Physics 210b

Lecture II - 3 - b - Central Limit Theorem and Beyond 2

Recall:

→ Central Limit Theorem, and proof.

$\langle \Delta x^2 \rangle < \infty$, IID are key elements.

→ CLT distribution (Gaussian)

is a particular case of an <u>L- stable</u> <u>distribution</u>

→ CLT / Diffusion is <u>self-similar</u>.

So, ask if other self-similar Levy-stable distributions?

i.e. $P(C_1 \Sigma) P(C_2 \Sigma) = P(C \Sigma)$
condition.

Now, $\hat{P}_n(k) = [\hat{p}(k)]^n$ — characteristic fctn

$$\begin{cases} Z_N = \text{[redacted]}\ X_N / a_N & \to \text{rescaling} \\ \mathrm{Pdf}(Z_N) = F_N(x) / a_N \end{cases}$$

$x = X_N / a_N.$

and:

— $\hat{F}_N(a_N k) = \hat{P}_N(k)$

Now, want $F_n(k) \xrightarrow[n\to\infty]{} F(k)$ [redacted].

some fixed fctn. (attractor)

let $\lim_{m\to\infty} \frac{a_{nm}}{a_m} = c_n$ (not cum)

~~Condition for function as~~ limiting case.

$$\boxed{\hat{F}(k\, c_n) = [\hat{F}(k)]^n}$$

scale.

self-sim.

so, need solve:

$$\boxed{\hat{F}(k\, u(\lambda)) = [\hat{F}(k)]^\lambda}$$

fixed fctn. condition

$$\Psi = \ln \hat{F}(k)$$

so

$$\Psi(k\mu(\lambda)) = \lambda \Psi(k)$$

$$k\left(\frac{d\mu}{d\lambda}\right)\Psi'(k\mu(\lambda)) = \Psi(k)$$

$$\mu(\lambda = 1) = 1 \qquad \text{(must.)}$$

$$k\mu'\Psi' = \Psi(k)$$

$$\boxed{\frac{d\Psi}{dk} = \Psi / \mu'(1)k}$$

$$\frac{k}{\Psi}\frac{d\Psi}{dk} = 1/\mu$$

$$k\frac{d\ln\Psi}{dk} = 1/\mu$$

self-sim.

$$\Psi(k) = \begin{cases} v_1|k|^\alpha & k>0 \\ v_2|k|^\alpha & k<0 \end{cases}$$

$$\hat{F}(k) = \begin{cases} \exp(v_1|k|^\alpha) & k>0 \\ \exp(v_2|k|^\alpha) & v<0 \end{cases}$$

or, in more detail,

$$\boxed{\hat{F}(k) = \exp\left[-a|k|^{\alpha}\left(1 - i\beta \tan\frac{\alpha\pi}{2}\,\mathrm{sgn}(k)\right)\right]}$$

skewness

Now, take $\beta = 0$ → Levy Distribution

$$\boxed{L_{\alpha}(a, k) = \hat{F}(k) = \exp\left(-a|k|^{\alpha}\right)}$$

$\alpha = 2$ $\quad \hat{F}(k) = \exp\left[-ak^2\right]$

→ Gaussian

$\alpha = 2$ is case of CLT

$$P(x) = \exp\left[-x^2/a\right]$$

$\alpha = 1$ $\quad \hat{F}(k) = e^{-a|k|}$

→ Cauchy, Lorentzian

$$P(x) = 1/(a^2 + x^2)$$

Note $\alpha = 2$ is max, and only Levy stable distribution with 2nd moment finite.

→ Large x expansion: $0 < \alpha < 2$

$$L_\alpha(a, x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx} e^{-a|k|^\alpha} dk$$

$$= \frac{1}{\pi}\int_0^{\infty} \cos(kx)\, e^{-a|k|^\alpha} dk$$

i.b.p

$$= \frac{a\alpha}{\pi}\int_0^{\infty} dk\, \frac{\sin k|x|}{|k|} \exp\left[-ak^\alpha\right] k^{\alpha-1} dk$$

$$= \frac{\alpha a}{\pi |x|^{1+\alpha}}\int_0^{\infty} \varepsilon^{\alpha-1} \sin\varepsilon \exp\left(\frac{-a\varepsilon^\alpha}{|x|^\alpha}\right) d\varepsilon$$

Now, $\varepsilon = k|x|$

$$d\varepsilon = dk\,|x|$$

$$L_\alpha(a, x) \underset{x\to\infty}{=} \frac{\alpha a}{\pi |x|^{1+\alpha}}\int_0^{\infty} \varepsilon^{\alpha-1} \sin\varepsilon\, d\varepsilon$$

so

$$L_\alpha(a, x) \sim (\ \ )\,/\,|x|^{1+\alpha}$$

$(\;) \sim \dfrac{a\alpha}{\pi}\Gamma(\alpha)\sin\pi\alpha$

Point: Levy Distributions $L_\alpha(a,x)$ have power law tails (fixed pt. fctn $\Leftrightarrow$ self-similarity)

$$L_\alpha(a,x) \simeq 1 \big/ |x|^{1+\alpha}$$

$$\underline{0<\alpha<2}$$

obviously need $\alpha=2$ for convergent 2nd moment.

- [redacted]· width, for N steps, $\sim N^{1/\alpha}$
- $\alpha > 2 \Rightarrow$ super-diffusive.

→ Now, obvious analogy question arises:

CLT : Diffusion :: Levy Distribution : ?

Gaussian $\qquad\qquad L_\alpha(a,X)$

→? Levy Process !

see Zaslavsky, Chapt. 15

- analogue / generalization of diffn.
  → Levy Flights

- time dependent

- Levy dist. at infinitesimal time

Now, C-K Eqn:

(t understood)

$$P(x_0, t_0 \,|\, x_N, t_N) = \int dx_1 \int dx_2 \cdots \int dx_{N-1}$$

$$* \left[P(x_0, t_0; x_1, t_1)\, P(x_1, t_1; x_2, t_2) \cdots P(x_{N-1}, t_{N-1}; x_N, t_N) \right]$$

now $t_{j+1} - t_j = \Delta t$

$t_N - t_0 = N \Delta t$ $\quad N \gg 1$

and assume process uniform in space and stationary in time, so:

so

$$P(x_j, t_j;\ x_{j+1}, t_{j+1}) = P(x_{j+1} - x_j, \Delta t)$$

$\Rightarrow$

$$P(x_N - x_0;\ N\Delta t) = \int dy_1 \cdots \int dy_N\, P(y_1, \Delta t) \cdots P(y_N, \Delta t)$$

Now,

$$P(q) = \int dy_j\, e^{iqy_j} P(y_j, \Delta t)$$

$$\left\{ \begin{aligned} P_N(q) &= \int dy^N \ldots\ e^{iqy^N} P(y^N, N\Delta t) \\ y^N &= \sum_{i=1}^{N} y_j = y_N - y_0 \end{aligned} \right.$$

<u>so</u>

$$P_N(q) = \left[P(q) \right]^N, \quad \text{as before.}$$

(identical steps)

Now, take generating fctn. to be <u>Levy Distribution.</u>

Now,

$$P(\xi) = P(\xi \mid \alpha, c)$$

here:

$$P(\xi) = P_\alpha(\xi, \Delta c)$$

$$= L_\alpha(\xi, \Delta c)$$

and need

$$P_N(\xi) = P_\alpha(\xi, c_N)$$

above consistent if:

$$c_N = \underbrace{N \Delta c}_{N \text{ increments}} = N \Delta t \frac{\Delta c}{\Delta t}$$

$$= (N \Delta t)\, \underset{\text{const}}{c}$$

$$= tc = ct.$$

$$\boxed{\begin{aligned} P_N \to P_\alpha(\xi, ct) &= \exp\left[-c N \Delta t\, |\xi|^\alpha\right] \\ &= \exp\left[-ct\, |\xi|^\alpha\right] \end{aligned}}$$

characteristic fctn of <u>Levy Process</u>

Characteristic fctn:

$$\boxed{P_\alpha(\xi, t) = \exp\left(-ct\,|\xi|^\alpha\right)}$$

$$\boxed{P_\alpha(x, t) = \int \frac{d\xi}{2\pi}\, e^{i\xi x}\, e^{-ct|\xi|^\alpha}}$$

- $\alpha = 2 \Rightarrow$ diffusion ✓
- $\langle x^2 \rangle \to \infty$, for $\alpha < 2$, at any t.

for $x \to \infty$

$$\boxed{P_\alpha(x, t) \sim t \,/\, |\Delta x|^{\alpha+1}}$$

- 'accelerating tail' distribution
  'expanding tail'
- i.e. P equal at $t > t'$
  $\Rightarrow (\Delta x) > (\Delta x)'$.

Next:

- Physics: Weeks, Swinney experiment
- how extend Fokker-Planck Theory?
    - ~ CTRW
    - ~ Fractional Kinetics. } Later

2/06

Notes [redacted] — Physics of Levy Flights, [CTRW
contd and Fractional Kinetics.]

Recently: – introduced L-stable distributions
– ex: Gaussians $\Leftrightarrow$ C.L.T.
– extended range of L-stable, not require finite variance
– used self-similarity

$$\Rightarrow \quad L_\alpha(a, k) = \exp\left[-a|k|^\alpha\right]$$

Levy Distrib $\qquad 0 < \alpha \leq 2$

$\alpha \rightarrow$ power law

$a \rightarrow$ akin extended strength parameter (i.e. D)

of course: $\displaystyle P_\alpha(x, a) = \int \frac{dk}{2\pi} e^{ikx} L_\alpha(a, k)$

Properties: $L_\alpha > 0$ $\qquad \left(\Psi(k\mu\alpha) = \lambda\Psi\right)$

$$\int dx\, P_\alpha(a, x) < \infty$$

obviously: $\displaystyle P_\alpha(x, a) \xrightarrow[x \to \infty]{} 1/|x|^{\alpha+1}$

$\alpha \to \mu$

$\alpha = 2 \rightarrow$ Gaussian.

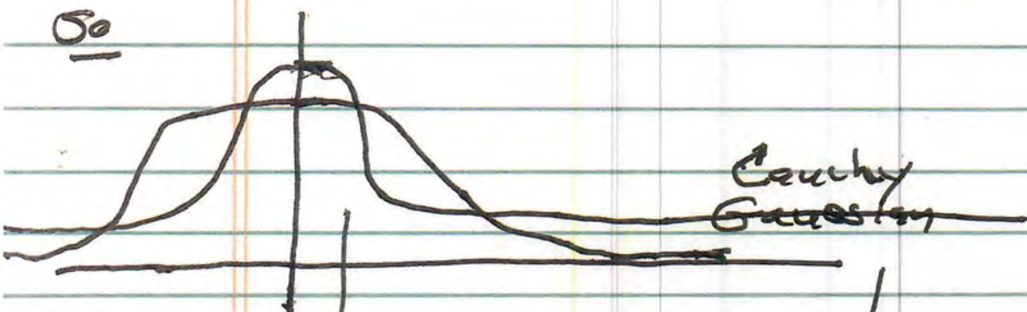
$$\frac{d\Psi}{d\ln k} = \Psi/\mu(1)\ln$$

B. D. Hughes

$\alpha = 1 \rightarrow$ Cauchy (Lorentzian)

$$P_1(x) = a/\pi(x^2+a^2)$$

So



fatter tail

$\alpha < 2$ sharper peak

And can consider time-dependent evolution, analogous to diffusion

- first, if discretize:

$$P_N(x) = \frac{1}{N^{1/\alpha}} L_\alpha\left(a, x/N^{1/\alpha}\right)$$

width $\sim N^{1/\alpha}$

$\alpha < 2$, $N^{1/\alpha} \gg N^{1/2} \rightarrow$ super-diffusive

- making time explicit:

$$P_\alpha(k, ct) = \exp\left[-ct|k|^\alpha\right]$$

$\alpha \to$ index
c ~~$\leftrightarrow$~~ $a \to$ strength

characteristic
Fctn.

$$P_\alpha(x, t) \underset{x\to\infty}{\longrightarrow} t / |x|^{\alpha+1}$$

N.B. Can see description of Levy process involves fractional calculus.

i.e.

$$P_\alpha(x, t) = \int \frac{e^{ikx}}{\sqrt{2\pi}} P_\alpha(k, t)$$

so $\partial_t P_\alpha(x, t) =$

$$\int \frac{e^{ikx}}{\sqrt{2\pi}} (-c|k|^\alpha) \exp\left[-ct|k|^\alpha\right]$$

$\Downarrow$

Fractional derivative for $\alpha \neq 2$.

if $\alpha = 2$
$c \Rightarrow D$

$$\partial_t P_{\alpha=2}(x,t) = \int \frac{e^{ikx}}{\sqrt{2\pi}} (-Dk^2) \exp\left[-Dtk^2\right]$$

$$= D\frac{\partial^2}{\partial x^2} \int \frac{e^{ikx}}{\sqrt{2\pi}} \exp\left[-Dtk^2\right]$$

$$= D\frac{\partial^2}{\partial x^2} P_2(x,t)$$

i.e. $\frac{\partial^2}{\partial x^2} \Leftrightarrow -k^2$

$\partial/\partial x \Leftrightarrow ik \Rightarrow (\partial/\partial x)^\alpha \Leftrightarrow (ik)^\alpha$

so $(ik)^\sigma \Leftrightarrow (\partial/\partial x)^\sigma$

meaning of fractional calculus
(i.e. integral transform defines
fractional derivative)

→ what does it all mean ??

– Levy Walks self-similar

i.e. even diffn $\Rightarrow \frac{x^2}{t} \to \frac{\lambda^2 x^2}{\delta t}$

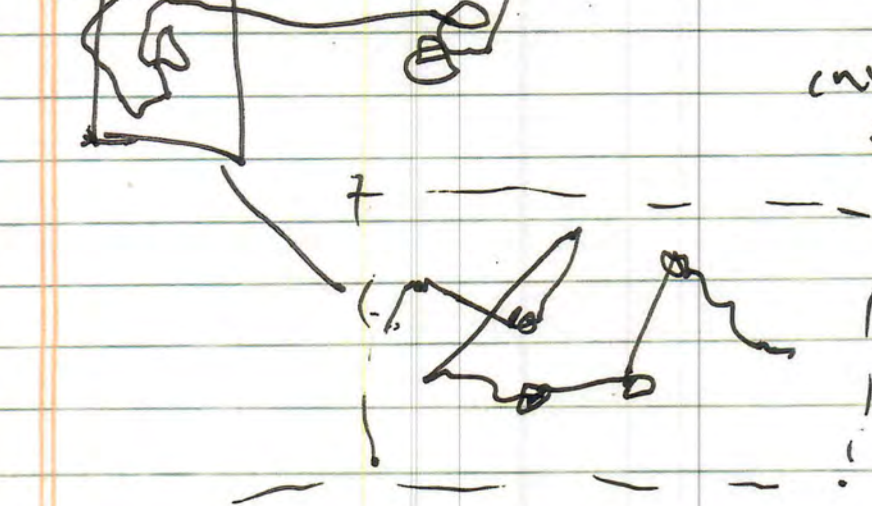
invariant under $\delta \sim \lambda^2$.

– some very large excursions (contrast standard random walk)

$\Rightarrow$ fat tail, large events ✓

weighted more

i.e.



large jumps → flights

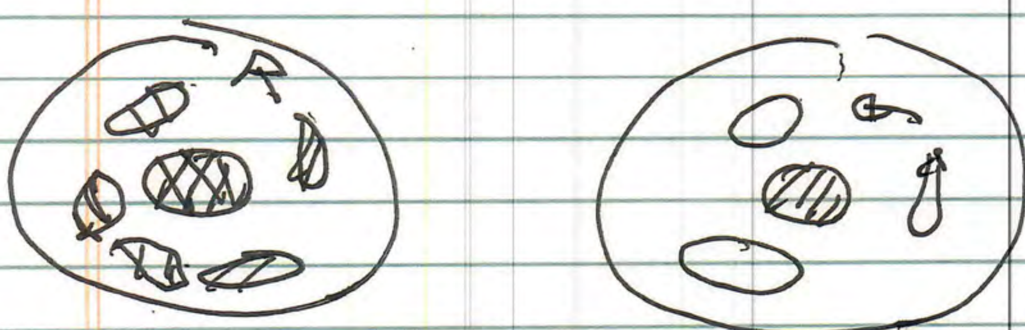
invariant under zoom.

- a bit of physics:
  Solomon, Weeks, ... Swinney et al. PRL '93
  a <u>must</u>

→ rotating tank, water pumped in, out via bottom.

→ sheared, counter-rotating azimuthal jet.

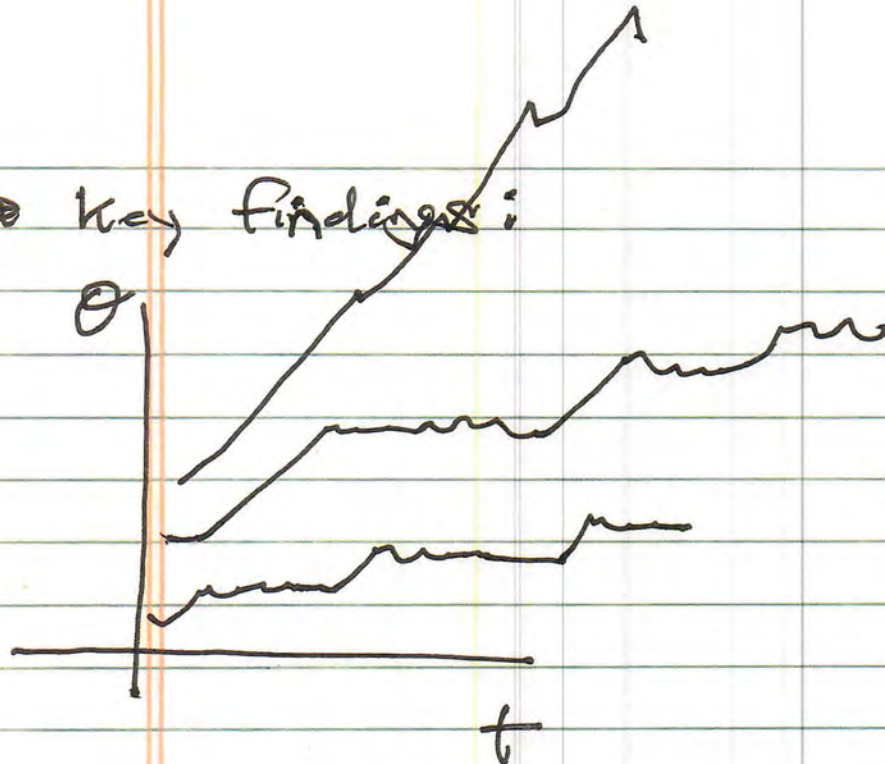
→ passive tracers injected



→ azimuthal vortex chain (6)

→ follow tracer trajectories, which are "chaotic" but no "turbulence" (flow chaotic, not turbulent)

→ key findings:



obviously:

→ are circulations in individual vortex.

tracer 'stuck' in individual vortex

→ are jumps or "flights" between vortices.

These are the large excursions of the Levy walk.

→ Can compute $\langle (\Theta - \Theta_0)^2 \rangle$

and $\langle (\Theta - \Theta_0)^2 \rangle \sim \underbrace{t^{1.6}}$

→ Levy Flight →

→ <u>super-diffusive</u>

$1/2 < H < 1$ (not calculated)

→ anomalous diffusion

N.B. → Anomalous Transport Confinement Application

$\Rightarrow D > D_{\text{coll.}}$

Anomalous Diffusion

$\langle \delta x^2 \rangle \sim t^{\alpha}$

$\alpha \neq 1$

→ Analogy → marker particles in - island chain (positrons) (chaos)

- Zonal flow + vortex

(novelty ?)

→ How Live and calculate in Levy World
— A Short Introduction?
— Coming...

— how calculate anything for diffusion processes, noise, etc.?

⇒ Fokker-Planck Theory → yields ~~this~~

— What is F-P Eqn?

~ recipe for turning (micro) or step pdf into eqn. for distribution

~ can calculate D, V, $P(x,t)$
$\underbrace{P(x,t)}_{t \to \infty}$

of course:

$$P(x, t+\Delta t) = \int d(\Delta x)\, \underset{\uparrow}{T}(\underline{x}, \underline{\Delta x}, \Delta t)\, P(\underline{x} - \underline{\Delta x}, t)$$

~~first~~ 1 step eqn.

transition probability

etc