Notes 2, section 3:

Hydrodynamic Equations and Correlation Functions II, Fluctuation-Dissipation Theorem

— have developed hydrodynamic model of magnetization response.

$$\rightarrow \frac{\partial M}{\partial t} + \underline{\nabla} \cdot \underline{J}_M = 0$$

$$\underline{J}_M = -D\, \underline{\nabla} M$$

$$\rightarrow \langle [M(\;), M(\;')] \rangle_{eq}$$

$$= \int \frac{d\omega}{\pi} \frac{d^3k}{(2\pi)^3} \chi''(k,\omega)\, e^{i\underline{k}\cdot(\underline{x}-\underline{x}')}\, e^{-i\omega(t-t')}$$

dynamic suscept. — correlation commutator.

$$= \int \frac{d\omega}{\pi} \int \frac{d^3k}{(2\pi)^3} \chi''(k,\omega)\, e^{i\underline{k}\cdot(\underline{x}-\underline{x}')}\, e^{-i\omega(t-t')}$$

$$\rightarrow \boxed{\chi''(k,\omega) = \frac{\chi k^2 D\, \omega}{\omega^2 + (k^2 D)^2}}$$

Dynamic Susceptibility

$$\mathrm{Re}\left[M/H_0\right] = \frac{\chi''}{\omega_c}$$

Now, proceed further:

- Sum rules for $\chi''(k,\omega)$
  (Get χ from χ'')
- Relaxation time, Dispersion Relation → generalizing constitutive relation
- Fluctuation-Dissipation Theorem.
  (Dynamic suscept. χ'' sets Fluctuation level).

④ Sum Rules — Relate Parts of Susceptibility.

Now: → Have derived and used χ''.

→ Need full.

so, in K-K spirit:

$$\boxed{\chi(k,z) = \int \frac{d\omega'}{\pi} \frac{\chi''(k,\omega')}{\omega' - z}}$$

i.e. $\chi(k, \omega + i0) = \chi'(k,\omega)$
$+ i\chi''(k,\omega)$

and

$$\frac{1}{\omega'-\omega} = \frac{P}{\omega'-\omega} - i\pi\delta(\omega'-\omega)$$

so, KK $\rightarrow$

$$\boxed{\chi'(k,\omega) = P\int\frac{d\omega'}{\pi}\,\frac{\chi''(k,\omega')}{\omega'-\omega}}$$

Now, have:

$$M(\underline{k},z) = \int\frac{d\omega'}{\pi i}\,\frac{\chi''(k,\omega')}{\omega'(\omega'-z)}\,H(\underline{k})$$

and

$$\chi(k,z) = \int\frac{d\omega'}{\pi}\,\frac{\chi''(k,\omega')}{\omega'-z}$$

$$\rightarrow \frac{-iz\,M(\underline{k},z)}{H_0(k)} = \int\frac{d\omega'}{\pi}\,\frac{\chi''(k,\omega')}{\omega'} - \chi(k,z).$$

so

$$\boxed{\frac{M(\underline{k}, z)}{H(\underline{k})} = \frac{-\chi(k, 0) + \chi(k, z)}{iz}}$$

(*) $\chi(k, 0) \Rightarrow$ zero frequency susceptibility.

Then,

$$M(\underline{k}) = \int d\underline{x}\, e^{-i\underline{k}\cdot\underline{x}} \langle M(\underline{x}, t=0) \rangle$$

from χ'' defn:

$$= \int \frac{d\omega'}{\pi} \frac{\chi''(k, \omega)}{\omega'} H(k)$$

$$= \chi'(k, 0)\, H(k)$$

from (K-K)

$$= \chi(k, 0)\, H(k), \quad \text{as } \chi'' \to 0 \text{ for } \omega \to 0$$

static, wave number dependent susceptibility.

$$\chi(k) = \chi(k, 0)$$

So,

$$\boxed{\begin{array}{l} \chi(k) = \int \frac{d\omega}{\pi} \frac{\chi''(k,\omega)}{\omega} \\ \text{and} \\ M(k) = \chi(k)\, H(k) \end{array}} \qquad (*)$$

$(*)$ → Sum rule:

$$\underbrace{\chi(k)}_{\text{static susceptibility}} = \underbrace{\int \frac{d\omega}{\pi} \frac{\chi''(k,\omega)}{\omega}}_{\text{integral over dynamic suscept.}}$$

of course: → Sum Rule from

K-K as $\omega \to 0$.

$$\rightarrow \chi = \left.\frac{\partial M}{\partial H}\right|_{H \to 0}.$$

thermo. derivative

Another Sum Rule:

$\left\{\begin{array}{l}\text{Relate } \chi'' \text{ moments (in } \omega) \text{ to} \\ \text{equal-time correlations/commutators,}\end{array}\right.$

Recall:

①

$$\langle [M(\underline{x},t), M(\underline{x}',t')] \rangle_{eq}$$

$$= \int \frac{d\omega}{\pi} \int \frac{d\underline{k}}{(2\pi)^3} \chi''(\underline{k},\omega)\, e^{i\underline{k}\cdot(\underline{x}-\underline{x}')} e^{-i\omega(t-t')}$$

② and

$$\frac{\partial M}{\partial t} + \underline{\nabla}\cdot\underline{J}^M = 0,$$

Take ∂_t ① :

$$\partial_t \langle [M(\underline{x},t), M(\underline{x}',t')] \rangle_{eq}$$

$$= - \langle [\underline{\nabla}\cdot\underline{J}^M(\underline{x},t), M(\underline{x}',t')] \rangle_{eq}$$

$$= -i \int \frac{d\omega}{\pi} \int \frac{d\underline{k}}{(2\pi)^3}\, \omega\, \chi''(\underline{k},\omega)\, e^{i\underline{k}\cdot(\underline{x}-\underline{x}')} e^{-i\omega(t-t')}$$

but:

$$\underline{J}_M = \sum_r \gamma \frac{s_r(t)}{2m} \left\{ \underline{p}_r(t), \delta(\underline{x}-\underline{x}_r(t)) \right\}$$

$$M(\underline{x},t) = \sum_r \gamma s_r(t)\, \delta(\underline{x}-\underline{x}_r(t))$$

crank:

$$\left\langle \left[\underline{J}_M(\underline{x},t), M(\underline{x}',t') \right] \right\rangle_{t'=t}$$

$$= \frac{\gamma^2}{4m}\, i\, \underline{\nabla}' \delta(\underline{x}-\underline{x}') \langle n(\underline{x},t) \rangle_{eq}$$

so

$$-\int \frac{d\omega}{\pi} \int \frac{d\underline{k}}{(2\pi)^3}\, \omega \chi''(k,\omega)\, e^{ik\cdot(\underline{x}-\underline{x}')} = \frac{\gamma^2 \langle n \rangle_{eq}}{4m} \nabla^2 \delta(\underline{x}-\underline{x}')$$

FT both sides $\Rightarrow$

$$\boxed{\int \frac{d\omega}{\pi}\, \omega\, \chi''(k,\omega) = \frac{n}{m} \frac{\gamma^2}{4} k^2}$$

another sum rule!

suscept. $\Leftrightarrow$ micro!

Why care?

$\Rightarrow$ Sum Rule imposes constraint!

Stay tuned...

How to use sum rule ?
⇒ constraints !

b.) Revisiting Constitutive Relation
⟷ Delay Time ! ?

~ Is constitutive relation the most general, Linearly ?

i.e. $\underline{J} = -D \, \underline{\nabla} M$?

~ Should there be some lag in response of current to magnetization ?

Yes ! ⟶ some indications in Theory and Experiments
(c.f. Kadanoff and Martin)

⟶ Relaxation Time Model for Constitutive Relation.

before $\langle \underline{J}_{M}(\underline{x},t) \rangle = -D \nabla \langle M(\underline{x},t) \rangle$

becomes: time delay

$$\partial_t \langle \underline{J}_M(\underline{x},t)\rangle + \frac{1}{\tau}\langle \underline{J}_M(\underline{x},t)\rangle = -\frac{D}{\tau}\underline{\nabla} M(\underline{x},t)$$

$t \ll \tau \Rightarrow$

$$\partial_t \underline{J}_M \sim -\frac{D}{\tau}\underline{\nabla} M$$

$t \gg \tau \Rightarrow$

$$\underline{J}_M \sim -D\,\underline{\nabla} M.$$

then

$$\partial_t \langle M\rangle = -\underline{\nabla}\cdot\langle \underline{J}_M\rangle$$

so raised order,

$$\partial_t^2 \langle M\rangle = -\underline{\nabla}\cdot\partial_t\langle \underline{J}_M\rangle$$

$$\left[\partial_t^2 + \frac{1}{\tau}\left(\partial_t - D\nabla^2\right)\right]\langle M(\underline{x},t)\rangle = 0$$

$t \geq 0$

→ Telegraph eqn.
replaces diffusion eqn.

→ i.e. $t < \tau$

M → wave equation.

$t \gg \tau$

M → diffusion equation

Now, impose i.c.:

static suscept.

$$\begin{cases} \langle M(\underline{x},0)\rangle = \chi H_0(\underline{x}) \\ \partial_t \langle M(\underline{x},t)\rangle \Big|_{t=0} = 0 \end{cases}$$

so, usual Laplace transform

⇒

$$\boxed{\frac{M(k,z)}{H(k)} = \frac{\chi(1-iz\tau)}{-iz + Dk^2 - \tau z^2}} \quad \dagger$$

Recall:

$$\int \frac{d\omega}{\pi}\, \omega\, \chi'' = \frac{n}{M}\, \frac{\gamma^2 k^2}{4}$$

etc.

$$\Rightarrow \lim_{z\to\infty} \frac{M(k,z)}{H(k)} = -\frac{\chi}{iz} - \frac{n\gamma^2 k^2}{4iz^3 M} + O\left(\frac{1}{z^4}\right)$$

Now, expanding $\dagger$ above for large z:

$(\dagger) \Rightarrow$

$$\frac{M(k,z)}{H(k)} = \frac{i\chi}{z} + \frac{iDk^2\chi}{\tau z^3}$$

So, time delay consistent with (model independent!) sum rule of:

$$\boxed{D = \frac{n\gamma^2}{4m}\frac{\tilde{\tau}}{\kappa}}$$

or

$$\boxed{\tilde{\tau} = \frac{4m\kappa D}{n\gamma^2}}$$ → sets time delay.

Note:

→ Derived w/o recourse to direct modelling!

→ Agreed with experiment cf: K + M

this brings us to:

c.) Fluctuation - Dissipation Theorem

(c.f. Landau - Lifshitz, "Statistical physics" on "Fluctuations")

→ FDT relates:

$$\text{Intensity} \leftrightarrow \left(\frac{\text{Dissipation}}{\text{susceptibility}}\right) T$$

i.e. Brownian Motion

$$\partial_t V + \underbrace{\frac{\beta}{m}}_{\text{drag}} V = \frac{\tilde{f}}{m}$$

fluctuation → Brownian Force due flctns.

$$\beta \langle \tilde{V}^2 \rangle = \langle \tilde{f} \tilde{V} \rangle$$

$$\sim \frac{\langle \tilde{f}^2 \rangle}{m} \tau_{ac}$$

$1/\tau_{ac} > \beta$

$$\langle \tilde{V}^2 \rangle = T/m$$

→

$$\beta T \sim \langle \tilde{f}^2 \rangle \tau_{ac}$$

$$\boxed{\underbrace{\langle \tilde{f}^2 \rangle \tau_{ac}}_{\text{Flctn}} \sim \underbrace{\beta T}_{\text{Dissp}^n}}$$

→ FDT embedded in $\left\{ \begin{array}{l} \text{Near Eqbrm} \\ \text{Linear Response} \end{array} \right.$

→ Point: $\text{Im}\,\chi = \chi''$ sets flctn levels

and so links Order parameter Correlation / Commutator to Susceptibility → $\boxed{\text{General}}$

→ Reflects Emission – Absorption Balance.

→ H. Nyquist: Circuit: $\Rightarrow$ Noise (flctn.) ↔ $\overline{V}_{\text{applied}}$ (Disspn) connection.

Callen, Welton.

Kubo 57.

K & M: "The fluctuation-dissipation theorem lies at the heart of

Ko.

[much recent] work on many particle physics."

General Derivation (Follows L+L).

→ preliminaries:

$$H \to H + \hat{V}$$

$$\hat{V} = -\hat{x}\, f(t)$$

force

$$\bar{x}(t) = \int_0^\infty \alpha(\tau)\, f(t-\tau)\, d\tau$$

mean susceptibility

$\alpha \Leftrightarrow \chi$

$$\left\{\begin{aligned} \bar{x}_\omega &= \alpha(\omega) f_\omega \\ \alpha &= \alpha'(\omega) + i\alpha''(\omega) \end{aligned}\right.$$

$$\left\{\begin{aligned} \alpha(-\omega) &= +\alpha^*(\omega) \\ \alpha'(-\omega) &= \alpha'(\omega) \\ \alpha''(-\omega) &= -\alpha''(\omega) \end{aligned}\right.$$

then,

$$f = \frac{1}{2}\left(f_0 e^{-i\omega t} + f_0^* e^{i\omega t}\right)$$

$$\bar{x} = \frac{1}{2}\left[\alpha(\omega) f_0 e^{-i\omega t} + \alpha(-\omega) f_0^* e^{i\omega t}\right]$$

then,

$$dE/dt = -\bar{x}\, df/dt$$

so

$$Q = \left\langle \frac{dE}{dt} \right\rangle = \left\langle -\left[\frac{1}{2}\alpha(\omega) f_0 e^{-i\omega t} + \frac{\alpha}{2}(-\omega) f_0^* e^{i\omega t}\right]\left[\frac{1}{2} f_0(-i\omega) e^{-i\omega t} + \frac{1}{2}(i\omega) f_0 e^{i\omega t}\right]\right\rangle$$

$$= \frac{1}{4} i\omega\left(\alpha^* - \alpha\right)|f_0|^2$$

$$= \frac{1}{2}\omega\alpha''(\omega)|f_0|^2$$

$$\boxed{Q = \frac{1}{2}\omega\alpha''(\omega)|f_0|^2}$$

→ The FDT (General):

Now, generally, the dissipation Q is due to transitions (i.e. net loss/gain) in/out of state due external perturbation

$$\left[\begin{array}{l} V = -f\hat{x} \quad , \text{ i.e.} \qquad n = \text{state} \\ Q = \sum_{M} \underset{\substack{\uparrow \\ \text{transition prob.}}}{W_{mn}} \hbar \omega_{mn} \end{array} \right.$$

Fermi Golden Rule:

$$W_{m,n} = \frac{\pi |f_0|^2}{2\hbar^2} |X_{m,n}|^2 \left[\delta(\omega + \omega_{m,n}) + \delta(\omega + \omega_{n,m}) \right]$$

so

$$\boxed{Q = \frac{\pi}{2\hbar} |f_0|^2 \sum_{n} |X_{n,m}|^2 \left[\delta(\omega + \omega_{m,n}) + \delta(\omega + \omega_{n,m}) \right] \omega_{m,n}}$$

So, above + previous:

$$Q = \frac{1}{2}\,\omega\,\alpha''(\omega)\,|f_0|^2$$

$$= \frac{\pi}{2\hbar}|f_0|^2 \sum_n |X_{n,m}|^2 \Big[\delta(\omega - \omega_{n,m}) + \delta(\omega + \omega_{n,m})\Big]\omega_{n,m}$$

So, can write
absorptive susceptibility as:

$$\boxed{\alpha''(\omega) = \frac{\pi}{\hbar}\sum_m |X_{n,m}|^2\Big[\delta(\omega - \omega_{n,m}) - \delta(\omega + \omega_{n,m})\Big]}$$

used:

$$Q = \frac{\pi}{2\hbar}|f_0|^2\,\omega \sum_n |X_{n,m}|^2\Big[\delta(\omega + \omega_{n,m}) - \delta(\omega + \omega_{n,m})\Big]$$

as $\delta \to 0$ except at arg $= 0$.

Now,

$$\underbrace{\alpha''(\omega)}_{\text{dissp'n}} = \frac{\pi}{\hbar}\sum_{m} |X_{n,m}|^2 \left\{ \delta(\omega+\omega_{n,m}) - \delta(\omega+\omega_{m,n}) \right\}$$

Similarly:

$$\underbrace{(x^2)_\omega}_{\text{fluct'n}} = \pi \sum_{m} |X_{n,m}|^2 \left[\delta(\omega+\omega_{n,m}) + \delta(\omega+\omega_{m,n}) \right]$$

fluct'n → emission/absorption
$m \to n$
$n \to m$.

Now, need average over distribution:

prob, state n

$$\langle (x^2)_\omega \rangle = \pi \sum_{\substack{n,m \\ \text{tot sum}}} \rho_n |X_{n,m}|^2 \left\{ \delta(\omega+\omega_{n,m}) + \delta(\omega+\omega_{m,n}) \right\}$$

$$\rho_n = \exp[(F_0 - E_n)/T]$$

$$\langle (X^2)_\omega \rangle = \pi \sum_{m,n} \rho_n |x_{n,m}|^2 \Big[\delta(\omega - \omega_{n,m}) + \delta(\omega + \omega_{m,n}) \Big]$$

can $n \to m$ in 2^{nd}

$$= \pi \sum_{m,n} (\rho_n + \rho_m) |x_{n,m}|^2 \delta(\omega + \omega_{n,m})$$

$$= \pi \sum_{m,n} \rho_n (1 + e^{\hbar\omega_{n,m}/T}) x_{n,m} \delta(\omega + \omega_{n,m})$$

$$\boxed{\langle (X^2)_\omega \rangle = \pi (1 + e^{-\hbar\omega/T}) \sum_{m,n} \rho_n |x_{n,m}|^2 \delta(\omega + \omega_{n,m})}$$

Similarly

$$\boxed{\langle \alpha'' \rangle = \frac{\pi}{\hbar} (1 - e^{-\hbar\omega/T}) \sum_{m,n} \rho_n |x_{n,m}|^2 \delta(\omega + \omega_{n,m})}$$

Then finally:

$$\boxed{\langle (x^2)_\omega \rangle = \hbar\, \alpha''(\omega) \coth \frac{\hbar\omega}{2T}}$$

$$\boxed{\langle x^2 \rangle = \frac{\hbar}{\pi} \int_0^\infty \alpha''(\omega) \coth \frac{\hbar\omega}{2T}\, d\omega}$$

→ Fluctn - Dissipn Thm General.

Alternatively:

$$(x^2)_\omega = |\alpha(\omega)|^2 (f^2)_\omega$$

so

$$\boxed{(f^2)_\omega = \frac{\hbar\, \alpha''(\omega)}{|\alpha(\omega)|^2} \coth \frac{\hbar\omega}{2T}}$$

Classically:

$\hbar\omega / 2T \ll 1$

$\coth x \sim \frac{1}{x}$ for $x \ll 1$

$\Rightarrow$

$$\boxed{\langle (x^2)_\omega \rangle = \frac{2\alpha''(\omega)\, T}{\omega}}$$

and

$$\boxed{\langle (f^2)_\omega \rangle = \frac{2\alpha''(\omega)\, T}{\omega\, |\alpha(\omega)|^2}}$$

Note:

- reln of ~~eqbrm~~ averaged quantities

- Flctn $\sim$ T dissp'n (prop)

- Flctns large for collective modes $\alpha \to 0$, disspn.

- Or, yet again:

$$\frac{\text{Flctn}}{\text{Dissprn}} \sim T$$

(Flctn ↳ emission; Dissprn ↳ absorption)

$T \to$ thermal avg.

N.B. Can show; (from FDT) for spin diffn coefficient:

$$\boxed{D\chi = \lim_{\omega\to 0}\left[\lim_{k\to 0}\frac{\omega}{k^2}\chi''(k,\omega)\right]}$$

$\to$ HW.