Garg - Sec. 4
Jackson - Appendix

Units and Dimensions

We have postponed discussion of units. Most of our studies have been formal. But at some point we would like to plug in numbers, compare with measurements and so on.

Two most common systems:

(i) Gaussian - CGS.
(ii) SI - MKS

CGS = cm, gram, second   MKS = m, kg, sec

denote the units used for mechanical quantities.

Gaussian is more natural:

1. $\vec{E}$ and $\vec{B}$ have same dimensions
2. All units are derived from CGS.

SI is more common (volts, amperes, coulombs)

1. $\vec{E}$ and $\vec{B}$ have different dimensions (and units)
2. Introduces one new basic unit: Ampere for current

Will not discuss how units are defined. See

https://www.nist.gov/si-redefinition/definitions-si-base-units

That $\vec{E}$ & $\vec{B}$ do/don't have same dimensions in CGS vs SI implies that formulae containing them change from one system to the other.

More detail:

For mechanics, formulae have the same form in MKS & CGS, eg

$\vec{F} = \frac{d\vec{p}}{dt}$ $\vec{L} = \vec{r} \times \vec{p}$, $\vec{p} = m\vec{v}$, etc.

For EM, formulae depend on system of units:

Gaussian	SI
$\vec{F} = \frac{q_1 q_2}{\lvert\vec{x}\rvert^2} \hat{x}$	$\vec{F} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{\lvert\vec{x}\rvert^2} \hat{x}$
$u = \frac{1}{8\pi} (\vec{E}^2 + \vec{B}^2)$	$u = \frac{1}{2} (\epsilon_0 \vec{E}^2 + \frac{1}{\mu_0} \vec{B}^2)$
$\vec{F} = q(\vec{E} + \frac{\vec{v}}{c} \times \vec{B})$	$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$

$\Rightarrow$ Translating between systems requires changes in Formulas.

Denote by [ ] dimensions (as usual) with $[\vec{x}] = L$, $[m] = M$, $[t] = T$ (with units that are measured in cm-g-s in CGS or m-kg-s in MKS).

We can see what the dimensions are of each quantity in each system. In particular the new (non-mechanical) quantities q (charge), $\vec{E}$ & $\vec{B}$.

Gaussian:

From $F = \frac{q^2}{\lvert\vec{x}\rvert^2}$, $[q] = \left([F=ma][x^2]\right)^{1/2} = [M]^{1/2}[L]^{3/2}[T]^{-1} = $ statcoulomb

$u = \frac{1}{8\pi}(\vec{E}^2 + \vec{B}^2)$ $[\vec{E}] = [\vec{B}] = \left[\frac{mv^2}{vol}\right]^{1/2} = [M]^{1/2}[L]^{-1/2}[T]^{-1} = $ statvolt·cm^{-1}

Sanity check: $F = qE \Rightarrow [M][L][T]^{-2} \stackrel{?}{=} \left([M]^{1/2}[L]^{3/2}[T]^{-1}\right) \cdot \left([M]^{1/2}[L]^{-1/2}[T]^{-1}\right)$ ✓✓

Other quantities trivially follow, eg $\vec{E} = -\vec{\nabla}\phi$ ($\phi = A^0$)

$\Rightarrow [\phi] = [\vec{E}][L] = [M]^{1/2}[L]^{1/2}[T]^{-1}$ $\left(= [\text{Force}]^{1/2}\right)$ = statvolt

SI

$[I]$ is a new basic dimension → Ampere

$I = \frac{dq}{dt} \Rightarrow [Q] = [I][T]$ → Coulomb

$F = \frac{1}{4\pi\epsilon_0}\frac{q^2}{|\vec{x}|^2} \rightarrow [M][L][T]^{-2} = [\epsilon_0]^{-1}[I]^2[T]^2[L]^{-2} \rightarrow [\epsilon_0] = [I]^2[T]^4[L]^{-3}[M]^{-1}$

$u = \frac{1}{8\pi}\left(\epsilon_0 \vec{E}^2 + \frac{1}{\mu_0}\vec{B}^2\right)$ → two relations

$$[\vec{E}] = \left([M][L]^{-1}[T]^{-2}\right)^{1/2}\left([I]^2[T]^4[L]^{-3}[M]^{-1}\right)^{-1/2} = [M][L][T]^{-3}[I]^{-1}$$

$$[\vec{B}]^2[\mu_0]^{-1} = [M][L]^{-1}[T]^{-2}$$

$F = q(\vec{E} + \vec{v}\times\vec{B})$ → two relations

$$[\vec{B}] = [\vec{E}][\vec{v}]^{-1} = [M][T]^{-2}[I]^{-1}$$

$$[\vec{E}] = [M][L][T]^{-2}[I]^{-1}[T]^{-1} = \text{same as above} \quad ✓✓$$

From u: $[\mu_0] = [B]^2[M]^{-1}[L][T]^2 = [M][L][T]^{-2}[I]^{-2}$

Check: $[\epsilon_0][\mu_0] = [L]^{-2}[T]^2$ consistent with $c^2 = \frac{1}{\epsilon_0\mu_0}$

Exercise: Check that Maxwell's Equations in SI are dimensionally consistent.

Translating between systems

We can go from an expression (say any of Maxwell's) in one system to another if we develop a dictionary.

For example

$F = \frac{q^2}{x^2}$ (Gauss) $\quad F = \frac{1}{4\pi\epsilon_0}\frac{q^2}{x^2}$ (SI)

Since F & $\vec{x}$ are mechanical quantities, they are the same in both systems. We infer $q_G^2 = \frac{q_{SI}^2}{4\pi\epsilon_0}$ or $q_G = \frac{q_{SI}}{\sqrt{4\pi\epsilon_0}}$

And from $u = \frac{1}{8\pi}\left(E_G^2 + B_G^2\right) = \frac{1}{2}\left(\epsilon_0 \vec{E}_{SI}^2 + \frac{1}{\mu_0}\vec{B}_{SI}^2\right)$

we get $\vec{E}_G = \sqrt{4\pi\epsilon_0}\,\vec{E}_{SI}$ and $\vec{B}_G = \sqrt{\frac{4\pi}{\mu_0}}\,\vec{B}_{SI}$

As above, $q, \vec{E}$ & $\vec{B}$ are sufficient to obtain the rest of the dictionary

The Lorentz force gives nothing new — except if we do not know a priori that $c^2 = \frac{1}{\epsilon_0\mu_0}$. $\vec{F} = q_G\left(\vec{E}_G + \frac{\vec{v}}{c}\times\vec{B}_G\right) = q_{SI}\left(\vec{E}_{SI} + \vec{v}\times\vec{B}_{SI}\right)$

$\Rightarrow q_G\vec{E}_G = q_{SI}\vec{E}_{SI}$ consistent with the above $q_G\vec{E}_G = \left(\frac{q_{SI}}{\sqrt{4\pi\epsilon_0}}\right)\left(\sqrt{4\pi\epsilon_0}\,\vec{E}_{SI}\right)$

and $\Rightarrow \frac{1}{c}q_G\vec{B}_G = q_{SI}\vec{B}_{SI} \Rightarrow \frac{1}{c}\left(\frac{q_{SI}}{\sqrt{4\pi\epsilon_0}}\right)\left(\sqrt{\frac{4\pi}{\mu_0}}\,\vec{B}_{SI}\right) = q_{SI}\vec{B}_{SI} \Rightarrow \frac{1}{c}\frac{1}{\sqrt{\mu_0\epsilon_0}} = 1 \Rightarrow c^2 = \frac{1}{\epsilon_0\mu_0}$

We can convert any formula Gaussian $\longleftrightarrow$ SI. For example, we derived Larmor's formula in Gaussian, so

$$P = \frac{2}{3}\frac{q_G^2}{c}a^2 \quad , \quad P \text{ \& } a \text{ are mechanical} \rightarrow P = \frac{2}{3}\frac{q_{SI}^2}{4\pi\epsilon_0\left(\sqrt{\epsilon_0\mu_0}\right)^{-1}}a^2 = \frac{1}{6\pi}\sqrt{\frac{\mu_0}{\epsilon_0}}\,q_{SI}^2\,a^2$$

Or Maxwell's equations, eg

$$\vec{\nabla}\times\vec{E}_G + \frac{1}{c}\frac{\partial\vec{B}_G}{\partial t} = 0 \Rightarrow \vec{\nabla}\times\left(\sqrt{4\pi\epsilon_0}\,\vec{E}_{SI}\right) + \sqrt{\epsilon_0\mu_0}\,\frac{\partial}{\partial t}\sqrt{\frac{4\pi}{\mu_0}}\,\vec{B}_{SI} = 0$$

$$\Rightarrow \vec{\nabla}\times\vec{E}_{SI} + \frac{\partial}{\partial t}\vec{B}_{SI} = 0$$

Student can derive the rest in SI from Gaussian.

Exercise: Derive the Poynting vector in SI by translating from Gaussian.

<u>Numerics.</u> How to convert quantities from one system of units to the other? And what about μ_0 & ϵ_0? As physicist I like Gaussian, but the "volt"meter does not give statvolts, nor statamperes, etc.

1. Since the only new dimension is [I] in SI, we should be able to translate any amount of anything between systems once we know how to translate current → or, equivalently, charge.

Of course we also need $1m = 10^2 cm$ $1kg = 10^3 g$, $1s = 1s$.

Consider Coulomb's law. In CGS

$F = \frac{q^2}{1 \times r^2}$ means two charges of 1 esu (= 1 statcoulomb) 1 cm apart experience a force of 1 dyn

esu is <u>derived</u>, very much like dyne or Erg.

(esu and statcoulomb are used interchangeably; franklin (fr) is also sometimes used - less common)

The same two charges (1 esu) at the same distance (1cm) experience the same force (!) in other systems. In SI the force is $1 dyn = 10^{-5}$ Newton, distance $1cm = 10^{-2} m$ and charge is $1 esu = x\, C$ (C = coulomb). So we have

$$10^{-5} = \frac{1}{4\pi\epsilon_0} \frac{x^2}{(10^{-2})^2} \quad \text{or} \quad x = \sqrt{4\pi\epsilon_0 10^{-9}}$$

With $\epsilon_0 = 8.854 \times 10^{-12}$ F/m

$$x = \sqrt{4\pi \times 8.854 \times 10^{-21}} = 3.336 \times 10^{-10} = 1/(2.998 \times 10^9)$$

Usually written $1 esu = 10^{-9}/3$ C or $1C = 3 \times 10^9$ esu

but "3" is 2.998, suspiciously the same digits that appear in c = speed of light

2. ϵ_0 & μ_0 vs c

Because $[I]$ is a new dimension in SI, formulae like

$$F \propto \frac{q^2}{r^2} \quad \text{and} \quad B \propto \frac{I}{r} \quad \text{or} \quad F/\ell \propto \frac{I^2}{r}$$

require introduction of dimensionful constants. For example, we saw

$$F = \frac{1}{4\pi\epsilon_0}\frac{q^2}{r^2} \quad \text{with } [\epsilon_0] = [I]^2[T]^4[L]^{-3}[M]^{-1}$$

And, say, $B = \frac{\mu_0 I}{2\pi r}$. But also $[\mu_0][\epsilon_0] = [T]^2[L]^{-2} \Rightarrow$ there is no need for two different conversion factors. This is clear just from dimensional analysis, has nothing to do with speed of light. (Had we introduced ϵ_0 & μ_0 with different coefficients in Coulomb and Ampere's laws, we would still get $\mu_0\epsilon_0 \propto 1/c^2$, but with some proportionality constant $\neq 1$). Since only one is needed, one may define one and measure the other, or define one, measure c, and infer the other from $\mu_0\epsilon_0 = 1/c^2$.

You probably know $\mu_0 = 4\pi \times 10^{-7}$ H/m by definition. This fixes $\epsilon_0 = 1/\mu_0 c^2 = 8.854 \times 10^{-12}$ F/m. But there is a better way to write this

$$4\pi\epsilon_0 = \left(\frac{4\pi}{\mu_0}\right)\cdot\frac{1}{c^2} \quad \text{so} \quad 4\pi\epsilon_0 = 10^7/c^2 \quad (\text{in F/m if } c \text{ in m/s}).$$

As we saw above $1\,\text{esu} = x\,C$ with $x = \sqrt{4\pi\epsilon_0 \cdot 10^{-9}}$

$$\Rightarrow x = \sqrt{\frac{10^7}{c^2}10^{-9}} = 1/10c = 1/(2.998 \times 10^9) \quad \text{as before.}$$

Exercise: Derive the conversion factors for 5 other quantities in the conversion tables 1.1 & 1.2 of Garg or table 4 of Appendix of Jackson.