Omega\, d\omega} \simeq \frac{q^2}{c}\frac{1}{4\pi}\, e^{-\frac{2}{3}\left(\frac{1}{\gamma^2}+\theta^2\right)^{3/2}\frac{\omega}{\omega_0}}\left[\left(\frac{4\omega}{\omega_0}\right)^{2/3}\left(\frac{1}{\gamma^2}+\theta^2\right)^{1/2}\left(\frac{\sqrt{2}\,\omega}{\omega_0}\right)^{1/3}\right.$$
$$\left. + \theta^2\left(\frac{\sqrt{2}\,\omega}{\omega_0}\right)^{\frac{4}{3}}\left(\frac{\sqrt{2}\,\omega}{\omega_0}\right)^{-1/3}\left(\frac{1}{\gamma^2}+\theta^2\right)^{-1/2}\right]$$
$$= \frac{\sqrt{2}\,q^2}{4\pi c}\, e^{-\frac{2}{3}\left(\frac{1}{\gamma^2}+\theta^2\right)^{3/2}\frac{\omega}{\omega_0}}\left(\frac{\omega}{\omega_0}\right)\left[\left(\frac{4}{\sqrt{2}}\right)^{2/3}\left(\frac{1}{\gamma^2}+\theta^2\right)^{1/2} + \frac{\theta^2}{\left(\frac{1}{\gamma^2}+\theta^2\right)^{1/2}}\right]$$
$$= \frac{3\sqrt{2}\,q^2}{2\pi c}\, e^{-2\left(1+\theta^2\gamma^2\right)^{3/2}\frac{\omega}{\omega_c}}\gamma^2\left(1+\gamma^2\theta^2\right)^{1/2}\frac{\omega}{\omega_c}\left[1 + \frac{1}{2}\frac{\gamma^2\theta^2}{1+\gamma^2\theta^2}\right]$$

The exponential gives a rapid falloff $e^{-2\left(1+\theta^2\gamma^2\right)^{3/2}\frac{\omega}{\omega_c}}$

where the critical frequency $\omega_c = 3\gamma^3\omega_0$ characterizes the frequency beyond which ($\omega \gtrsim \omega_c$) radiation is negligible even for $\theta = 0$.

Note also that for small ω the polarization is largely $\parallel$. Likewise, at $\theta=0$ only $\epsilon_{\parallel}$ contributes.

Jackson has a nice plot:

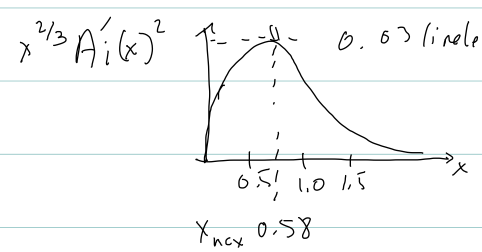


where $\omega_c \equiv 3\gamma^3\omega_0$ is defined by $z=1$ at $\theta=0$, the critical frequency beyond which there is negligible radiation for any angle.

Added: At $\theta=0$ we have $z = \left(\frac{\omega}{2\omega_0}\right)^{2/3}\frac{1}{\gamma^2} = \left(\frac{\omega}{2\gamma^3\omega_0}\right)^{2/3} = \left(\frac{3}{2}\frac{\omega}{\omega_c}\right)^{2/3}$

and

$$\left.\frac{d^2I}{d\Omega\, d\omega}\right|_{\theta=0} = 4\frac{q^2}{c}\gamma^2 \left(\frac{3\omega}{2\omega_c}\right)^{2/3}\left[Ai'\left(\left(\frac{3\omega}{2\omega_c}\right)^{2/3}\right)\right]^2$$



This explains the $\theta \simeq 0$ behavior of curve above.