[Garg 15]

Electrostatics. Spherical Harmonics. Multipole Expansion

Electrostatics: $\frac{\partial \vec{E}}{\partial t}=0$, $\vec{B}=0$, $\vec{J}=0$ $\frac{\partial \rho}{\partial t}=0$ in Maxwell's Eqs:

$$\vec{\nabla}\times\vec{E}=0 \qquad \vec{\nabla}\cdot\vec{E}=4\pi\rho$$

$$\Downarrow$$

$$\vec{E}=-\vec{\nabla}\phi \quad (\phi=A^0,\ \vec{A}=0). \quad \Rightarrow \quad \nabla^2\phi=-4\pi\rho \quad \text{Poisson Equation}$$

We have seen the solution in terms of Green functions:

$$\phi(\vec{x})=\phi_{hom}(\vec{x})+\int G(\vec{x}-\vec{x}')\,\rho(\vec{x}')\,d^3x' \quad \text{where} \quad \nabla^2 G(\vec{x})=-4\pi\,\delta^{(3)}(\vec{x}),\ \nabla^2\phi_{hom}=0$$

and had determined $G(\vec{x})$ by Fourier transform. We can also infer G from our knowledge of Coulombs law:

$$\phi(\vec{x})=\frac{q}{|\vec{x}|} \quad \text{is for} \quad \rho(\vec{x})=q\,\delta^{(3)}(\vec{x}) \Longrightarrow \nabla^2\phi=-4\pi q\,\delta^{(3)}(\vec{x})$$

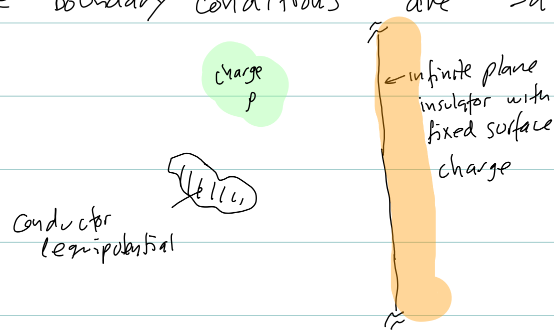
$$\Rightarrow \quad G(\vec{x})=\frac{1}{|\vec{x}|}$$

So

$$\boxed{\phi(\vec{x})=\phi_{hom}(\vec{x})+\int\frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}\,d^3x'}$$

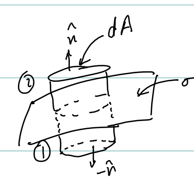
Boundary value problems: often concerned with region of space with boundaries on which we know something about $\vec{E}$. Then ϕ_{hom} is chosen to ensure these "boundary conditions" are satisfied.

As in



Conducting boundary: ϕ = constant

Surface charge density: $\Delta\vec{E}\cdot\hat{n}=4\pi\sigma$ (pillbox:



Gauss: $\int\vec{E}\cdot\hat{n}\,dA=4\pi Q$

$\Rightarrow (\vec{E}_2-\vec{E}_1)\cdot\hat{n}=\frac{4\pi Q}{dA}=4\pi\sigma$)

Regardless of the presence of charges, the central problem in electrostatics is then to solve

$$\nabla^2 \phi = 0 \quad \text{(Laplace Equation)}$$

subject to boundary conditions:

(i) ϕ specified (Dirichlet)

or

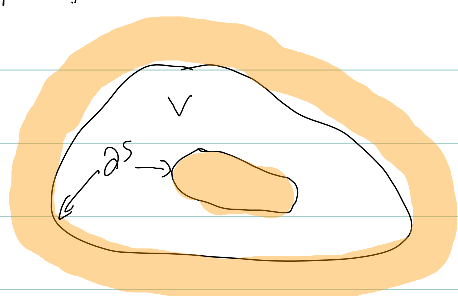
(ii) $\frac{\partial \phi}{\partial n}$ specified (Neumann)

(or (iii) mixed)

Uniqueness of solution of Poisson with boundaries: [Garg 16]

If ϕ_1 & ϕ_2 are two solutions to $\nabla^2 \phi = 4\pi\rho$ with ϕ or $\frac{\partial \phi}{\partial n}$ specified on $S = \partial V$, then $\psi = \phi_2 - \phi_1$ satisfies $\nabla^2 \psi = 0$ and

$\psi = 0$ or $\frac{\partial \psi}{\partial n} = 0$ on ∂V.



By Gauss's theorem

$$\int_V \vec{\nabla} \cdot (\psi \vec{\nabla} \psi) dV = \int_{\partial V} (\psi \vec{\nabla} \psi) \cdot \hat{n} \, dS = \int_{\partial V} \psi \frac{\partial \psi}{\partial n} \, dS$$

The RHS vanishes by assumption. The LHS

is $\int_V dV \left(\vec{\nabla} \psi \cdot \vec{\nabla} \psi + \underbrace{\psi \nabla^2 \psi}_{=0 \text{ by assumption}} \right) \Rightarrow \int_V dV \, |\vec{\nabla} \psi|^2 = 0 \Rightarrow \vec{\nabla} \psi = 0$

$\Rightarrow \psi = \text{Constant} \Rightarrow \boxed{\phi_2 = \phi_1 + \text{Constant}}$

(Note, for 2 functions ψ_1 & ψ_2

$$\int_V (\psi_1 \nabla^2 \psi_2 + \vec{\nabla} \psi_1 \cdot \vec{\nabla} \psi_2) \, dV = \int_{\partial V} \psi_1 \frac{\partial \psi_2}{\partial n} dS$$ is "Green's 1st identity"

subtracting $1 \leftrightarrow 2$ $\Rightarrow$ $$\int (\psi_1 \nabla^2 \psi_2 - \psi_2 \nabla^2 \psi_1) \, dV = \int_{\partial V} \left(\psi_1 \frac{\partial \psi_2}{\partial n} - \psi_2 \frac{\partial \psi_1}{\partial n} \right) dS$$ is "Green's 2nd identity" or Green's theorem)

Solving Laplace (PDE's) : separation of variables

Cartesian: $\phi(x,y,z) = X(x)\, Y(y)\, Z(z)$

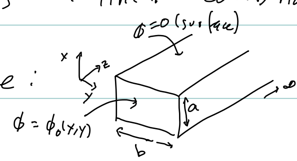
$$\frac{1}{\phi}\nabla^2\phi = \frac{1}{X}X''(x) + \frac{1}{Y}Y''(y) + \frac{1}{Z}Z''(z) = 0$$

The three functions of different arguments can add up to zero only if each one is a constant:

$$\frac{1}{X}X'' = \alpha^2 \quad \frac{1}{Y}Y'' = \beta^2 \quad \frac{1}{Z}Z'' = \gamma^2$$

with $\alpha^2+\beta^2+\gamma^2 = 0$, $X \propto e^{\pm\alpha x}$, $Y \propto e^{\pm\beta y}$ $\quad Z = e^{\pm\gamma z}$

The boundary conditions (b.c.'s) limit values of (α,β,γ). The solution is a linear combination.

Example:  We need to specify $\phi(x,y,0) = \phi_0(x,y)$

with $\phi_0(0,y) = \phi_0(a,y) = 0 = \phi_0(x,0) = \phi_0(x,b)$

So take $\alpha^2<0$, $\beta^2<0$ above for oscillatory functions. Changing $\alpha^2\to-\alpha^2$, $\beta^2\to-\beta^2$ above

so $\gamma^2 = \alpha^2+\beta^2 \Rightarrow \phi \sim e^{\pm i\alpha x}\, e^{\pm i\beta y}\, e^{\pm\gamma z} \quad \gamma = \sqrt{\alpha^2+\beta^2}$

b.c.s at $z=0$, $x=0,a$ $\Rightarrow \left(Ae^{i\alpha x} + Be^{-i\alpha x}\right)\Big|_{0,a} = 0 \Rightarrow B = -A$ and $\sin(\alpha a) = 0 \quad \alpha = \frac{n\pi}{a} \quad n = 1,2,\ldots$

$$\Rightarrow \phi(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\left(A_{nm}\cosh(\gamma_{nm}z) + B_{nm}\sinh(\gamma_{nm}z)\right)$$

where $\gamma_{nm} = \pi\sqrt{\left(\frac{n}{a}\right)^2 + \left(\frac{m}{b}\right)^2}$. The b.c. at $z=0$ gives

$$\phi_0(x,y) = \sum_{n,m} A_{nm}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)$$

With $\int_0^{\pi} d\xi\, \sin(n\xi)\sin(n'\xi) = \frac{\pi}{2}\delta_{nn'} \quad (n>0, n'>0) \Rightarrow A_{nm} = \frac{4}{ab}\int_0^a dx \int_0^b dy\, \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\phi_0(x,y)$

Finally as $z\to\infty$ we do not want $|\vec{E}| \sim e^{+\gamma z} \to \infty$ choose $B_{nm} = -A_{nm}$

CLEAR THAT THE SOLUTION is MOST GENERAL, BUT IMPLEMENTING BOUNDARY CONDITIONS COMPLICATED UNLESS RECTANGULAR SYMMETRY IN PROBLEM $\longrightarrow$ consider also curvilinear coordinates

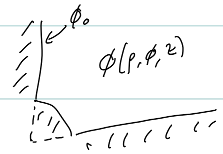
Cylindrical: $\phi(\rho,\phi,z) = R(\rho)\,\Phi(\varphi)\,Z(z)$ (use φ = angle, ϕ = potential)

$$\frac{1}{\rho R}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + \frac{1}{\rho^2\Phi}\Phi'' + \frac{1}{Z}Z'' = 0$$

$$\Rightarrow Z'' = \alpha^2 Z \quad , \quad \rho^2\left[\frac{1}{\rho R}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + \alpha^2\right] = -\frac{1}{\Phi}\Phi'' = \beta^2$$

$$\Rightarrow \Phi'' = -\beta^2\Phi \quad \Rightarrow \quad \left[\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} + \alpha^2 - \frac{\beta^2}{\rho^2}\right]R = 0$$

Φ periodic (except for "conical" configurations

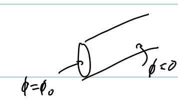
eg  $\phi(\rho,\phi,z)$ (z out of plane).)

$\Rightarrow \beta = m$, $m \in \mathbb{Z}$, $\Phi = e^{\pm i m\varphi}$.

$Z = e^{\pm\alpha z}$

R equation is Bessel's (see 203A , cavities & wave guides)

$\rightarrow R(\rho) = J_m(\alpha\rho)$. Won't review that $\rightarrow$ see last quarter notes.

(Expansion in zeroes $J_m(\xi_{mn}) = 0$, say , if $\phi(\rho = a, \varphi, 0) = 0$ 

$\phi(\rho,\varphi,z) = \sum_{n,m} C_{nm} J_m\left(\xi_{mn}\frac{\rho}{a}\right) e^{im\varphi} e^{-\xi_{mn} z/a}$ ...)

[Garg App. A]

Spherical Coordinates $\phi(r,\theta,\varphi)$

Appropriate for problems with spherical symmetry: spherical boundaries.

Recall, a scalar has $\phi'(\vec{x}') = \phi(\vec{x})$ with $\vec{x}' = R\vec{x}$ $\quad R^T R = 1$

With $R = 1 + \epsilon$, ϵ-infinitesimal $R^T R = 1 \Rightarrow \epsilon^T = -\epsilon$

with $\phi'(\vec{x}) = \phi(R^{-1}\vec{x})$ we have $\delta\phi = \phi(R^{-1}\vec{x}) - \phi(\vec{x}) = \phi(\vec{x} - \epsilon\vec{x}) - \phi(\vec{x}) = -\epsilon_{ij} x_j \partial_i \phi$

$\Rightarrow$ the infinitesimal rotation is generated by $-\epsilon_{ij} x_j \partial_i = \epsilon_{ji} x_j \partial_i$

Now an antisymmetric 3x3 matrix has $\frac{3\cdot 2}{2} = 3$ independent components, so we can parametrize ϵ_{ij} as $\epsilon_{ij} = \epsilon^a \epsilon_{aij}$ $\quad \epsilon^a$, $a=1,2,3$ the parameters (infinitesimal)

(and ϵ_{aij} = completely antisymmetric tensor with $\epsilon_{123} = +1$).

$$\Rightarrow \quad \delta\phi = \epsilon^a \epsilon_{aij} x_i \partial_j = \vec{\epsilon}\cdot(\vec{x}\times\vec{\nabla})$$

This should ring a bell! In QM $\vec{L} = \vec{r}\times\vec{p} = -i\hbar\, \vec{r}\times\vec{\nabla}$

Setting $\hbar = 1$ (because we are not doing QM) ie, $\vec{L} \equiv -i\, \vec{r}\times\vec{\nabla}$ then

$$\delta\phi = i\,\vec{\epsilon}\cdot\vec{L}\phi$$

We can use our knowledge from QM here:

$$[L_i, L_j] = i\epsilon_{ijk} L_k$$

(Proof: $[L_i, L_j] = (-i)^2 \epsilon_{imn}\epsilon_{jpq} [x_m\partial_n, x_p\partial_q] = (-i)^2 \epsilon_{imn}\epsilon_{jpq} (\delta_{np} x_m \partial_q - \delta_{qm} x_p \partial_n)$

$= (-i)^2 \left((\delta_{iq}\delta_{mj} - \cancel{\delta_{ij}\delta_{mq}}) x_m\partial_q - (\delta_{ip}\delta_{nj} - \cancel{\delta_{ij}\delta_{np}}) x_p \partial_n \right)$

$= (-i)^2 (x_j\partial_i - x_i\partial_j)$

$= (-i)^2 \epsilon_{jik}\epsilon_{kmn} x_m \partial_n$

$= -i\, \epsilon_{jik} L_k = i\, \epsilon_{ijk} L_k$).

$L^2 \equiv \vec{L}\cdot\vec{L}$, $[L^2, L_i] = 0$ $\qquad$ (Proof $L_j[L_j, L_i] + [L_j, L_i]L_j = \epsilon_{jik}(L_j L_k + L_k L_j) = 0$).

$L^{\pm} \equiv \frac{1}{\sqrt{2}}(L_1 \pm i L_2) \Rightarrow [L^+, L^-] = \frac{1}{2}\left(-i[L_1, L_2] + i[L_2, L_1]\right) = L_3$

$[L^{\pm}, L_3] = \frac{1}{\sqrt{2}}\left([L_1, L_3] \pm i[L_2, L_3]\right) = \frac{1}{\sqrt{2}}(-iL_2 \mp L_1) = \mp L^{\pm}$

We can simultaneously diagonalize L^2 and one of $L_{1,2,3}$, say L_3:

The eigenvectors are $Y_{\ell m}$, "spherical harmonics"

Before we find them, lets connect this to $\nabla^2\phi = 0$

Note that
$$L^2 = (-i)^2(\vec{r}\times\vec{\nabla})\cdot(\vec{r}\times\vec{\nabla}) = -\epsilon_{mij}x_i\partial_j\epsilon_{mpq}x_i\partial_j x_p\partial_q$$
$$= -(\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp})x_i\partial_j x_p\partial_q$$
$$= -x_i\partial_j x_i\partial_j + x_i\partial_j x_j\partial_i$$
$$= -\vec{x}\cdot\vec{\nabla} - r^2\nabla^2 + 3\vec{x}\cdot\vec{\nabla} + x_i x_j\partial_j\partial_i$$

$x_j\partial_j x_i\partial_i = x_i x_j\partial_i\partial_j + \vec{x}\cdot\vec{\nabla} \Longrightarrow$
$$= (\vec{x}\cdot\vec{\nabla})^2 + (\vec{x}\cdot\vec{\nabla}) - r^2\nabla^2$$

Now $\vec{x}\cdot\vec{\nabla} = r\frac{\partial}{\partial r}$ so
$$\frac{1}{r^2}L^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial r} - \nabla^2$$
$$= \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \nabla^2$$
$$\Rightarrow \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\vec{L}^2 = \frac{1}{r}\frac{\partial^2}{\partial r^2}r - \frac{1}{r^2}L^2$$
$$\Rightarrow L^2 = \frac{\partial^2}{\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} \Rightarrow Y_{\ell m}\text{'s are functions of } \theta \,\&\, \varphi, \ \underline{Y_{\ell m}(\theta,\varphi)}$$

As we will see $L^2 Y_{\ell m} = \ell(\ell+1)Y_{\ell m}$ $\quad \ell = 1, 2, \ldots$

So $\phi = \frac{1}{r}R_\ell(r)\,Y_{\ell m}$ so $\nabla^2\phi = 0 \Rightarrow \frac{1}{R}R'' - \frac{\ell(\ell+1)}{r^2} = 0 \Rightarrow R'' - \frac{\ell(\ell+1)}{r^2}R = 0$

This is homogeneous in $r \Rightarrow R = r^\alpha$ gives $\alpha(\alpha-1) = \ell(\ell+1) \Rightarrow \alpha = \ell+1$, $\alpha = -\ell$

$$\boxed{\phi(r,\theta,\varphi) = \sum_{\ell,m} c_{\ell m} r^\ell Y_{\ell m} + d_{\ell m} r^{-\ell-1} Y_{\ell m}}$$

By the standard argument, the $Y_{\ell m}$'s form an orthonormal set. They are normalized,
$$\int d\Omega\, Y^*_{\ell' m'}(\theta,\varphi)\,Y_{\ell m}(\theta,\varphi) = \delta_{\ell\ell'}\delta_{mm'}$$

and form a complete basis (in the space of normalizable functions on the unit sphere), so
$$c_{\ell m} r^\ell + d_{\ell m} r^{-(\ell+1)} = \int d\Omega\, Y^*_{\ell m}(\theta,\varphi)\,\phi(r,\theta,\varphi)$$

Either determine this at two radii to solve for both $c_{\ell m}$ and $d_{\ell m}$, OR, very commonly, use some additional condition, eg, regularity at origin ($r=0 \Rightarrow d_{\ell m}=0$) or at $r=\infty$ ($\Rightarrow c_{\ell m}=0$ for $\ell>0$).

Finding eigenvalues: $L^2\psi = \lambda\psi \quad L_3\psi = \lambda'\psi$.

To streamline notation, use QM notation $L_3|\lambda'\rangle = \lambda'|\lambda'\rangle$. We really should write $|\lambda,\lambda'\rangle$ for $L^2|\lambda,\lambda'\rangle = \lambda|\lambda,\lambda'\rangle$ and $L_3|\lambda,\lambda'\rangle = \lambda'|\lambda,\lambda'\rangle$, but for now we concentrate on λ' for fixed λ, so omit λ in $|\lambda,\lambda'\rangle$.

We also need an inner product $\langle\psi|\chi\rangle = \int d\Omega\ \psi^*\chi$. Note that $\vec{L}$ is hermitian w.r.t. this inner product: the hermitian conjugate

$$\langle\chi|L^\dagger|\psi\rangle \equiv \langle\psi|L|\chi\rangle^*$$

has

$$\langle\chi|L^\dagger|\psi\rangle = \langle\chi|L|\psi\rangle$$

Note also $L_\pm^\dagger = L_\mp$.

$\Rightarrow$ We will show that $|\lambda'\rangle$'s come in discrete sets with $\lambda' = -l, -l+1, \ldots, l-1, l$ for some integer l.

First note: $L_3 L_\pm|\lambda'\rangle = ([L_3, L_\pm] + L_\pm L_3)|\lambda'\rangle = (\lambda'\pm 1)\, L_\pm|\lambda'\rangle$

$\Rightarrow L_\pm|\lambda'\rangle$ is an eigenvector of L_3 with eigenvalue $(\lambda'\pm 1)$

So we have a chain $\ldots, L_-^2|\lambda'\rangle, L_-|\lambda'\rangle, |\lambda'\rangle, L_+|\lambda'\rangle, L_+^2|\lambda'\rangle, \ldots$

This will terminate if $L_+|l\rangle = 0$ for some $\lambda' = l$. Assume this.

Introduce proportionality constants into $L_+|\gamma\rangle \propto |\gamma+1\rangle$ (use "γ" because the prime in λ' is tiring).

and assume $|\gamma\rangle$ is normalized: $\langle\gamma|\gamma\rangle = 1$ (any γ)

$$L_+|\gamma\rangle = C_\gamma|\gamma+1\rangle \quad \text{and} \quad L_-|\gamma\rangle = D_\gamma|\gamma-1\rangle$$

These are not independent: $\|L_+|\gamma\rangle\|^2 = C_\gamma^2\langle\gamma+1|\gamma+1\rangle = \langle\gamma|L_-L_+|\gamma\rangle = C_\gamma D_{\gamma+1}\langle\gamma|\gamma\rangle$

$$\Rightarrow C_\gamma = D_{\gamma+1}$$

Now we use $[L_+, L_-]\,|\gamma\rangle = L_3|\gamma\rangle = \gamma|\gamma\rangle$

or $\quad C_{\gamma-1}D_\gamma - C_\gamma D_{\gamma+1} = \gamma$

$\Rightarrow \quad C_{\gamma-1}^2 - C_\gamma^2 = \gamma$

and $L_+|l\rangle = 0$ is $C_l = 0$. So we have

$$C_{l-1}^2 - 0 = l \qquad \Rightarrow \qquad C_{l-1}^2 = l$$

$$C_{l-2}^2 - C_{l-1}^2 = l-1 \qquad\qquad C_{l-2}^2 = 2l-1$$

$$\vdots \qquad\qquad \vdots$$

$$C_{l-(k+1)}^2 - C_{l-k}^2 = l-k \qquad\qquad C_{l-(k+1)}^2 = (k+1)l - \underbrace{(1+\dots+k)}_{\frac{1}{2}k(k+1)}$$

$$\vdots$$

$$\Rightarrow \quad C_{l-(k+1)}^2 = \tfrac{1}{2}(k+1)(2l-k)$$

This should not be negative: if we take $2l$ = integer then for $k = 2l$

$$C_{-l-1} = D_{-l} = 0 \quad \Rightarrow \quad L_-|-l\rangle = 0$$

$\Rightarrow$ The set of functions is $|-l\rangle, |-l+1\rangle, \dots, |l-1\rangle, |l\rangle \Rightarrow$ call them $|m\rangle$

for l = integer or half-integer. $\quad$ $m = -l, \dots, l$

Let's get back to $L^2 = L_1^2 + L_2^2 + L_3^2$

Note that $L_+L_- + L_-L_+ = \frac{1}{2}\left[(L_1 + iL_2)(L_1 - iL_2) + (L_1 - iL_2)(L_1 + iL_2)\right] = L_1^2 + L_2^2$

so $L^2 = L_+L_- + L_-L_+ + L_3^2$. But

$$L_+L_-|m\rangle = C_{m-1}D_m|m\rangle = C_{m-1}^2|m\rangle \qquad L_-L_+|m\rangle = C_m^2|m\rangle$$

$\Rightarrow \quad L^2|m\rangle = \left(C_{m-1}^2 + C_m^2 + m^2\right)|m\rangle$. But $\quad C_{l-k}^2 = \frac{1}{2}k(2l+1-k) \Rightarrow C_m^2 = \frac{1}{2}(l-m)(l+1+m)$

$C_m^2 + C_{m-1}^2 = \frac{1}{2}(l-m)(l+1+m) + \frac{1}{2}(l+1-m)(l+m) = l(l+1) - m^2 \quad \Rightarrow \quad L^2|m\rangle = l(l+1)|m\rangle$

So all our functions have the same L^2 eigenvalue, and are fully (properly)

labeled $|l,m\rangle$, with $L^2|l,m\rangle = l(l+1)|l,m\rangle$, $L_3|l,m\rangle = m|l,m\rangle$

$l = \frac{1}{2}\mathbb{Z}_+, \quad m = -l, -l+1, \dots, l-1, l$

Find eigenfunctions.

$$L_\pm = \frac{1}{\sqrt{2}}(L_1 \pm i L_2) = \frac{(-i)}{\sqrt{2}}\left[(y\partial_z - z\partial_y) \pm i(z\partial_x - x\partial_z)\right]$$

$$= \frac{1}{\sqrt{2}}\left[\mp(x \pm iy)\partial_z \pm z(\partial_x \pm i\partial_y)\right]$$

Note that $(\partial_x + i\partial_y)(x+iy) = 0$, $\quad (\partial_x - i\partial_y)(x-iy) = 0$

$$\frac{1}{\sqrt{2}}(\partial_x + i\partial_y)\frac{(x-iy)}{\sqrt{2}} = 1 \qquad \frac{1}{\sqrt{2}}(\partial_x - i\partial_y)\frac{(x+iy)}{\sqrt{2}} = 1$$

So $L_+ \left(\frac{x+iy}{\sqrt{2}}\right)^{n_+}\left(\frac{x-iy}{\sqrt{2}}\right)^{n_-} z^{n_3}$ = replace $z^{n_3} \to n_3 z^{n_3-1}\left[-\left(\frac{x+iy}{\sqrt{2}}\right)\right]$

and $\left(\frac{x-iy}{\sqrt{2}}\right)^{n_-} \to n_-\left(\frac{x-iy}{\sqrt{2}}\right)^{n_--1} z$

$$= -n_3\left(\frac{x+iy}{\sqrt{2}}\right)^{n_++1}\left(\frac{x-iy}{\sqrt{2}}\right)^{n_-} z^{n_3-1} + n_-\left(\frac{x+iy}{\sqrt{2}}\right)^{n_+}\left(\frac{x-iy}{\sqrt{2}}\right)^{n_--1} z^{n_3+1}$$

Also $L_3 = i(x\partial_y - y\partial_x)$

To streamline notation, let $x_\pm = \frac{x \pm iy}{\sqrt{2}}$ and $\partial_\pm = \frac{\partial_x \mp i\partial_y}{\sqrt{2}}$

(so $\partial_\pm x_\pm = 1$, $\partial_\pm x_\mp = 0$).

Then $x = \frac{1}{\sqrt{2}}(x_+ + x_-)$ $\quad y = -\frac{i}{\sqrt{2}}(x_+ - x_-)$ $\quad \partial_x = \frac{1}{\sqrt{2}}(\partial_+ + \partial_-)$ $\quad \partial_y = \frac{i}{\sqrt{2}}(\partial_+ - \partial_-)$

so $L_3 = \frac{i}{2}\left[i(x_+ + x_-)(\partial_+ - \partial_-) + i(x_+ - x_-)(\partial_+ + \partial_-)\right] = x_+\partial_+ - x_-\partial_-$

and in this notation $L_\pm = \mp x_\pm \partial_z \pm z\partial_\mp$ counts ± 1 for each power of $x_\pm$, 0 for z

This suggests $|l,l\rangle = N_l\, x_+^l$ with N_l a normalization constant

Clearly $L_+|l,l\rangle = 0$, $L_3|l,l\rangle = l\,|l,l\rangle$

$$L^2|l,l\rangle = \left(L_+L_- + L_-L_+ + L_z^2\right) N_l\, x_+^l = N_l\left(L_+(-lz\, x_+^{l-1}) + 0 + l^2 x_+^l\right)$$

$$= N_l\left[-l(-x_+^l) + 0 + l^2 x_+^l\right] = l(l+1)\,|l,l\rangle$$

Bingo! We have $|l, l-k\rangle = N_{l-k}\, L_-^k\, x_+^l$

For example $|l,l-1\rangle = N_{l-1} L_- x_+^l = N_{l-1}(x_-\partial_z - z\partial_+)x_+^l = -l N_{l-1}\, z\, x_+^{l-1}$

and $|l,l-2\rangle = N_{l-2} L_-(-lz\,x_+^{l-1}) = N_{l-2}(-l)\left(x_- x_+^{l-1} - (l-1) z^2 x_+^{l-2}\right)$

In terms of θ, φ, $x_\pm = \frac{1}{\sqrt{2}} r\sin\theta(\cos\varphi \pm i\sin\varphi) = \frac{r}{\sqrt{2}}\sin\theta\, e^{\pm i\varphi}$

$$z = r\cos\theta$$

Since $|l,k\rangle$ defined above is proportional to r^l for all functions, we can take it out and replace $|l,k\rangle \to \frac{1}{r^l}|l,k\rangle$, a function of θ and φ only.

This gives Y_{lm}, up to normalization, $\boxed{Y_{lm}(\theta,\varphi) = \frac{1}{r^l} N_{lm} L_-^{l-m} x_+^l}$

Normalization: $\int d\Omega\, Y^*_{l'm'} Y_{lm} = \delta_{ll'}\delta_{mm'}$

Example: $l=1$. From the above $\frac{x_+}{r}, z, \frac{x_-}{r}$

So $Y_{1\pm1} = N_{\pm1}\sin\theta\, e^{\pm i\varphi}$ $\quad Y_{10} = N_0\cos\theta$

$$\int d\Omega\, |Y_{1\pm1}|^2 = 2\pi|N_{1\pm1}|^2\int_0^\pi \sin\theta d\theta \sin^2\theta = \frac{8\pi}{3}|N_{1\pm1}|^2 \qquad |N_{1\pm1}| = \sqrt{\frac{3}{8\pi}}$$

$$\int d\Omega\, |Y_{10}|^2 = 2\pi|N_0|^2\int_0^\pi \sin\theta\, d\theta \cos^2\theta = \frac{4\pi}{3}|N_{10}|^2 \qquad |N_{10}| = \sqrt{\frac{3}{4\pi}}$$

The phase is by convention: $Y_{10} = \sqrt{\frac{3}{4\pi}}\cos\theta$ and

get $Y_{1+1} = \frac{1}{|C_1|}L_+ Y_{10}$ $\quad Y_{1-1} = \frac{1}{|C_{-1}|}L_- Y_{10}$

$\Rightarrow$ $Y_{1\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{\pm i\varphi}$

☆ Note that the eigensystems with $2l$ = odd integer, i.e., $l = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$ are periodic in φ with period 4π, not 2π: $e^{im(\varphi+2\pi)} \to -e^{im\varphi}$. Hence they do not play a role in solving Laplace's equation, but they do play a role in other physics (spinors).

☆ Note: derivation of eigenvalues only depends on commutation relations and normalizable vectors. Applies equally to $\vec{L}$ = matrices. Then $\vec{L}$ are $l\times l$ on space of $|l,m\rangle$ vectors.

This presentation emphasizes the connection to the rotation group and angular momentum. There are many other ways to introduce Y_{lm}'s and many additional developments. Here we list some facts

- $Y_{lm}(-\hat{r}) = (-1)^l \, Y_{lm}(\hat{r})$
- $Y^*_{lm}(\hat{r}) = (-1)^m \, Y_{l-m}(\hat{r})$
- Completeness : $\displaystyle\sum_{l=0}^{\infty} \sum_{m=-l}^{l} Y^*_{lm}(\theta,\varphi) \, Y_{lm}(\theta',\varphi') = \delta(\cos\theta - \cos\theta') \, \delta(\varphi-\varphi')$
- Addition theorem : let $\cos\theta_{12} = \hat{r}_1 \cdot \hat{r}_2$ (and recall Y_{l0} is φ independent).

$$Y_{l0}(\theta_{12}, 0) = \sqrt{\frac{4\pi}{2l+1}} \sum_{m=-l}^{l} Y^*_{lm}(\hat{r}_1) \, Y_{lm}(\hat{r}_2)$$

- Relation to Legendre polynomials $Y_{l0}(\theta,\varphi) = \sqrt{\frac{2l+1}{4\pi}} \, P_l(\cos\theta)$

(and, more generally, to Associated Legendre functions

$$Y_{lm}(\theta,\varphi) = \sqrt{\frac{2l+1}{4\pi}} \sqrt{\frac{(l-m)!}{(l+m)!}} \, P_l^m(\cos\theta) \, e^{im\varphi} \quad)$$

Generating function for $P_l(x)$:

$$\frac{1}{\sqrt{1+t^2-2tx}} = \sum_{l=0}^{\infty} t^l P_l(x) \qquad x \in [-1, 1]$$

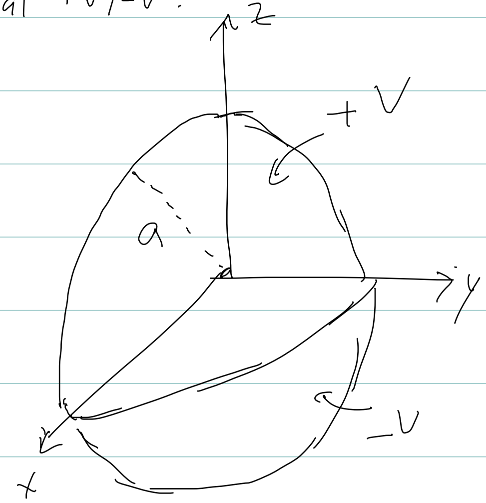
- Generating function for P_l + addition theorem $\Rightarrow$

$$\frac{1}{|\vec{r} - \vec{r}'|} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y^*_{lm}(\hat{r}') \, Y_{lm}(\hat{r})$$

where $r_< = \min(|\vec{r}|, |\vec{r}'|)$ and $r_> = \max(|\vec{r}|, |\vec{r}'|)$. I have switched notation from $\vec{x}$ to $\vec{r}$ to stay closer to textbook (Garg).

Example:

Conducting Sphere of radius a with upper/lower hemispheres at potential +V/-V.



Has azymuthal symmetry so only $m=0$ contributes to expansion of ϕ:

$$\phi(r,\theta) = \sum_{l=0}^{\infty} \left(c_{l0} r^l + d_{l0} r^{-l-1}\right) Y_{l0}$$

For ϕ inside sphere $d_{l0} = 0$ (regularity at origin).

Inverting

$$c_{l0} r^l = \int d\Omega \, Y^*_{l0} \phi(r,\theta)$$

or, evaluating at $r=a$

$$c_{l0} = \frac{1}{a^l} 2\pi \int_{-1}^{1} d(\cos\theta) \sqrt{\frac{2l+1}{4\pi}} P_l(\cos\theta) \times \begin{cases} +V & \cos\theta > 1 \\ -V & \cos\theta < 1 \end{cases}$$

$$= \frac{V}{a^l} 2\pi \sqrt{\frac{2l+1}{4\pi}} \int_0^1 dx \left[P_l(x) - P_l(-x) \right]$$

This vanishes for l = even, and we need to do the integral

$$\int_0^1 dx \, P_l(x) \quad \text{for odd } l.$$

From the generating function:

$$\int_0^1 dx \sum t^\ell P_\ell(x) = \sum_{\ell=0}^{\infty} t^\ell \int_0^1 dx\, P_\ell(x)$$

which equals

$$\int_0^1 dx \frac{1}{\sqrt{1+t^2-2tx}} = \frac{2}{(-2t)} \sqrt{1+t^2-2tx}\Big|_0^1 = \frac{1}{t}\left(\sqrt{1+t^2} - (1-t)\right)$$

Use the Taylor expansion

$$(1+x)^s = \sum_{k=0}^{\infty} \frac{1}{k!} s(s-1)\cdots(s-k+1) x^k \qquad \text{with } s = \frac{1}{2},\ x = t^2$$

So

$$\frac{1}{t}\sqrt{1+t^2} - \frac{1}{t} = \sum_{k=1}^{\infty} \frac{1}{k!} \frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\cdots\left(\frac{1}{2}-k+1\right) t^{2k-1}$$

and so $(\ell = 2k-1)$

$$\int_0^1 P_{2\ell-1}(x)\, dx = \frac{(-1)^{k+1}}{2^k k!} \underbrace{1 \cdot 1 \cdot 3 \cdots (2k-3)}_{(2k-3)!!}$$

So we have

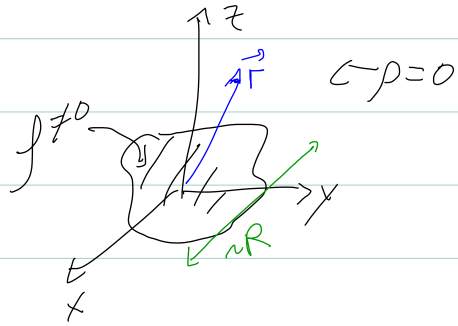
$$\phi(r,\theta) = V \sum_{\ell=1}^{\infty} \sqrt{4\pi}\sqrt{4\ell-1}\left(\frac{r}{a}\right)^{2\ell-1} \frac{(-1)^{\ell+1}}{2^\ell \ell!} (2\ell-3)!!\; Y_{2\ell-1,0}(\theta,\varphi)$$

$$= V \sum_{\ell=1}^{\infty} (4\ell-1)\left(\frac{r}{a}\right)^{2\ell-1} \frac{(-1)^{\ell+1}}{2^\ell \ell!} (2\ell-3)!!\; P_{2\ell-1}(\cos\theta)$$

$$= V\left[\frac{3}{2}\frac{r}{a} P_1(\cos\theta) - \frac{7}{8}\left(\frac{r}{a}\right)^3 P_3(\cos\theta) + \frac{11}{16}\left(\frac{r}{a}\right)^5 P_5(\cos\theta) \cdots\right]$$

Multipole expansion [Garg 19]

Localized charge distribution ρ:



Interested in $\phi(\vec{r})$ outside (and far from) ρ.

$$\phi(\vec{r}) = \sum_a \frac{q_a}{|\vec{r}-\vec{r}_a|} = \int d^3r' \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$$

Physical idea of multipole expansion: at $r \gg R$ we should have $\phi \approx \frac{q}{r}$, where $q = \int d^3r' \rho(\vec{r}\,')$. Corrections are by an expansion in

D/r: $\phi \sim \frac{q}{r}\left(1 + c_1 \frac{R}{r} + c_2 \frac{R^2}{r^2} + \cdots\right)$.

- - -

Expand in powers or $|\vec{r}\,'|/|\vec{r}| < 1$

We can do this in one swoop by using Y_{lm} expansion of $1/|\vec{r}-\vec{r}\,'|$. But let's get some intuition of what this is by expanding by hand first. We'll use a Taylor series for $\vec{r}\,'$ about $\vec{0}$:

$$f(\vec{x}) = f(0) + x_i \partial_i f(0) + \frac{1}{2!} x_i x_j \partial_i \partial_j f(0) + \cdots$$

$$\Rightarrow \frac{1}{|\vec{r}\,'-\vec{r}|} = \frac{1}{r} + x_i' \partial_i' \frac{1}{|\vec{r}\,'-\vec{r}|}\bigg|_{\vec{r}\,'=0} + \frac{1}{2!} x_i' x_j' \partial_i' \partial_j' \frac{1}{|\vec{r}\,'-\vec{r}|}\bigg|_{\vec{r}\,'=0} + \cdots$$

$$= \frac{1}{r} + x_i' f_{1i}(\vec{r}) + x_i' x_j' f_{2ij}(\vec{r}) + \cdots$$

Stick this into $\int \frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} d^3r' = \frac{1}{r} \underbrace{\int \rho(\vec{r}\,') d^3r'}_{\text{mono-pole} = \text{charge}} + f_{1i}(\vec{r}) \underbrace{\int \rho(\vec{r}\,') x_i' d^3r'}_{\text{"dipole" moment}} + f_{2ij} \underbrace{\int \rho x_i' x_j' d^3r'}_{\text{"quadrupole" moment}}$

More precisely:

Define $q = \int \rho(\vec{r}\,')d^3r'$ monopole = charge

$\vec{d} = \int \rho(\vec{r}\,')\vec{r}\,' d^3r'$ "dipole moment"

$D_{ij} = \int \rho(\vec{r}\,')(3x_i'x_j' - r'^2\delta_{ij})$ "quadrupole moment"

The extra term $\delta_{ij}r'^2$ in the definition of D_{ij} is included so that $\delta^{ij}D_{ij} = 0$ (ie, it's traceless). This can be done freely because the coefficient $f_{2ij}(\vec{r})$ is traceless as well, as we will see. The "3" is just an arbitrary normalization in the definition of D_{ij}.

Note that under rotations,

$q \leftrightarrow$ scalar $(l=0)$ $\vec{d} \leftrightarrow$ vector $(l=1)$, $D_{ij} \leftrightarrow l=2$

ie, 2-index symmetric tensors transform in 1-to-1 correspondance with $|l,m\rangle$ for $l=2$. Note both have 5 (independent) components. This is why we subtract the trace in the definition of D_{ij}: a 3×3 matrix M_{ij} in general has 9 components:

		components	
Trace:	$M_{ii} \longleftrightarrow l=0$	1	sum = 9
Anti-symmetric:	$M_{ij}-M_{ji} \longleftrightarrow l=1$	3	
Symmetric-traceless	$M_{ij}+M_{ji} - \frac{2}{3}\delta_{ij}M_{kk} \longleftrightarrow l=2$	5	$(=3\times3\ \checkmark)$

Compute $f_{1,2}$:

$$\partial_i' f(\vec{r}\,'-\vec{r})\Big|_{\vec{r}\,'=0} = -\partial_i f(\vec{r}\,'-\vec{r})\Big|_{\vec{r}\,'=0} = -\partial_i f(-\vec{r})$$

and $\partial_i \frac{1}{|\vec{r}|} = \partial_i \frac{1}{\sqrt{x_j x_j}} = -\frac{1}{2}\frac{2x_i}{|\vec{r}|^3}$ and $\partial_i\partial_j \frac{1}{|\vec{r}|} = -\frac{\delta_{ij}}{|r|^3} + \frac{3x_ix_j}{|r|^5}$

$$f_{1i} = \frac{x_i}{r^3} \qquad f_{2ij} = \frac{3x_ix_j - \delta_{ij}r^2}{r^5}$$ note $\delta^{ij}f_{2ij} = 0$ as advertised

So we have

$$\phi(\vec{r}) = \frac{q}{r} + \frac{\vec{d}\cdot\vec{r}}{r^3} + \frac{1}{2}\frac{x_i x_j - \frac{1}{3}\delta_{ij} r^2}{r^5} D_{ij} + \cdots$$

or

$$\boxed{\phi(\vec{r}) = \frac{q}{r} + \frac{\vec{d}\cdot\vec{n}}{r^2} + \frac{n_i n_j}{2r^3} D_{ij} + \cdots}$$

where $\vec{n} = \frac{\vec{r}}{r}$ has $n^2 = 1$.

Clearly $|\vec{d}| \lesssim qR$ $\quad |D_{ij}| \lesssim qR^2$ as expected.

Systematize:

$$\phi(\vec{r}) = \int d^3r' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} = \int d^3r' \rho(\vec{r}') \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{4\pi}{2l+1} \frac{r'^l}{r^{l+1}} Y^*_{lm}(\hat{r}') Y_{lm}(\hat{r})$$

$$= \sum_{l=0}^{\infty} \frac{1}{r^{l+1}} \sqrt{\frac{4\pi}{2l+1}} \sum_{m=-l}^{l} q_{lm} Y_{lm}(\hat{r})$$

where $q_{lm} \equiv \sqrt{\frac{4\pi}{2l+1}} \int d^3r' \rho(\vec{r}') r'^l Y^*_{lm}(\hat{r}')$ "2^l-pole moment"

Note that $\phi(\vec{r})^* = \phi(\vec{r}) \Rightarrow \sum_{m=-l}^{l} q_{lm} Y_{lm}(\hat{r}) = \sum_{m=-l}^{l} q^*_{lm} Y^*_{lm}(\hat{r}) = \sum_{m=-l}^{l} q^*_{lm} (-1)^m Y_{l,-m}(\hat{r})$

$\Rightarrow q_{lm} = (-1)^m q^*_{l,-m}$ which is verified by its definition in terms of the integral over the real charge density $\rho(\vec{r}')$.

Relation to $\vec{d}$: $\quad r' Y^*_{1\pm1}(\hat{r}') = \mp\sqrt{\frac{3}{8\pi}}(x' \pm i y')^* = \mp\sqrt{\frac{3}{8\pi}}(x' \mp i y')$

$$\Rightarrow q_{1\pm1} = \mp\sqrt{\frac{4\pi}{3}\cdot\frac{3}{8\pi}} \int d^3r' \rho(\vec{r}')(x' \mp i y') = \mp\frac{1}{\sqrt{2}}(d_x \mp i d_y)$$

Similarly $q_{10} = d_z$

Exercise: show 19.25 in Garg

$$q_{20} = D_{zz} \qquad q_{2\pm1} = \mp \frac{1}{\sqrt{6}}\left(D_{xz} \mp i D_{yz}\right)$$

$$q_{2\pm2} = \frac{1}{2\sqrt{6}}\left(D_{xx} - D_{yy} \mp 2 i D_{xy}\right)$$

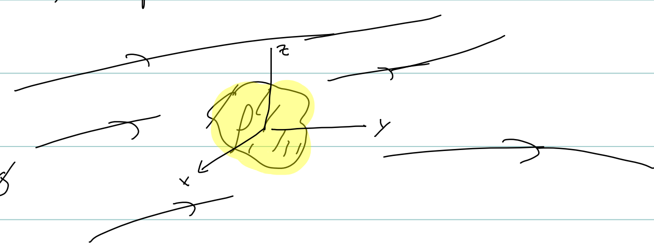
Assignment: Read about Earnshaw's theorem in Garg.

In a charge free region $\langle \phi \rangle$ over a sphere equals $\phi|_{center}$.

Charge distributions in external Fields [Garg 20]

A charge distribution characterized by $q, \vec{d}, D_{ij} \ldots$ that give its field away, in an external $\vec{E}$ field experiences forces and torques. What are they?

Take now $\phi(\vec{r})$ = potential due to external sources



$$\vec{E} = -\vec{\nabla}\phi$$

The energy in this configuration is (with $q_a = d^3r' \rho(\vec{r}\,')$)

$$U = \sum_a q_a \phi(\vec{r}_a) = \int d^3\vec{r}\,' \rho(\vec{r}\,') \phi(\vec{r}\,')$$

Now choose coordinate system with origin within ρ (heck, choose the same one as was used to define multi-poles): expanding $\phi(\vec{r}\,')$

$$\phi(\vec{r}\,') = \phi(0) + x_i' \partial_i \phi\big|_0 + \tfrac{1}{2} x_i' x_j' \partial_i \partial_j \phi\big|_0 + \ldots$$

so

$$U = \int d^3\vec{r}\,' \rho(\vec{r}\,') \left[\phi(0) + \ldots \right] = q\,\phi(0) + d_i \partial_i \phi\big|_0 + \tfrac{1}{6} D_{ij} \partial_i \partial_j \phi\big|_0 + \ldots$$

(using $\nabla^2 \phi\big|_0 = 0$ since the external field is due to remote charges).

Examine result: contributions to U:

(i) Lowest: $q\phi(0)$

If we were to move the distribution to a new location, $\vec{r}$, we would have instead $q\phi(\vec{r})$. The force on this is

$$\vec{F} = -\vec{\nabla}U(\vec{r}) = q(-\vec{\nabla}\phi) = q\vec{E}(\vec{r})$$

No surprise!

(ii) 1st correction: $d_i \partial_i \phi|_0 = -\vec{d}\cdot\vec{E}(0)$

As above, a rigid translation $\rightarrow -\vec{d}\cdot\vec{E}(\vec{r})$

Force $\vec{F} = -\vec{\nabla}(-\vec{d}\cdot\vec{E}) = d_i \vec{\nabla} E_i$

or $F_j = d_i \partial_j E_i = d_i(\partial_j E_i - \partial_i E_j) + (\vec{d}\cdot\vec{\nabla})E_j$

But in static situation $\vec{\nabla}\times\vec{E} = 0 \Rightarrow \vec{F} = (\vec{d}\cdot\vec{\nabla})\vec{E}$

Even if $\partial_i E_j = 0$ ($\vec{E}$ = uniform) there is a torque

$$\vec{N} = \int \vec{r}\,' \times \left(d^3r'\rho(r')\vec{E}(\vec{r}\,')\right) = \int d^3r'\rho(\vec{r}\,')\,\vec{r}\,' \times \left(\vec{E}(0) + x_i' \partial_i \vec{E}\big|_0 + \cdots\right)$$

lowest term

$$\vec{N} = \vec{d} \times \vec{E}(0)$$

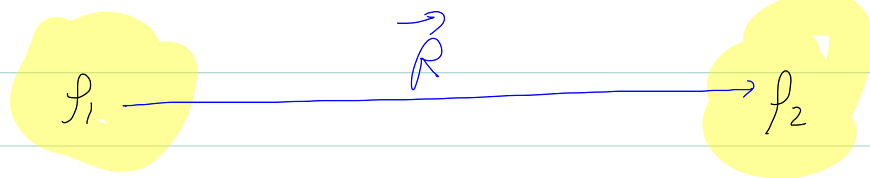
(iii) 2nd corr: $\frac{1}{6} D_{ij}\partial_i\partial_j\phi\big|_0 = -\frac{1}{6} D_{ij}\partial_i E_j(0)$

$$F_k = \frac{1}{6} D_{ij}\,\partial_i\partial_k E_j$$

and $N_i = \epsilon_{ijk}\left(\int d^3r'\rho(r')\,x_j' x_m'\right)\partial_m E_k = \frac{1}{3}\epsilon_{ijk} D_{jm}\,\partial_m E_k$

where we used $\vec{\nabla}\times\vec{E} = 0$ to include the δ_{jm} term at no price.

Charge on charge:



Energy of configuration: use ρ_1 as source of external field and ρ_2 in presence of this

$$\phi = \frac{q_1}{r} + \frac{\vec{d}_1 \cdot \vec{r}}{r^3} + \frac{1}{2} D^{(1)}_{ij} \frac{r_i r_j}{r^5} + \cdots \qquad (1)$$

and

$$U = q_2 \phi(0) - \vec{d}_2 \cdot \vec{E}(0) - \frac{1}{6} D^{(2)}_{ij} \partial_i E_j(0) + \cdots \qquad (2)$$

Here "0" is in the ρ_2 distribution, so take that as $\vec{R}$ from "center" of ρ_1; set $\vec{r} = \vec{R}$ in (1). Then stick (1) into (2). Expanding in $1/R$ we have $U = U^{(0)} + U^{(1)} + U^{(2)} + \cdots$ with

$$U^{(0)} = q_2 \left[\frac{q_1}{R} + \cdots \right] = \frac{q_1 q_2}{R}$$

the potential energy of two point charges

Next

$$U^{(1)} = q_2 \left[\frac{\vec{d}_1 \cdot \vec{R}}{R^3} \right] + \left[-\vec{d}_2 \cdot \vec{E} \right]$$

where $\vec{E} = -\vec{\nabla}\left(\frac{q_1}{r}\right)\Big|_{\vec{r}=\vec{R}}$

$$= q_1 \frac{\vec{R}}{R^3}$$

$$\Rightarrow \quad U^{(1)} = q_2 \frac{\vec{d}_1 \vec{R}}{R^3} - q_1 \frac{\vec{d}_2 \cdot \vec{R}}{R^3}$$

(Looks asymmetric, but isn't: exchange $q_1 \leftrightarrow q_2$ also exchanges $\vec{R} \leftrightarrow -\vec{R}$).

At order $1/R^3$ there are $q D_{ij}$ terms and $d_1 d_2$ terms.

The latter are

$$V^{(2)} = -\vec{d}_2 \cdot \vec{E}$$

with $\vec{E} = -\vec{\nabla} \frac{\vec{d}_1 \cdot \vec{r}}{r^3}\Big|_{\vec{R}} = \frac{3 R_i R_j - \delta_{ij} R^2}{R^5} d_{1,i}$

$$\Rightarrow \quad U^{(2)} = \frac{\vec{d}_1 \cdot \vec{d}_2 R^2 - 3(\vec{R} \cdot \vec{d}_1)(\vec{R} \cdot \vec{d}_2)}{R^5}$$

Min for $\vec{d}_1 \parallel \vec{d}_2 \parallel \vec{R}$, max for $\vec{d}_1 \parallel \vec{R}$, $\vec{d}_2 \parallel (-\vec{R})$ or viceversa.