Physics 210 B

Lecture 4a: Transport Coefficients and Viscous Fluids

Recall:

- Boltzmann Eqn.
- $dS/dt \geq 0$
- Fluid Equations (Ideal) - Macroscopic Conservation

$$\frac{\partial n}{\partial t} + \underline{\nabla} \cdot (n \underline{V}) = 0$$

$$m \partial_t (n \underline{V}) = - \underline{\nabla} \cdot \underline{\underline{\Pi}} \rightarrow \partial_t (\rho \underline{V}) = - \underline{\nabla} \cdot \underline{\underline{\Pi}}$$

$$\underline{\underline{\Pi}} = \underbrace{mn \underline{V}\, \underline{V}}_{\text{Reynolds stress}} + p \underline{\underline{I}}$$ (ideal stress tensor)

$$\Rightarrow \boxed{\rho \left(\partial_t \underline{V} + \underline{V} \cdot \underline{\nabla}\, \underline{V} \right) = - \underline{\nabla} p}$$

Euler equation.

– Ideal Fluids : $f = f_{eq}$

$f_{eq} \rightarrow$ annihilates $C(f)$

$\rightarrow$ local Maxwellian

$l_{mfp} / L \rightarrow 0$.

$\boxed{\text{Inhomogeneity critical !}}$

But $\Rightarrow$ if $T(\underline{x})$, $\underline{V}(\underline{x})$, $n(\underline{x})$ etc

f_{eq} does not solve Boltzmann equation !

i.e.

$$\partial_t f + \underline{v} \cdot \underline{\nabla} f = C(f)$$

$$\underbrace{\cancel{\partial_t f_{eq}}}_{0} + \underline{v} \cdot \underbrace{\underline{\nabla}}_{\neq 0} f_{eq} \neq C(f_{eq}) = 0$$

so will need a correction, i.e.

$$f = f_{eq} + \delta f$$

so – Magnitude ? ⇒ Calculate δf

– Implications ? ⇒ impact/ contribution to fluid equations.

⇒ flux

so

Calculating δf:

– assign time scales

$$\partial_t f + \underline{v} \cdot \underline{\nabla} f = C(f)$$

$\partial_t f \to \omega$

$\underline{v} \cdot \underline{\nabla} f \to v_{Th}/L \to$ Gradient scale (subtlety)

$C(f) \to \nu \to$ Collision frequency

Recall basic ordering:

$$d < \bar{r} < l_{mfp} < L$$

$\bar{r} \sim n^{-1/3}$

$\nu = v_{Th}/l_{mfp}$

$l_{mfp} = 1/n\sigma$

$\nu \approx v_{Th}/l_{mfp}$.

so $v_{Th}/L < \nu$.

$\Rightarrow$ f_{eq} is zeroth order solution.

δf systematics $\Rightarrow$ Chapman-Enskog Expansion

Simple Perspective:

$$\underbrace{\frac{\partial f}{\partial t} + \underline{v} \cdot \underline{\nabla} f}_{O(v_{Th}/L)} = \underbrace{C(f)}_{O(\nu)}$$

$\Rightarrow$ really integral equation.

Clearest in context of: Krook Model

(c.f. really "Crook" Model – H. Grad)

$$\boxed{C(f) = -\nu\,(f - f_{eq})}$$

$\nu(|v|)$ sometimes

$\downarrow$

decay to local Maxwellian on collision time scale.

N.B.: why "Crook"?

what is unsatisfactory?

So $\underline{v} \cdot \underline{\nabla} f = C(f) = -\nu(f - f_{eq})$

$$f = f_{eq} + \delta f$$

$$\underset{\substack{\downarrow \\ O(v/L)}}{\underline{v} \cdot \nabla (f_{eq} + \delta f)} = \underset{\substack{\downarrow \\ O(\nu)}}{-\nu (f - f_{eq})}$$

l.o. $0 = -\nu(f - f_{eq}) \Rightarrow C(f) = 0$

$f = f_{eq}$ l.o. solution is local Maxwellian

i.e. $C(f) = 0 \Rightarrow f = f_{eq}$

$$f - f_{eq} = \delta f$$

then:

1^{st} Order:

$$\underline{v} \cdot \nabla (f_{eq} + \delta f) = -\nu \, \delta f$$

so

$$\delta f = - \frac{\underline{v} \cdot \nabla f_{eq}}{\nu}$$

$$\cong O\underbrace{\left(\frac{l_{mfp}}{L}\right)}_{\ll 1} f_{eq}$$

$$\frac{\delta f}{f_{eq}} \sim \frac{l_{mfp}}{L}$$

correction to f_{eq}!

so $$\delta f \cong - \frac{\underline{v} \cdot \underline{\nabla} f_{eq}}{\nu}$$

$$\boxed{f \doteq f_{eq} - \frac{\underline{v} \cdot \nabla}{\nu} f_{eq}}$$

Could continue to $O(l_{mfp}/L)^2$, etc.

Better: - Moment Truncation (coming)

- Rigorous Calculation - Later.

⇒ What does it mean ?

Now, in calculating stresses, or flux, need: $\boxed{\text{Include } \delta f}$

so de

$$\langle V_\alpha' V_\beta' \rangle = \int d^3v \, (f_0 + \delta f) \, V_\alpha' V_\beta'$$

$\boxed{\text{new}}$

$\rho \delta_{\alpha,\beta}$ viscous stress

$\sim -\eta \, \underline{\nabla} \, \overline{\underline{V}}$

viscosity   (tensor)

(transport coefficient)

N.B Transport Coefficient:

$$(\text{Flux}) \sim [\text{Coeff}] \, [\text{Gradient}]$$

so, for $\underline{\nabla} \cdot \underline{V} = 0$,

$$\underset{\text{Flux}}{mn \langle V'_\alpha V'_\beta \rangle} = -\underset{\text{coeff}}{\eta} \underset{\text{Gradient}}{\frac{\partial \overline{V}_\alpha}{\partial x_\beta}}$$

recall $\rightarrow \partial$ $\alpha \neq \beta$ for ideal fluid

viscosity

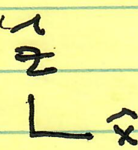
$mn D \sim \eta$

$D \sim l_{mfp} V_{th}$

Simple Calculation – Shear Viscosity

Seek $\underline{1}$ element in viscous stress tensor,

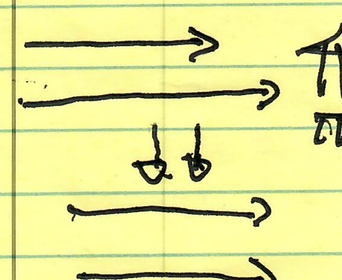
$\Pi_{z,x} \equiv \hat{z}$ direction flux of $\hat{x}$ - direction momentum



$\hat{z}$ flux

$$\Pi_{zx} = \int d^3v \; V_z \, (m V_x f)$$

$\hat{x}$ momentum



so, from Chapman – Enskog Expansion:

$$f = f_{eq} + \delta f$$

$$\Pi_{z,x} = \int d^3v \; v_z \left(m v_x \left(f_{eq} + \delta f \right) \right)$$

Take $\underline{V} = V(z)\,\hat{x}$, so:

(#)

$$f_{eq} \cong \frac{n_0}{v_{th}^3} \exp\left(\frac{-v_y^2}{2v_{th}^2} \right) \exp\left(\frac{-v_z^2}{2v_{th}^2} \right) \times$$

$$\exp\left(\frac{-(v_x - V(z))^2}{2v_{th}^2} \right)$$

~ local, shifted Maxwellian.

odd in v_z !

$$\Pi_{z,x} = \int d^3v \; v_z \left(m v_x \left(\cancel{f_{eq}} + \delta f \right) \right)$$

$$= \int d^3v \; v_z \, m v_x \left(- \frac{\underline{v} \cdot \underline{\nabla}}{\nu} f_{eq} \right)$$

– only δf drives/survives the viscous stress.

$$\Pi_{z,x} = \int d^3v \; v_z\, n m v_x \left(-\frac{v_z}{\nu}\frac{\partial}{\partial z} f_{eq}\right)$$

$$\frac{\partial f_{eq}}{\partial z} \cong \frac{n}{v_{th}^3}\left\{\frac{v_x}{v_{th}^2}\frac{\partial V(z)}{\partial z}\exp\left[\quad\right] - \cancel{\frac{2V(z)}{v_{th}^2}\frac{\partial V(z)}{\partial z}}\exp\left[\quad\right]\right\}$$

NL → dropped.

interested in linear response of flux to gradient — week distortion from Maxwellian

$$\frac{\partial f_{eq}}{\partial z} \cong f_0 \frac{v_x}{v_{th}^2}\frac{\partial V(z)}{\partial z}$$

Aside: Alternatively,

$$f_{eq} \cong \frac{n}{v_{th}^3}\exp\left[-\frac{(\underline{v}-V(z)\hat{x})^2}{2v_{th}^2}\right]$$

$$\cong \frac{n}{v_{th}^3}\exp\left[-\frac{(v^2-2v_xV(z)+V(z)^2)}{2v_{th}^2}\right]$$

$$\cong f_{eq}\left(1 + \frac{v_x \bar{V}}{v_{th}^2}\right)$$

c.f.: Upcoming on linear response theory.

Plugging it all in:

$$\Pi_{z,x} = \int d^3x \; v_z \, m v_x \left(\frac{-v_z v_x}{\nu v_{th}^2} \right) f_{eq} \, \frac{\partial \bar{V}_x(z)}{\partial z}$$

all even

$$\cong -(\#) \, \frac{m n}{\nu} \, v_{th}^2 \, \frac{\partial \bar{V}_x(z)}{\partial z}$$

— Basic result for shear viscosity

$$\boxed{\begin{aligned} \Pi_{z,x} &= -(\#) \, m n D \, \frac{\partial \bar{V}_x(z)}{\partial z} \\ &= -(\#) \, \rho D \, \frac{\partial \bar{V}_x(z)}{\partial z} \end{aligned}}$$

$D = v_{th} \, l_{mfp}$

Comments:

- for $\underline{\nabla}\cdot\underline{V} = 0$

$$\Pi_{jk} = \eta\left(\frac{\partial V_i}{\partial x_k} + \frac{\partial V_k}{\partial x_i} - \frac{2}{3}\delta_{jk}\frac{\partial V_\ell}{\partial x_\ell}\right)$$

$\eta = \rho\nu$ - shear viscosity

↳ kinematic viscosity (diffusion coeff)

$\eta \Rightarrow$ friction in/between sliding layers

so <u>Navier-Stokes Equation</u>:

= Euler + viscous stress

$\Leftrightarrow$ from $f = f_{eq} \rightarrow f = f_{eq} + \delta f$

δf: 1st order in Chapman-Enskog

$$\boxed{\frac{\partial \underline{V}}{\partial t} + \underline{V}\cdot\underline{\nabla}\,\underline{V} = -\frac{\underline{\nabla} P}{\rho} + \frac{\eta}{\rho}\nabla^2\underline{V}}$$

$\underline{\nabla}\cdot\underline{V} = 0$

N.B. – $\nu \to 0 \iff \frac{l_{mfp}}{L} \to 0$

is singular perturbation !

– N.S. for small $\nu \iff$

high Re $\iff$ different from

$\nu = 0$, Euler.

– for compressible fluid: ζ

$\zeta \neq$ compressive viscosity

$$\rho \left[\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \underline{\nabla}\, \underline{v} \right] = - \underline{\nabla} p + \mu \nabla^2 \underline{v} + \left(\zeta + \frac{1}{3}\mu \right) \underline{\nabla} (\underline{\nabla} \cdot \underline{v})$$

– some ν: $\mu = \rho \nu$

H_2O $\nu = .01$ cm²/sec Mercury
air $\nu = .15$ cm²/sec $\nu = .001$ cm²/sec
Glycerine $\nu = 6.8$ cm²/sec

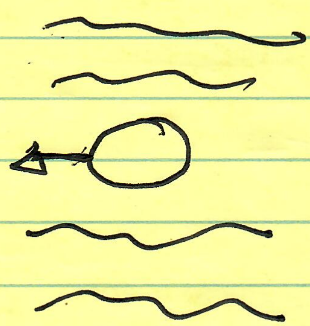
N.B. Why air more viscous than water (at equal temperatures).

- predictably, heat transport, χ etc. similarly.

- Why bother? - i.e. $l_{mfp}/L \ll 1$

→ channel for internal dissipation

→ flow structure [sphere in fluid]

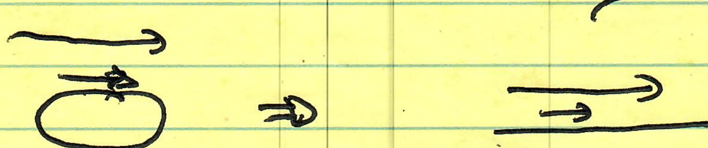


→ far from sphere, ideal dynamics is good approx

→ Euler works.

Potential flow

near



No slip boundary condition!

i.e. $\left. V_{tan} \right|_{surf} = 0$

bndry layer:

⇒ interpolates between potential flow and surface

→ $w \sim (\nu R/u)^{1/2}$

→ region where $|\underline{V}|$ small, but $\partial \underline{V}/\partial \underline{x}$ large!

⇒ viscous stress.

Fluids dynamics very sensitive to Boundary Layers.