Physics 2D — Final Solutions — Fall 2020

Problem 1

electron has kinetic energy $511{,}000\,eV = m_e c^2$

$K = (\gamma - 1) m_e c^2 = m_e c^2 \Rightarrow \gamma m_e c^2 = 2 m_e c^2 \Rightarrow$

$$\Rightarrow \boxed{\gamma = 2} \quad \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \Rightarrow \frac{1}{\gamma^2} = 1 - \frac{v^2}{c^2} \Rightarrow \frac{v^2}{c^2} = 1 - \frac{1}{\gamma^2}$$

$$\Rightarrow \frac{v^2}{c^2} = 1 - \frac{1}{4} = \frac{3}{4} \Rightarrow \boxed{\frac{v}{c} = \frac{\sqrt{3}}{2} = 0.866} \text{ (a)}$$

$$p = \gamma m_e v \Rightarrow p = \gamma m_e c^2 \frac{v}{c} \cdot \frac{1}{c} = \cancel{2} \times 511{,}000\,eV \times \frac{\sqrt{3}}{\cancel{2}} \times \frac{1}{c}$$

$$\Rightarrow \boxed{p = 885{,}078\ eV/c} \text{ (b)}$$

(c) Let d be the distance earth – sun. Time that the light takes is $\Delta t_{light} = \frac{d}{c} = 500\,s$

Time that electron takes is $\Delta t = \frac{d}{v} = \frac{d}{c} \cdot \frac{c}{v} = 577\,s$ in earth's frame

In the rest frame of the electron Δt is the proper time, i.e.

$$\Delta t_p = \frac{\Delta t}{\gamma} = \frac{d}{v}\frac{1}{\gamma} = \frac{d}{c} \cdot \frac{c}{v} \cdot \frac{1}{\gamma} \Rightarrow$$

$$\Rightarrow \Delta t_p = \Delta t_{light} \cdot \frac{2}{\sqrt{3}} \cdot \frac{1}{2} = \frac{\Delta t_{light}}{\sqrt{3}}$$

$$\Rightarrow \boxed{\Delta t_p = 288.7\,s} \text{ (c)}$$

Problem 2

Wien's law, $\lambda_m T = \dfrac{hc}{4.96\,k_B}$. $T = 300\,K \Rightarrow$

$$\Rightarrow \lambda_m = \frac{12{,}400 \times 11{,}600}{4.96 \times 300}\ \mathring{A} \Rightarrow \boxed{\lambda_m^{emit} = 96{,}667\ \mathring{A}}\ (a)$$

(b) The earth absorbs maximum radiation at wavelength where the suns emits it, i.e. same formula as above with $T = 5800\,K$

$$\Rightarrow \boxed{\lambda_m^{absorb} = \lambda_m^{emit} \times \frac{300}{5800}\ \mathring{A} = 5000\ \mathring{A}}\ (b)$$

(c) The total radiation emitted by the earth is the same as what is incident on the earth. So

$$P_{tot} = \sigma T^4 \times 4\pi R_e^2$$

$$P_{tot} = 5.67 \times 10^{-8} \times 300^4 \times 4\pi \times 6378^2 \times 10^6\ W$$

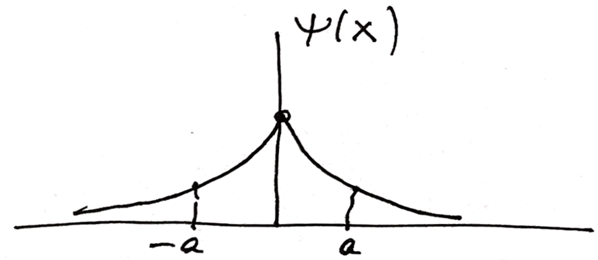
$$\boxed{P_{tot} = 2.35 \times 10^{17}\ W}$$

Problem 3

$\Psi(x) = C e^{-|x|/a}$. Use that Ψ is even.

$$\frac{1}{2} = \int_0^\infty dx\, |\Psi(x)|^2 = C^2 \int_0^\infty dx\, e^{-2x/a} = -C^2 \frac{a}{2} e^{-2x/a}\Big]_0^\infty = C^2 \frac{a}{2}$$

$$\Rightarrow \frac{1}{2} = C^2 \frac{a}{2} \Rightarrow \boxed{C = \frac{1}{\sqrt{a}}} \text{ (a)}$$



(b) $\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$

$\langle x \rangle = 0$ by symmetry

$$\langle x^2 \rangle = \frac{2}{a} \int_0^\infty dx\, x^2 |\Psi(x)|^2 = \frac{2}{a} \int_0^\infty dx\, x^2 e^{-2x/a}$$

Use formula $\int_0^\infty dx\, x^n e^{-\lambda x} = \frac{n!}{\lambda^{n+1}} \Rightarrow$

$$\Rightarrow \langle x^2 \rangle = \frac{2}{a} \cdot \frac{2!}{\left(\frac{2}{a}\right)^3} = \frac{2a^3}{a} \cdot \frac{2}{2^3} = \frac{a^2}{2}$$

$$\Rightarrow \boxed{\Delta x = \frac{a}{\sqrt{2}} = 0.71a} \text{ (b)}$$

(c) $$P(|x| < a) = 2 \int_0^a dx\, |\Psi(x)|^2 = 2 \int_0^a dx\, \frac{1}{a} e^{-2x/a} =$$

$$= \frac{2}{a}\left(-\frac{a}{2}\right) e^{-2x/a}\Big]_0^a = 1 - e^{-2} = 0.865$$

$$\boxed{P(|x| < a) = 0.865} \text{ (c)}$$

Problem 4

energies for harmonic oscillator are

$$E_n = \hbar\omega\left(n+\tfrac{1}{2}\right)$$

n has to be ± for absorption or emission. So

$$\hbar\omega = \frac{hc}{\lambda} = \frac{12,400}{5000}\ eV = 2.48\,eV$$

⇒ ground state energy $\boxed{E_0 = \frac{\hbar\omega}{2} = 1.24\,eV}$ (a)

(b) Classical amplitude A obeys

$$\frac{1}{2} m_e\omega^2 A^2 = \frac{\hbar\omega}{2} \Rightarrow A^2 = \frac{\hbar}{m_e\omega} = \frac{\hbar^2}{m_e\,\hbar\omega}$$

using $\frac{\hbar^2}{m_e} = 7.62\,eV\,\mathring{A}^2$ ⇒ $\boxed{A = \sqrt{\frac{7.62}{2.48}}\,\mathring{A} = 1.75\,\mathring{A}}$ (b)

(c) Wave function for ground state is

$$\psi(x) = C\,e^{-\frac{m\omega}{2\hbar}x^2} = C\,e^{-\frac{x^2}{2A^2}}$$

$$\frac{|\psi(x=A)|^2}{|\psi(0)|^2} = e^{-1} = 0.368 \Rightarrow$$

$$\boxed{\frac{|\psi(0)|^2}{|\psi(A)|^2} = 2.72 \text{ times more likely}}$$ (c)

Problem 5

(a) $E_1 = \frac{\hbar^2\pi^2}{2m_e L^2} = \frac{3.81\pi^2}{4^2} eV = 2.35 eV$

$\Rightarrow$ $\boxed{E_1 = 2.35 eV \text{ assuming } \infty \text{ well}}$ (a)

(b) Wavefunction penetrates on right side a distance $\delta = 1/\alpha$

$$\alpha = \sqrt{\frac{2m}{\hbar^2}(U-E)} = \sqrt{\frac{1}{3.81}(20-2.35)}\ Å^{-1} =$$

$= 2.152 Å^{-1}$ $\Rightarrow$ $\boxed{\delta = 0.465 Å}$

so the effective length of the well is

$$L_{eff} = L + \delta = 4.465 Å$$

so a better estimate for the ground state energy is

$$E_1' = \frac{\hbar^2\pi^2}{2m_e L_{eff}^2} = 2.35 \times \left(\frac{4}{4.465}\right)^2 eV \Rightarrow$$

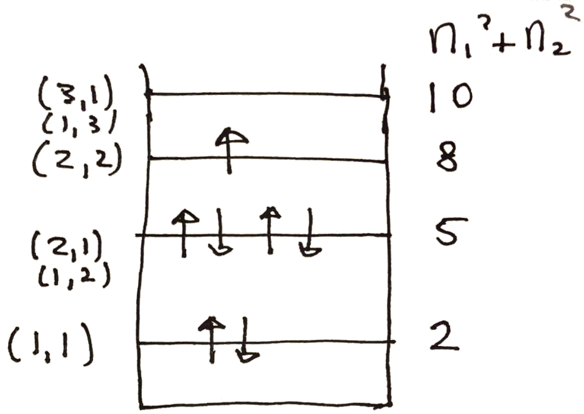
$\boxed{E_1' = 1.89 eV}$

(c) Transmission coefficient

$$T = e^{-2\sqrt{\frac{2m}{\hbar^2}(U-E_1')}\cdot 1Å} = e^{-4.36} = 0.013$$

$\Rightarrow$ $\boxed{\text{probability of escape in 1 attempt is } 1.3\%}$

Problem 6



$$E_{n_1,n_2} = \frac{\hbar^2\pi^2}{2m_eL^2}(n_1^2+n_2^2) \equiv E_0(n_1^2+n_2^2)$$

There can be 2 electrons in each state

Largest wavelength photon is absorbed for transition with smallest ΔE

That is the transition $(2,2)$ to $(1,3)$ or $(3,1)$

$$\Delta E = E_0(10-8) = 2E_0 = \frac{hc}{\lambda}$$

$$E_0 = \frac{\hbar^2\pi^2}{2m_eL^2} = \frac{3.81\,\pi^2}{25}\,eV = 1.504\,eV$$

$$\Rightarrow \boxed{\lambda = \frac{12{,}400}{3.008}\,\text{Å} = 4122\,\text{Å}} \quad (a)$$

(b) Next largest: $(2,1) \to (2,2)$

$$\Delta E = E_0(8-5) = 3E_0 = \frac{hc}{\lambda}$$

$$\Rightarrow \boxed{\lambda = 2748\,\text{Å}} \quad (b)$$

(c) $E_{gr\,state} = E_0(2\times2 + 4\times5 + 1\times8) = 32\,E_0$

$$\Rightarrow \boxed{E_{gr\,st} = 48.13\,eV}$$

Problem 7

$L = 4\hbar$ according to Bohr $\Rightarrow n = 4$

(a) de Broglie wavelength is $\lambda = \frac{h}{p} = \frac{hr}{pr}$

since $L = pr$ is the angular momentum, and $\hbar = h/2\pi$

$\lambda = \frac{hr}{4\hbar} = \frac{2\pi r}{4} = \frac{\pi r}{2}$. Since $r = r_n = a_0 n^2$ with

$a_0 = 0.528\,\text{Å}$ and $n = 4$ $\Rightarrow$ $\lambda = \frac{\pi}{2} a_0 \cdot 16 = 8\pi a_0 = 2\pi a_0 \times 4$

$\Rightarrow \boxed{\lambda = 13.30\,\text{Å}}$ (a)

(b) For Schrödinger, $L = \sqrt{\ell(\ell+1)}\,\hbar$. Since $n = 4$, maximum allowed ℓ is $\ell = 3$, so

$\boxed{L = \sqrt{12}\,\hbar = 3.46\,\hbar}$ (b)

Schrödinger ~~knows~~ knows that $n = 4$ since he measured the electron's energy

$\boxed{E_4 = -\frac{E_0}{4^2} = -0.85\,\text{eV}}$ with $E_0 = 13.6$ eV (b)

(c) Bohr says the most likely value for the distance is $r_n = a_0 n^2$,

so $\boxed{r_4 = 8.46\,\text{Å}}$

Schrödinger says: If $\ell = 3$, which gives L closest to Bohr's value and largest $r \Rightarrow$ $R(r) \propto r^3 e^{-r/4a_0} \Rightarrow P(r) \propto r^8 e^{-r/2a_0}$

Most likely r is for $P'(r) = 0 \Rightarrow$

$8r^7 - \frac{r^8}{2a_0} = 0 \Rightarrow \boxed{r = 16 a_0 = 8.46\,\text{Å}}$ agrees with Bohr's.

Problem 8

$n=3,\ l=2,\ m_l=-1$

For $n=2$, $l=1$ or $l=0$. Selection rule is $\Delta l = \pm 1$, and $\Delta m_l = 0, \pm 1$. So $l=1$ for final state, and $m_l = -1$ or $m_l = 0$ for final state

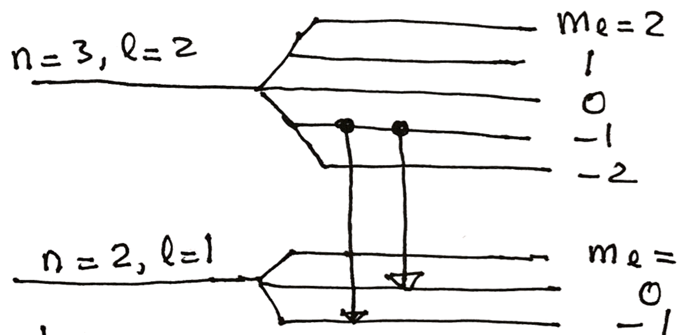
$$\Delta E = \frac{hc}{\lambda} = E_0\left(\frac{1}{4}-\frac{1}{9}\right) = \frac{5}{36}E_0 \Rightarrow$$

$$\lambda = \frac{36}{5}\times\frac{12{,}400}{13.6}\ \text{Å} \Rightarrow \boxed{\lambda = 6565\ \text{Å}} \quad (a)$$

Possible final states: $\boxed{\begin{matrix} n=2,\ l=1,\ m_l=-1 \\ n=2,\ l=1,\ m_l=0 \end{matrix}}$ (a)

(b) In a magnetic field, energy shifts by

$\Delta E_B = +\mu_B B m_l$



(i) For $m_l=-1 \rightarrow m_l=-1$,
Same as before: $\lambda = 6565\ \text{Å}$

(ii) For $m_l=-1 \rightarrow m_l=0$

$$\Delta E = E_0\left(\frac{1}{4}-\frac{1}{9}\right) - \mu_B B = \frac{hc}{\lambda'}$$

$$= \frac{5}{36}E_0 - 5.79\times10^{-5}\times 50\,\text{eV} = 1.888\,\text{eV} - 2.895\times10^{-3}\,\text{eV} = 1.886\,\text{eV}$$

$$\Rightarrow \boxed{\lambda' = 6575\ \text{Å}} \quad (b)$$

(c) (i) There are 2 states with $m_l=-1$: $l=1, m_l=-1$ and $l=2, m_l=-1$
There is 1 state with $m_l=-2$. So $\boxed{\text{prob}(m_l=-1)/\text{prob}(m_l=-2)=2}$

With a magnetic field, energy of $m_l=-2$ state is lower $\Rightarrow$ becomes more probable

$$\boxed{\frac{\text{prob}(m_l=-1)}{\text{prob}(m_l=-2)} = 2e^{-\Delta E_B/k_BT} = 2e^{-0.112} = 1.79}$$

Problem 9

$$\psi(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{\pi x}{L}\right)$$

$$\langle Q \rangle = \int dx\, \psi^*(x)\, [Q]\, \psi(x)$$

$$\Delta Q = \sqrt{\langle Q^2\rangle - \langle Q\rangle^2}$$

$$[p] = \frac{\hbar}{i}\frac{d}{dx} \quad ; \quad [p^2] = -\hbar^2 \frac{d^2}{dx^2}$$

(a) $$[p]\psi(x) = \frac{\pi}{L}\sqrt{\frac{2}{L}}\cos\left(\frac{\pi x}{L}\right) \neq q\,\psi(x)$$

$\Rightarrow$ $\boxed{[p] \text{ is not sharp, it is fuzzy} \Rightarrow \text{no eigenvalue}}$

$$[p^2]\psi(x) = \frac{\hbar^2\pi^2}{L^2}\psi(x) \Rightarrow$$

$\boxed{[p^2] \text{ is sharp. Its eigenvalue is } \frac{\hbar^2\pi^2}{L^2}}$

(b) Since $[Q] = [p^2]$ is a sharp observable,

$$\boxed{\langle p^2\rangle = \frac{\hbar^2\pi^2}{L^2}, \quad \Delta(p^2) = 0}$$

For $[Q] = [p]$,

$$\boxed{\begin{gathered}\langle p\rangle = 0 \text{ by symmetry} \\ \Delta p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \frac{\hbar\pi}{L}\end{gathered}}$$