Garg Chap 10: Radiation From Localized Sources.

Private Notes

$\partial\cdot A = 0 \to \partial^2 A_\mu = \frac{4\pi}{c} j_\mu \to A_\mu = \frac{1}{c}\int d^3x' \, \frac{j_\mu(\vec r', t_r)}{|\vec r - \vec r'|}$ $\quad t_r = t - \frac{|\vec r-\vec r'|}{c}$

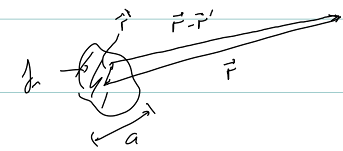
$\left(A_0 = \phi = \int d^3x' \, \frac{\rho(\vec r', t_r)}{|\vec r - \vec r'|}\right)$

FT's $\quad j_\mu = \int \frac{d\omega}{2\pi} e^{-i\omega t}\,\tilde j_\mu(\vec r,\omega) \quad \left((c\rho_\omega(\vec r), \vec j_\omega(\vec r)) \text{ in Garg}\right)$.

$\quad = \int \frac{d^3q\, d\omega}{(2\pi)^4} e^{i(\vec q\cdot\vec r - \omega t)}\, \tilde{j}(\vec q,\omega) \quad (j_{\vec q\omega} \text{ in Garg})$.

$\vec E = -\vec\nabla\phi - \frac{\partial \vec A}{c\,\partial t} \Rightarrow \tilde{\vec E} = -\vec\nabla\phi(\omega) + \frac{i\omega}{c}\vec A = -\vec\nabla\tilde\phi + ik\tilde A \quad k \equiv \frac{\omega}{c}$,

Far



use $|\vec r - \vec r'| = r - \hat r\cdot\vec r' + O(\frac{1}{r})$

Garg $\vec r =$ "$\vec R$" $\quad \to = R - \hat R\cdot\vec r' + O(\frac{1}{R})$ $\quad$ WHY DOES HE DO THIS? yikes.

Far zone approx: $a \ll R$, $\lambda \ll R$.

$\lambda = \frac{c}{\omega}$ and currents ~ go like $v \sim a\omega$ so NR approx is $\frac{a\omega}{c} \ll 1 \Leftrightarrow a \ll \lambda$.

$$\tilde A_\mu(\omega,\vec r) = \int dt\, e^{i\omega t} A_\mu(\vec r,t) = \frac{1}{c}\int d^3x' \frac{1}{|\vec r-\vec r'|}\underbrace{\int dt\, e^{i\omega t} j_\mu(\vec r', t_r)\, e^{\frac{-i\omega|\vec r-\vec r'|}{c}}}_{\tilde j_\mu(\vec r',\omega)} \quad e^{ik|\vec r-\vec r'|}$$

$$= \frac{1}{c}\int d^3x' \frac{e^{ik|\vec r-\vec r'|}}{|\vec r-\vec r'|}\tilde j_\mu(\vec r',\omega)$$

$$\approx \frac{1}{c}\int d^3x' \frac{e^{ik(R-\hat R\cdot\vec r')}}{R}\tilde j_\mu(\vec r',\omega)$$

$$= \frac{1}{cR}e^{ikR}\,\tilde j_\mu(\vec k,\omega) \quad \text{with } \vec k = k\hat R$$

(Note: LHS is $A_\mu(\vec r,\omega)$ (ie FT in time only)

while RHS is $\tilde j_\mu(\vec k,\omega)$ (ie FT in time & space)

Fields: $\vec{E}(\vec{r},\omega) = -\vec{\nabla}\phi(\omega) + ik\vec{A}(\omega)$

$$= -\vec{\nabla}\left(\frac{e^{ikR}}{R}\tilde{\rho}(k\hat{R},\omega)\right) + ik\frac{e^{ikR}}{R}\frac{1}{c}\tilde{J}(\vec{k},\omega)$$

Since $\vec{\nabla}\frac{1}{R} \sim \frac{1}{R^2}$ and $\frac{1}{R}\vec{\nabla}\hat{R} \sim \frac{1}{R^2}$ keep only $\frac{1}{R}\vec{\nabla}e^{ikR} = \frac{e^{ikR}}{R}ik\hat{R}$

$$= ik\frac{e^{ikR}}{R}\left[-\hat{R}\rho(\vec{k},\omega) + \frac{1}{c}\vec{J}(\vec{k},\omega)\right]$$

$$\vec{B}(\vec{r},\omega) = \vec{\nabla}\times\vec{A}(\vec{r},\omega) = \frac{ik}{c}\frac{e^{ikR}}{R}\hat{R}\times\vec{J}(\vec{k},\omega)$$

$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} = 0 \Rightarrow \quad -i\omega\rho(\vec{q},\omega) + i\vec{q}\cdot\vec{J}(\vec{q},\omega) = 0$

Here is $\omega\rho(\vec{k},\omega) = k\hat{R}\cdot\vec{J}(\vec{k},\omega) = \omega\frac{1}{c}\hat{R}\cdot\vec{J}(\vec{k},\omega)$

$$\Rightarrow \vec{E}(\vec{R},\omega) = ik\frac{e^{ikR}}{cR}\underbrace{\left[\vec{J}(\vec{k},\omega) - \hat{R}\hat{R}\cdot\vec{J}(\vec{k},\omega)\right]}_{\equiv\vec{J}^{\perp}} = ik\frac{e^{ikR}}{cR}\tilde{J}^{\perp}(\vec{k},\omega)$$

$\vec{J} = \vec{J}^{\perp} + \vec{J}^{\parallel}$

$\vec{B} = ik\frac{e^{ikR}}{cR}\hat{R}\times\vec{J}^{\perp}(k,\omega) = \hat{R}\times\vec{E}(\vec{r},\omega)$ (only used $R \gg r'$ so far, no assumption on ω (or λ) yet $\Rightarrow$ ok for relativistic sources $\Rightarrow$ apply to point particle recover old results).

Time domain: $\vec{E}(\vec{R},t) = \int\frac{d\omega}{2\pi}e^{-i\omega t}\vec{E}(\vec{R},\omega)$

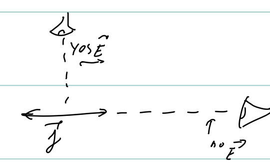
$$= \int\frac{d\omega}{2\pi}e^{-i\omega t}\,ik\frac{e^{ikR}}{cR}\int d^3r'\,e^{-ik\hat{R}\cdot\vec{r}'}\vec{J}^{\perp}(\vec{r}',\omega)$$

$$= \frac{1}{cR}\int d^3r'\int\frac{d\omega}{2\pi}ik\,e^{-i\omega(t-\frac{R}{c}+\hat{R}\cdot\vec{r}'/c)}\vec{J}^{\perp}(\vec{r}',\omega)$$

The factor of $ik = \frac{\omega}{c}$ is $-\frac{\partial}{\partial t} \Rightarrow \quad = -\frac{1}{c^2R}\int d^3r'\frac{\partial}{\partial t}\vec{J}^{\perp}\left(\vec{r}', t-\frac{R}{c}+\frac{\hat{R}\cdot\vec{r}'}{c}\right)$

$$= -\frac{1}{c^2R}\int d^3r'\frac{\partial}{\partial t}\vec{J}^{\perp}(\vec{r}', t_r)$$

$\Rightarrow$ $\vec{E}$ is along $\vec{J}^{\perp}$, no $\vec{E}$ if $\vec{J}$ points along line of sight



$\Rightarrow$ Steady currents do not radiate ($J = qv\delta^{(3)} \rightarrow \dot{J} = q\dot{v}\delta$ $\hookrightarrow$ acceleration)

Power $\vec{S} = \frac{c}{4\pi} \vec{E}\times\vec{B} = \frac{c}{4\pi}\hat{R} E^2$

$$\frac{dP}{d\Omega} = R^2 \hat{R}\cdot\vec{S} = \frac{1}{4\pi c^3}\left[\frac{\partial}{\partial t}\int d^3r' \, \vec{J}(\vec{r}', t_r)\right]^2 =$$

Spectrum: We already have $\vec{E}(\hat{R},\omega) = ik\frac{e^{ikR}}{cR}\tilde{\vec{J}}^{\perp}(\vec{k},\omega)$

Now $\frac{dI}{d\Omega} = \int_{-\infty}^{\infty} dt \frac{dP}{d\Omega} = \frac{c}{4\pi}R^2\int_{-\infty}^{\infty} dt\, E^2(t) \underset{\text{Parseval's}}{=} \frac{c}{4\pi}R^2\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}|E(\omega)|^2 = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{c}{4\pi}\left(\frac{k^2}{c^2}\right)|\vec{J}^{\perp}(\vec{k},\omega)|^2$

$= 2\int_0^{\infty} i\,d\omega$

$$\Rightarrow \boxed{\frac{dI}{d\Omega\, d\omega} = \frac{\omega^2}{4\pi^2 c^3}|\tilde{\vec{J}}^{\perp}(\vec{k},\omega)|^2}$$

(I = intensity, is "$\mathcal{E}$" in Garg).

(This is for the case "Burst" in Garg, that must mean "localized in time").

Periodic: Fundamental $\omega_0 \rightarrow \frac{dP}{d\Omega} = \sum_n c_n e^{in\omega_0 t}$ (Note: ***different*** treatment than in Garg)

$$\frac{1}{T}\int_0^T dt\, e^{-im\omega_0 t}\frac{dP}{d\Omega} = c_m \quad \left(T = \frac{2\pi}{\omega_0}\right).$$

Now, instead of $\psi(\vec{r},t) = \int\frac{d\omega}{2\pi}e^{-i\omega t}\psi(\vec{r},\omega) \xrightarrow{\text{have}} \sum_n e^{-in\omega_0 t}\psi_n(\vec{r})$

So formulas are translated ie $\vec{E}_n(\vec{R}) = i k_n \frac{e^{ik_n R}}{cR}\vec{J}^{\perp}_n(k_n\hat{R})$ $\quad k_n = \frac{n\omega_0}{c}$

And Parseval is now

$$\langle E^2\rangle = \frac{1}{T}\int_0^T dt\, E^2(t) = \sum_{n,m}\int_0^T\frac{dt}{T}E_n E_m e^{-i(n+m)\omega_0 t} = \sum_n E_n E_{-n} = E_0^2 \overset{\to 0}{} + \sum_{n=1}^{\infty} 2E_n E_n^*$$

$$\Rightarrow \left\langle\frac{dP}{d\Omega}\right\rangle = \frac{cR^2}{4\pi}2\sum_{n=1}^{\infty}|\vec{E}_n|^2 = \frac{c}{2\pi}\sum_n\left|\frac{ik_n}{c}e^{ik_nR}\vec{J}^{\perp}_n(k_n\hat{R})\right|^2$$

$$\text{or} \quad \boxed{\frac{dP_n}{d\Omega} = \frac{n^2\omega_0^2}{2\pi c^3}\left|\vec{J}^{\perp}_n\left(\frac{n\omega_0}{c}\hat{R}\right)\right|^2}$$

Connect with Garg:

A Fourier exp can be written as a FT:

$$f(t) = \sum_n f_n e^{-in\omega_0 t} = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega t}\sum_n 2\pi\,\delta(\omega - n\omega_0) f_n$$

$$\Rightarrow \left\langle\frac{dP}{d\Omega}\right\rangle = \int d\omega\, e^{-i\omega t}\sum_n\frac{\omega^2}{2\pi c^3}\left|\tilde{\vec{J}}^{\perp}_n\left(\frac{\omega}{c}\hat{R}\right)\right|^2\delta(\omega - \omega_0 n)$$

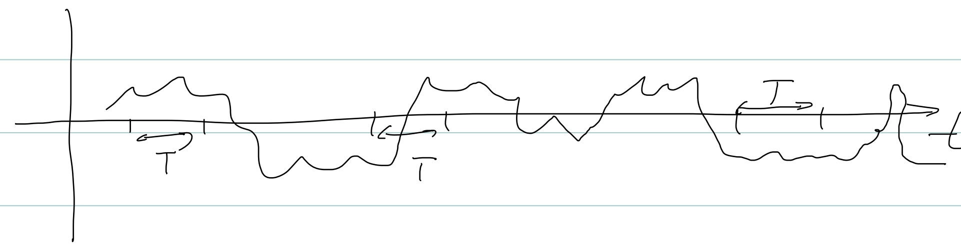
$$\Rightarrow \frac{dP}{d\Omega\, d\omega} = \frac{\omega^2}{2\pi c^3}\sum_n\left|\vec{J}^{\perp}_n\left(\frac{\omega}{c}\hat{R}\right)\right|^2\delta(\omega - n\omega_0)$$

Garg has a $\cos\omega_0 t = \frac{1}{2}e^{i\omega_0 t} + \text{c.c.} \Rightarrow$ an extra $\left(\frac{1}{2}\right)^2 \rightarrow \frac{\omega^2}{8\pi c^3}\cdots$

Stochastic

Underlying process (unspecified) gives rise to random movement of charges in the confined region. $\vec{J}$ becomes a random variable.

If $f(t)$ is random (assume $\langle f(t) \rangle = 0$) we can take one instance of the function $f(t)$ and look at widely separated intervals (t_1, t_1+T), (t_2, t_2+T), .. with $t_1 \ll t_2 \ll$ .. with T large, but $T \ll t_{n+1} - t_n$ and each segment will be a sample of $f(t)$ over an interval $(0,T)$ taken from a random distribution.



(This plot has some $\langle f \rangle \neq 0$, but if T gets larger then eventually $\langle f \rangle = 0$).

Then we can use a single function to compute correlations:

$$\langle f(t) f(t+\tau) \rangle = \lim_{T\to\infty} \frac{1}{T} \int_0^T dt\, f(t) f(t+\tau)$$

Note: I am not sure how to prove this. I'd like

$\langle f(t) f(t+\tau) \rangle = N \int [df]\, \mu(f)\, f(t) f(t+\tau)$ for some measure $\mu(f)$, say, $\mu = e^{-\int_{-\infty}^{\infty} \left(\frac{df}{dt}\right)^2 dt}$. End note.

We assume J^1 is stochastic. To obtain the spectrum, start from the above formula

$$\frac{dI}{d\Omega d\omega} = \frac{\omega^2}{4\pi^2 c^3} \langle |\tilde{J}^1(\vec{k}, \omega)|^2 \rangle$$

where we have taken the expectation value of the stochastic variable.

Now we undo the time FT:

$$\langle |\vec{j}^\perp(\vec{k},\omega)|^2 \rangle = \left\langle \int_{-\infty}^{\infty} dt_1\, e^{i\omega t_1}\, \vec{j}^\perp(\vec{k},t_1) \cdot \left(\int_{-\infty}^{\infty} dt_2\, e^{i\omega t_2}\, \vec{j}^\perp(\vec{k},t_2) \right)^* \right\rangle$$

change variables $t_1 = t + \frac{1}{2}\tau \qquad t_2 = t - \frac{1}{2}\tau$

$$dt_1\, dt_2 = \left(\frac{\partial(t_1,t_2)}{\partial(t,\tau)} \right) dt\, d\tau = \begin{vmatrix} 1 & 1 \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} dt\, d\tau = dt\, d\tau$$

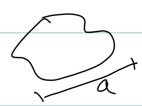
$$\cdots = \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} dt\; e^{i\omega\tau}\, \underbrace{\langle \vec{j}(\vec{k}, t+\tfrac{1}{2}\tau) \cdot \vec{j}^{*\perp}(\vec{k}, t-\tfrac{1}{2}\tau) \rangle}_{\equiv G^\perp_{jj}(\tau)}$$

So
$$\int_{-\infty}^{\infty} dt\, \frac{dP}{d\Omega\, d\omega} = \int_{-\infty}^{\infty} dt \left[\frac{\omega^2}{4\pi^2 c^3} \int_{-\infty}^{\infty} d\tau\, e^{i\omega\tau}\, G^\perp_{jj}(\tau) \right]$$

and we take a leap of faith equating the integrands and interpreting the LHS as an instantaneous $\frac{dP}{d\Omega\, d\omega}(t)$:

$$\frac{dP}{d\Omega\, d\omega} = \frac{\omega^2}{4\pi^2 c^3} \int_{-\infty}^{\infty} d\tau\, e^{i\omega\tau}\, G^\perp_{jj}(\tau)$$

The long wavelength = non-relativistic = electric dipole approximation



if typical velocity is v then since motion is within size R we will have (fundamental) frequency $\omega R = v$ $\Rightarrow$ the emitted spectrum has

$$\lambda = \frac{2\pi c}{\omega} \sim \frac{ac}{v} \Rightarrow a/\lambda \sim \beta. \text{ So } \beta \ll 1 \iff \lambda \gg a$$

Now $\tilde{\vec{j}}(\omega, \vec{k}) = \int d^3\vec{r}\, e^{i\vec{k}\cdot\vec{r}}\, \tilde{\vec{j}}(\vec{r}, \omega)$

The multipole expansion is $e^{i\vec{k}\cdot\vec{r}} = 1 + \vec{k}\cdot\vec{r} + \cdots$, ie small $|\vec{k}\cdot\vec{r}| \sim \frac{R}{\lambda}$

Lowest order: $\tilde{\vec{j}}^{(0)}(\omega) = \int d^3r\, \tilde{\vec{j}}(\vec{r}, \omega)$ ("0" means $(\vec{k}\cdot\vec{r})^0$).

Interpretation: with $\vec{j}(\vec{r}, t) = \sum_a q_a \vec{v}_a\, \delta^{(3)}(\vec{r} - \vec{r}_a(t))$

$\Rightarrow$ $\tilde{\vec{j}}^{(0)}(\omega) = \int dt\, d^3r\, e^{i\omega t} \sum_a q_a \dot{\vec{r}}_a\, \delta^3(\vec{r} - \vec{r}_a(t)) = \int dt\, e^{i\omega t} \sum_a q_a \dot{\vec{r}}_a = -i\omega \int dt\, e^{i\omega t} \sum_a q_a \vec{r}_a = -i\omega \int dt\, e^{i\omega t}\, \vec{d}(t)$

or $\boxed{\tilde{\vec{j}}^{(0)}(\omega) = -i\omega\, \vec{d}(\omega)}$

Then $\vec{E}(\vec{R}, \omega) = \frac{ik e^{ikR}}{cR} \tilde{\vec{j}}(\vec{k}, \omega) \simeq k^2 \vec{d}^{\perp}(\omega) \frac{e^{ikR}}{R}$

(For discrete particles this is as before, from Lienard Wiechert in NR limit:

$$\vec{E}(\vec{r}, t) = \int d\omega\, e^{-i\omega t} \left[\frac{\omega^2}{c^2} \frac{1}{R} e^{i\frac{\omega}{c}R}\, \vec{d}^{\perp}(\omega) \right] = -\frac{1}{R} \frac{1}{c^2} \frac{d^2}{dt^2} \int d\omega\, e^{-i\omega(t - \frac{R}{c})}\, \vec{d}^{\perp}(\omega) = -\frac{1}{Rc^2} \frac{d^2}{dt^2} \vec{d}^{\perp}(t_{ret})$$

Now $\vec{d}(t) = q\vec{r}(t) \rightarrow \vec{d}^{\perp} = -\hat{R} \times (\hat{R} \times \vec{d})$ and the correspondance follows).

Dipole Spectrum: $$\frac{d^2 I}{d\Omega\, d\omega} = \frac{\omega^2}{4\pi^2 c^3} |\vec{j}^{\perp}|^2 = \frac{\omega^4}{4\pi^2 c^3} |\vec{d}^{\perp}|^2 = \frac{\omega^4}{4\pi^2 c^3} |\hat{R} \times (\hat{R} \times \vec{d})|^2$$

and $\frac{dI}{d\omega} = \frac{\omega^4}{4\pi^2 c^3} \int d\Omega\, |\vec{d} - \hat{R}\, \hat{R}\cdot\vec{d}|^2 = \frac{\omega^4}{4\pi^2 c^3} \vec{d}_i \vec{d}_j \int d\Omega \left(\delta_{ij} - \hat{R}_i \hat{R}_j \right) = \frac{\omega^4}{4\pi^2 c^3} d_i d_j\, 4\pi \left(\delta_{ij} - \frac{1}{3}\delta_{ij} \right)$

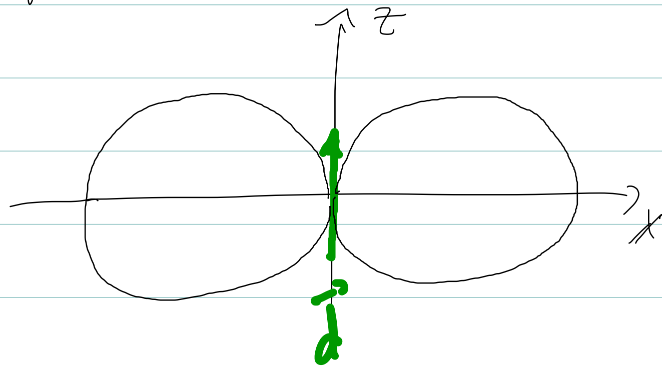
$$\Rightarrow \frac{dI}{d\omega} = \frac{2}{3\pi} \frac{\omega^4 d^2(\omega)}{c^3}$$

What kind of pattern? It depends on $\vec{d}$. Example: take $\vec{d} = \hat{z} d$,

$\Rightarrow |\vec{d} - \hat{R}(\hat{R}\cdot\vec{d})|^2 = d^2 - (\hat{R}\cdot\vec{d})^2 = d^2(1 - \cos^2\theta) = \sin^2\theta\, d^2$

so $\quad \dfrac{d^2 I}{d\Omega\, d\omega} = \dfrac{\omega^2}{4\pi^2 c^3}\, d^2 \sin^2\theta$

The radiation pattern (whereby $\dfrac{dI}{d\omega\, d\Omega}$ is represented as distance from origin) is



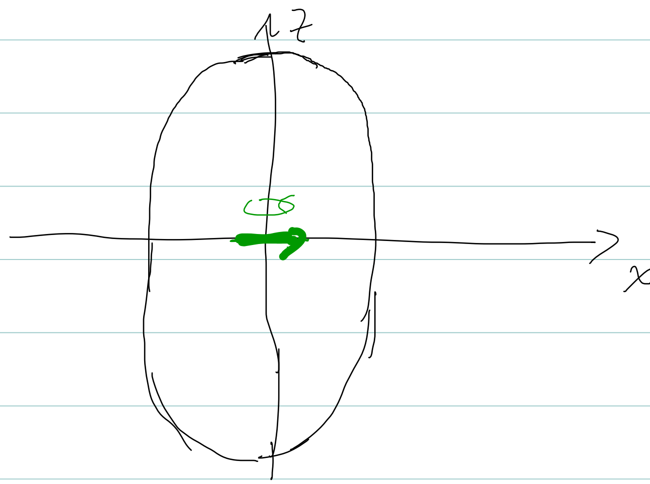
$\vec{d} \propto \hat{z}$

$l = 1$

$m = 0$

For $\;l=1,\ m=1 \quad \vec{d}(\omega) = d_\omega \dfrac{(\hat{x}+i\hat{y})}{\sqrt{2}} \qquad \hat{n}\cdot\vec{d} = \dfrac{d}{\sqrt{2}} \sin\theta\,(\cos\varphi + i\sin\varphi) = \dfrac{d}{\sqrt{2}} \sin\theta\, e^{i\varphi}$

$$|\vec{d}|^2 - |\hat{n}\cdot\vec{d}|^2 = |d_\omega|^2\left(1 - \tfrac{1}{2}\sin^2\theta\right) = \tfrac{1}{2}|d_\omega|^2\left(1 + \cos^2\theta\right)$$

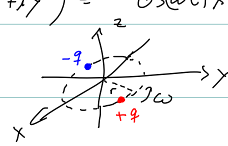


$\vec{d} \propto \hat{x} + i\hat{y}$

$l=1,\ m=1$

Note that $\mathrm{Re}\left(e^{-i\omega t}\,(\hat{x}+i\hat{y})\right) = \cos(\omega t)\,\hat{x} + \sin(\omega t)\,\hat{y} \;\to\;$ circular motion in xy plane.

Dipole:



Next order: $\tilde{\vec{j}}^{(1)}(\vec{k},\omega) = \int (-i\vec{k}\cdot\vec{r})\,\vec{j}(\vec{r},\omega)\,d^3r$

$$\tilde{j}^{(1)}_j(\vec{k},\omega) = -i\frac{\omega}{c}\hat{R}_i \underbrace{\int r_i j_j(\vec{r},\omega)\,d^3r}$$

what is this 2-index object?

Answ: use $\vec{j}(\vec{r},t) = \sum_a q_a \vec{v}_a \delta^3(\vec{r}-\vec{r}_a(t))$ again (postpone FT to ω):

$$\int r_i \sum_a q_a v_{aj} \delta^3(\vec{r}-\vec{r}_a) = \sum_a q_a r_{ai} v_{aj}$$

Now $\frac{d}{dt} r_i r_j = v_i r_j + r_i v_j$

$$\Rightarrow r_i v_j = \frac{1}{2}(r_i v_j + r_j v_i) + \frac{1}{2}(r_i v_j - r_j v_i)$$
$$= \frac{1}{2}\frac{d}{dt}(r_i r_j) + \frac{1}{2}\epsilon_{ijk}\epsilon_{kmn} r_m v_n$$

The 1st term gives

$$\frac{d}{dt}\sum_a q_a r_{ai} r_{aj} = \frac{d}{dt}\frac{1}{3}\underbrace{\sum_a q_a (3r_{ai}r_{aj} - r_a^2\delta_{ij})}_{D_{ij}} + \frac{1}{3}\delta_{ij}\frac{1}{3}\frac{d}{dt}\sum_a q_a r_a^2$$

The 2nd gives $\sum_a q_a \epsilon_{kmn} r_{am} v_{an} = \sum_a q_a (\vec{r}_a \times \vec{v}_a) = 2c\vec{m}$ (magnetic dipole moment).

So, with $\vec{D} = \hat{R}_i D_{ij}$ and taking $-i\omega \to \frac{d}{dt}$

$$\tilde{\vec{j}}^{(1)}(\vec{k},\omega) = \int dt\, e^{i\omega t}\left[\dot{\vec{m}}\times\hat{R} + \frac{1}{6c}\ddot{\vec{D}} + \underbrace{\frac{1}{6c}\left(\frac{d^2}{dt^2}\sum_a q_a r_a^2\right)\hat{R}}_{\text{only } j^{\parallel}}\right]$$

When taking $j^{\perp}$, the last term never contributes: drop it.

$$\Rightarrow \vec{E}^{(1)}(\vec{R},\omega) = \frac{ike^{ikR}}{cR} j^{\perp}(\vec{k},\omega) = \frac{i\omega e^{ikR}}{c^2 R}\left[-\frac{\omega^2}{6c}\vec{D}(\omega) - i\omega\,\vec{m}(\omega)\times\hat{R}\right]$$

or in time domain $\omega \to i\frac{\partial}{\partial t}$ an e^{ikR} is $t \to t_{ret}$. Putting (0)+(1) together:

$$\vec{E}(\hat{R},t) = \frac{1}{c^2 R}\left[\hat{R}\times(\hat{R}\times\ddot{\vec{d}}) + \hat{R}\times\ddot{\vec{m}} + \frac{1}{6c}\hat{R}\times(\hat{R}\times\dddot{\vec{D}})\right]_{ret}$$

$$\vec{B}(\hat{R},t) = \hat{R}\times\vec{E} = \frac{1}{c^2 R}\left[-\hat{R}\times\ddot{\vec{d}} + \hat{R}\times(\hat{R}\times\ddot{\vec{m}}) - \frac{1}{6c}\hat{R}\times\dddot{\vec{D}}\right]_{ret}$$

Terminology: radiation from $\vec{d}, \vec{D}, \ldots$ is E1, E2, ...

from $\vec{m}, \ldots$ is M1, M2, ..

$\frac{dP}{d\Omega}$: patterns have interference between those terms

$P = \int \frac{dP}{d\Omega} d\Omega$ is sum of individual powers?

To see this we need averages of powers of $\hat{R}$: let $\langle \cdot \rangle = \frac{1}{4\pi} \int d\Omega \cdot$

$$\langle 1 \rangle = 1 \quad , \quad \langle \hat{R} \rangle = 0 = \langle \hat{R}_{i_1} \hat{R}_{i_2} \cdots \hat{R}_{i_{2n+1}} \rangle$$

$$\langle \hat{R}_i \hat{R}_j \rangle = \frac{1}{3} \delta_{ij}$$

(Proof: $\langle \hat{R}_i \hat{R}_j \rangle = c\, \delta_{ij}$ by rotational symmetry $\Rightarrow \langle 1 \rangle = 3c \Rightarrow c = 1/3$ ✓)

$$\langle \hat{R}_i \hat{R}_j \hat{R}_k \hat{R}_l \rangle = \frac{1}{15} \left(\delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right) \qquad \left(\langle 1 \rangle = \tfrac{1}{15}(9+3+3) \right)$$

So for $\frac{dP}{d\omega}$ (or P) need $\langle E^2 \rangle$; keep in mind $\vec{D}_i = \hat{R}_j D_{ij}$. So the number of $\hat{R}_i$'s in $\vec{d}$ has $\vec{m}$ or $\vec{D}$ cross terms is odd. Moreover, the $\vec{m}$ crossterm with $\vec{D}$ has four $\hat{R}_i$'s $\rightarrow$ product of δ_{ij}'s and ϵ_{ijk}'s that need to contract with $m_k D_{ij}$: only possibility is $\epsilon^{ijk} m_k D_{ij} = 0$ $(D_{ij} = D_{ji})$

$$\Rightarrow \frac{dP}{d\omega} \sim d^2 , \; m^2 \text{ and } D_{ij}^2 \text{ terms}$$

In fact: $$P = \frac{2}{3c^2} \ddot{\vec{d}}^2 + \frac{2}{3c^2} \ddot{\vec{m}}^2 + \frac{1}{180 c^5} \dddot{D}_{ij} \dddot{D}_{ij}$$

Do $\vec{d}$ first: $$\frac{dP}{d\Omega} = R^2 \frac{c}{4\pi} \vec{E}^2 = \frac{1}{4\pi c^3} \left| \hat{R} \times \ddot{\vec{d}} \right|^2 = \frac{1}{4\pi c^3} \left(|\ddot{\vec{d}}|^2 - |\hat{R} \cdot \ddot{\vec{d}}|^2 \right)$$

$$\Rightarrow P = \frac{1}{c^3} \left(|\ddot{\vec{d}}|^2 - \langle \hat{R}_i \hat{R}_j \rangle \ddot{d}_i^* \ddot{d}_j \right) = \frac{1}{c^3} \left(|\ddot{\vec{d}}|^2 - \frac{1}{3} \delta_{ij} \ddot{d}_i^* \ddot{d}_j \right) = \frac{2}{3c^3} \ddot{\vec{d}}$$

Clearly $\ddot{\vec{m}}$ is the same.

Exercise: Do $\dddot{D}^2$ term

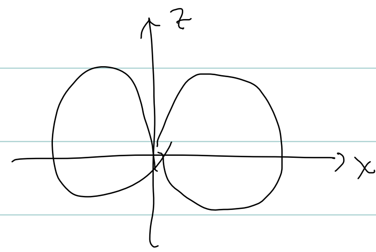
See text for pure quadrupole patterns.

$$\frac{dP}{d\Omega} = \frac{c}{4\pi}\left|\frac{1}{c^2}\cdot\frac{1}{6c}\hat{n}\times(\hat{R}\times\dddot{\vec{D}})\right|^2 = \frac{1}{144\pi c^5}\left(|\dddot{\vec{D}}|^2 - |\hat{n}\cdot\dddot{\vec{D}}|^2\right)$$

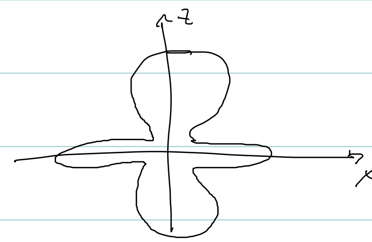
$$= \frac{1}{144\pi c^5}\left(\hat{R}_i\hat{R}_j\delta_{mn}\dddot{D}_{im}\dddot{D}_{jn} - \hat{R}_i\hat{R}_j\hat{R}_m\hat{R}_n\dddot{D}_{im}\dddot{D}_{jn}\right)$$

Then pick particular D_{ij} (traceless, symmetric). Text picks them to be q_{2m}'s.

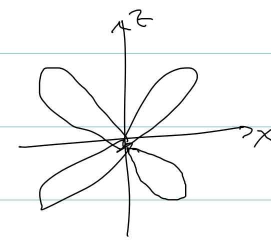
Pictures:



$l=2$ $m=2$

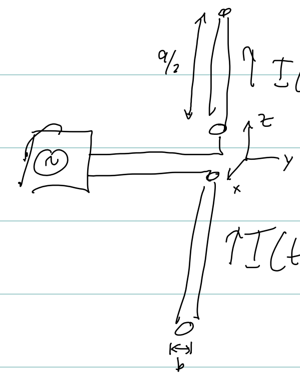


$l=2$ $m=1$



$l=2$ $m=0$

Antennas

Centerfed linear antenna: 

(antenna on z-axis, origin at middle point).

Crude model: $I(t) = I_0 \cos\omega_0 t$ independent of z! (unrealistic).

$$\text{Recall } -i\omega\vec{d}(\omega) = \vec{J}^{(0)}(\omega) = \int d^3r\, \vec{J}(\vec{r},\omega)$$

and $$\vec{J}(\vec{r},\omega) = \int dt\, e^{i\omega t}\, J(\vec{r},t)$$

so $$\vec{J}^{(0)}(\omega) = \int dt\, e^{i\omega t}\, \hat{z}\, I(t) a = \hat{z}\tfrac{1}{2} a I_0 \int dt\; e^{i\omega t}\left(e^{i\omega_0 t} - e^{-i\omega_0 t}\right)$$
$$= \pi a I_0 \hat{z}\left(\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right)$$

$$\dot{\vec{d}} = \int dt\; e^{i\omega t}\; \vec{J}^{(0)}(\omega) = a I_0 \hat{z} \cos(\omega_0 t)$$

$$\Rightarrow \quad |\vec{E}(\vec{R},t)| = \left|\frac{1}{c^2 R}\, \ddot{\vec{d}}_{ret}\right| = \frac{a I_0 \omega_0}{c^2 R} \sin(\omega_0 t) \sin\theta \qquad \left(\sin\theta = \sqrt{1-(\hat{z}\cdot\hat{R})^2} = \sqrt{1-\cos^2\theta}\right).$$

So the power follows:

$$\frac{dP}{d\Omega} = R^2 \frac{c}{4\pi}|\vec{E}|^2 = \frac{a^2 I_0^2 \omega_0^2}{4\pi c^3} \sin^2(\omega_0 t_{ret}) \sin\theta$$

or in terms of wave-length $\lambda = \frac{2\pi c}{\omega_0}$, $\frac{dP}{d\Omega} = \frac{\pi}{c}\left(\frac{a}{\lambda}\right)^2 I_0^2 \sin^2(\omega_0 t_{ret}) \sin^2\theta$

and averaging over one cycle, $\sin^2(\omega_0 t) \to \frac{1}{2}$ $\quad \overline{\frac{dP}{d\Omega}} = \frac{\pi}{c}\left(\frac{a}{\lambda}\right)^2 I_0^2 \sin^2\theta$

We could have guessed the $\left(\frac{a}{\lambda}\right)^2$ (leading multipole, $d \sim \frac{a}{\lambda}$)
the $\sin^2\theta$ ($\vec{d}$ along $\hat{z}$) and the I_0^2 ($\dot{d} \sim I_0$). The $\frac{1}{c}$ is from dimensional analysis. Left with "π", for which we needed a calculation.

A more realistic model needs $I(|z|=a/2)=0$. For a very thin antenna we take $\vec{J} \propto \delta(x)\delta(y)$. So we propose

$$\vec{J} = I_m \sin\left(\tfrac{1}{2}k_0 a - k_0|z|\right)\cos(\omega_0 t)\,\delta(y)\delta(x)\,\hat{z}$$

I_m is the max current: at center, the fed current is $I_0 = I_m \sin(\tfrac{1}{2}k_0 a)$

Calculate:
$$\vec{J}^{(0)}(\omega) = \int dt\, e^{i\omega t}\int d^3r\, \vec{J}(\vec{r},t)$$
$$= \hat{z} I_m \int dt\, e^{i\omega t}\cos(\omega_0 t)\int_{-a/2}^{a/2} dz\, \sin\left(\tfrac{1}{2}k_0 a - k_0|z|\right)$$
$$= \hat{z} I_m \int dt\, e^{i\omega t}\cos(\omega_0 t)\,\frac{2}{k_0}\left(1-\cos\left(\tfrac{1}{2}k_0 a\right)\right)$$

$$\dot{\vec{d}} = \int \frac{d\omega}{2\pi} e^{-i\omega t}\vec{J}^{(0)}(\omega) = \hat{z} I_m \int \frac{d\omega}{2\pi} e^{-i\omega t}\int dt'\, e^{i\omega t'}\cos(\omega_0 t')\,\frac{2}{k_0}\left[1-\cos\left(\tfrac{1}{2}k_0 a\right)\right]$$
$$= \hat{z} I_m \cos(\omega_0 t)\,\frac{2}{k_0}\left[1-\cos\left(\tfrac{1}{2}k_0 a\right)\right]$$

$$\ddot{\vec{d}}_{ret} = -\hat{z} I_m \frac{2\omega_0}{k_0}\sin(\omega_0 t_{ret})\left[1-\cos\left(\tfrac{1}{2}k_0 a\right)\right]$$

Then $$|\vec{E}| = \left|\frac{1}{c^2 R}\ddot{\vec{d}}_{ret}\right| = \frac{2 I_m}{cR}|\sin(\omega_0 t_{ret})|\left[1-\cos\left(\tfrac{1}{2}k_0 a\right)\right]\sin\theta$$

and $$\overline{\frac{dP}{d\Omega}} = R^2\frac{c}{4\pi}\overline{|E|^2} = \frac{1}{2\pi c} I_m^2\left(1-\cos\left(\tfrac{1}{2}k_0 a\right)\right)^2\sin^2\theta$$

For small $k_0 a$ (dipole approximation, which we are using) $1-\cos(\tfrac{1}{2}k_0 a) \simeq \tfrac{1}{2}(\tfrac{1}{2}k_0 a)^2$

$$\Rightarrow \overline{\frac{dP}{d\Omega}} = \frac{1}{128\pi c} I_m^2 (k_0 a)^4 \sin^2\theta$$

This problem can be solved without use of multipole expansion: recall

Recall for monochromatic source

$$\frac{dP_n}{d\Omega} = \frac{n^2\omega_0^2}{2\pi c^3}\left|\vec{J}_n^{\perp}\left(n\frac{\omega_0}{c}\hat{R}\right)\right|^2$$

where we only need $n=1$, and $\vec{J}_n^{\perp}(\vec{k}) = \frac{1}{T}\int_0^T dt\, e^{in\omega_0 t}\,\vec{J}(\vec{k},t) = \frac{1}{T}\int_0^T dt\, e^{in\omega_0 t}\int d^3r\, e^{-i\vec{k}\cdot\vec{r}}\,\vec{J}(\vec{r},t)$

So $$\vec{J}_1^{\perp}(\vec{k}) = \frac{1}{T}\int_0^T dt\, e^{-i\omega_0 t}\int d^3r\, e^{-i\vec{k}\cdot\vec{r}}\,\hat{z} I_m \sin\left(\tfrac{1}{2}k_0 a - k_0|z|\right)\delta(x)\delta(y)\cos(\omega_0 t)$$
$$= \frac{1}{T}\int_0^T dt\, e^{i\omega_0 t}\cos(\omega_0 t)\;\hat{z} I_m \int_{-a/2}^{a/2} dz\, e^{-ik_z z}\sin\left(\tfrac{1}{2}k_0 a - k_0|z|\right)$$

The time integral gives $\frac{1}{2}$ (and there is a $\vec{J}_{-1}$ which also has $\frac{1}{2}$, but we have accounted for it, recall $2\sum_{n>0} E_n E_n^* \cdots$).

Need $\int_{-a/2}^{a/2} dz\, e^{-ik_z z} \sin(\frac{1}{2}k_0 a - k_0|z|)$

$$= \int_0^{a/2} dz\, e^{-ik_z z} \sin(\tfrac{1}{2}k_0 a - k_0 z) + \int_{-a/2}^{0} dz\, e^{-ik_z z} \sin(\tfrac{1}{2}k_0 a + k_0 z)$$

$$= \mathrm{Im}\left[\int_0^{a/2} dz\, e^{-ik_z z} e^{i(\frac{1}{2}k_0 a - k_0 z)} + \int_{-a/2}^{0} dz\, e^{-ik_z z} e^{i(\frac{1}{2}k_0 a + k_0 z)}\right]$$

$$= \mathrm{Im}\left[e^{i\frac{1}{2}k_0 a} \frac{i}{k_0+k_z}\left(e^{-i(k_z+k_0)\frac{a}{2}} - 1\right) + e^{ik_0\frac{a}{2}} \frac{i}{k_z-k_0}\left(1 - e^{-i(k_z-k_0)\frac{a}{2}}\right)\right]$$

$$= -\mathrm{Re}\left[\frac{1}{k_z+k_0}\left(e^{-ik_z\frac{a}{2}} - e^{ik_0 a/2}\right) + \frac{1}{k_z-k_0}\left(e^{ik_0\frac{a}{2}} - e^{-ik_z\frac{a}{2}}\right)\right]$$

$$= -\frac{1}{k_z+k_0}\left(\cos(k_z\tfrac{a}{2}) - \cos(k_0\tfrac{a}{2})\right) - \frac{1}{k_z-k_0}\left(\cos(\tfrac{k_0 a}{2}) - \cos(\tfrac{k_z a}{2})\right)$$

$$= \left(\cos(k_z\tfrac{a}{2}) - \cos(k_0\tfrac{a}{2})\right)\frac{-2k_0}{k_z^2-k_0^2}$$

This is for arbitrary $\vec{k}$. But we need only $\vec{k} = \hat{n}k_0 \rightarrow k_z = \cos\theta\, k_0$

$$= \left(\cos(\tfrac{k_0 a}{2}\cos\theta) - \cos(\tfrac{k_0 a}{2})\right)\frac{-2k_0}{k_0^2(\cos^2\theta - 1)} = \frac{2}{k_0\sin^2\theta}\left(\cos(\tfrac{k_0 a}{2}\cos\theta) - \cos(\tfrac{k_0 a}{2})\right)$$

so $\vec{J}_1(k_0\hat{n}) = \hat{z}\, I_m \frac{1}{k_0} \frac{\left(\cos(\frac{k_0 a}{2}\cos\theta) - \cos(\frac{k_0 a}{2})\right)}{\sin^2\theta}$

For $|\vec{J}^{\perp}|^2$ need $|\hat{z} - \hat{n}\cdot\hat{z}\,\hat{n}|^2 = 1 - \cos^2\theta = \sin^2\theta$

It follows that (use $\frac{\omega_0^2}{k_0^2} = c^2$)

$$\frac{dP}{d\Omega} = \frac{I_m^2}{2\pi c} \frac{\left(\cos(\frac{k_0 a}{2}\cos\theta) - \cos(\frac{k_0 a}{2})\right)^2}{\sin^2\theta}$$

For $k_0 a \ll 1$, $\cos(\frac{k_0 a}{2}\cos\theta) - \cos(\frac{k_0 a}{2}) = -\frac{1}{2}(\frac{k_0 a}{2}\cos\theta)^2 + \frac{1}{2}(\frac{k_0 a}{2})^2 = \frac{(k_0 a)^2}{8}\sin^2\theta$

$\Rightarrow \frac{dP}{d\Omega} = \frac{I_m^2}{128\pi c}(k_0 a)^4 \sin^2\theta$ as before for dipole approximation.

Radiation pattern: we've seen in dipole approx we have $\frac{dP}{d\Omega} \sim \sin^2\theta$

But (for this model) in exact case we have

$$\frac{dP}{d\Omega} \propto \frac{\left(\cos\left(\frac{k_0 a}{2}\cos\theta\right) - \cos\left(\frac{k_0 a}{2}\right)\right)^2}{\sin^2\theta}$$

At small θ $\quad \cos\left(\frac{k_0 a}{2}\left(1-\frac{1}{2}\theta^2\right)\right) = \cos\left(\frac{k_0 a}{2}\right)\cos\left(-\frac{k_0 a}{4}\theta^2\right) + \sin\left(\frac{k_0 a}{2}\right)\sin\left(\frac{k_0 a}{4}\theta^2\right)$

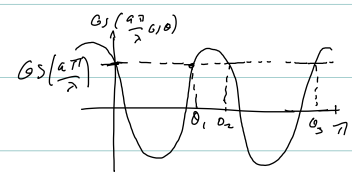
$$\simeq \cos\left(\frac{k_0 a}{2}\right) + \sin\left(\frac{k_0 a}{2}\right)\frac{k_0 a}{4}\theta^2$$

So $\frac{dP}{d\Omega} \propto \theta^2$ so $\theta = 0$ (and π) is a null-direction

just as in dipole approximation.

Additional null directions arise if $k_0 a$ is not small

that is $\frac{k_0 a}{2} = \frac{1}{2}\left(\frac{2\pi}{\lambda}\right)a = \frac{\pi a}{\lambda}$:



Define radiation resistance R_{rad} of antenna such that the power radiated is the "dissipated" power: $P = \frac{1}{2} I_0^2 R_{rad}$

In the dipole approximation, $I_0 = I_m \sin\left(\frac{k_0 a}{2}\right) \approx I_m \frac{k_0 a}{2}$

and $P = \int \frac{dP}{d\Omega} d\Omega = \int \frac{I_m^2}{128\pi c} (k_0 a)^4 \sin^2\theta \, d\Omega$

$$(k_0 a)^2 \frac{4\left(I_m \frac{k_0 a}{2}\right)^2}{128\pi c} 2\pi \int_{-1}^{1} d\xi \, (1 - \xi^2)$$

$$= \frac{(k_0 a)^2}{12 c} I_0^2 \qquad R_{rad} = \frac{(k_0 a)^2}{6c}$$

[Units? : recall $F = \frac{q^2}{r^2} \Rightarrow [P] = [q]^2 [L]^{-2} \cdot [L][T]^{-1} = [q]^2 [L]^{-1} [T]^{-1} = [I]^2 [T][L]^{-1} \checkmark$]

So what is $\frac{1}{c}$ in ohms? We can look in tables, or use our translation instructions.

Recall: $q_G^2 = \frac{q_{SI}^2}{4\pi\epsilon_0} \rightarrow I_G^2 = \frac{I_{SI}^2}{4\pi\epsilon_0} \quad c^2 = \frac{1}{\epsilon_0 \mu_0}$

$$P = \frac{(k_0 a)^2}{12} \sqrt{\epsilon_0 \mu_0} \frac{I_0^2}{4\pi\epsilon_0} = \frac{(k_0 a)^2}{48\pi} \sqrt{\frac{\mu_0}{\epsilon_0}} I_0^2$$

The quantity $Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 377$ ohms is known as the "impedance of the vacuum."

and $R_{rad} = Z_0 \frac{(k_0 a)^2}{24\pi} = \frac{Z_0}{24\pi}\left(\frac{2\pi a}{\lambda}\right)^2 = \frac{\pi}{6} Z_0 \left(\frac{a}{\lambda}\right)^2 = 197\Omega \left(\frac{a}{\lambda}\right)^2$

Since $a \ll \lambda$ in this approximation, this is a fairly small number, hence low radiation efficiency. For $\frac{a}{\lambda} \simeq 0.2$ $R_{rad} \simeq 8\Omega$, while $\frac{a}{\lambda} \simeq \frac{1}{2}$ ("half-wave antenna") $R_{rad} = 49\Omega$. Better yet $a \simeq \lambda$ ("full-wave antenna") but then need to integrate $\int \frac{dP}{d\Omega} d\Omega$ without $\frac{a}{\lambda} \ll 1$ approximation.

Near Zone Fields: Very brief word on this. The region

$R \ll \lambda$ is called "near-zone". For NR sources one has in addition $a \ll \lambda$.

Then in $A_\mu(\vec{r},t) = \frac{1}{c}\int d^3r' \frac{j_\mu(\vec{r}',t_{ret})}{|\vec{r}-\vec{r}'|}$

one has $t_{ret} = t - \frac{R}{c} \to t$ and $A_\mu(\vec{r},t) = \frac{1}{c}\int d^3r' \frac{j_\mu(\vec{r}',t)}{|\vec{r}-\vec{r}'|}$

the instantaneous field.

One can justify this ($t_{ret} \to t$) more precisely by going to Fourier space:

Recall $\tilde{A}_\mu(\vec{r},\omega) = \frac{e^{ikR}}{cR}\vec{\tilde{j}}(\vec{k},\omega)$

Now $e^{ikR} \simeq 1 + ikR + \cdots$ with $kR = 2\pi\frac{R}{\lambda} \ll 1$. Replace $e^{ikR} \to 1$. Done.