Lecture Notes on Classical Mechanics
(A Work in Progress)

Daniel Arovas
Department of Physics
University of California, San Diego

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0.1 Preface

These lecture notes are based on material presented in both graduate and undergraduate mechanics classes which I have taught on several occasions during the past 30 years at UCSD (Physics 110A-B and Physics 200A-B).

The level of these notes is appropriate for an advanced undergraduate or a first year graduate course in classical mechanics. In some instances, I've tried to collect the discussion of more advanced material into separate sections, but in many cases this proves inconvenient, and so the level of the presentation fluctuates.

My presentation and choice of topics has been influenced by many books as well as by my own professors. I've reiterated extended some discussions from other texts, such as Barger and Olsson's treatment of the gravitational swing-by effect, and their discussion of rolling and skidding tops. The figures were, with very few exceptions, painstakingly made using Keynote and/or SM.

My only request, to those who would use these notes: please contact me if you find errors or typos, or if you have suggestions for additional material. My email address is `darovas@ucsd.edu`. I plan to update and extend these notes as my time and inclination permit.

Chapter 0

Reference Materials

Here I list several resources, arranged by topic. My personal favorites are marked with a diamond ($\diamond$).

0.1 Lagrangian Mechanics

- $\diamond$ Alain J. Brizard, *An Introduction to Lagrangian Mechanics*, 2nd ed. (World Scientific, 2015)
- $\diamond$ A. L. Fetter and J. D. Walecka, *Nonlinear Mechanics* (Dover, 2006)
- L. D. Landau and E. M. Lifshitz, *Mechanics*, 3rd ed. (Butterworth-Heinemann, 1976)
- O. D. Johns, *Analytical Mechanics for Relativity and Quantum Mechanics* (Oxford, 2005)
- D. T. Greenwood, *Classical Mechanics* (Dover, 1997)
- H. Goldstein, C. P. Poole, and J. L. Safko, *Classical Mechanics*, 3rd ed. (Addison-Wesley, 2001)
- V. Barger and M. Olsson, *Classical Mechanics : A Modern Perspective* (McGraw-Hill, 1994)

0.2 Hamiltonian Mechanics

- $\diamond$ R. Z. Sagdeev, D. A. Usikov, and G. M. Zaslavsky *Nonlinear Physics* (Harwood, 1998)

- ◇ E. Ott, *Chaos in Dynamical Systems*, 2nd ed. (Cambridge, 2002)
- • J. V. José and E. J. Saletan, *Mathematical Methods of Classical Mechanics* (Springer, 1997)
- • V. I. Arnold *Introduction to Dynamics* (Cambridge, 1982)
- • V. I. Arnold, V. V. Kozlov, and A. I. Neishtadt, *Mathematical Aspects of Classical and Celestial Mechanics* (Springer, 2006)
- • I. Percival and D. Richards, *Introduction to Dynamics* (Cambridge, 1982)
- • W. Dittrich and M. Reuter, *Classical and Quantum Dynamics* (Springer, 2001)

0.3 Mathematics

- ◇◇ C. M. Bender and S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers* (Springer, 1999)
- ◇◇ S. H. Strogatz, *Nonlinear Dynamics and Chaos* (CRC Press, 2000)
- ◇ I. M. Gelfand and S. V. Fomin, *Calculus of Variations* (Dover, 1991)
- ◇ R. Weinstock, *Calculus of Variations* (Dover, 1974)
- ◇ V. I. Arnold, *Ordinary Differential Equations* (MIT Press, 1973)
- ◇ V. I. Arnold, *Geometrical Methods in the Theory of Ordinary Differential Equations* (Springer, 1988)

Chapter 1

Introduction to Dynamics

1.1 What is Dynamics?

Loosely speaking, dynamics is the study of mathematical models of "what happens next," which is to say how systems evolve in time. There are four main elements to dynamics:

(i) The **initial conditions**, or "how things are now."

(ii) The **equations of motion**, which encode how a given system evolves. There are two broad classes to speak of: *difference equations* which describe evolution in discrete time steps, and *differential equations* which describe continuous time evolution.

(iii) There may be a *random* component to the evolution, in which case the equations of motion are said to be **stochastic**.

(iv) The *solution* of the equations of motion, given the initial conditions, tells us the **motion** of the system, *i.e.* "how things will be in the future." For stochastic systems, we cannot compute the motion itself, but rather only statistical properties thereof, such as the average position(s) at some future time.

Our main concern will be in applying these mathematical models to physical mechanical systems: balls and springs, celestial bodies, spinning tops, *etc.*, which are the purview of *classical mechanics*. In classical mechanics, the equations of motion describe continuous time dynamics of each system's various *degrees of freedom* in the form of coupled second order ordinary differential equations[1], which are nothing more than Newton's second law $F = ma$. In one space dimension, for example, we have $m\,d^2x/dt^2 = F(x)$. Such systems are special, and constitute a restricted class of the general family of continuous time dynamical systems. For example, if the forces are derivable from a potential energy function, then there is

[1]The degrees of freedom are the positional coordinates for point particles and the orientational coordinates for rigid bodies. In the case of nonrigid continuous systems, like strings, membranes, and elastic media, the equations of motion are partial differential equations involving both space and time. Continuum mechanics is discussed below in chapter 10.

a conserved quantity, which is the total energy[2]. We will derive the equations of motion, *i.e.* Newton's laws, using a powerful variational principle known as the *principle of extremal action*, which lies at the foundation of Lagrange's approach to mechanics. A related and even more powerful approach, due to Hamilton, is the subject of graduate level mechanics courses.

Lets start by considering some examples.

1.1.1 Simple difference equation

Consider the difference equation

$$x_{n+1} = x_n + \alpha \quad , \tag{1.1}$$

where $x_n \in \mathbb{R}$ is the position of a point object at discrete time step n, and $\alpha \in \mathbb{R}$ is a real number. The initial conditions are specified by x_0, which is the position at discrete time step $n = 0$.

Clearly the position advances by α with each step, and thus the motion of the system is given by

$$x_n = x_0 + n\alpha \quad . \tag{1.2}$$

1.1.2 Another difference equation: Fibonacci numbers

Next, consider the difference equation

$$x_{n+1} = x_n + x_{n-1} \quad . \tag{1.3}$$

The initial conditions are now specified by two values, x_0 and x_1. Given these, we can compute $x_2 = x_1 + x_0$, $x_3 = x_2 + x_1 = 2x_1 + x_0$, *etc.* Can we obtain a general expression for x_n? Yes we can! Let's try a solution of the form $x_n = A\lambda^n$ where A and λ are as yet undetermined. We stick this into eqn. 1.3 and obtain the relation

$$\lambda^2 - \lambda - 1 = 0 \quad , \tag{1.4}$$

which has two solutions,

$$\lambda_\pm = \tfrac{1}{2}(1 \pm \sqrt{5}) = \big\{1 + \phi \, , \, -\phi\big\} \quad , \tag{1.5}$$

where $\phi = \frac{1}{2}(\sqrt{5} - 1) = 0.618034\ldots$ is the golden mean. Thus we write

$$x_n = A_+ \, \lambda_+^n + A_- \, \lambda_-^n \quad . \tag{1.6}$$

Imposing the initial conditions by setting $n = 0$ and $n = 1$ then yields the relations

$$\begin{pmatrix} 1 & 1 \\ \lambda_+ & \lambda_- \end{pmatrix} \begin{pmatrix} A_+ \\ A_- \end{pmatrix} = \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \quad , \tag{1.7}$$

and thus

$$\begin{pmatrix} A_+ \\ A_- \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \lambda_+ & \lambda_- \end{pmatrix}^{-1} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \frac{1}{\lambda_+ - \lambda_-} \begin{pmatrix} x_1 - \lambda_- x_0 \\ \lambda_+ x_0 - x_1 \end{pmatrix} \quad . \tag{1.8}$$

[2]More precisely, the conserved quantity is the *Hamiltonian* H, which may differ from the total energy E, as we shall discuss in §4.13.2 below.

The full motion of the system is then given by

$$x_n = \frac{1}{1+2\phi}\Big\{(1+\phi)^n\big[\phi x_0 + x_1\big] + (-\phi)^n\big[(1+\phi)x_0 + x_1\big]\Big\} \tag{1.9}$$

For the initial conditions $x_0 = 0$ and $x_1 = 1$, we obtain

$$x_n = \frac{1}{1+2\phi}\Big\{(1+\phi)^n - (-\phi)^n\Big\} = F_n \quad , \tag{1.10}$$

i.e. the n^{th} Fibonacci number: $F_n = \{1, 1, 2, 3, 5, 8, 13, \ldots\}$ starting from $n = 1$. Did you know that such a closed form expression for all the Fibonacci numbers can be derived?

1.1.3 Stochastic difference equation: diffusion

Now consider the stochastic difference equation

$$x_{n+1} = x_n + \sigma_n \quad , \tag{1.11}$$

where the $\{\sigma_n\}$ are independent, identically distributed ('IID' in statistics parlance) random numbers whose distribution is given by

$$\mathsf{Prob}[\sigma_n = \varepsilon] = p\,\delta_{\varepsilon,+1} + q\,\delta_{\varepsilon,-1} = \begin{cases} p & \text{if } \varepsilon = +1 \\ q & \text{if } \varepsilon = -1 \end{cases} \quad , \tag{1.12}$$

with $p \in [0,1]$. Since there are only two possibilities for each σ_n, the sum of their probabilities must be unity, *i.e.* $p + q = 1$, which fixes $q = 1 - p$. This system corresponds to a one-dimensional random walk, where the probability of a step to the right, *i.e.* $x_{n+1} = x_n + 1$, is p, and the probability of a step to the left, *i.e.* $x_{n+1} = x_n - 1$, is q. The initial conditions are given by the value of x_0. Clearly we have

$$x_n = x_0 + \sigma_1 + \sigma_2 + \ldots + \sigma_n = x_0 + \sum_{j=1}^{n} \sigma_j \quad . \tag{1.13}$$

We can now compute averages with respect to the random distribution:

$$\begin{aligned} \langle x_n \rangle &= x_0 + \sum_{j=1}^{n} \langle \sigma_j \rangle \\ \langle x_n^2 \rangle &= x_0^2 + 2x_0 \sum_{j=1}^{n} \langle \sigma_j \rangle + \sum_{j=1}^{n}\sum_{k=1}^{n} \langle \sigma_j \sigma_k \rangle \quad . \end{aligned} \tag{1.14}$$

We will need

$$\begin{aligned} \langle \sigma_j \rangle &= \sum_{\varepsilon=\pm 1} \varepsilon\, \mathsf{Prob}[\sigma_j = \varepsilon] \\ &= p - q = 2p - 1 \end{aligned} \tag{1.15}$$

and

$$\langle \sigma_j \sigma_k \rangle = \begin{cases} 1 & \text{if } j = k \\ (2p-1)^2 & \text{if } j \neq k \end{cases} \quad , \tag{1.16}$$

because $\sigma_j^2 = 1 \;\Rightarrow\; \langle \sigma_j^2 \rangle = 1$ and $\langle \sigma_j \sigma_k \rangle = \langle \sigma_j \rangle \langle \sigma_k \rangle = (2p-1)^2$ for $j \neq k$ (IID). Thus,

$$\begin{aligned} \langle x_n \rangle &= x_0 + (2p-1)n \\ \langle x_n \rangle^2 &= x_0^2 + 2x_0(2p-1)n + (2p-1)^2 n^2 \\ \langle x_n^2 \rangle &= x_0^2 + 2x_0(2p-1)n + (2p-1)^2 n(n-1) + n \end{aligned} \tag{1.17}$$

and

$$\begin{aligned} \big\langle (\Delta x_n)^2 \big\rangle &\equiv \big\langle \big[x_n - \langle x_n \rangle \big]^2 \big\rangle \\ &= \langle x_n^2 \rangle - \langle x_n \rangle^2 = 4p(1-p)n \quad . \end{aligned} \tag{1.18}$$

When $p \neq q$ the random walk is *biased*, and there is an unequal probability of stepping to the right and to the left. Thus $\langle x_n \rangle = x_0 + (2p-1)n$ on average changes by $\langle \sigma_j \rangle = p - q = 2p - 1$ during each time step. If we start at x_0 and then execute our random walk, after n time steps we know that we will end up at some point x_n between $x_0 - n$ (all steps to the left) and $x_0 + n$ (all steps to the right). Where we end up will likely change each time we rerun the experiment, but if we average over a great many experiments, we will obtain $\langle x_n \rangle = x_0 + (2p-1)n$. But while the average of the difference Δx_n between x_n and its mean $\langle x_n \rangle$ vanishes, the average of its square $\big\langle (\Delta x_n)^2 \big\rangle$ grows linearly with n. The *root mean square* variation then grows as $n^{1/2}$, *viz.*

$$\Delta x_n^{\mathsf{RMS}} = \sqrt{\big\langle (\Delta x_n)^2 \big\rangle} = 2\sqrt{pqn} \quad . \tag{1.19}$$

This is an example of *diffusion*.

1.1.4 Nonlinear discrete dynamics: the logistic map

Consider the simple case of a general one-dimensional *map*,

$$x_{n+1} = g(x_n) \quad , \tag{1.20}$$

where $g(x) : \mathbb{R} \to \mathbb{R}$ is a real function of a real number. A *fixed point* of this map satisfies $g(x) = x$. Some maps have no fixed points, such as $g(x) = x + 1$. For $g(x) = x$, *every* point is a fixed point. This last example is highly nongeneric; generically the set of fixed points - if there are any fixed points at all - is discrete.

Let's focus in on what happens when x is close to some fixed point x^* and write $x_n = x^* + u_n$ with $|u_n| \ll 1$. Then

$$u_{n+1} = g(x^* + u_n) - x^* = g'(x^*)\, u_n + \tfrac{1}{2}\, g''(x^*)\, u_n^2 + \ldots \quad . \tag{1.21}$$

Here we have used Taylor's theorem to expand $g(x^* + u_n)$ in powers of the small quantity u_n. If we drop all the terms in the Taylor series which are beyond linear in u_n , we obtain the equation $u_{n+1} = \kappa\, u_n$, where $\kappa = g'(x^*)$. The solution is $u_n = \kappa^n u_0$ and we conclude

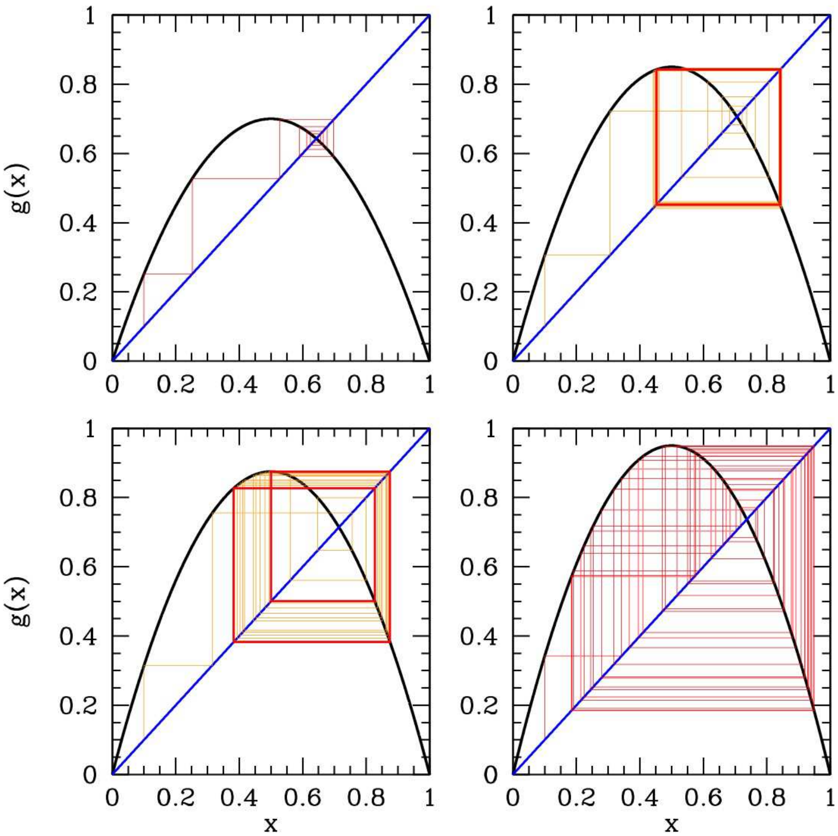

Figure 1.1: Cobweb diagram showing iterations of the logistic map $g(x) = rx(1-x)$ for $r = 2.8$ (upper left), $r = 3.4$ (upper right), $r = 3.5$ (lower left), and $r = 3.8$ (lower right). Note the single stable fixed point for $r = 2.8$, the stable two-cycle for $r = 3.4$, the stable four-cycle for $r = 3.5$, and the chaotic behavior for $r = 3.8$.

- If $|g'(x^*)| < 1$ then $|u_{n+1}| < |u_n|$ and the magnitude of u_n decreases exponentially with n:

$$u_n = (\pm 1)^n e^{-\alpha n} \, u_0 \quad , \tag{1.22}$$

  where $\alpha = -\log |g(x^*)| > 0$ and we take the $+$ sign if $g(x^*) > 0$ and the $-$ sign if $g(x^*) < 0$. The approximation to neglect higher order terms in the Taylor series expansion of $g(x^* + u_n)$ gets better and better as n increases. A fixed point x^* with $|g'(x^*)| < 1$ is called a *stable fixed point* (SFP).

- If $|g(x^*)| > 1$, then $|u_{n+1}| > |u_n|$ and the magnitude of u_n increases exponentially with n. Successive iterations of the map move us further and further away from x^*. However, at some point the higher order terms which we've neglected in the Taylor expansion of $g(x^* + u_n)$ become non-negligible, and the behavior is no longer exponential. A fixed point x^* for which $|g(x^*)| > 1$ is called an *unstable fixed point* (UFP).

Perhaps the most important and most studied of the one-dimensional maps is the logistic map, where $g(x) = rx(1-x)$, defined on the interval $x \in [0,1]$, with $r \in [0,4]$. There is a fixed point at $x = 0$ which is stable for $r < 1$ and unstable for $r > 1$. When $r > 1$, a new fixed point is present, at $x^* = 1 - r^{-1}$ if

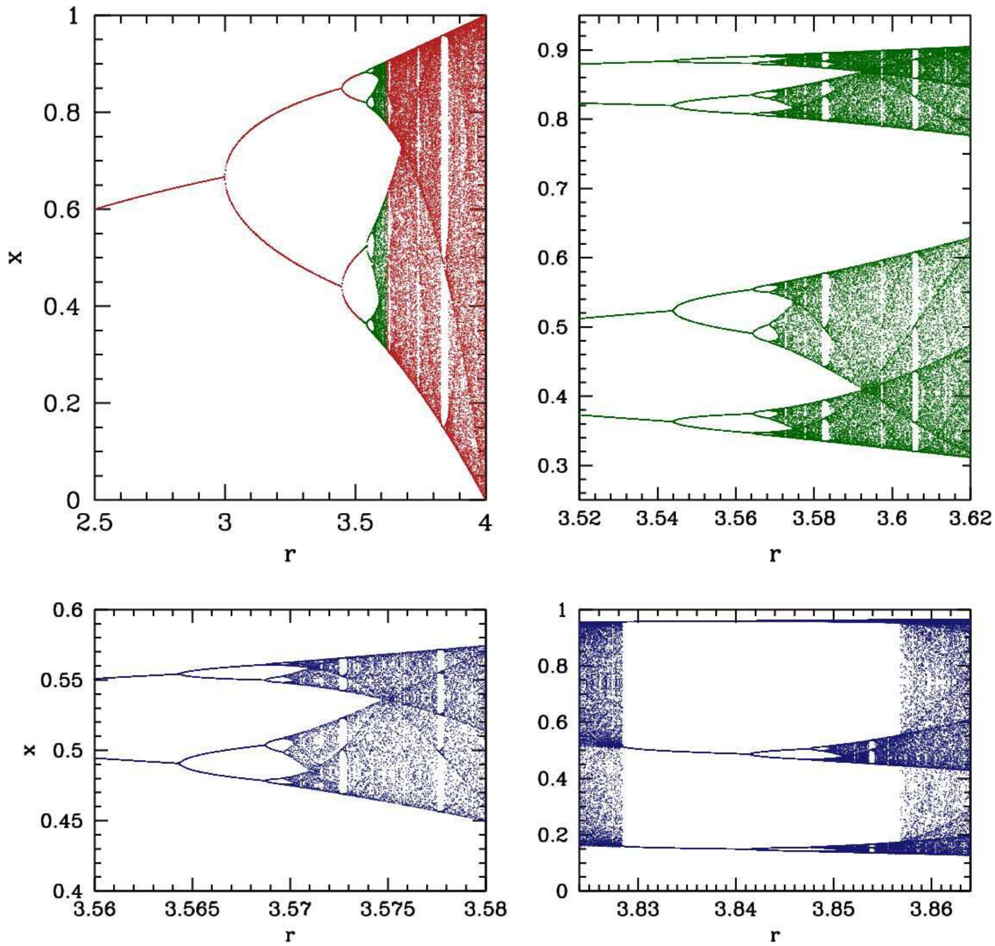

Figure 1.2: Iterates of the logistic map $g(x) = rx(1-x)$.

$r > 1$. We then have $g'(x^*) = 2 - r$, so the fixed point is stable if $r \in (1, 3)$. What happens for $r > 3$? We can explore the behavior of the iterated map by drawing a *cobweb diagram*, shown in fig. 17.9. We sketch, on the same graph, the curves $y = x$ (in blue) and $y = g(x)$ (in black). Starting with a point x on the line $y = x$, we move vertically until we reach the curve $y = g(x)$. To iterate, we then move horizontally to the line $y = x$ and repeat the process. We see that for $r = 3.4$ the fixed point x^* is unstable, but there is a stable two-cycle, defined by the equations

$$\begin{aligned} x_2 &= rx_1(1-x_1) \\ x_1 &= rx_2(1-x_2) \; . \end{aligned} \tag{1.23}$$

The second iterate of $g(x)$ is then

$$g^{(2)}(x) = g\big(g(x)\big) = r^2 x(1-x)\big(1 - rx + rx^2\big) \; . \tag{1.24}$$

Setting $x = g^{(2)}(x)$, we obtain a cubic equation. Since $x - x^*$ must be a factor, we can divide out by this monomial and obtain a quadratic equation for x_1 and x_2. We find

$$x_{1,2} = \frac{1 + r \pm \sqrt{(r+1)(r-3)}}{2r} \; . \tag{1.25}$$

How stable is this 2-cycle? We find

$$\frac{d}{dx} g^{(2)}(x)\bigg|_{x_{1,2}} = r^2(1-2x_1)(1-2x_2) = -r^2 + 2r + 4 \,. \tag{1.26}$$

The condition that the 2-cycle be stable is then

$$-1 < r^2 - 2r - 4 < 1 \qquad \Longrightarrow \qquad r \in \big[3\,,\,1+\sqrt{6}\,\big] \,. \tag{1.27}$$

At $r = 1 + \sqrt{6} = 3.4494897\ldots$ there is a bifurcation to a 4-cycle, as can be seen in fig. 17.10.

In the 1970s, Mitchell Feigenbaum described how this system exhibits an increasingly dense cascade of period doubling transitions in which a 2^n-cycle becomes unstable and is replaced by a 2^{n+1}-cycle at $r = r_n$. The value $n = \infty$ is reached for a finite value $r_\infty = 3.5699456\ldots$. We will study this system in more detail in chapter 17.

1.1.5 Dynamical systems

A *dynamical system* in n variables is a set of n coupled ordinary differential equations. It's general form can be written as

$$\frac{d\boldsymbol{\varphi}}{dt} = \boldsymbol{V}(\boldsymbol{\varphi}) \tag{1.28}$$

where[3]

$$\boldsymbol{\varphi} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_n \end{pmatrix} \quad , \qquad \boldsymbol{V}(\boldsymbol{\varphi}) = \begin{pmatrix} V_1(\varphi_1, \ldots, \varphi_n) \\ V_2(\varphi_1, \ldots, \varphi_n) \\ \vdots \\ V_n(\varphi_1, \ldots, \varphi_n) \end{pmatrix} \tag{1.29}$$

In general $\boldsymbol{\varphi} \in \mathcal{M}$ lives on a *manifold* $\mathcal{M}$, which is an n-dimensional topological space which is locally diffeomorphic to $\mathbb{R}^n$. But for our purposes we can ignore all the fancy math vernacular and just consider $\boldsymbol{\varphi} \in \mathbb{R}^n$ is some n-tuple of real numbers[4]. The vector $\boldsymbol{V}(\boldsymbol{\varphi})$ is called the *velocity vector* at the point $\boldsymbol{\varphi}$. As $\boldsymbol{V}(\boldsymbol{\varphi})$ specifies a vector at each point $\boldsymbol{\varphi} \in \mathcal{M}$, we call $\boldsymbol{V}$ a *vector field*. The solution $\boldsymbol{\varphi}(t)$ to these coupled ODEs, subject to some set of initial conditions $\boldsymbol{\varphi}(0)$, is what we mean by the motion of the system, also called an *integral curve*. Thus, an integral curve is a set of points $\{t, \boldsymbol{\varphi}(t)\} \in \mathbb{R} \times \mathcal{M}$. The collection of points $\{\boldsymbol{\varphi}(t) \,|\, t \in \mathbb{R}\}$ is a curve in $\mathcal{M}$ itself, known as a *phase curve*. The difference is that a phase curve does not include the time coordinate. (See fig. 1.3.)

There's a helpful theorem which says that if $\boldsymbol{V}(\boldsymbol{\varphi})$ is a smooth vector field over some open set $\mathcal{D} \subset \mathcal{M}$, then for any $\boldsymbol{\varphi}(0) \in \mathcal{D}$ the initial value problem (*i.e.* the dynamical system plus its initial conditions) has a solution on some finite time interval $t \in [-\tau, +\tau]$, and furthermore that solution is unique. Moreover, this solution may be extended forward and backward in time either indefinitely or until $\boldsymbol{\varphi}(t)$ reaches the boundary of $\mathcal{D}$. A corollary of this theorem guarantees that different trajectories never intersect. Some

[3]It is important that the dynamical system as defined here is *autonomous*, *i.e.* $\boldsymbol{V}(\boldsymbol{\varphi})$ is a function only of the coordinates $\{\varphi_1, \ldots, \varphi_n\}$ and not on t itself - at least not explicitly.

[4]The mathy language just means that we could consider $\boldsymbol{\varphi}$ to live on a torus, or on the surface of a sphere, or on some complicated twisty higher dimensional space with lots of holes and handles.

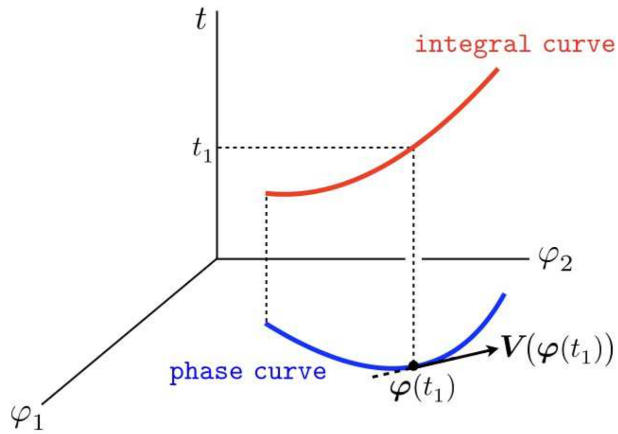

Figure 1.3: Integral curve *vs.* phase curve.

aspects of dynamical systems in low dimensions (*i.e.* $n = 1$ and $n = 2$) are discussed in chs. 11 through 13 of these lecture notes.

Note that any n^{th} order ODE, of the general form

$$\frac{d^n x}{dt^n} = F\bigg(x\,,\frac{dx}{dt}\,,\ldots\,,\frac{d^{n-1}x}{dt^{n-1}}\bigg) \quad , \tag{1.30}$$

may be represented by the first order system $\dot{\boldsymbol{\varphi}} = \boldsymbol{V}(\boldsymbol{\varphi})$. To see this, define $\varphi_k \equiv d^{k-1}x/dt^{k-1}$, with $k = 1, \ldots, n$. Thus, for $j < n$ we have $\dot{\varphi}_j = \varphi_{j+1}$, with $\dot{\varphi}_n = F$. In other words,

$$\overbrace{\frac{d}{dt}\begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_{n-1} \\ \varphi_n \end{pmatrix}}^{\dot{\boldsymbol{\varphi}}} = \overbrace{\begin{pmatrix} \varphi_2 \\ \vdots \\ \varphi_n \\ F\big(\varphi_1, \ldots, \varphi_n\big) \end{pmatrix}}^{\boldsymbol{V}(\boldsymbol{\varphi})} \quad . \tag{1.31}$$

Fixed points

A *fixed point* of a dynamical system is a point $\boldsymbol{\varphi}^*$ such that $\boldsymbol{V}(\boldsymbol{\varphi}^*) = 0$. Thus, if we start at time zero with $\boldsymbol{\varphi}(0) = \boldsymbol{\varphi}^*$, the system will remain at that point in phase space. But suppose we deviate just a teensy bit from the fixed point. We write $\boldsymbol{\varphi}(t) = \boldsymbol{\varphi}^* + \boldsymbol{\epsilon}(t)$. Since

$$V_j(\boldsymbol{\varphi}^* + \boldsymbol{\epsilon}) = \sum_{k=1}^{n} \frac{\partial V_j}{\partial \varphi_k}\bigg|_{\boldsymbol{\varphi}^*} \epsilon_k + \mathcal{O}(\boldsymbol{\epsilon}^2) \quad , \tag{1.32}$$

we have to lowest order in $\boldsymbol{\epsilon}$ the system

$$\frac{d\epsilon_j}{dt} = \sum_{k=1}^{n} M_{jk}\,\epsilon_k + \mathcal{O}(\boldsymbol{\epsilon}^2) \qquad , \qquad M_{jk} = \frac{\partial V_j}{\partial \varphi_k}\bigg|_{\boldsymbol{\varphi}^*} \quad . \tag{1.33}$$

The matrix M is real but not necessarily symmetric, so its eigenvalues can either be pure real or occur in complex conjugate pairs. The fixed point $\boldsymbol{\varphi}^*$ is then stable if all the eigenvalues of M have negative real parts. In this case, the vector $\boldsymbol{\epsilon}(t)$ collapses to zero exponentially at late times. Formally, the solution of the linearized dynamics is given by

$$\boldsymbol{\epsilon}(t) = \exp(Mt)\,\boldsymbol{\epsilon}(0) \quad . \tag{1.34}$$

In general, the right eigenvectors of M will not be the same as the left eigenvectors of M. Indeed it may be that M has fewer than n linearly independent eigenvectors – such is the case when M has nontrivial *Jordan blocks*, which is a nongeneric state of affairs. Assuming that M does have n linearly independent right eigenvectors R^α_j and n linearly independent left eigenvectors L^α_j, where R^α_j is the $j^{\rm th}$ component of the $\alpha^{\rm th}$ right eigenvector. Thus,

$$\sum_{j=1}^n L^\alpha_j M_{jk} = \lambda_\alpha\, L^\alpha_k \qquad , \qquad \sum_{k=1}^n M_{jk} R^\alpha_k = \lambda_\alpha\, R^\alpha_j \quad , \tag{1.35}$$

as well as the orthonormality and completeness relations

$$\sum_{j=1}^n L^\alpha_j\, R^\beta_j = \delta^{\alpha\beta} \qquad , \qquad \sum_{\alpha=1}^n R^\alpha_j\, L^\alpha_k = \delta_{jk} \quad . \tag{1.36}$$

Furthermore, we may decompose M into its eigenvectors as follows:

$$M_{jk} = \sum_{\alpha=1}^n \lambda_\alpha\, R^\alpha_j\, L^\alpha_k \quad . \tag{1.37}$$

Thus, if we write $\boldsymbol{\epsilon}(t)$ in terms of the right eigenvectors of M, *i.e.*

$$\epsilon_j(t) = \sum_{\alpha=1}^n C_\alpha(t)\, R^\alpha_j \quad , \tag{1.38}$$

then

$$C_\alpha(t) = C_\alpha(0)\, \exp(\lambda_\alpha t) \quad . \tag{1.39}$$

Thus, for $\mathsf{Re}\,(\lambda_\alpha) > 0$, $C_\alpha(t)$ grows with increasing time, indicating that the fixed point is unstable. A stable fixed point therefore requires $\mathsf{Re}\,(\lambda_\alpha) < 0$ for all $\alpha \in \{1,\ldots,n\}$.

Attractors, strange attractors, and dynamical chaos

An *attractor* of a dynamical system $\dot{\boldsymbol{\varphi}} = \boldsymbol{V}(\boldsymbol{\varphi})$ is the set of $\boldsymbol{\varphi}$ values that the system evolves to after a sufficiently long time. For $n = 1$ the only possible attractors are stable fixed points. For $n = 2$, we have, generically, two different classes of stable fixed points, called stable nodes and stable spirals. But there are also stable *limit cycles*, which are one-dimensional curves along which the motion is trapped. For $n > 2$ the situation is qualitatively different, and a fundamentally new type of set, the *strange attractor*, emerges.

A strange attractor is basically a bounded set on which nearby orbits diverge exponentially (*i.e.* there exists at least one positive Lyapunov exponent). To envision such a set, consider a flat rectangle, like a piece of chewing gum. Now fold the rectangle over, stretch it, and squash it so that it maintains its original volume. Keep doing this. Two points which started out nearby to each other will eventually, after a sufficiently large number of folds and stretches, grow far apart. Formally, a strange attractor is a *fractal*, and may have *noninteger Hausdorff dimension.* (We won't discuss fractals and Hausdorff dimension here.)

The Lorenz Model

The canonical example of a strange attractor is found in the Lorenz model. E. N. Lorenz, in a seminal paper from the early 1960's, reduced the essential physics of the coupled *partial* differential equations describing Rayleigh-Benard convection (a fluid slab of finite thickness, heated from below – in Lorenz's case a model of the atmosphere warmed by the ocean) to a set of twelve coupled nonlinear *ordinary* differential equations. Lorenz's intuition was that his weather model should exhibit recognizable patterns over time. What he found instead was that in some cases, changing his initial conditions by a part in a thousand rapidly led to totally different behavior. This *sensitive dependence on initial conditions* is a hallmark of chaotic systems.

The essential physics/mathematics of Lorenz's $n = 12$ system is elicited by the reduced $n = 3$ system,

$$\begin{aligned} \dot{X} &= -\sigma X + \sigma Y \\ \dot{Y} &= rX - Y - XZ \\ \dot{Z} &= XY - bZ \ , \end{aligned} \tag{1.40}$$

where σ, r, and b are all real and positive. Here t is the familiar time variable (appropriately scaled), and (X, Y, Z) represent linear combinations of physical fields, such as global wind current and poleward temperature gradient. These equations possess a symmetry under $(X, Y, Z) \to (-X, -Y, Z)$, but what is most important is the presence of nonlinearities in the second and third equations.

Typically the system is studied for fixed σ and b as a function of the single control parameter r. Clearly $(X, Y, Z) = (0, 0, 0)$ is a fixed point for all $\{\sigma, b, r\}$. It is quite easy to show that this fixed point is stable provided $0 < r < 1$. For $r > 1$, a new pair of solutions emerges, with

$$X^* = Y^* = \pm\sqrt{b(r-1)} \qquad , \qquad Z^* = r - 1 \quad . \tag{1.41}$$

One can then show that these fixed points are stable for $r \in \big[1, r_c\big]$, where

$$r_c = \frac{\sigma(\sigma + b + 3)}{\sigma - b - 1} \ . \tag{1.42}$$

These fixed points correspond to steady convection in the fluid model.

The Lorenz system has commonly been studied with $\sigma = 10$ and $b = \frac{8}{3}$. For these parameters, one has $r_c = \frac{470}{19} \approx 24.74$. In addition to the new pair of fixed points, a strange attractor appears for $r > r_s \simeq 24.06$. The capture by the strange attractor is shown in Fig. 17.15. In the narrow interval $r \in [24.06, 24.74]$ there are then *three* stable attractors, two of which correspond to steady convection and the third to chaos. Over this interval, there is also hysteresis. *I.e.* starting with a convective state for

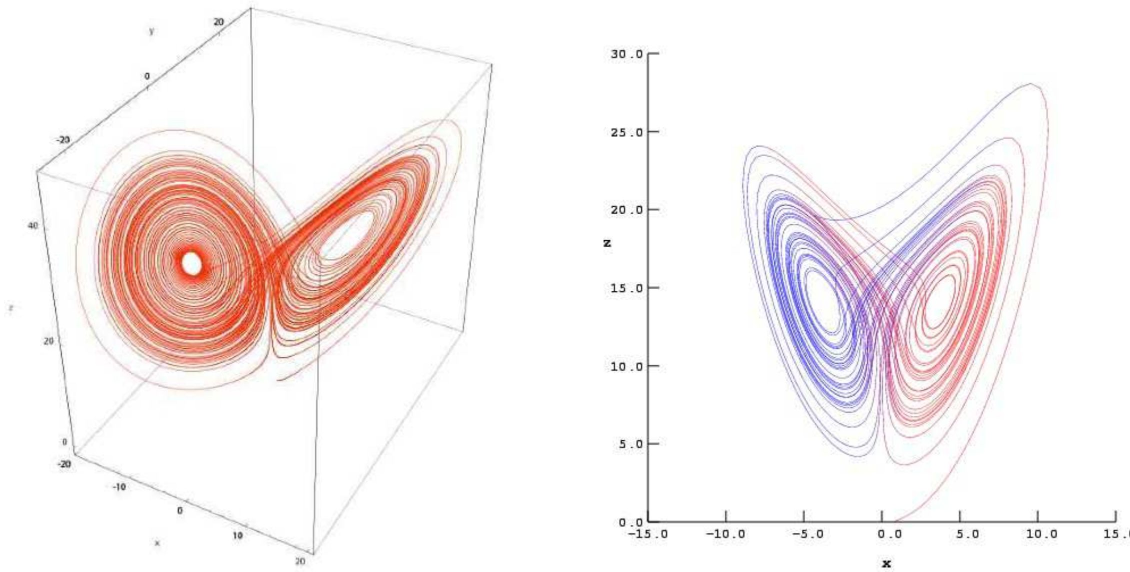

Figure 1.4: Left: Evolution of the Lorenz equations for $\sigma = 10$, $b = \frac{8}{3}$, and $r = 28$, with initial conditions $(X_0, Y_0, Z_0) = (0, 1, 0)$, showing the 'strange attractor'. Right: The Lorenz attractor, projected onto the (X, Z) plane. (Source: Wikipedia)

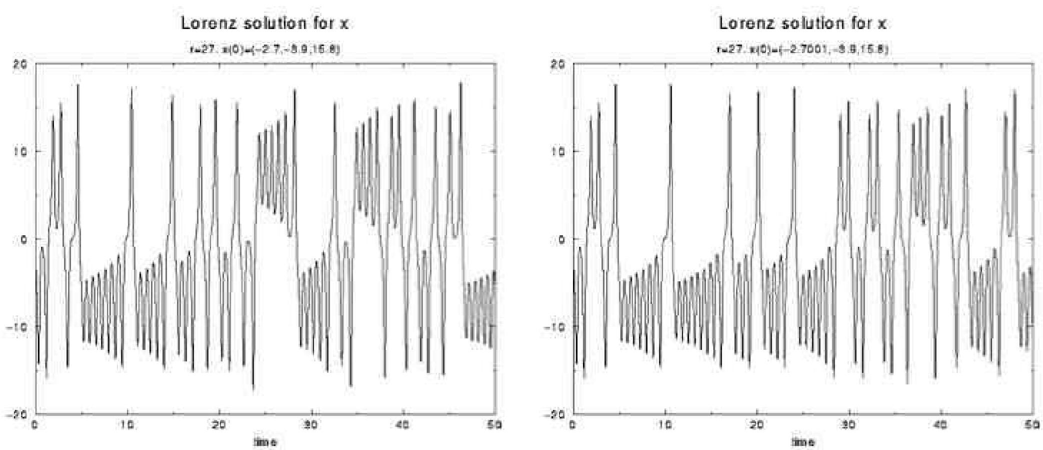

Figure 1.5: $X(t)$ for the Lorenz equations with $\sigma = 10$, $b = \frac{8}{3}$, $r = 28$, and initial conditions $(X_0, Y_0, Z_0) = (-2.7, -3.9, 15.8)$, and initial conditions $(X_0, Y_0, Z_0) = (-2.7001, -3.9, 15.8)$.

$r < 24.06$, the system remains in the convective state until $r = 24.74$, when the convective fixed point becomes unstable. The system is then driven to the strange attractor, corresponding to chaotic dynamics. Reversing the direction of r, the system remains chaotic until $r = 24.06$, when the strange attractor loses its own stability. Fig. 17.16 shows the chaotic evolution of the coordinate $X(t)$ for the case where $r = 28$. Note how, for the chosen parameters, $X(t)$ spends time oscillating about $X \approx -8$ and $X \approx +8$, but jumps randomly between these two regions, sometimes executing a single excursional spike into the opposite region.

Dynamical systems with $n = 1$

The simplest class of dynamical systems are those for which phase space is one-dimensional, *i.e.* $n = 1$. We then have

$$\frac{du}{dt} = f(u) \quad , \tag{1.43}$$

where there is a single coordinate u and the velocity function is $f(u)$. The dynamics are exceedingly simple to describe graphically. Simply sketch the function $f(u)$ *versus* u. In regions where $f(u) > 0$, $\dot{u} > 0$ and u moves to the right, *i.e.* to greater values. In regions where $f(u) < 0$, $\dot{u} < 0$ and u moves to the left. At any point $f(u) = 0$, the motion stops and $\dot{u} = 0$. Such a point is called a *fixed point* of the dynamics. Suppose $f(u^*) = 0$ and we write $u = u^* + \varepsilon$ with $|\varepsilon| \ll 1$. Then

$$\frac{d\varepsilon}{dt} = f(u^* + \varepsilon) = f'(u^*)\,\varepsilon + \tfrac{1}{2} f''(u^*)\,\varepsilon^2 + \mathcal{O}(\varepsilon^3) \quad . \tag{1.44}$$

Working to lowest nontrivial order, we see that if $f'(u^*) < 0$ then $\varepsilon(t)$ will collapse to zero exponentially (stable fixed point), but if $f'(u^*) > 0$ then $\varepsilon(t)$ will grow (unstable fixed point) until eventually we are no longer justifies in dropping higher order terms in the Taylor expansion. The fate of $u(t)$ is thus to be attracted to the first stable fixed point encountered, or to flow off to infinity.

A particularly simple example is the *logistic equation*,

$$\dot{N} = rN\Big(1 - \frac{N}{K}\Big) \quad , \tag{1.45}$$

with $r > 0$, which has the solution

$$N(t) = \frac{KN_0}{N_0 + \big(K - N_0\big)\exp(-rt)} \quad , \tag{1.46}$$

where the initial conditions are given by $N(0) \equiv N_0$. Note that $N = 0$ is an unstable fixed point and $N = K$ is a stable fixed point. Regardless of the initial value, as $t \to \infty$, $N(t)$ approaches the SFP, $N(+\infty) = K$. Conversely, if we run time backwards we approach the UFP, $N(-\infty) = 0$.

Note that in our discussion of the one-dimensional map $x_{n+1} = g(x_n)$ in §12.3, whether or not a fixed point x^* was stable or unstable depended on whether $|g'(x^*)|$ was greater or less than 1. Do you understand the difference between the two?

1.1.6 One-dimensional mechanics : simple examples

Ballistic motion

We now consider the second order ordinary differential equation

$$\frac{d^2x}{dt^2} = a_0 \quad , \tag{1.47}$$

which describes a particle undergoing constant acceleration a_0. Some notation:

$$\dot{x} \equiv \frac{dx}{dt} \quad , \quad \ddot{x} \equiv \frac{d^2x}{dt^2} \quad , \quad \overset{\vdots\vdots}{x} = \frac{d^7x}{dt^7} \quad , \quad \textit{etc.} \tag{1.48}$$

Defining $v \equiv \dot{x}$, we then have $\dot{v} = a_0$, which we can integrate to obtain $v(t) = v(0) + a_0 t$. We now have

$$\dot{x} = \frac{dx}{dt} = v(0) + a_0 t \quad , \tag{1.49}$$

which we integrate to obtain the motion of the system,

$$x(t) = x(0) + v(0)\, t + \tfrac{1}{2} a_0 t^2 \quad . \tag{1.50}$$

Simple harmonic motion

Consider next the second order ODE

$$\frac{d^2x}{dt^2} = -\omega^2 x \quad , \tag{1.51}$$

i.e. $\ddot{x} = -\omega^2 x$. With $v \equiv \dot{x}$ we may write this as two coupled first order ODEs, *viz.*

$$\frac{d}{dt} \overbrace{\begin{pmatrix} x \\ v \end{pmatrix}}^{\boldsymbol{\varphi}} = \begin{pmatrix} v \\ -\omega^2 x \end{pmatrix} = \overbrace{\begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}}^{M} \overbrace{\begin{pmatrix} x \\ v \end{pmatrix}}^{\boldsymbol{\varphi}} \quad , \tag{1.52}$$

i.e. $\dot{\boldsymbol{\varphi}} = M\boldsymbol{\varphi}$. This is a linear set of coupled first order ODEs in the components of the vector $\boldsymbol{\varphi}$. In terms of the components, $\dot{x} = v$ and $\dot{v} = -\omega^2 x$. Provided the matrix M is time-independent[5], we can solve $\dot{\boldsymbol{\varphi}} = M\boldsymbol{\varphi}$ as if $\boldsymbol{\varphi}$ were a simple scalar:

$$\boldsymbol{\varphi}(t) = \exp(Mt)\, \boldsymbol{\varphi}(0) \quad . \tag{1.53}$$

But what do we mean by the exponential of the matrix Mt? We give meaning to the expression $\exp(Mt)$ through its Taylor expansion:

$$\exp(Mt) = \mathbf{1} + Mt + \tfrac{1}{2} M^2 t^2 + \tfrac{1}{6} M^3 t^3 + \ldots \quad . \tag{1.54}$$

Notice that

$$M^2 = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} = \begin{pmatrix} -\omega^2 & 0 \\ 0 & -\omega^2 \end{pmatrix} = -\omega^2\, \mathbf{1} \quad . \tag{1.55}$$

Thus, $M^{2k} = (-\omega^2)^k\, \mathbf{1}$ and $M^{2k+1} = (-\omega^2)^k M$, which entails

$$\begin{aligned} \exp(Mt) &= \sum_{k=0}^{\infty} \frac{M^{2k}\, t^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{M^{2k+1}\, t^{2k+1}}{(2k+1)!} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k\, (\omega t)^{2k}}{(2k)!}\, \mathbf{1} + \frac{1}{\omega} \sum_{k=0}^{\infty} \frac{(-1)^k\, (\omega t)^{2k+1}}{(2k+1)!}\, M \\ &= \cos(\omega t)\, \mathbf{1} + \omega^{-1} \sin(\omega t)\, M = \begin{pmatrix} \cos(\omega t) & \omega^{-1} \sin(\omega t) \\ -\omega \sin(\omega t) & \cos(\omega t) \end{pmatrix} \quad . \end{aligned} \tag{1.56}$$

[5] More precisely, provided that $M(t)$ commutes with $M(t')$ for all t and t'.

Thus, the motion is

$$\boldsymbol{\varphi}(t) = \begin{pmatrix} x(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} \cos(\omega t) & \omega^{-1}\sin(\omega t) \\ -\omega\sin(\omega t) & \cos(\omega t) \end{pmatrix} \begin{pmatrix} x_0 \\ v_0 \end{pmatrix} \quad , \tag{1.57}$$

which is to say

$$\begin{aligned} x(t) &= \cos(\omega t)\, x_0 + \omega^{-1}\sin(\omega t)\, v_0 \\ v(t) &= -\omega\sin(\omega t)\, x_0 + \cos(\omega t)\, v_0 \quad . \end{aligned} \tag{1.58}$$

One can now check explicitly that $\dot{x}(t) = v(t)$ and $\dot{v}(t) = -\omega^2 x(t)$.

Uniform force with linear frictional damping

We consider motion in the $\hat{\boldsymbol{z}}$ direction in the presence of a uniform gravitational field and frictional damping. The equation of motion is

$$m\,\frac{d^2 z}{dt} = -mg - \gamma\frac{dz}{dt} \tag{1.59}$$

which may be rewritten as a first order equation for $v = \dot{z}$, *viz.*

$$\begin{aligned} \frac{dv}{v + mg/\gamma} &= -\frac{\gamma}{m}dt \\ d\log(v + mg/\gamma) &= -(\gamma/m)dt \quad . \end{aligned} \tag{1.60}$$

Integrating then gives

$$\begin{aligned} \log\left(\frac{v(t) + mg/\gamma}{v(0) + mg/\gamma}\right) &= -\gamma t/m \\ v(t) &= -\frac{mg}{\gamma} + \left(v(0) + \frac{mg}{\gamma}\right) e^{-\gamma t/m} \quad . \end{aligned} \tag{1.61}$$

Note that the solution to the first order ODE $m\dot{v} = -mg - \gamma v$ entails one constant of integration, $v(0)$.

One can further integrate to obtain the motion

$$z(t) = z(0) + \frac{m}{\gamma}\left(v(0) + \frac{mg}{\gamma}\right)(1 - e^{-\gamma t/m}) - \frac{mg}{\gamma}\,t \quad . \tag{1.62}$$

The solution to the *second* order ODE $m\ddot{z} = -mg - \gamma\dot{z}$ thus entails *two* constants of integration: $v(0)$ and $z(0)$. Notice that as t goes to infinity the velocity tends towards the asymptotic value $v = -v_\infty$, where $v_\infty = mg/\gamma$. This is known as the *terminal velocity*. Indeed, solving the equation $\dot{v} = 0$ gives $v = -v_\infty$. The initial velocity is effectively "forgotten" on a time scale $\tau \equiv m/\gamma$.

Electrons moving in solids under the influence of an electric field also achieve a terminal velocity. In this case the force is not $F = -mg$ but rather $F = -eE$, where $-e$ is the electron charge ($e > 0$) and E is the electric field. The terminal velocity is then obtained from

$$v_\infty = eE/\gamma = e\tau E/m \quad . \tag{1.63}$$

The *current density* is a product:

$$\text{current density} = (\text{number density}) \times (\text{charge}) \times (\text{velocity}) \quad ,$$

thus

$$j = n \cdot (-e) \cdot (-v_\infty) = \frac{ne^2\tau}{m}\, E \quad . \tag{1.64}$$

The ratio j/E is called the *conductivity* of the metal, σ. According to our theory, $\sigma = ne^2\tau/m$. This is one of the most famous equations of solid state physics! The dissipation is caused by electrons scattering off impurities and lattice vibrations ("phonons"). In high purity copper at low temperatures ($T \lesssim 4\,\text{K}$), the *scattering time* τ is about a nanosecond ($\tau \approx 10^{-9}\,\text{s}$).

Uniform force with quadratic frictional damping

At higher velocities, the frictional damping is proportional to the *square* of the velocity. The frictional force is then $F_\text{f} = -cv^2\,\text{sgn}\,(v)$, where $\text{sgn}\,(v)$ is the *sign* of v: $\text{sgn}\,(v) = +1$ if $v > 0$ and $\text{sgn}\,(v) = -1$ if $v < 0$. (Note one can also write $\text{sgn}\,(v) = v/|v|$ where $|v|$ is the *absolute value*.) Why all this trouble with $\text{sgn}\,(v)$? Because it is important that the frictional force *dissipate* energy, and therefore that F_f be *oppositely directed* with respect to the velocity v. We will assume that $v < 0$ always, hence $F_\text{f} = +cv^2$.

Notice that there is a terminal velocity, since setting $\dot{v} = -g + (c/m)v^2 = 0$ gives $v = \pm v_\infty$, where $v_\infty = \sqrt{mg/c}$. One can write the equation of motion as

$$\frac{dv}{dt} = \frac{g}{v_\infty^2}(v^2 - v_\infty^2) \tag{1.65}$$

and using

$$\frac{1}{v^2 - v_\infty^2} = \frac{1}{2v_\infty}\left\{\frac{1}{v - v_\infty} - \frac{1}{v + v_\infty}\right\} \tag{1.66}$$

we obtain

$$\begin{aligned} \frac{dv}{v^2 - v_\infty^2} &= \frac{1}{2v_\infty}\frac{dv}{v - v_\infty} - \frac{1}{2v_\infty}\frac{dv}{v + v_\infty} \\ &= \frac{1}{2v_\infty} d\log\left(\frac{v_\infty - v}{v_\infty + v}\right) = \frac{g}{v_\infty^2}\, dt \quad . \end{aligned} \tag{1.67}$$

Assuming $v(0) = 0$, we integrate to obtain

$$\log\left(\frac{v_\infty - v(t)}{v_\infty + v(t)}\right) = \frac{2gt}{v_\infty} \tag{1.68}$$

which may be massaged to give the final result

$$v(t) = -v_\infty \tanh(gt/v_\infty) \quad . \tag{1.69}$$

Recall that the *hyperbolic tangent* function $\tanh(x)$ is given by

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \quad . \tag{1.70}$$

Thus, as in the previous example, as $t \to \infty$ one has $v(t) \to -v_\infty$, *i.e.* $v(\infty) = -v_\infty$.

Digression: To gain an understanding of the constant c, consider a flat surface of area S moving through a fluid at velocity v ($v > 0$). During a time Δt, all the fluid molecules inside the volume $\Delta V = S \cdot v\,\Delta t$ will have executed an elastic collision with the moving surface. Since the surface is assumed to be much more massive than each fluid molecule, the center of mass frame for the surface-molecule collision is essentially the frame of the surface itself. If a molecule moves with velocity u is the laboratory frame, it moves with velocity $u - v$ in the center of mass (CM) frame, and since the collision is elastic, its final CM frame velocity is reversed, to $v - u$. Thus, in the laboratory frame the molecule's velocity has become $2v - u$ and it has suffered a change in velocity of $\Delta u = 2(v - u)$. The total momentum change is obtained by multiplying Δu by the total mass $M = \varrho\,\Delta V$, where ϱ is the mass density of the fluid. But then the total momentum imparted to the fluid is

$$\Delta P = 2(v - u) \cdot \varrho\, S\, v\, \Delta t \tag{1.71}$$

and the force on the fluid is

$$F = \frac{\Delta P}{\Delta t} = 2S\, \varrho\, v(v - u) \quad . \tag{1.72}$$

Now it is appropriate to average this expression over the microscopic distribution of molecular velocities u, and since on average $\langle u \rangle = 0$, we obtain the result $\langle F \rangle = 2S\varrho v^2$, where $\langle \cdots \rangle$ denotes a microscopic average over the molecular velocities in the fluid. (There is a subtlety here concerning the effect of fluid molecules striking the surface from either side – you should satisfy yourself that this derivation is sensible!) Newton's Third Law then states that the frictional force imparted to the moving surface by the fluid is $F_{\rm f} = -\langle F \rangle = -cv^2$, where $c = 2S\varrho$. In fact, our derivation is too crude to properly obtain the numerical prefactors, and it is better to write $c = \mu\varrho S$, where μ is a dimensionless constant which depends on the *shape* of the moving object.

1.1.7 Stochastic differential equation: Langevin's equation

Consider a particle of mass m subjected to both dissipation as well as external forcing with both a constant and a random fluctuating component. We'll examine this system in one dimension to gain an understanding of the essential physics. We write

$$\dot{v} + \gamma v = g + \zeta(t) \quad . \tag{1.73}$$

Here, v is the particle's velocity, γ is the damping rate due to friction, $g = F/m$ is the acceleration due to the constant external force, and $\zeta(t)$ is a *stochastic random force* (per unit mass). This equation, known as the *Langevin equation*, describes a ballistic particle in a uniform force field being buffeted by random forcing events. The Langevin equation is an example of a *stochastic differential equation* (*i.e.* a *stochastic dynamical system*), *i.e.* a differential equation where the evolution depends on one or more random functions. Stochastic differential equations are found in many areas of statistical physics and in the mathematical theory of finance as well, where they describe the time evolution of financial instruments. In the current context, think of a particle of dust as it moves in the atmosphere, in which case $|g|$ would then represent the acceleration due to gravity and $\zeta(t)$ the random acceleration due to collisions with the air molecules. For a sphere of radius a moving in a fluid of dynamical viscosity η, hydrodynamics gives $\gamma = 6\pi\eta a/m$. It is illustrative to compute γ in some setting. Consider a micron sized droplet ($a = 10^{-4}$ cm) of some liquid of density $\rho \sim 1.0\,\mathrm{g/cm^3}$ moving in air at $T = 20^\circ$ C. The viscosity of air is $\eta = 1.8 \times 10^{-4}\ \mathrm{g/cm \cdot s}$ at this temperature[6]. If the droplet density is constant, then $\gamma = 9\eta/2\rho a^2 = 8.1 \times 10^4\,\mathrm{s}^{-1}$, hence the time scale for viscous relaxation of the particle is $\tau = \gamma^{-1} = 12\,\mu$s. We should stress that the viscous damping on the particle is of course also due to the fluid (*e.g.*, air) molecules, in some average 'coarse-grained' sense. The random component $\zeta(t)$ thus represents the fluctuations with respect to this average.

We can easily integrate this equation:

$$\frac{d}{dt}\left(v\,e^{\gamma t}\right) = g\,e^{\gamma t} + \zeta(t)\,e^{\gamma t} \quad \Rightarrow \quad v(t) = v(0)\,e^{-\gamma t} + \gamma^{-1} g\left(1 - e^{-\gamma t}\right) + \int\limits_0^t ds\,\zeta(s)\,e^{\gamma(s-t)} \tag{1.74}$$

Note that the solution $v(t)$ depends on the random function $\zeta(t)$[7]. We can therefore only compute averages in order to characterize the motion of the system. One important feature of the above solution is that we see the system "loses memory" of its initial condition $u(0)$ on a time scale γ^{-1}.

The first average we will compute is that of u itself. In so doing, we assume that $\zeta(t)$ has zero mean: $\big\langle \zeta(t) \big\rangle = 0$. Then

$$\big\langle v(t) \big\rangle = v(0)\,e^{-\gamma t} + \gamma^{-1} g\left(1 - e^{-\gamma t}\right) \quad . \tag{1.75}$$

On the time scale γ^{-1}, the initial conditions $v(0)$ are effectively forgotten, and asymptotically for $t \gg \gamma^{-1}$ we have $\big\langle v(t) \big\rangle \to \gamma^{-1} g$, which is the terminal velocity.

Next, consider

$$\big\langle v^2(t) \big\rangle = \big\langle v(t) \big\rangle^2 + \int\limits_0^t ds_1 \int\limits_0^t ds_2\, e^{\gamma(s_1 - t)}\, e^{\gamma(s_2 - t)} \,\big\langle \zeta(s_1)\,\zeta(s_2) \big\rangle \quad . \tag{1.76}$$

We now need to know the *autocorrelator* $\big\langle \zeta(s_1)\,\zeta(s_2) \big\rangle$ of the random function $\zeta(s)$. We assume that this is a function only of the time difference $\Delta s = s_1 - s_2$, *viz.*

$$\big\langle \zeta(s_1)\,\zeta(s_2) \big\rangle = \phi(s_1 - s_2) \quad . \tag{1.77}$$

The function $\phi(s)$ is the autocorrelation function of the random force. A macroscopic object moving in a fluid is constantly buffeted by fluid particles over its entire perimeter. These different fluid particles are almost completely uncorrelated, hence $\phi(s)$ is basically nonzero except on a very small time scale τ_ϕ, which is the time a single fluid particle spends interacting with the object. We can take $\tau_\phi \to 0$ and approximate $\phi(s) \approx \varGamma\,\delta(s)$. As we shall now see, we can determine the value of the constant $\varGamma$ from equilibrium thermodynamic considerations.

[6]The cgs unit of viscosity is the *Poise* (P). $1\,\mathrm{P} = 1\,\mathrm{g/cm{\cdot}s}$.

[7]Mathematically, we say that $v(t)$ is a *functional* of $\zeta(s)$.

With this form for $\phi(s)$, we can easily calculate the equal time velocity autocorrelation:

$$\big\langle v^2(t)\big\rangle = \big\langle v(t)\big\rangle^2 + \Gamma\int\limits_0^t ds\, e^{2\gamma(s-t)} = \big\langle v(t)\big\rangle^2 + \frac{\Gamma}{2\gamma}\left(1 - e^{-2\gamma t}\right) \quad . \tag{1.78}$$

Consider the case where $F = 0$. We demand that the object thermalize at fluid temperature T at late times $t \gg \gamma^{-1}$, when $\langle v(t)\rangle \to 0$ and the particle has effectively forgotten all about its initial conditions. Thus, we impose the equipartition condition

$$\big\langle\, \tfrac{1}{2} M v^2(t)\,\big\rangle = \tfrac{1}{2} k_{\rm B} T \qquad \Rightarrow \qquad \Gamma = \frac{2\gamma k_{\rm B} T}{M} \quad . \tag{1.79}$$

This fixes the value of Γ. We can now compute the general momentum autocorrelator:

$$\big\langle v(t)\, v(t')\big\rangle - \big\langle v(t)\big\rangle\big\langle v(t')\big\rangle = \int\limits_0^t ds \int\limits_0^{t'} ds'\, e^{\gamma(s-t)}\, e^{\gamma(s'-t')}\, \big\langle \zeta(s)\,\zeta(s')\big\rangle = \frac{\Gamma}{2\gamma}\, e^{-\gamma|t-t'|} \quad , \tag{1.80}$$

which is valid for $|t-t'|$ finite, and in the limit where t and t' each tend to infinity.

Since we have in eqn. 1.74 the full solution for the velocity $u(t)$, we can use it to compute the position $x(t) = x(0) + \int\limits_0^t ds\, v(s)$ and its statistical properties. Let's compute the position $x(t)$. We find

$$x(t) = \big\langle x(t)\big\rangle + \int\limits_0^t ds \int\limits_0^s ds_1\, \zeta(s_1)\, e^{\gamma(s_1 - s)} \quad , \tag{1.81}$$

where

$$\big\langle x(t)\big\rangle = x(0) + \gamma^{-1}\big(v(0) - \gamma^{-1} g\big)\left(1 - e^{-\gamma t}\right) + \gamma^{-1} g t \quad . \tag{1.82}$$

Note that for $\gamma t \ll 1$ we have $\big\langle x(t)\big\rangle = x(0) + v(0)\, t + \frac{1}{2} g t^2 + \mathcal{O}(t^3)$, as is appropriate for ballistic particles moving under the influence of a constant force. This long time limit of course agrees with our earlier evaluation for the terminal velocity, $\big\langle v(\infty)\big\rangle \equiv v_\infty = \gamma^{-1} g$. We next compute the position autocorrelation:

$$\begin{aligned}\big\langle x(t)\, x(t')\big\rangle - \big\langle x(t)\big\rangle\big\langle x(t')\big\rangle &= \int\limits_0^t ds \int\limits_0^{t'} ds'\, e^{-\gamma(s+s')} \int\limits_0^s ds_1 \int\limits_0^{s'} ds_1'\, e^{\gamma(s_1+s_2)}\, \big\langle \zeta(s_1)\,\zeta(s_2)\big\rangle \\ &= \frac{\Gamma}{\gamma^2}\, \mathsf{min}(t,t') + \mathcal{O}(1) \quad .\end{aligned}$$

In particular, at late times the equal time autocorrelator is

$$\big\langle x^2(t)\big\rangle - \big\langle x(t)\big\rangle^2 = \frac{\Gamma t}{\gamma^2} \equiv 2Dt \quad , \tag{1.83}$$

up to terms of order unity. Here, $D = \Gamma/2\gamma^2 = k_{\rm B} T/\gamma m$ is the *diffusion constant*. For a liquid droplet of radius $a = 1\,\mu$m moving in air at $T = 293\,$K, for which $\eta = 1.8 \times 10^{-4}$ P, we have

$$D = \frac{k_{\rm B} T}{6\pi\eta a} = \frac{(1.38 \times 10^{-16}\, {\rm erg/K})\,(293\,{\rm K})}{6\pi\,(1.8 \times 10^{-4}\,{\rm P})\,(10^{-4}\,{\rm cm})} = 1.19 \times 10^{-7}\,{\rm cm^2/s} \quad . \tag{1.84}$$

This result presumes that the droplet is large enough compared to the intermolecular distance in the fluid that one can adopt a continuum approach and use the Navier-Stokes equations, and then assuming a laminar flow.

If we consider molecular diffusion, the situation is quite a bit different. The diffusion constant is then $D = \ell^2/2\tau$, where ℓ is the mean free path and τ is the collision time. Elementary kinetic theory gives that the mean free path ℓ, collision time τ, number density n, and total scattering cross section σ are related by[8] $\ell = \bar{v}\tau = 1/\sqrt{2}\,n\sigma$, where $\bar{v} = \sqrt{8k_{\rm B}T/\pi m}$ is the average particle speed. Approximating the particles as hard spheres, we have $\sigma = 4\pi a^2$, where a is the hard sphere radius. At $T = 293\,{\rm K}$, and $p = 1\,{\rm atm}$, we have $n = p/k_{\rm B}T = 2.51 \times 10^{19}\,{\rm cm}^{-3}$. Since air is predominantly composed of $\rm N_2$ molecules, we take $a = 1.90 \times 10^{-8}\,{\rm cm}$ and $m = 28.0\,{\rm amu} = 4.65 \times 10^{-23}\,{\rm g}$, which are appropriate for $\rm N_2$. We find an average speed of $\bar{v} = 471\,{\rm m/s}$ and a mean free path of $\ell = 6.21 \times 10^{-6}\,{\rm cm}$. Thus, $D = \frac{1}{2}\ell\bar{v} = 0.146\,{\rm cm}^2/{\rm s}$. Though much larger than the diffusion constant for large droplets, this is still too small to explain certain common experiences. Suppose we set the characteristic distance scale at $d = 10\,{\rm cm}$ and we ask how much time a point source would take to diffuse out to this radius. The answer is $\Delta t = d^2/2D = 343\,{\rm s}$, which is between five and six minutes. Yet if someone in the next seat emits a foul odor, you detect the offending emission in on the order of a second. What this tells us is that diffusion isn't the only transport process involved in these and like phenomena. More important are *convection* currents which distribute the scent much more rapidly.

1.1.8 Newton's laws of motion

Aristotle held that objects move because they are somehow impelled to seek out their natural state. Thus, a rock falls because rocks belong on the earth, and flames rise because fire belongs in the heavens. To paraphrase Wolfgang Pauli, such notions are so vague as to be "not even wrong." It was only with the publication of Newton's *Principia* in 1687 that a theory of motion which had detailed predictive power was developed.

Newton's three Laws of Motion may be stated as follows:

I. A body remains in uniform motion unless acted on by a force.

II. Force equals rate of change of momentum: $\boldsymbol{F} = d\boldsymbol{p}/dt$.

III. Any two bodies exert equal and opposite forces on each other.

Newton's First Law states that a particle will move in a straight line at constant (possibly zero) velocity if it is subjected to no forces. Now this cannot be true in general, for suppose we encounter such a "free" particle and that indeed it is in uniform motion, so that $\boldsymbol{r}(t) = \boldsymbol{r}(0) + \boldsymbol{v}(0)t$. Now $\boldsymbol{r}(t)$ is measured in some coordinate system, and if instead we choose to measure $\boldsymbol{r}(t)$ in a different coordinate system whose origin $\boldsymbol{R}$ moves according to the function $\boldsymbol{R}(t)$, then in this new "frame of reference" the position of our particle will be

$$\begin{aligned}\boldsymbol{r}'(t) &= \boldsymbol{r}(t) - \boldsymbol{R}(t) \\ &= \boldsymbol{r}(0) + \boldsymbol{v}(0)\,t - \boldsymbol{R}(t) \quad .\end{aligned} \tag{1.85}$$

[8]The scattering time τ is related to the particle density n, total scattering cross section σ, and mean speed $\bar{v}$ through the relation $n\sigma\bar{v}_{\rm rel}\tau = 1$, which says that on average one scattering event occurs in a cylinder of cross section σ and length $\bar{v}_{\rm rel}\tau$. Here $\bar{v}_{\rm rel} = \sqrt{\bar{v}}$ is the mean relative speed of a pair of particles.

If the acceleration $d^2\boldsymbol{R}/dt^2$ is nonzero, then merely by shifting our frame of reference we have apparently falsified Newton's First Law – a free particle does *not* move in uniform rectilinear motion when viewed from an accelerating frame of reference. Thus, together with Newton's Laws comes an assumption about the existence of frames of reference – called *inertial frames* – in which Newton's Laws hold. A transformation from one frame $\mathcal{K}$ to another frame $\mathcal{K}'$ which moves at constant velocity $\boldsymbol{V}$ relative to $\mathcal{K}$ is called a *Galilean transformation*. The equations of motion of classical mechanics are *invariant* (do not change) under Galilean transformations.

At first, the issue of inertial and noninertial frames is confusing. Rather than grapple with this, we will try to build some intuition by solving mechanics problems assuming we *are* in an inertial frame. The earth's surface, where most physics experiments are done, is *not* an inertial frame, due to the centripetal accelerations associated with the earth's rotation about its own axis and its orbit around the sun. In this case, not only is our coordinate system's origin – somewhere in a laboratory on the surface of the earth – accelerating, but the coordinate axes themselves are rotating with respect to an inertial frame. The rotation of the earth leads to fictitious "forces" such as the Coriolis force, which have large-scale consequences. For example, hurricanes, when viewed from above, rotate counterclockwise in the northern hemisphere and clockwise in the southern hemisphere. Later on in the course we will devote ourselves to a detailed study of motion in accelerated coordinate systems.

Newton's "quantity of motion" is the momentum $\boldsymbol{p}$, defined as the product $\boldsymbol{p} = m\boldsymbol{v}$ of a particle's mass m (how much stuff there is) and its velocity (how fast it is moving). In order to convert the Second Law into a meaningful equation, we must know how the force $\boldsymbol{F}$ depends on the coordinates (or possibly velocities) themselves. This is known as a *force law*. Examples of force laws include:

$$\begin{aligned}
\text{Constant force :} \quad & \boldsymbol{F} = -m\boldsymbol{g} \\
\text{Hooke's Law :} \quad & F = -kx \\
\text{Gravitation :} \quad & \boldsymbol{F} = -GMm\,\hat{\boldsymbol{r}}/r^2 \\
\text{Lorentz force :} \quad & \boldsymbol{F} = q\,\boldsymbol{E} + q\,\frac{\boldsymbol{v}}{c} \times \boldsymbol{B} \\
\text{Fluid friction } (v \text{ small}) \text{ :} \quad & \boldsymbol{F} = -b\,\boldsymbol{v} \quad .
\end{aligned}$$

Note that for an object whose mass does not change we can write the Second Law in the familiar form $\boldsymbol{F} = m\boldsymbol{a}$, where $\boldsymbol{a} = d\boldsymbol{v}/dt = d^2\boldsymbol{r}/dt^2$ is the acceleration. Most of our initial efforts will lie in using Newton's Second Law to solve for the motion of a variety of systems.

The Third Law is valid for the extremely important case of *central forces* which we will discuss in great detail later on. Newtonian gravity – the force which makes the planets orbit the sun – is a central force. One consequence of the Third Law is that in free space two isolated particles will accelerate in such a way that $\boldsymbol{F}_1 = -\boldsymbol{F}_2$ and hence the accelerations are parallel to each other, with

$$\frac{a_1}{a_2} = -\frac{m_2}{m_1} \quad , \tag{1.86}$$

where the minus sign is used here to emphasize that the accelerations are in opposite directions. We can also conclude that the *total momentum* $\boldsymbol{P} = \boldsymbol{p}_1 + \boldsymbol{p}_2$ is a constant, a result known as the *conservation of momentum*.

Aside : inertial *vs.* **gravitational mass**

In addition to postulating the Laws of Motion, Newton also deduced the gravitational force law, which says that the force $\boldsymbol{F}_{ij}$ exerted by a particle i by another particle j is

$$\boldsymbol{F}_{ij} = -G m_i m_j \frac{\boldsymbol{r}_i - \boldsymbol{r}_j}{|\boldsymbol{r}_i - \boldsymbol{r}_j|^3} \quad , \tag{1.87}$$

where G, the *Cavendish constant* (first measured by Henry Cavendish in 1798), takes the value

$$G = (6.6726 \pm 0.0008) \times 10^{-11} \mathrm{N} \cdot \mathrm{m}^2/\mathrm{kg}^2 \quad . \tag{1.88}$$

Notice Newton's Third Law in action: $\boldsymbol{F}_{ij} + \boldsymbol{F}_{ji} = 0$. Now a very important and special feature of this "inverse square law" force is that a spherically symmetric mass distribution has the same force on an external body as it would if all its mass were concentrated at its center. Thus, for a particle of mass m near the surface of the earth, we can take $m_i = m$ and $m_j = M_{\mathrm{e}}$, with $\boldsymbol{r}_i - \boldsymbol{r}_j \simeq R_{\mathrm{e}} \hat{\boldsymbol{r}}$ and obtain

$$\boldsymbol{F} = -mg\hat{\boldsymbol{r}} \equiv -m\boldsymbol{g} \tag{1.89}$$

where $\hat{\boldsymbol{r}}$ is a radial unit vector pointing from the earth's center and $g = GM_{\mathrm{e}}/R_{\mathrm{e}}^2 \simeq 9.8\,\mathrm{m/s}^2$ is the acceleration due to gravity at the earth's surface. Newton's Second Law now says that $\boldsymbol{a} = -\boldsymbol{g}$, *i.e.* objects accelerate as they fall to earth. However, it is not *a priori* clear why the *inertial mass* which enters into the definition of momentum should be the same as the *gravitational mass* which enters into the force law. Suppose, for instance, that the gravitational mass took a different value, m'. In this case, Newton's Second Law would predict

$$\boldsymbol{a} = -\frac{m'}{m}\,\boldsymbol{g} \tag{1.90}$$

and unless the ratio m'/m were *the same number* for *all* objects, then bodies would fall with *different accelerations*. The experimental fact that bodies in a vacuum fall to earth at the same rate demonstrates the equivalence of inertial and gravitational mass, *i.e.* $m' = m$.

1.1.9 Crossed electric and magnetic fields

Consider now a three-dimensional example of a particle of charge q moving in mutually perpendicular $\boldsymbol{E}$ and $\boldsymbol{B}$ fields. We'll throw in gravity for good measure. We take $\boldsymbol{E} = E\hat{\boldsymbol{x}}$, $\boldsymbol{B} = B\hat{\boldsymbol{z}}$, and $\boldsymbol{g} = -g\hat{\boldsymbol{z}}$. The equation of motion is Newton's 2nd Law again:

$$m\,\ddot{\boldsymbol{r}} = m\boldsymbol{g} + q\boldsymbol{E} + \frac{q}{c}\,\dot{\boldsymbol{r}} \times \boldsymbol{B} \quad . \tag{1.91}$$

The RHS (right hand side) of this equation is a vector sum of the forces due to gravity plus the Lorentz force of a moving particle in an electromagnetic field. In component notation, we have

$$\begin{aligned} m\ddot{x} &= qE + \frac{qB}{c}\,\dot{y} \\ m\ddot{y} &= -\frac{qB}{c}\,\dot{x} \\ m\ddot{z} &= -mg \quad . \end{aligned} \tag{1.92}$$

The equations for coordinates x and y are coupled, while that for z is independent and may be immediately solved to yield

$$z(t) = z(0) + \dot{z}(0)\,t - \tfrac{1}{2}gt^2 \quad . \tag{1.93}$$

The remaining equations may be written in terms of the velocities $v_x = \dot{x}$ and $v_y = \dot{y}$:

$$\begin{aligned} \dot{v}_x &= \omega_{\rm c}(v_y + u_{\rm D}) \\ \dot{v}_y &= -\omega_{\rm c}\, v_x \quad , \end{aligned} \tag{1.94}$$

where $\omega_{\rm c} = qB/mc$ is the *cyclotron frequency* and $u_{\rm D} = cE/B$ is the *drift speed* for the particle. As we shall see, these are the equations for a harmonic oscillator. The solution is

$$\begin{aligned} v_x(t) &= v_x(0)\,\cos(\omega_{\rm c}t) + \big(v_y(0) + u_{\rm D}\big)\,\sin(\omega_{\rm c}t) \\ v_y(t) &= -u_{\rm D} + \big(v_y(0) + u_{\rm D}\big)\cos(\omega_{\rm c}t) - v_x(0)\,\sin(\omega_{\rm c}t) \quad . \end{aligned} \tag{1.95}$$

Integrating again, the full motion is given by:

$$\begin{aligned} x(t) &= x(0) + A\,\sin\delta + A\,\sin(\omega_{\rm c}t - \delta) \\ y(r) &= y(0) - u_{\rm D}\,t - A\cos\delta + A\,\cos(\omega_{\rm c}t - \delta) \quad , \end{aligned} \tag{1.96}$$

where

$$A = \frac{1}{\omega_{\rm c}}\sqrt{\dot{x}^2(0) + \big(\dot{y}(0) + u_{\rm D}\big)^2} \quad , \quad \delta = \tan^{-1}\left(\frac{\dot{y}(0) + u_{\rm D}}{\dot{x}(0)}\right) \quad . \tag{1.97}$$

Thus, in the full solution of the motion there are *six* constants of integration:

$$x(0) \quad , \quad y(0) \quad , \quad z(0) \quad , \quad A \quad , \quad \delta \quad , \quad \dot{z}(0) \quad . \tag{1.98}$$

Of course instead of A and δ one may choose as constants of integration $\dot{x}(0)$ and $\dot{y}(0)$.

Pause for reflection

In mechanical systems, for each coordinate, or "degree of freedom," there exists a corresponding second order ODE. The full solution of the motion of the system entails two constants of integration for each degree of freedom.

1.2 Motion in One Space Dimension

1.2.1 Equations of motion for potential systems

For one-dimensional mechanical systems, Newton's second law reads

$$m\ddot{x} = F(x) \quad . \tag{1.99}$$

A system is *conservative* if the force is derivable from a potential: $F = -dU/dx$. The total energy,

$$E = T + U = \tfrac{1}{2}m\dot{x}^2 + U(x) \quad , \tag{1.100}$$

is then conserved. This may be verified explicitly:

$$\frac{dE}{dt} = \frac{d}{dt}\Big[\tfrac{1}{2}m\dot{x}^2 + U(x)\Big] = \Big[m\ddot{x} + U'(x)\Big]\,\dot{x} = 0 \quad . \tag{1.101}$$

Conservation of energy allows us to reduce the equation of motion from second order to first order:

$$\frac{dx}{dt} = \pm\sqrt{\frac{2}{m}\Big(E - U(x)\Big)} \quad . \tag{1.102}$$

Note that the constant E is a constant of integration. The $\pm$ sign above depends on the direction of motion. Points $x(E)$ which satisfy

$$E = U(x) \quad \Rightarrow \quad x(E) = U^{-1}(E) \quad , \tag{1.103}$$

where U^{-1} is the inverse function, are called *turning points*. When the total energy is E, the motion of the system is bounded by the turning points, and confined to the region(s) $U(x) \le E$. We can integrate eqn. 1.102 to obtain

$$t(x) - t(x_0) = \pm\sqrt{\frac{m}{2}}\int\limits_{x_0}^{x}\frac{dx'}{\sqrt{E - U(x')}} \quad . \tag{1.104}$$

This is to be inverted to obtain the function $x(t)$. Note that there are now *two* constants of integration, E and x_0. Since

$$E = E_0 = \tfrac{1}{2}mv_0^2 + U(x_0) \quad , \tag{1.105}$$

we could also consider x_0 and v_0 as our constants of integration, writing E in terms of x_0 and v_0. Thus, there are two *independent* constants of integration.

For motion confined between two turning points $x_\pm(E)$, the period of the motion is given by

$$T(E) = \sqrt{2m}\int\limits_{x_-(E)}^{x_+(E)}\frac{dx'}{\sqrt{E - U(x')}} \quad . \tag{1.106}$$

1.2.2 The simple harmonic oscillator

In the case of the harmonic oscillator, we have $U(x) = \tfrac{1}{2}kx^2$, hence

$$\frac{dt}{dx} = \pm\sqrt{\frac{m}{2E - kx^2}} \quad . \tag{1.107}$$

The turning points are $x_{\pm}(E) = \pm\sqrt{2E/k}$, for $E \geq 0$. To solve for the motion, let us substitute

$$x = \sqrt{\frac{2E}{k}}\,\sin\theta \quad . \tag{1.108}$$

We then find

$$dt = \sqrt{\frac{m}{k}}\,d\theta \quad , \tag{1.109}$$

with solution

$$\theta(t) = \theta_0 + \omega t \quad , \tag{1.110}$$

where $\omega = \sqrt{k/m}$ is the harmonic oscillator frequency. Thus, the motion of the system is given by

$$x(t) = \sqrt{\frac{2E}{k}}\,\sin(\omega t + \theta_0) \qquad , \qquad v(t) = \sqrt{\frac{2E}{m}}\,\cos(\omega t + \theta_0) \quad . \tag{1.111}$$

Note the two constants of integration, E and θ_0.

1.2.3 One-dimensional mechanics as a dynamical system

Rather than writing the equation of motion as a single second order ODE, we can instead write it as two coupled first order ODEs, *viz.*

$$\begin{aligned} \frac{dx}{dt} &= v \\ \frac{dv}{dt} &= \frac{1}{m}\,F(x) \quad . \end{aligned} \tag{1.112}$$

This may be written in matrix-vector form, as

$$\frac{d}{dt}\begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} v \\ \frac{1}{m}\,F(x) \end{pmatrix} \quad . \tag{1.113}$$

This is an example of a *dynamical system*, described by the general form

$$\frac{d\boldsymbol{\varphi}}{dt} = \boldsymbol{V}(\boldsymbol{\varphi}) \quad , \tag{1.114}$$

where $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_n)$ is an n-dimensional vector in *phase space*. For the model of eqn. 1.113, we evidently have $n = 2$. The object $\boldsymbol{V}(\boldsymbol{\varphi})$ is called a *vector field*. It is itself a vector, existing at every point in phase space, $\mathbb{R}^n$. Each of the components of $\boldsymbol{V}(\boldsymbol{\varphi})$ is, in general, a function of *all* n components of $\boldsymbol{\varphi}$:

$$V_j = V_j(\varphi_1, \ldots, \varphi_n) \qquad (j = 1, \ldots, n) \quad . \tag{1.115}$$

Solutions to the equation $\dot{\boldsymbol{\varphi}} = \boldsymbol{V}(\boldsymbol{\varphi})$ are called *integral curves*. Each such integral curve $\boldsymbol{\varphi}(t)$ is uniquely determined by n constants of integration, which may be taken to be the initial value $\boldsymbol{\varphi}(0)$. The collection of all integral curves is known as the *phase portrait* of the dynamical system.

In plotting the phase portrait of a dynamical system, we need to first solve for its motion, starting from arbitrary initial conditions. In general this is a difficult problem, which can only be treated numerically. But for conservative mechanical systems in $d = 1$, it is a trivial matter! The reason is that energy conservation completely determines the phase portraits. The velocity becomes a unique double-valued function of position, $v(x) = \pm\sqrt{\frac{2}{m}\big(E - U(x)\big)}$. The phase curves are thus curves of constant energy.

1.2.4 Sketching phase curves

To plot the phase curves,

(i) Sketch the potential $U(x)$.

(ii) Below this plot, sketch $v(x;E) = \pm\sqrt{\frac{2}{m}\big(E - U(x)\big)}$.

(iii) When E lies at a local extremum of $U(x)$, the system is at a *fixed point*.

  (a) For E slightly above $E_{\rm min}$, the phase curves are ellipses.

  (b) For E slightly below $E_{\rm max}$, the phase curves are (locally) hyperbolae.

  (c) For $E = E_{\rm max}$ the phase curve is called a *separatrix*[9].

(iv) When $E > U(\infty)$ or $E > U(-\infty)$, the motion is *unbounded*.

(v) Draw arrows along the phase curves: to the right for $v > 0$ and left for $v < 0$.

The period of the orbit $T(E)$ has a simple geometric interpretation. The area $\mathcal{A}$ in phase space enclosed by a bounded phase curve is

$$\mathcal{A}(E) = \oint\limits_E dx\, v = \sqrt{\frac{8}{m}} \int\limits_{x_-(E)}^{x_+(E)} dx'\, \sqrt{E - U(x')} \quad . \tag{1.116}$$

Thus, the period is proportional to the rate of change of $\mathcal{A}(E)$ with E:

$$T = m\,\frac{\partial\mathcal{A}}{\partial E} \quad . \tag{1.117}$$

[9]We might as well define separatrices to be phase curves for energies corresponding to *both* local minima as well as local maxima. For $E = E_{\rm min}$, there is a phase curve corresponding to the point $(x^*, 0)$, where x^* is the location of the local minimum in $U(x)$. For E just below $E_{\rm min}$, there is no phase curve in the vicinity of x^*, while for E just above $E_{\rm min}$, the phase curves in the vicinity of x^* are ellipses. When $U(x^*) = E_{\rm max}$ is a local maximum, the phase curves in the vicinity of x^* are hyperbolae. Precisely at $x = x^*$, the phase curves cross in a diabolical point resembling the letter X. Thus, in both cases corresponding to $E = E_{\rm min}$ and $E = E_{\rm max}$, *the separatrix phase curves are not (one-dimensional) manifolds*. At $E = E_{\rm min}$, the phase curve corresponds to a point, which is zero-dimensional, while at $E = E_{\rm max}$, the phase curve contains a diabolical point, at which the curve is also no longer locally homeomorphic to $\mathbb{R}^1$. For all other energies, the phase-curves are 1-manifolds, corresponding to the image of the map $t \mapsto \boldsymbol{\varphi}(t)$ from the time manifold $\mathbb{R}$ to the n-dimensional phase space manifold $\mathcal{M}^n$ (typically $\mathbb{R}^n$).

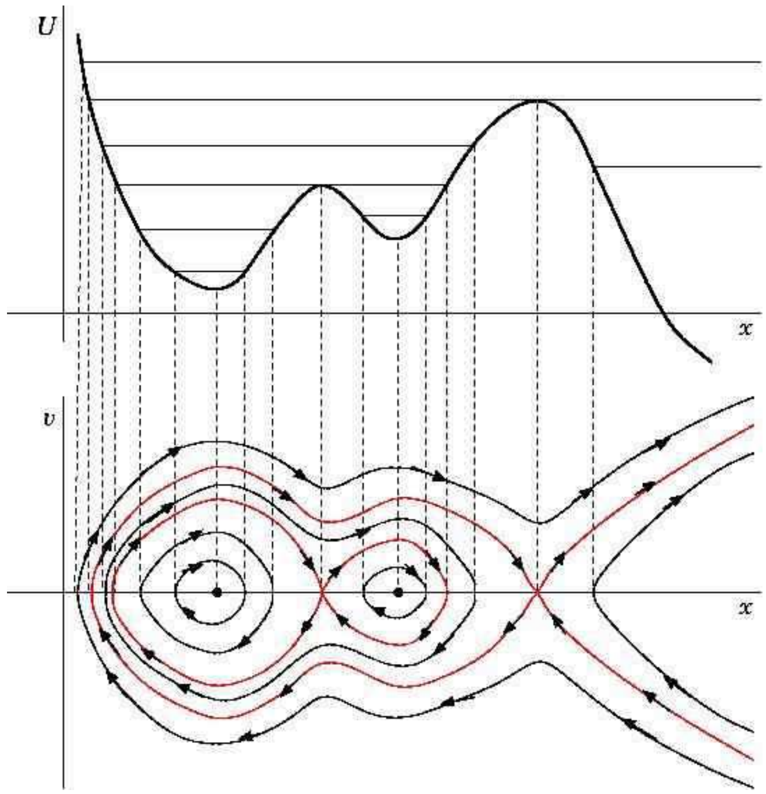

Figure 1.6: A potential $U(x)$ and the corresponding phase portraits (with separatrices in red).

1.2.5 Linearized dynamics in the vicinity of a fixed point

A fixed point (x^*, v^*) of the dynamics satisfies $U'(x^*) = 0$ and $v^* = 0$. Taylor's theorem then allows us to expand $U(x)$ in the vicinity of x^*:

$$U(x) = U(x^*) + U'(x^*)\,(x - x^*) + \tfrac{1}{2}U''(x^*)\,(x - x^*)^2 + \tfrac{1}{6}U'''(x^*)\,(x - x^*)^3 + \ldots \quad . \tag{1.118}$$

Since $U'(x^*) = 0$ the linear term in $\delta x = x - x^*$ vanishes. If δx is sufficiently small, we can ignore the cubic, quartic, and higher order terms, leaving us with

$$U(\delta x) \approx U_0 + \tfrac{1}{2}k\,(\delta x)^2 \quad , \tag{1.119}$$

where $U_0 = U(x^*)$ and $k = U''(x^*)$. The solutions to the motion in this potential are:

$$\begin{aligned} U''(x^*) > 0 \ : \ \delta x(t) &= \delta x_0 \,\cos(\omega t) + \frac{\delta v_0}{\omega}\,\sin(\omega t) \\ \delta v(t) &= -\omega\,\delta x_0\,\sin(\omega t) + \delta v_0\,\cos(\omega t) \end{aligned} \tag{1.120}$$

and

$$\begin{aligned} U''(x^*) < 0 \ : \ \delta x(t) &= \delta x_0 \,\cosh(\gamma t) + \frac{\delta v_0}{\gamma}\,\sinh(\gamma t) \\ \delta v(t) &= \gamma\,\delta x_0\,\sinh(\gamma t) + \delta v_0\,\cosh(\gamma t) \quad , \end{aligned} \tag{1.121}$$

where $\omega = \sqrt{k/m}$ for $k > 0$ and $\gamma = \sqrt{-k/m}$ for $k < 0$. The energy is

$$E = U_0 + \tfrac{1}{2}m\,(\delta v_0)^2 + \tfrac{1}{2}k\,(\delta x_0)^2 \quad . \tag{1.122}$$

For a separatrix, we have $E = U_0$ and $U''(x^*) < 0$. From the equation for the energy, we obtain $\delta v_0 = \pm\gamma\,\delta x_0$. Let's take $\delta v_0 = -\gamma\,\delta x_0$, so that the initial velocity is directed toward the unstable fixed point (UFP). *I.e.* the initial velocity is negative if we are to the right of the UFP ($\delta x_0 > 0$) and positive if we are to the left of the UFP ($\delta x_0 < 0$). The motion of the system is then

$$\delta x(t) = \delta x_0\, \exp(-\gamma t) \quad . \tag{1.123}$$

The particle gets closer and closer to the unstable fixed point at $\delta x = 0$, but it takes an infinite amount of time to actually get there. Put another way, the time it takes to get from δx_0 to a closer point $\delta x < \delta x_0$ is

$$t = \gamma^{-1} \log\left(\frac{\delta x_0}{\delta x}\right) \quad . \tag{1.124}$$

This diverges logarithmically as $\delta x \to 0$. Generically, then, *the period of motion along a separatrix is infinite*.

Linearization for general dynamical systems

Linearizing in the vicinity of such a fixed point, we wrote $\delta x = x - x^*$ and $\delta v = v - v^*$, obtaining

$$\frac{d}{dt}\begin{pmatrix} \delta x \\ \delta v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -m^{-1}\,U''(x^*) & 0 \end{pmatrix}\begin{pmatrix} \delta x \\ \delta v \end{pmatrix} + \ldots\,, \tag{1.125}$$

This is a *linear* equation, which we can solve completely. The result for a general n-component dynamical system $\dot{\boldsymbol{\varphi}} = \boldsymbol{V}(\boldsymbol{\varphi})$ is given in eqn. 1.33. The linearized dynamics in the vicinity of a fixed point $\boldsymbol{\varphi}^*$, where $\boldsymbol{V}(\boldsymbol{\varphi}^*) = 0$, is given by $\dot{\boldsymbol{\varphi}} = M\boldsymbol{\varphi}$, where the components of the $n \times n$ matrix M are given by $M_{jk} = (\partial V_j/\partial \varphi_k)\big|_{\boldsymbol{\varphi}^*}$.

Consider now the general linear equation $\dot{\boldsymbol{\varphi}} = M\boldsymbol{\varphi}$, where M is a fixed real matrix, *i.e.* one which is independent of time t. Formally, the solution is $\boldsymbol{\varphi}(t) = \exp(Mt)\,\boldsymbol{\varphi}(0)$. Now whenever we have a problem involving matrices, we should instantly start thinking about eigenvalues and eigenvectors. Invariably, the eigenvalues and eigenvectors will prove to be useful, if not essential, in solving the problem. The eigenvalue equation is

$$M\,\boldsymbol{\psi}_\alpha = \lambda_\alpha\,\boldsymbol{\psi}_\alpha \quad . \tag{1.126}$$

Here $\boldsymbol{\psi}_\alpha$ is the α^{th} *right eigenvector*[10] of M. The eigenvalues are roots of the characteristic equation, *i.e.* solutions to the equation $P(\lambda) = 0$, where $P(\lambda) = \det(\lambda \cdot \mathbb{I} - M)$. Let's expand $\boldsymbol{\varphi}(t)$ in terms of the right eigenvectors of M:

$$\boldsymbol{\varphi}(t) = \sum_\alpha C_\alpha(t)\,\boldsymbol{\psi}_\alpha \quad . \tag{1.127}$$

[10]If M is symmetric, the right and left eigenvectors are the same. If M is not symmetric, the right and left eigenvectors differ, although the set of corresponding eigenvalues is the same. We assume that the matrix M has no nontrivial Jordan blocks.

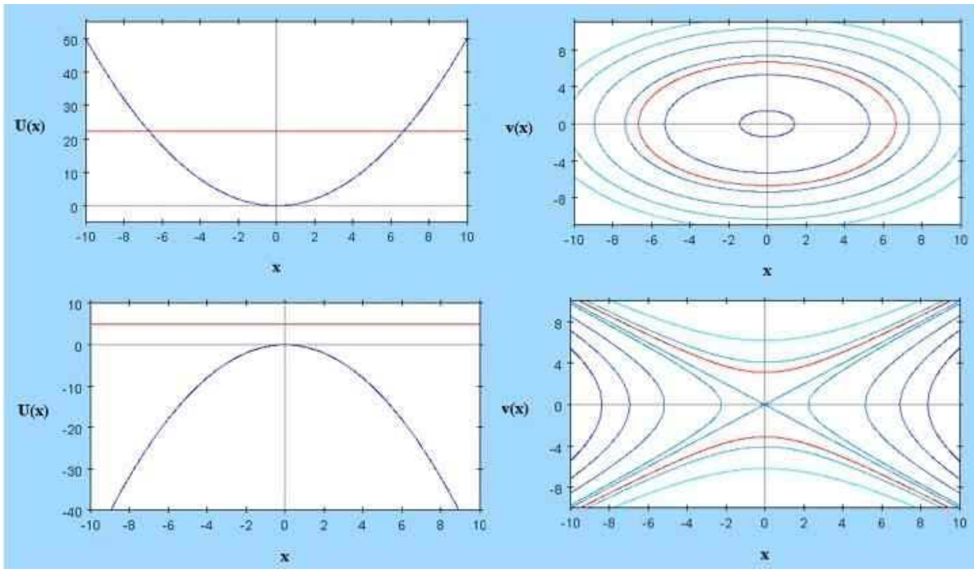

Figure 1.7: Phase curves in the vicinity of centers and saddles.

Assuming, for the purposes of this discussion, that M is nondegenerate, *i.e.* its eigenvectors span $\mathbb{R}^n$, the dynamical system can be written as a set of *decoupled* first order ODEs for the coefficients $C_\alpha(t)$:

$$\dot{C}_\alpha = \lambda_\alpha \, C_\alpha \quad , \tag{1.128}$$

with solutions

$$C_\alpha(t) = C_\alpha(0) \, \exp(\lambda_\alpha t) \quad . \tag{1.129}$$

If $\mathsf{Re}\,(\lambda_\alpha) > 0$, $C_\alpha(t)$ flows off to infinity, while if $\mathsf{Re}\,(\lambda_\alpha) < 0$, $C_\alpha(t)$ flows to zero. If $|\lambda_\alpha| = 1$, then $C_\alpha(t)$ oscillates with frequency $\mathsf{Im}(\lambda_\alpha)$.

For a two-dimensional matrix, it is easy to show – an exercise for the reader – that

$$P(\lambda) = \lambda^2 - T\lambda + D \quad , \tag{1.130}$$

where $T = \mathsf{Tr}(M)$ and $D = \mathsf{det}(M)$. The eigenvalues are then

$$\lambda_\pm = \tfrac{1}{2}T \pm \tfrac{1}{2}\sqrt{T^2 - 4D} \quad . \tag{1.131}$$

We'll defer study of the general case. For now, we focus on our conservative mechanical system of eqn. 1.125. The trace and determinant of the above matrix are $T = 0$ and $D = m^{-1}\,U''(x^*)$. Thus, there are only two (generic) possibilities: *centers*, when $U''(x^*) > 0$, and *saddles*, when $U''(x^*) < 0$. Examples of each are shown in fig. 1.6.

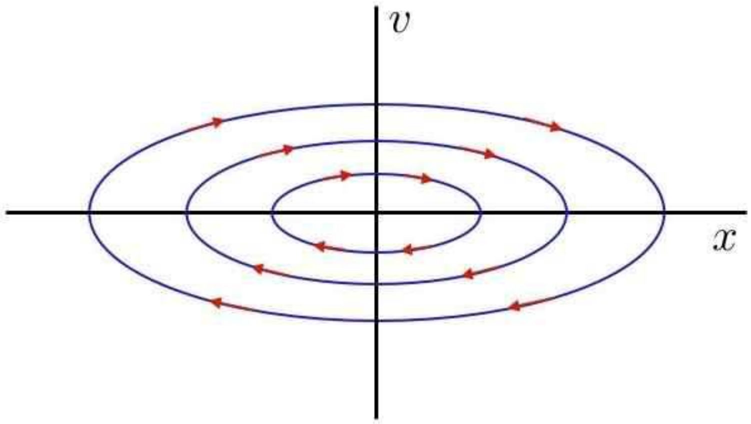

Figure 1.8: Phase curves for the harmonic oscillator.

1.3 Examples of Conservative One-Dimensional Systems

1.3.1 Harmonic oscillator

The potential energy of the harmonic oscillator in $d = 1$ dimension is $U(x) = \frac{1}{2}kx^2$. The equation of motion is

$$m\,\frac{d^2x}{dt^2} = -\frac{dU}{dx} = -kx \quad , \tag{1.132}$$

where m is the mass and k the force constant (of a spring). With $v = \dot{x}$, this may be written as the $N = 2$ system,

$$\frac{d}{dt}\begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}\begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} v \\ -\omega^2\, x \end{pmatrix} \quad , \tag{1.133}$$

where $\omega = \sqrt{k/m}$ has the dimensions of frequency (inverse time). The solution is well known:

$$\begin{aligned} x(t) &= x_0\,\cos(\omega t) + \frac{v_0}{\omega}\,\sin(\omega t) \\ v(t) &= v_0\,\cos(\omega t) - \omega\, x_0\,\sin(\omega t) \quad . \end{aligned} \tag{1.134}$$

The phase curves are ellipses:

$$\omega_0\, x^2(t) + \omega_0^{-1}\, v^2(t) = C \quad , \tag{1.135}$$

where C is a constant, independent of time. A sketch of the phase curves and of the phase flow is shown in fig. 14.1. Note that the x and v axes have different dimensions.

Energy is conserved:

$$E = \tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2 \quad . \tag{1.136}$$

Therefore we may find the length of the semimajor and semiminor axes by setting $v = 0$ or $x = 0$, which gives

$$x_{\rm max} = \sqrt{\frac{2E}{k}} \quad , \quad v_{\rm max} = \sqrt{\frac{2E}{m}} \quad . \tag{1.137}$$

The area of the elliptical phase curves is thus

$$\mathcal{A}(E) = \pi\, x_{\rm max}\, v_{\rm max} = \frac{2\pi E}{\sqrt{mk}} \quad . \tag{1.138}$$

The period of motion is therefore

$$T(E) = m\,\frac{\partial \mathcal{A}}{\partial E} = 2\pi\sqrt{\frac{m}{k}} \quad , \tag{1.139}$$

which is independent of E.

1.3.2 Pendulum

Next, consider the simple pendulum, composed of a mass point m affixed to a massless rigid rod of length ℓ. The potential is $U(\theta) = -mg\ell\cos\theta$, hence

$$m\ell^2\,\ddot{\theta} = -\frac{dU}{d\theta} = -mg\ell\sin\theta \quad . \tag{1.140}$$

This is equivalent to

$$\frac{d}{dt}\begin{pmatrix}\theta\\ \omega\end{pmatrix} = \begin{pmatrix}\omega\\ -\omega_0^2\,\sin\theta\end{pmatrix} \quad , \tag{1.141}$$

where $\omega = \dot{\theta}$ is the angular velocity, and where $\omega_0 = \sqrt{g/\ell}$ is the natural frequency of small oscillations. The conserved energy is

$$E = \tfrac{1}{2}\,m\ell^2\,\dot{\theta}^2 + U(\theta) \quad . \tag{1.142}$$

Assuming the pendulum is released from rest at $\theta = \theta_0$,

$$\frac{2E}{m\ell^2} = \dot{\theta}^2 - 2\omega_0^2\,\cos\theta = -2\omega_0^2\,\cos\theta_0 \quad . \tag{1.143}$$

The period for motion of amplitude θ_0 is then

$$T\big(\theta_0\big) = \frac{\sqrt{8}}{\omega_0}\int\limits_0^{\theta_0}\frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}} = \frac{4}{\omega_0}\,\mathrm{K}\big(\sin^2\tfrac{1}{2}\theta_0\big) \quad , \tag{1.144}$$

where $\mathrm{K}(z)$ is the complete elliptic integral of the first kind. Expanding $\mathrm{K}(z)$, we have

$$T\big(\theta_0\big) = \frac{2\pi}{\omega_0}\left\{1 + \tfrac{1}{4}\,\sin^2\big(\tfrac{1}{2}\theta_0\big) + \tfrac{9}{64}\,\sin^4\big(\tfrac{1}{2}\theta_0\big) + \ldots\right\} \quad . \tag{1.145}$$

For $\theta_0 \to 0$, the period approaches the usual result $2\pi/\omega_0$, valid for the linearized equation $\ddot{\theta} = -\omega_0^2\,\theta$. As $\theta_0 \to \frac{\pi}{2}$, the period diverges logarithmically.

The phase curves for the pendulum are shown in fig. 14.2. The small oscillations of the pendulum are essentially the same as those of a harmonic oscillator. Indeed, within the small angle approximation,

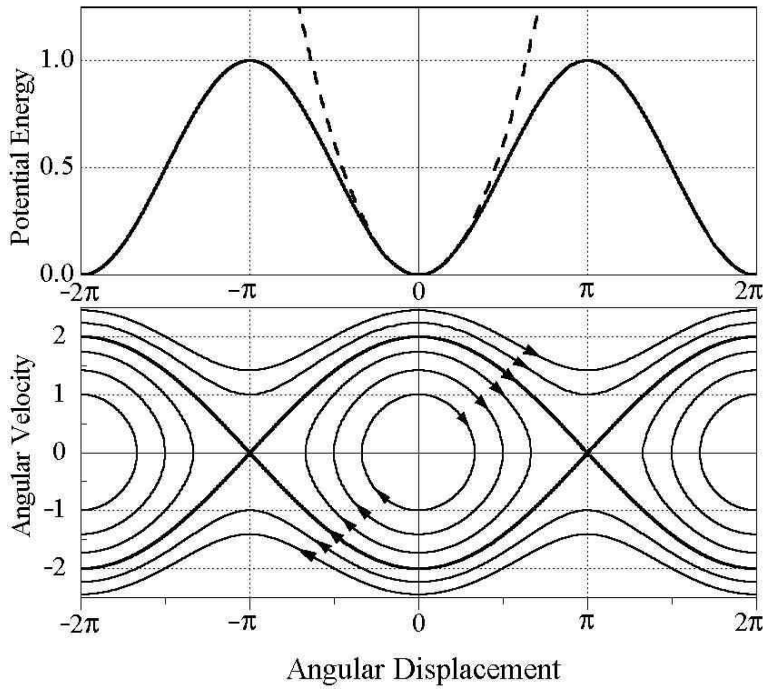

Figure 1.9: Phase curves for the simple pendulum. The *separatrix* divides phase space into regions of rotation and libration.

$\sin\theta \approx \theta$, and the pendulum equations of motion are exactly those of the harmonic oscillator. These oscillations are called *librations*. They involve a back-and-forth motion in real space, and the phase space motion is contractable to a point, in the topological sense. However, if the initial angular velocity is large enough, a qualitatively different kind of motion is observed, whose phase curves are *rotations*. In this case, the pendulum bob keeps swinging around in the same direction, because, as we'll see in a later lecture, the total energy is sufficiently large. The phase curve which separates these two topologically distinct motions is called a *separatrix*.

1.3.3 Other potentials

Using a phase plotter[11] it is possible to explore the phase curves for a wide variety of potentials. Three examples are shown in the following pages. The first is the effective potential for the Kepler problem,

$$U_{\text{eff}}(r) = -\frac{k}{r} + \frac{\ell^2}{2\mu r^2} \quad , \tag{1.146}$$

about which we shall have much more to say when we study central forces. Here r is the separation between two gravitating bodies of masses m_1 and m_2, $\mu = m_1 m_2/(m_1 + m_2)$ is the 'reduced mass', ℓ is the angular momentum perpendicular to the fixed plane of the motion, and $k = G m_1 m_2$ where G is the

[11] The phase plotter used here was written by Benjamin Schmidel.

Cavendish constant. We can then write

$$U_{\text{eff}}(r) = U_0 \left\{ -\frac{1}{x} + \frac{1}{2x^2} \right\} \quad , \tag{1.147}$$

where $x \equiv r/a$ is the radial coordinate measured in units of $a \equiv \ell^2/\mu k$ (which has dimensions of length), and where $U_0 \equiv k/a = \mu k^2/\ell^2$. Thus, if distances are measured in units of a and the potential in units of U_0, the dimensionless potential may be written in dimensionless form as $\mathcal{U}(x) = -\frac{1}{x} + \frac{1}{2x^2}$.

The second is the hyperbolic secant potential,

$$U(x) = -U_0 \,\text{sech}^2(x/a) \quad , \tag{1.148}$$

which, in dimensionless form, is $\mathcal{U}(x) = -\text{sech}^2(x)$, after measuring distances in units of a and potential in units of U_0.

The final example is

$$U(x) = U_0 \left\{ \cos\left(\frac{x}{a}\right) + \frac{x}{2a} \right\} \quad . \tag{1.149}$$

Again measuring x in units of a and U in units of U_0, we arrive at $\mathcal{U}(x) = \cos(x) + \frac{1}{2}x$.

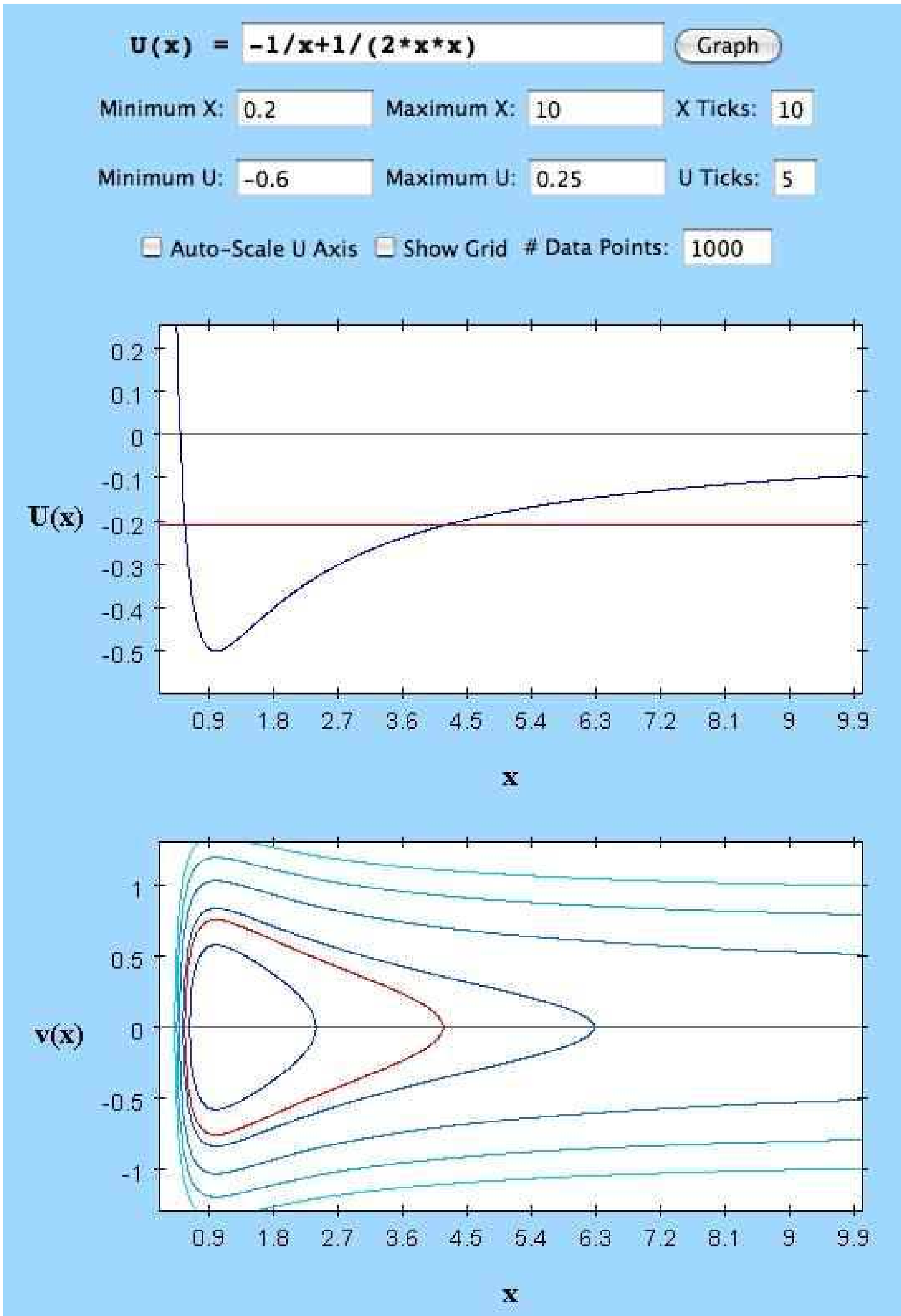

Figure 1.10: Phase curves for the Kepler effective potential $U(x) = -x^{-1} + \frac{1}{2}x^{-2}$.

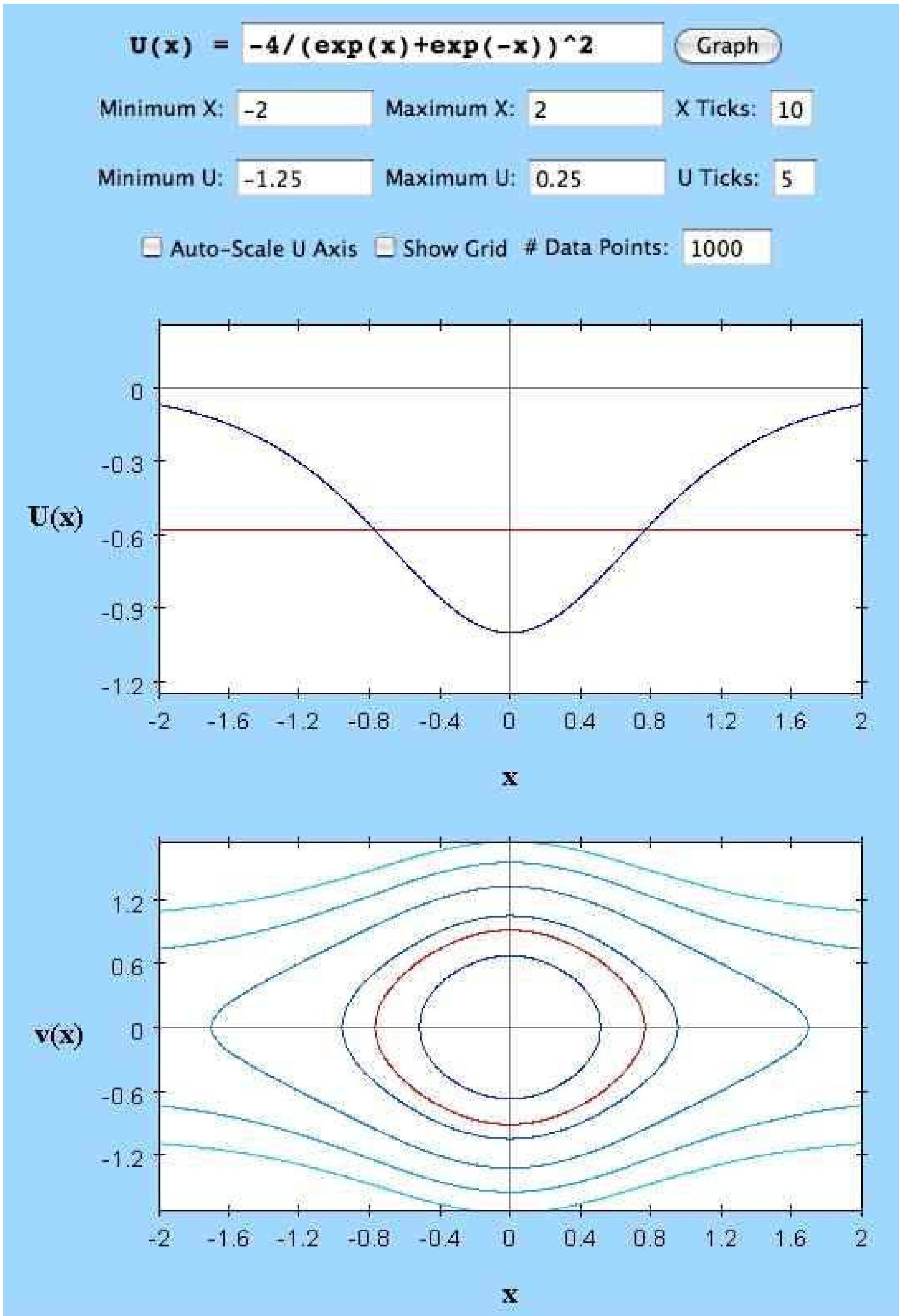

Figure 1.11: Phase curves for the potential $U(x) = -\text{sech}^2(x)$.

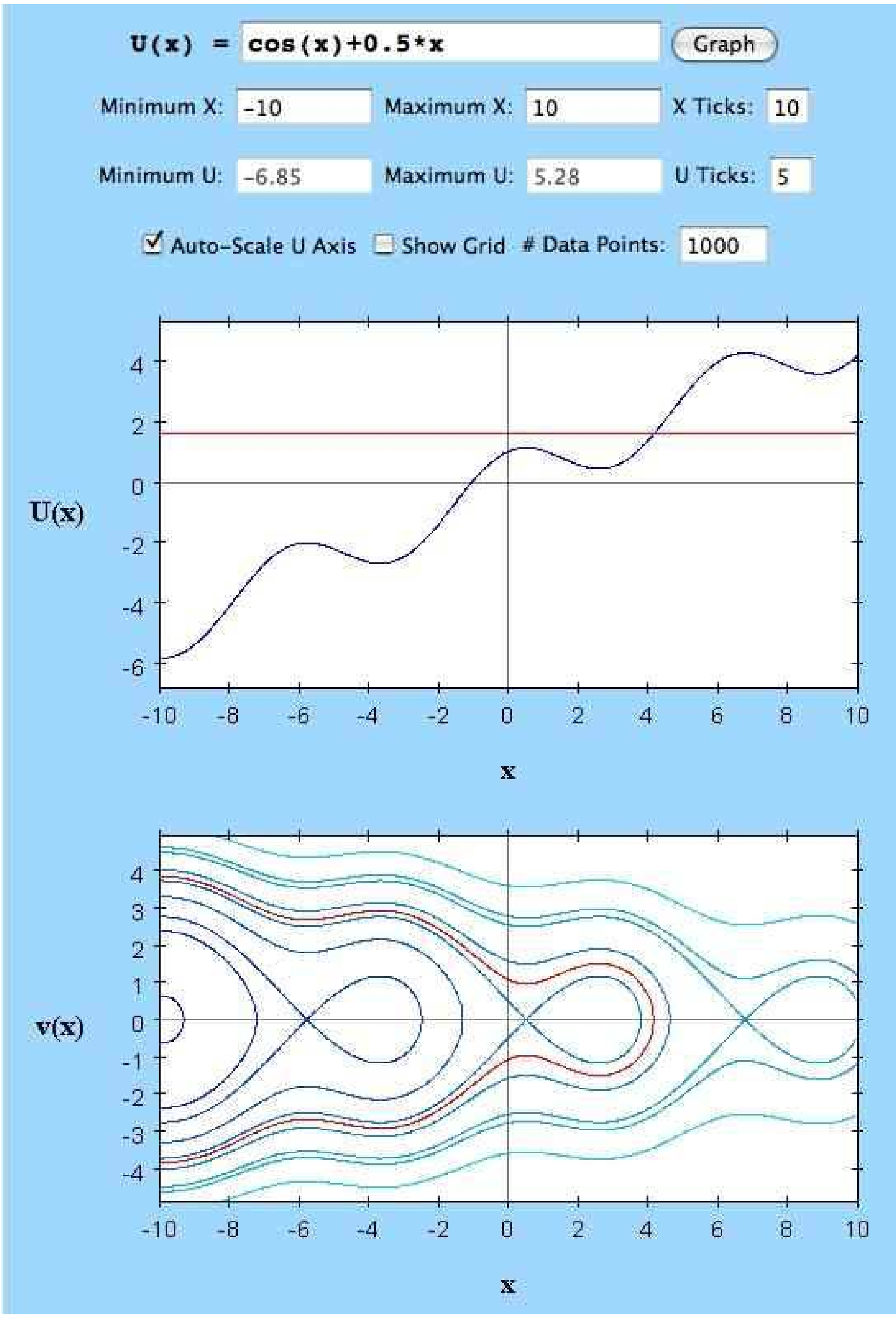

Figure 1.12: Phase curves for the potential $U(x) = \cos(x) + \frac{1}{2}x$.

Chapter 2

Linear Oscillations

2.1 Harmonic Motion

Harmonic motion is ubiquitous in physics. The reason is that any potential energy function, when expanded in a Taylor series in the vicinity of a local minimum, is a harmonic function:

$$U(q) = U(q^*) + \sum_{j=1}^{N} \overbrace{\frac{\partial U}{\partial q_j}\bigg|_{q=q^*}}^{\boldsymbol{\nabla} U(q^*)=0} (q_j - q_j^*) + \frac{1}{2} \sum_{j,k=1}^{N} \frac{\partial^2 U}{\partial q_j \, \partial q_k}\bigg|_{q=q^*} (q_j - q_j^*)\,(q_k - q_k^*) + \ldots \quad , \tag{2.1}$$

where $q = \{q_1, \ldots, q_N\}$ are the *generalized coordinates* of a system of point particles – more on this when we discuss Lagrangians. In one dimension, we have one coordinate, which we shall call x, and we expand the potential $U(x)$ about an extremum using Taylor's theorem, *viz.*

$$U(x) = U(x^*) + \tfrac{1}{2}\, U''(x^*)\,(x - x^*)^2 + \ldots \quad . \tag{2.2}$$

Provided the deviation $\eta = x - x^*$ is small enough in magnitude, the remaining terms in the Taylor expansion may be ignored. Newton's Second Law then gives

$$m\,\ddot{\eta} = -U''(x^*)\,\eta + \mathcal{O}(\eta^2) \quad . \tag{2.3}$$

This, to lowest order, is the equation of motion for a harmonic oscillator. If $U''(x^*) > 0$, the equilibrium point $x = x^*$ is *stable*, since for small deviations from equilibrium the restoring force pushes the system back toward the equilibrium point. When $U''(x^*) < 0$, the equilibrium is *unstable*, and the forces push one further away from equilibrium.

2.2 Damped Harmonic Oscillator

In the real world, there are frictional forces, which we here will approximate by $F = -\gamma v$. We begin with the homogeneous equation for a damped harmonic oscillator,

$$\frac{d^2x}{dt^2} + 2\beta\,\frac{dx}{dt} + \omega_0^2\,x = 0 \quad , \tag{2.4}$$

where $\gamma = 2\beta m$. To solve, write $x(t) = \sum_n C_n \, e^{-i\omega_n t}$. This renders the differential equation 2.4 an *algebraic* equation for the two eigenfrequencies ω_i, each of which must satisfy

$$\omega^2 + 2i\beta\omega - \omega_0^2 = 0 \quad , \tag{2.5}$$

hence

$$\omega_\pm = -i\beta \pm (\omega_0^2 - \beta^2)^{1/2} \quad . \tag{2.6}$$

The most general solution to eqn. 2.4 is then

$$x(t) = C_+ \, e^{-i\omega_+ t} + C_- \, e^{-i\omega_- t} \tag{2.7}$$

where $C_\pm$ are arbitrary constants. Notice that the eigenfrequencies are in general complex, with a negative imaginary part (so long as the damping coefficient β is positive). Thus $e^{-i\omega_\pm t}$ decays to zero as $t \to \infty$.

2.2.1 Classes of damped harmonic motion

We identify three classes of motion:

Underdamped motion ($\omega_0^2 > \beta^2$)

The solution for underdamped motion is

$$\begin{aligned} x(t) &= A\cos(\nu t)\, e^{-\beta t} + B\sin(\nu t)\, e^{-\beta t} \\ \dot{x}(t) &= (-\beta A + \nu B)\, \cos(\nu t)\, e^{-\beta t} - (\nu A + \beta B)\, \sin(\nu t)\, e^{-\beta t} \quad , \end{aligned} \tag{2.8}$$

where $\nu = \sqrt{\omega_0^2 - \beta^2}$, and where A and B are constants determined by initial conditions,

$$x_0 = A \qquad , \qquad \dot{x}_0 = -\beta A + \nu B \quad .$$

Simultaneously solving these two equations in the two unknowns yields

$$A = x_0 \qquad , \qquad B = \frac{\beta}{\nu}\, x_0 + \frac{1}{\nu}\, \dot{x}_0 \quad . \tag{2.9}$$

Overdamped motion ($\omega_0^2 < \beta^2$)

The solution in the case of overdamped motion is

$$\begin{aligned} x(t) &= C\cosh(\lambda t)\, e^{-\beta t} + D\sinh(\lambda t)\, e^{-\beta t} \\ \dot{x}(t) &= (-\beta C + \lambda D)\cosh(\lambda t)\, e^{-\beta t} + (\lambda C - \beta D)\sinh(\lambda t)\, e^{-\beta t} \quad , \end{aligned} \tag{2.10}$$

where $\lambda = \sqrt{\beta^2 - \omega_0^2}$ and where C and D are constants determined by the initial conditions

$$x_0 = C \qquad , \qquad \dot{x}_0 = -\beta C + \lambda D \quad . \tag{2.11}$$

Solving for the two unknowns, we have

$$C = x_0 \qquad , \qquad D = \frac{\beta}{\lambda}\, x_0 + \frac{1}{\lambda}\, \dot{x}_0 \quad . \tag{2.12}$$

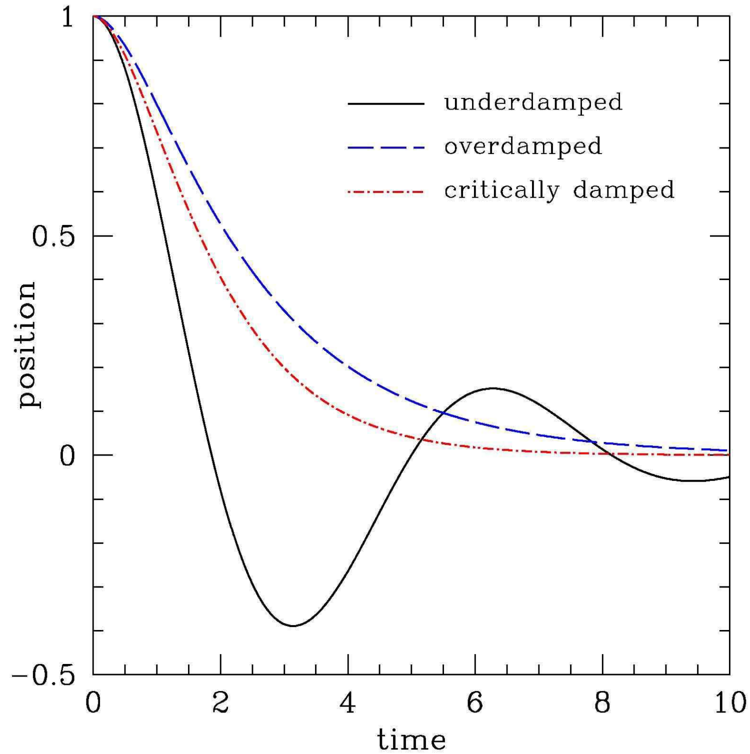

Figure 2.1: Three classifications of damped harmonic motion. The initial conditions are $x(0) = 1$, $\dot{x}(0) = 0$.

Critically damped motion ($\omega_0^2 = \beta^2$)

The solution in the case of critically damped motion is

$$\begin{aligned} x(t) &= E\,e^{-\beta t} + F t\,e^{-\beta t} \\ \dot{x}(t) &= (-\beta E + F)\,e^{-\beta t} - \beta F\,e^{\beta t} \quad . \end{aligned} \tag{2.13}$$

Thus, $x_0 = E$ and $\dot{x}_0 = -\beta E + F$, and

$$E = x_0 \qquad , \qquad F = \dot{x}_0 + \beta x_0 \quad . \tag{2.14}$$

The screen door analogy

The three types of behavior are depicted in fig. 2.1. To concretize these cases in one's mind, it is helpful to think of the case of a screen door or a shock absorber. If the hinges on the door are underdamped, the door will swing back and forth (assuming it doesn't have a rim which smacks into the door frame) several times before coming to a stop. If the hinges are overdamped, the door may take a very long time to close. To see this, note that for $\beta \gg \omega_0$ we have

$$\sqrt{\beta^2 - \omega_0^2} = \beta \left(1 - \frac{\omega_0^2}{\beta^2}\right)^{1/2} = \beta - \frac{\omega_0^2}{2\beta} - \frac{\omega_0^4}{8\beta^3} + \ldots \quad , \tag{2.15}$$

which leads to

$$\begin{aligned} i\omega_+ &= \beta - \sqrt{\beta^2 - \omega_0^2} = \frac{\omega_0^2}{2\beta} + \frac{\omega_0^4}{8\beta^3} + \ldots \\ i\omega_- &= \beta + \sqrt{\beta^2 - \omega_0^2} = 2\beta - \frac{\omega_0^2}{2\beta} - + \ldots \quad . \end{aligned} \tag{2.16}$$

Thus, we can write

$$x(t) = C\, e^{-t/\tau_1} + D\, e^{-t/\tau_2} \quad , \tag{2.17}$$

with

$$\tau_1 = \frac{1}{\beta - \sqrt{\beta^2 - \omega_0^2}} \approx \frac{2\beta}{\omega_0^2} \qquad , \qquad \tau_2 = \frac{1}{\beta + \sqrt{\beta^2 - \omega_0^2}} \approx \frac{1}{2\beta} \quad . \tag{2.18}$$

Thus $x(t)$ is a sum of exponentials, with decay times $\tau_{1,2}$. For $\beta \gg \omega_0$, we have that τ_1 is much larger than τ_2 – the ratio is $\tau_1/\tau_2 \approx 4\beta^2/\omega_0^2 \gg 1$. Thus, on time scales on the order of τ_1, the second term has completely damped away. The decay time τ_1, though, is very long, since β is so large. So a highly overdamped oscillator will take a very long time to come to equilbrium.

2.2.2 Remarks on the case of critical damping

Define the first order differential operator

$$\mathcal{D}_t = \frac{d}{dt} + \beta \quad . \tag{2.19}$$

The solution to $\mathcal{D}_t\, x(t) = 0$ is $\tilde{x}(t) = A\, e^{-\beta t}$, where A is a constant. Note that the *commutator* of $\mathcal{D}_t$ and t is unity:

$$\big[\mathcal{D}_t\,, t\big] = 1 \quad , \tag{2.20}$$

where $[A, B] \equiv AB - BA$. The simplest way to verify eqn. 2.20 is to compute its action upon an arbitrary function $f(t)$:

$$\begin{aligned} \big[\mathcal{D}_t\,,\, t\big]\, f(t) &= \left(\frac{d}{dt} + \beta\right) t\, f(t) - t \left(\frac{d}{dt} + \beta\right) f(t) \\ &= \frac{d}{dt}\big(t\, f(t)\big) - t\, \frac{d}{dt}\, f(t) = f(t) \quad . \end{aligned} \tag{2.21}$$

We know that $x(t) = \tilde{x}(t) = A\, e^{-\beta t}$ satisfies $\mathcal{D}_t\, x(t) = 0$. Therefore

$$\begin{aligned} 0 &= \mathcal{D}_t \big[\mathcal{D}_t\,, t\big]\, \tilde{x}(t) \\ &= \mathcal{D}_t^2 \Big(t\, \tilde{x}(t)\Big) - \mathcal{D}_t\, t\, \overbrace{\mathcal{D}_t\, \tilde{x}(t)}^{0} = \mathcal{D}_t^2 \Big(t\, \tilde{x}(t)\Big) \quad . \end{aligned} \tag{2.22}$$

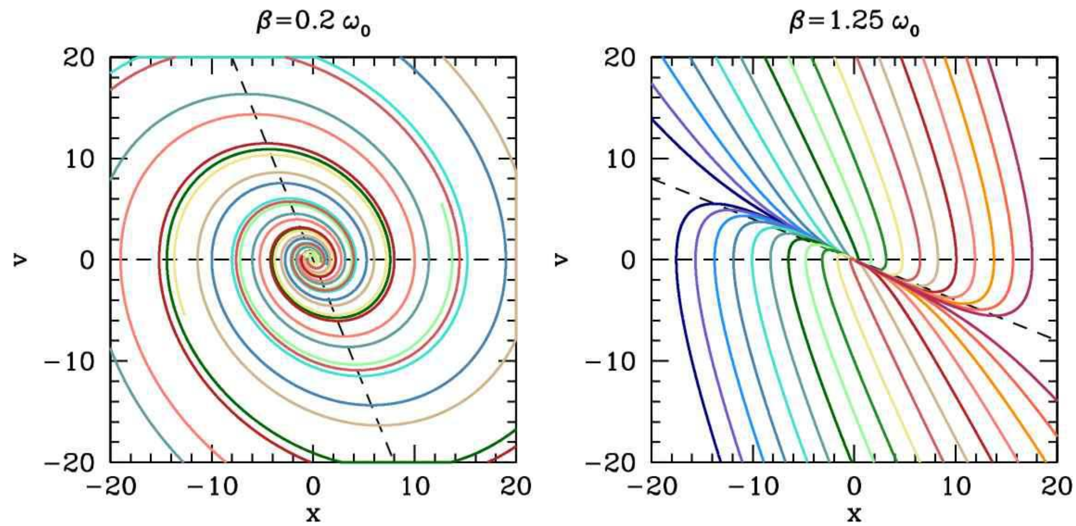

Figure 2.2: Phase curves for the damped harmonic oscillator. Left panel: underdamped motion. Right panel: overdamped motion. Note the *nullclines* along $v = 0$ and $v = -(\omega_0^2/2\beta)x$, which are shown as dashed lines.

We already know that $\mathcal{D}_t^2\,\tilde{x}(t) = \mathcal{D}_t\,\mathcal{D}_t\,\tilde{x}(t) = 0$. The above equation establishes that the second independent solution to the second order ODE $\mathcal{D}_t^2\,x(t) = 0$ is $x(t) = t\,\tilde{x}(t)$. Indeed, we can keep going, and show that

$$\mathcal{D}_t^n\left(t^{n-1}\,\tilde{x}(t)\right) = 0 \quad . \tag{2.23}$$

Thus, the n independent solutions to the n^{th} order ODE

$$\left(\frac{d}{dt} + \beta\right)^n x(t) = 0 \tag{2.24}$$

are

$$x_k(t) = A\,t^k\,e^{-\beta t} \quad , \quad k = 0, 1, \ldots, n-1 \quad . \tag{2.25}$$

2.2.3 Phase portraits for the damped harmonic oscillator

Expressed as a dynamical system, the equation of motion $\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$ is written as two coupled first order ODEs, *viz.*

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= -\omega_0^2\,x - 2\beta v \quad . \end{aligned} \tag{2.26}$$

In the theory of dynamical systems, a *nullcline* is a curve along which one component of the phase space velocity $\dot{\boldsymbol{\varphi}}$ vanishes. In our case, there are two nullclines: $\dot{x} = 0$ and $\dot{v} = 0$. The equation of the first nullcline, $\dot{x} = 0$, is simply $v = 0$, *i.e.* the first nullcline is the x-axis. The equation of the second nullcline, $\dot{v} = 0$, is $v = -(\omega_0^2/2\beta)\,x$. This is a line which runs through the origin and has negative slope. Everywhere along the first nullcline $\dot{x} = 0$, we have that $\dot{\boldsymbol{\varphi}}$ lies parallel to the v-axis. Similarly, everywhere along the second nullcline $\dot{v} = 0$, we have that $\dot{\boldsymbol{\varphi}}$ lies parallel to the x-axis. The situation is depicted in fig. 2.2.

2.3 Damped Harmonic Oscillator with Forcing

When forced, the equation for the damped oscillator becomes

$$\frac{d^2x}{dt^2} + 2\beta\,\frac{dx}{dt} + \omega_0^2\,x = f(t) \quad , \tag{2.27}$$

where $f(t) = F(t)/m$. Since this equation is linear in $x(t)$, we can, without loss of generality, restrict out attention to harmonic forcing terms of the form

$$f(t) = f_0\cos(\Omega t + \varphi_0) = \text{Re}\left[f_0\,e^{-i\varphi_0}\,e^{-i\Omega t}\right] \tag{2.28}$$

where Re stands for "real part". Here, Ω is the forcing frequency.

Consider first the complex equation

$$\frac{d^2z}{dt^2} + 2\beta\,\frac{dz}{dt} + \omega_0^2\,z = f_0\,e^{-i\varphi_0}\,e^{-i\Omega t} \quad . \tag{2.29}$$

We try a solution $z(t) = z_0\,e^{-i\Omega t}$. Plugging in, we obtain the algebraic equation

$$z_0 = \frac{f_0\,e^{-i\varphi_0}}{\omega_0^2 - 2i\beta\Omega - \Omega^2} \equiv A(\Omega)\,e^{i\delta(\Omega)}\;f_0 e^{-i\varphi_0} \quad . \tag{2.30}$$

The amplitude $A(\Omega)$ and phase shift $\delta(\Omega)$ are given by the equation

$$A(\Omega)\,e^{i\delta(\Omega)} = \frac{1}{\omega_0^2 - 2i\beta\Omega - \Omega^2} \quad . \tag{2.31}$$

A basic fact of complex numbers:

$$\frac{1}{a - ib} = \frac{a + ib}{a^2 + b^2} = \frac{e^{i\tan^{-1}(b/a)}}{\sqrt{a^2 + b^2}} \quad . \tag{2.32}$$

Thus,

$$A(\Omega) = \Big((\omega_0^2 - \Omega^2)^2 + 4\beta^2\Omega^2\Big)^{-1/2} \qquad , \qquad \delta(\Omega) = \tan^{-1}\left(\frac{2\beta\Omega}{\omega_0^2 - \Omega^2}\right) \quad . \tag{2.33}$$

Now since the coefficients β and ω_0^2 are real, we can take the complex conjugate of eqn. 2.29, and write

$$\begin{aligned} \ddot{z} + 2\beta\,\dot{z} + \omega_0^2\,z &= f_0\,e^{-i\varphi_0}\,e^{-i\Omega t} \\ \ddot{z}^* + 2\beta\dot{z}^* + \omega_0^2\,z^* &= f_0\,e^{+i\varphi_0}\,e^{+i\Omega t} \quad , \end{aligned} \tag{2.34}$$

where z^* is the complex conjugate of z. We now add these two equations and divide by two to arrive at

$$\ddot{x} + 2\beta\,\dot{x} + \omega_0^2\,x = f_0\,\cos(\Omega t + \varphi_0) \quad . \tag{2.35}$$

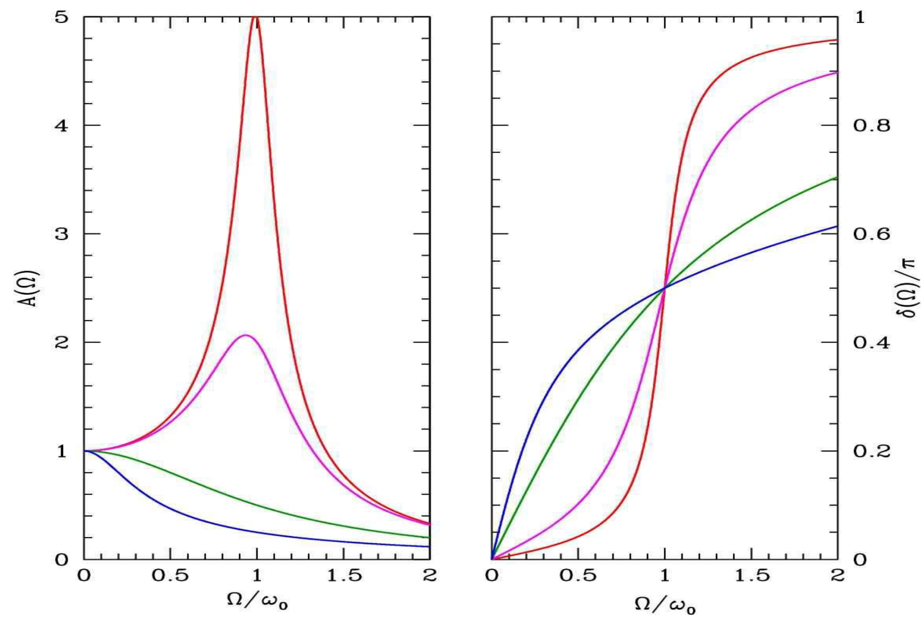

Figure 2.3: Amplitude and phase shift *versus* oscillator frequency (units of ω_0) for β/ω_0 values of 0.1 (red), 0.25 (magenta), 1.0 (green), and 2.0 (blue).

Therefore, the real, physical solution we seek is

$$\begin{aligned} x_{\rm inh}(t) &= \mathsf{Re}\Big[A(\varOmega)\, e^{i\delta(\varOmega)} \cdot f_0\, e^{-i\varphi_0}\, e^{-i\varOmega t}\Big] \\ &= A(\varOmega)\, f_0\, \cos\big(\varOmega t + \varphi_0 - \delta(\varOmega)\big) \quad . \end{aligned} \tag{2.36}$$

The quantity $A(\varOmega)$ is the *amplitude* of the response (in units of f_0), while $\delta(\varOmega)$ is the (dimensionless) *phase lag* (typically expressed in radians).

The maximum of the amplitude $A(\varOmega)$ occurs when $A'(\varOmega) = 0$. From

$$\frac{dA}{d\varOmega} = -\frac{2\varOmega}{\big[A(\varOmega)\big]^3}\,\big(\varOmega^2 - \omega_0^2 + 2\beta^2\big) \quad , \tag{2.37}$$

we conclude that $A'(\varOmega) = 0$ for $\varOmega = 0$ and for $\varOmega = \varOmega_{\rm R}$, where

$$\varOmega_{\rm R} = \sqrt{\omega_0^2 - 2\beta^2} \quad . \tag{2.38}$$

The solution at $\varOmega = \varOmega_{\rm R}$ pertains only if $\omega_0^2 > 2\beta^2$, of course, in which case $\varOmega = 0$ is a local minimum and $\varOmega = \varOmega_{\rm R}$ a local maximum. If $\omega_0^2 < 2\beta^2$ there is only a local maximum, at $\varOmega = 0$. See fig. 2.3.

Since equation 2.27 is linear, we can add a solution to the homogeneous equation to $x_{\rm inh}(t)$ and we will still have a solution. Thus, the most general solution to eqn. 2.27 is Therefore, the real, physical solution we seek is

$$\begin{aligned} x(t) &= x_{\rm inh}(t) + x_{\rm hom}(t) \\ &= \text{Re}\Big[A(\Omega)\, e^{i\delta(\Omega)} \cdot f_0\, e^{-i\varphi_0}\, e^{-i\Omega t}\Big] + C_+\, e^{-i\omega_+ t} + C_-\, e^{-i\omega_- t} \\ &= \overbrace{A(\Omega)\, f_0 \cos\big(\Omega t + \varphi_0 - \delta(\Omega)\big)}^{x_{\rm inh}(t)} + \overbrace{\begin{cases} C\, e^{-\beta t}\, \cos(\nu t) + D\, e^{-\beta t}\, \sin(\nu t) \\ C\, e^{-\beta t}\, \cosh(\lambda t) + D\, e^{-\beta t}\, \sinh(\lambda t) \end{cases}}^{x_{\rm hom}(t)} \quad . \end{aligned} \tag{2.39}$$

When $\omega_0^2 > \beta^2$, we choose the top expression for $x_{\rm hom}(t)$, with $\nu = \sqrt{\omega_0^2 - \beta^2}$. When $\omega^2 < \beta^2$, we choose the bottom expression for $x_{\rm hom}(t)$, with $\lambda = \sqrt{\beta^2 - \omega_0^2}$.

The quantity $x_{\rm hom}(t)$ in eqn. 2.39 is the solution to the homogeneous equation, *i.e.* with $f(t) = 0$. This involves two constants of integration, C and D, which are then determined by imposing initial conditions on $x(0)$ and $\dot{x}(0)$ – two constants of integration always arise in the solution of a second order ODE, whether or not it is homogeneous. That is, C and D are adjusted so as to satisfy $x(0) = x_0$ and $\dot{x}_0 = v_0$. However, due to their $e^{-\beta t}$ prefactor, these terms decay to zero once t reaches a relatively low multiple of β^{-1}. They are called *transients*, and may be set to zero if we are only interested in the long time behavior of the system. This means, incidentally, that the initial conditions are effectively forgotten over a time scale on the order of β^{-1}.

For $\Omega_{\rm R} > 0$, one defines the *quality factor*, Q, of the oscillator by $Q = \Omega_{\rm R}/2\beta$. Q is a rough measure of how many periods the unforced oscillator executes before its initial amplitude is damped down to a small value. For a forced oscillator driven near resonance, and for weak damping, Q is also related to the ratio of average energy in the oscillator to the energy lost per cycle by the external source. To see this, let us compute the energy lost per cycle,

$$\begin{aligned} \Delta E &= m \int\limits_0^{2\pi/\Omega} \! dt\, \dot{x}\, f(t) \\ &= -m \int\limits_0^{2\pi/\Omega} \! dt\, \Omega\, A\, f_0^2\, \sin(\Omega t + \varphi_0 - \delta)\, \cos(\Omega t + \varphi_0) \\ &= \pi A\, f_0^2\, m\, \sin\delta = 2\pi\beta\, m\, \Omega\, A^2(\Omega)\, f_0^2 \quad , \end{aligned} \tag{2.40}$$

since $\sin\delta(\Omega) = 2\beta\Omega\, A(\Omega)$. The oscillator energy, averaged over the cycle, is

$$\langle E \rangle = \frac{\Omega}{2\pi} \int\limits_0^{2\pi/\Omega} \! dt\, \tfrac{1}{2} m \big(\dot{x}^2 + \omega_0^2\, x^2\big) = \tfrac{1}{4} m\, (\Omega^2 + \omega_0^2)\, A^2(\Omega)\, f_0^2 \quad . \tag{2.41}$$

Thus, we have

$$\frac{2\pi \langle E \rangle}{\Delta E} = \frac{\Omega^2 + \omega_0^2}{4\beta\Omega} \quad . \tag{2.42}$$

Thus, for $\Omega \approx \Omega_{\rm R}$ and $\beta^2 \ll \omega_0^2$, we have

$$Q \approx \frac{2\pi \langle E \rangle}{\Delta E} \approx \frac{\omega_0}{2\beta} \quad . \tag{2.43}$$

2.3.1 Resonant forcing

When the damping β vanishes, the response diverges at resonance. The solution to the resonantly forced oscillator

$$\ddot{x} + \omega_0^2\, x = f_0 \cos(\omega_0\, t + \varphi_0) \tag{2.44}$$

is given by

$$x(t) = \frac{f_0}{2\omega_0}\, t\, \sin(\omega_0\, t + \varphi_0) + \overbrace{A\, \cos(\omega_0\, t) + B\, \sin(\omega_0\, t)}^{x_{\rm hom}(t)} \quad . \tag{2.45}$$

The amplitude of this solution grows linearly due to the energy pumped into the oscillator by the resonant external forcing. In the real world, nonlinearities can mitigate this unphysical, unbounded response.

2.3.2 R-L-C circuits

Consider the R-L-C circuit of fig. 2.4. When the switch is to the left, the capacitor is charged, eventually to a steady state value $Q = CV$. At $t = 0$ the switch is thrown to the right, completing the R-L-C circuit. Recall that the sum of the voltage drops across the three elements must be zero:

$$L\,\frac{dI}{dt} + IR + \frac{Q}{C} = 0 \quad . \tag{2.46}$$

We also have $\dot{Q} = I$, hence

$$\frac{d^2Q}{dt^2} + \frac{R}{L}\,\frac{dQ}{dt} + \frac{1}{LC}\,Q = 0 \quad , \tag{2.47}$$

which is the equation for a damped harmonic oscillator, with $\omega_0 = (LC)^{-1/2}$ and $\beta = R/2L$.

The boundary conditions at $t = 0$ are $Q(0) = CV$ and $\dot{Q}(0) = 0$. Under these conditions, the full solution at all times is

$$\begin{aligned} Q(t) &= CV\, e^{-\beta t} \left(\cos \nu t + \frac{\beta}{\nu}\, \sin \nu t \right) \\ I(t) &= -CV\, \frac{\omega_0^2}{\nu}\, e^{-\beta t}\, \sin \nu t \quad , \end{aligned} \tag{2.48}$$

again with $\nu = \sqrt{\omega_0^2 - \beta^2}$.

If we put a time-dependent voltage source in series with the resistor, capacitor, and inductor, we would have

$$L\,\frac{dI}{dt} + IR + \frac{Q}{C} = V(t) \quad , \tag{2.49}$$

which is the equation of a *forced* damped harmonic oscillator.

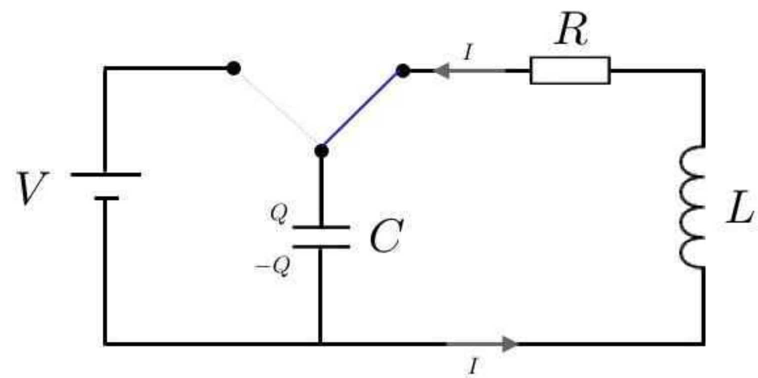

Figure 2.4: An R-L-C circuit which behaves as a damped harmonic oscillator.

2.3.3 Examples

Third order linear ODE with forcing

The problem is to solve the equation

$$\mathcal{L}_t\, x \equiv \dddot{x} + (a+b+c)\,\ddot{x} + (ab+ac+bc)\,\dot{x} + abc\,x = f_0\cos(\Omega t) \quad . \tag{2.50}$$

The key to solving this is to note that the differential operator $\mathcal{L}_t$ factorizes:

$$\begin{aligned}\mathcal{L}_t &= \frac{d^3}{dt^3} + (a+b+c)\,\frac{d^2}{dt^2} + (ab+ac+bc)\,\frac{d}{dt} + abc \\ &= \Big(\frac{d}{dt}+a\Big)\Big(\frac{d}{dt}+b\Big)\Big(\frac{d}{dt}+c\Big) \quad ,\end{aligned} \tag{2.51}$$

which says that the third order differential operator appearing in the ODE is in fact a product of first order differential operators. Since

$$\frac{dx}{dt} + \alpha x = 0 \quad \Longrightarrow \quad x(t) = A\,e^{-\alpha x} \quad , \tag{2.52}$$

we see that the homogeneous solution takes the form

$$x_{\rm h}(t) = A\,e^{-at} + B\,e^{-bt} + C\,e^{-ct} \quad , \tag{2.53}$$

where A, B, and C are constants.

To find the inhomogeneous solution, we solve $\mathcal{L}_t\, x = f_0\, e^{-i\Omega t}$ and take the real part. As before, we assume the inhomogeneous solution $x(t)$ oscillates with the driving frequency Ω, and we write $x(t) = x_0\, e^{-i\Omega t}$, which entails

$$\mathcal{L}_t\, x_0\, e^{-i\Omega t} = (a-i\Omega)\,(b-i\Omega)\,(c-i\Omega)\,x_0\, e^{-i\Omega t} \tag{2.54}$$

and thus

$$x_0 = \frac{f_0\, e^{-i\Omega t}}{(a-i\Omega)(b-i\Omega)(c-i\Omega)} \equiv A(\Omega)\, e^{i\delta(\Omega)}\, f_0\, e^{-i\Omega t} \quad ,$$

where

$$\begin{aligned} A(\Omega) &= \Big[(a^2+\Omega^2)\,(b^2+\Omega^2)\,(c^2+\Omega^2)\Big]^{-1/2} \\ \delta(\Omega) &= \tan^{-1}\Big(\frac{\Omega}{a}\Big) + \tan^{-1}\Big(\frac{\Omega}{b}\Big) + \tan^{-1}\Big(\frac{\Omega}{c}\Big) \quad . \end{aligned} \tag{2.55}$$

Thus, the most general solution to $L_t\, x(t) = f_0 \cos(\Omega t)$ is

$$x(t) = A(\Omega)\, f_0 \, \cos\big(\Omega t - \delta(\Omega)\big) + A\, e^{-at} + B\, e^{-bt} + C\, e^{-ct} \quad . \tag{2.56}$$

Note that the phase shift increases monotonically from $\delta(0) = 0$ to $\delta(\infty) = \frac{3}{2}\pi$.

Mechanical analog of RLC circuit

Consider the electrical circuit in fig. 2.5. Our task is to construct its mechanical analog. To do so, we invoke Kirchoff's laws around the left and right loops:

$$\begin{aligned} L_1\, \dot I_1 + \frac{Q_1}{C_1} + R_1\,(I_1 - I_2) &= 0 \\ L_2\, \dot I_2 + R_2\, I_2 + R_1\,(I_2 - I_1) &= V(t) \quad . \end{aligned} \tag{2.57}$$

Let $Q_1(t)$ be the charge on the left plate of capacitor C_1, and define

$$Q_2(t) = \int\limits_0^t \! dt'\, I_2(t') \quad . \tag{2.58}$$

Then Kirchoff's laws may be written

$$\begin{aligned} \ddot Q_1 + \frac{R_1}{L_1}\,(\dot Q_1 - \dot Q_2) + \frac{1}{L_1 C_1}\, Q_1 &= 0 \\ \ddot Q_2 + \frac{R_2}{L_2}\, \dot Q_2 + \frac{R_1}{L_2}\,(\dot Q_2 - \dot Q_1) &= \frac{V(t)}{L_2} \quad . \end{aligned} \tag{2.59}$$

Now consider the mechanical system in fig. 2.6. The blocks have masses M_1 and M_2. The friction coefficient between blocks 1 and 2 is b_1, and the friction coefficient between block 2 and the floor is b_2. Here we assume a velocity-dependent frictional force $F_{\rm f} = -b\dot x$, rather than the more conventional constant $F_{\rm f} = -\mu\, W$, where W is the weight of an object. Velocity-dependent friction is applicable when the relative velocity of an object and a surface is sufficiently large. There is a spring of spring constant k_1 which connects block 1 to the wall. Finally, block 2 is driven by a periodic acceleration $f_0 \, \cos(\omega t)$. We now identify

$$X_1 \leftrightarrow Q_1 \quad , \quad X_2 \leftrightarrow Q_2 \quad , \quad b_1 \leftrightarrow \frac{R_1}{L_1} \quad , \quad b_2 \leftrightarrow \frac{R_2}{L_2} \quad , \quad k_1 \leftrightarrow \frac{1}{L_1 C_1} \quad , \tag{2.60}$$

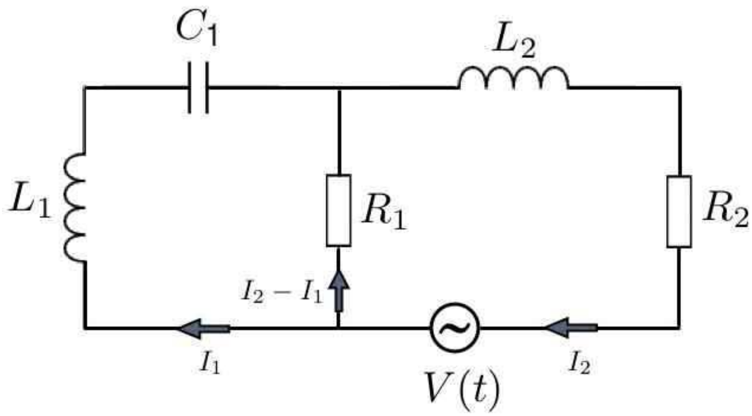

Figure 2.5: A driven L-C-R circuit, with $V(t) = V_0 \cos(\omega t)$.

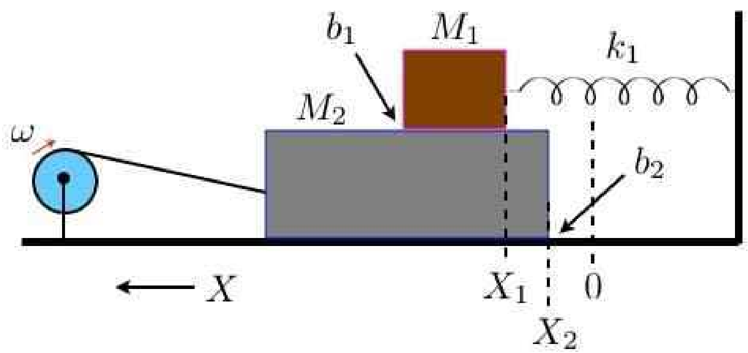

Figure 2.6: The equivalent mechanical circuit for fig. 2.5.

as well as $f(t) \leftrightarrow V(t)/L_2$.

The solution again proceeds by Fourier transform. We write

$$V(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \hat{V}(\omega)\, e^{-i\omega t} \tag{2.61}$$

and

$$\begin{Bmatrix} Q_1(t) \\ \hat{I}_2(t) \end{Bmatrix} = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \begin{Bmatrix} \hat{Q}_1(\omega) \\ \hat{I}_2(\omega) \end{Bmatrix} e^{-i\omega t} \tag{2.62}$$

The frequency space version of Kirchoff's laws for this problem is

$$\overbrace{\begin{pmatrix} -\omega^2 - i\omega\, R_1/L_1 + 1/L_1 C_1 & R_1/L_1 \\ & \\ i\omega\, R_1/L_2 & -i\omega + (R_1 + R_2)/L_2 \end{pmatrix}}^{\hat{G}(\omega)} \begin{pmatrix} \hat{Q}_1(\omega) \\ \\ \hat{I}_2(\omega) \end{pmatrix} = \begin{pmatrix} 0 \\ \\ \hat{V}(\omega)/L_2 \end{pmatrix} \tag{2.63}$$

The homogeneous equation has eigenfrequencies given by the solution to $\det \hat{G}(\omega) = 0$, which is a cubic equation. Correspondingly, there are three initial conditions to account for: $Q_1(0)$, $I_1(0)$, and $I_2(0)$. As in the case of the single damped harmonic oscillator, these transients are damped, and for large times may be ignored. The solution then is

$$\begin{pmatrix} \hat{Q}_1(\omega) \\ \\ \hat{I}_2(\omega) \end{pmatrix} = \begin{pmatrix} -\omega^2 - i\omega\, R_1/L_1 + 1/L_1\, C_1 & R_1/L_1 \\ \\ i\omega\, R_1/L_2 & -i\omega + (R_1 + R_2)/L_2 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \\ \hat{V}(\omega)/L_2 \end{pmatrix} \quad . \tag{2.64}$$

To obtain the time-dependent $Q_1(t)$ and $I_2(t)$, we must compute the Fourier transform back to the time domain.

2.4 Green's Functions

2.4.1 General solution of forced damped harmonic oscillator

For a general forcing function $f(t)$, we solve by Fourier transform. Recall that a function $F(t)$ in the time domain has a Fourier transform $\hat{F}(\omega)$ in the frequency domain. The relation between the two is:[1]

$$F(t) = \int\limits_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega t}\, \hat{F}(\omega) \quad \Longleftrightarrow \quad \hat{F}(\omega) = \int\limits_{-\infty}^{\infty} dt\, e^{+i\omega t}\, F(t) \quad . \tag{2.65}$$

We can convert the differential equation 2.27 to an algebraic equation in the frequency domain for the Fourier transform $\hat{x}(\omega)$, *viz.*

$$\ddot{x}(t) + 2\beta\dot{x}(t) + \omega_0^2 x(t) = f(t) \qquad \Leftrightarrow \qquad \hat{x}(\omega) = \hat{G}(\omega)\, \hat{f}(\omega) \quad , \tag{2.66}$$

where

$$\hat{G}(\omega) = \frac{1}{\omega_0^2 - 2i\beta\omega - \omega^2} \tag{2.67}$$

is the *Green's function* in the frequency domain. The general solution is written

$$x(t) = \int\limits_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega t}\, \hat{G}(\omega)\, \hat{f}(\omega) + x_{\rm h}(t) \quad , \tag{2.68}$$

[1] Different texts often use different conventions for Fourier and inverse Fourier transforms. Sometimes the factor of $(2\pi)^{-1}$ is associated with the time integral, and sometimes a factor of $(2\pi)^{-1/2}$ is assigned to both frequency and time integrals. The convention I use is obviously the best.

where $x_\mathrm{h}(t) = \sum_i C_i \, e^{-i\omega_i t}$ is a solution to the homogeneous equation. We may also write the above integral over the time domain:

$$x(t) = \int\limits_{-\infty}^{\infty} dt' \, G(t-t') \, f(t') + x_\mathrm{h}(t) \tag{2.69}$$

$$G(s) = \int\limits_{-\infty}^{\infty} \frac{d\omega}{2\pi} \, e^{-i\omega s} \, \hat{G}(\omega) \tag{2.70}$$
$$= \nu^{-1} \exp(-\beta s) \, \sin(\nu s) \, \Theta(s)$$

where $\Theta(s)$ is the *step function*,

$$\Theta(s) = \begin{cases} 1 & \text{if } s \geq 0 \\ 0 & \text{if } s < 0 \end{cases} \tag{2.71}$$

where once again $\nu \equiv \sqrt{\omega_0^2 - \beta^2}$. In the overdamped case, we write $\nu = i\lambda$ with $\lambda = \sqrt{\beta^2 - \omega_0^2}$, and we then have

$$G(s) = \lambda^{-1} \exp(-\beta s) \, \sinh(\lambda s) \, \Theta(s) \quad . \tag{2.72}$$

Example: force pulse

Consider a pulse force

$$f(t) = f_0 \, \Theta(t) \, \Theta(T-t) = \begin{cases} f_0 & \text{if } 0 \leq t \leq T \\ 0 & \text{otherwise.} \end{cases} \tag{2.73}$$

In the underdamped regime, for example, and ignoring the transients, we find the solution

$$x(t) = \frac{f_0}{\omega_0^2} \left\{ 1 - e^{-\beta t} \cos \nu t - \frac{\beta}{\nu} e^{-\beta t} \sin \nu t \right\} \tag{2.74}$$

if $0 \leq t \leq T$ and

$$x(t) = \frac{f_0}{\omega_0^2} \left\{ \left(e^{-\beta(t-T)} \cos \nu(t-T) - e^{-\beta t} \cos \nu t \right) + \frac{\beta}{\nu} \left(e^{-\beta(t-T)} \sin \nu(t-T) - e^{-\beta t} \sin \nu t \right) \right\} \tag{2.75}$$

if $t > T$.

2.4.2 General linear autonomous inhomogeneous ODEs

This method immediately generalizes to the case of general autonomous linear inhomogeneous ODEs of the form

$$\frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \ldots + a_1 \frac{dx}{dt} + a_0 \, x = f(t) \quad . \tag{2.76}$$

We can write this as

$$\mathcal{L}_t \, x(t) = f(t) \quad , \tag{2.77}$$

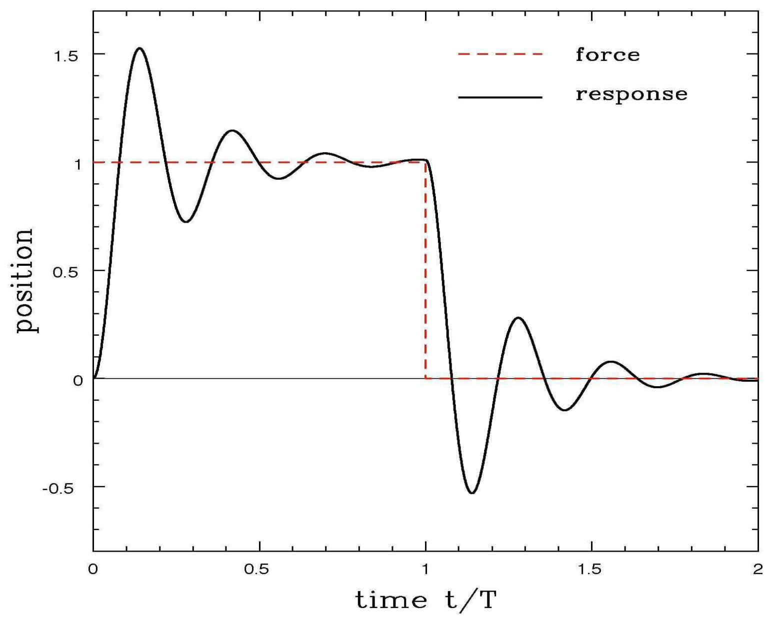

Figure 2.7: Response of an underdamped oscillator to a pulse force.

where $\mathcal{L}_t$ is the n^{th} order differential operator

$$\mathcal{L}_t = \frac{d^n}{dt^n} + a_{n-1}\,\frac{d^{n-1}}{dt^{n-1}} + \ldots + a_1\,\frac{d}{dt} + a_0 \quad . \tag{2.78}$$

The general solution to the inhomogeneous equation is given by

$$x(t) = x_{\rm h}(t) + \int\limits_{-\infty}^{\infty} dt'\, G(t,t')\, f(t') \quad , \tag{2.79}$$

where $G(t,t')$ is the Green's function. Note that $\mathcal{L}_t\, x_{\rm h}(t) = 0$. Thus, in order for eqns. 2.77 and 2.79 to be true, we must have

$$\mathcal{L}_t\, x(t) = \overbrace{\mathcal{L}_t\, x_{\rm h}(t)}^{\text{this vanishes}} + \int\limits_{-\infty}^{\infty} dt'\, \mathcal{L}_t\, G(t,t')\, f(t') = f(t) \quad , \tag{2.80}$$

which means that

$$\mathcal{L}_t\, G(t,t') = \delta(t-t') \quad , \tag{2.81}$$

where $\delta(t-t')$ is the Dirac δ-function. Some properties of $\delta(x)$:

$$\int\limits_a^b dx\, f(x)\, \delta(x-y) = \begin{cases} f(y) & \text{if } a < y < b \\ 0 & \text{if } y < a \text{ or } y > b \end{cases} \quad . \tag{2.82}$$

$$\delta\big(g(x)\big) = \sum_{\substack{x_i \text{with} \\ g(x_i)=0}} \frac{\delta(x-x_i)}{\big|g'(x_i)\big|} \quad , \tag{2.83}$$

valid for any functions $f(x)$ and $g(x)$. The sum in the second equation is over the zeros x_i of $g(x)$.

Incidentally, the Dirac δ-function enters into the relation between a function and its Fourier transform, in the following sense. We have

$$f(t) = \int\limits_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega t}\, \hat{f}(\omega) \qquad , \qquad \hat{f}(\omega) = \int\limits_{-\infty}^{\infty} dt\, e^{+i\omega t}\, f(t) \quad . \tag{2.84}$$

Substituting the second equation into the first, we have

$$f(t) = \int\limits_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega t} \int\limits_{-\infty}^{\infty} dt'\, e^{i\omega t'}\, f(t') = \int\limits_{-\infty}^{\infty} dt' \left\{ \int\limits_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{i\omega(t'-t)} \right\} f(t') \quad , \tag{2.85}$$

which is indeed correct because the term in brackets is a representation of $\delta(t-t')$:

$$\int\limits_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{i\omega s} = \delta(s) \quad . \tag{2.86}$$

If the differential equation $\mathcal{L}_t\, x(t) = f(t)$ is defined over some finite t interval with prescribed boundary conditions on $x(t)$ at the endpoints, then $G(t,t')$ will depend on t and t' separately. For the case we are considering, the interval is the entire real line $t \in (-\infty, \infty)$, and $G(t,t') = G(t-t')$ is a function of the single variable $t-t'$.

Note that $\mathcal{L}_t = \mathcal{L}\big(\frac{d}{dt}\big)$ may be considered a function of the differential operator $\frac{d}{dt}$. If we now Fourier transform the equation $\mathcal{L}_t\, x(t) = f(t)$, we obtain

$$\begin{aligned} \int\limits_{-\infty}^{\infty} dt\, e^{i\omega t}\, f(t) &= \int\limits_{-\infty}^{\infty} dt\, e^{i\omega t} \left\{ \frac{d^n}{dt^n} + a_{n-1}\, \frac{d^{n-1}}{dt^{n-1}} + \ldots + a_1\, \frac{d}{dt} + a_0 \right\} x(t) \\ &= \int\limits_{-\infty}^{\infty} dt\, e^{i\omega t} \left\{ (-i\omega)^n + a_{n-1}\, (-i\omega)^{n-1} + \ldots + a_1\, (-i\omega) + a_0 \right\} x(t) \quad , \end{aligned} \tag{2.87}$$

where we integrate by parts on t, assuming the boundary terms at $t = \pm\infty$ vanish, *i.e.* $x(\pm\infty) = 0$, so that, inside the t integral,

$$e^{i\omega t} \left(\frac{d}{dt}\right)^{k} x(t) \to \left[\left(-\frac{d}{dt}\right)^{k} e^{i\omega t} \right] x(t) = (-i\omega)^k\, e^{i\omega t}\, x(t) \quad . \tag{2.88}$$

Thus, if we define

$$\hat{\mathcal{L}}(\omega) = \sum_{k=0}^{n} a_k\, (-i\omega)^k \quad , \tag{2.89}$$

then we have

$$\hat{\mathcal{L}}(\omega)\, \hat{\boldsymbol{x}}(\omega) = \hat{f}(\omega) \quad , \tag{2.90}$$

where $a_n \equiv 1$. According to the Fundamental Theorem of Algebra, the n^{th} degree polynomial $\hat{\mathcal{L}}(\omega)$ may be uniquely factored over the complex ω plane into a product over n roots:

$$\hat{\mathcal{L}}(\omega) = (-i)^n \, (\omega - \omega_1)(\omega - \omega_2) \cdots (\omega - \omega_n) \quad . \tag{2.91}$$

If the $\{a_k\}$ are all real, then $\big[\hat{\mathcal{L}}(\omega)\big]^* = \hat{\mathcal{L}}(-\omega^*)$, hence if Ω is a root then so is $-\Omega^*$. Thus, the roots appear in pairs which are symmetric about the imaginary axis. *I.e.* if $\Omega = a + ib$ is a root, then so is $-\Omega^* = -a + ib$.

The general solution to the homogeneous equation is

$$x_{\text{h}}(t) = \sum_{i=1}^{n} A_i \, e^{-i\omega_i t} \quad , \tag{2.92}$$

which involves n arbitrary complex constants A_i. The susceptibility, or Green's function in Fourier space, $\hat{G}(\omega)$ is then

$$\hat{G}(\omega) = \frac{1}{\hat{\mathcal{L}}(\omega)} = \frac{i^n}{(\omega - \omega_1)(\omega - \omega_2) \cdots (\omega - \omega_n)} \quad , \tag{2.93}$$

and the general solution to the inhomogeneous equation is again given by

$$x(t) = x_{\text{h}}(t) + \int\limits_{-\infty}^{\infty} \! dt' \, G(t - t') \, f(t') \quad , \tag{2.94}$$

where $x_{\text{h}}(t)$ is the solution to the homogeneous equation, *i.e.* with zero forcing, and where

$$\begin{aligned} G(s) &= \int\limits_{-\infty}^{\infty} \! \frac{d\omega}{2\pi} \, e^{-i\omega s} \, \hat{G}(\omega) \\ &= i^n \int\limits_{-\infty}^{\infty} \! \frac{d\omega}{2\pi} \, \frac{e^{-i\omega s}}{(\omega - \omega_1)(\omega - \omega_2) \cdots (\omega - \omega_n)} = \sum_{j=1}^{n} \frac{e^{-i\omega_j s}}{i \, \mathcal{L}'(\omega_j)} \, \Theta(s) \quad , \end{aligned} \tag{2.95}$$

where we assume that $\mathsf{Im}\,(\omega_j) < 0$ for all j. The integral above was done using Cauchy's theorem and the calculus of residues – a beautiful result from the theory of complex functions.

As an example, consider the familiar case

$$\hat{\mathcal{L}}(\omega) = \omega_0^2 - 2i\beta\omega - \omega^2 = -(\omega - \omega_+)\,(\omega - \omega_-) \quad , \tag{2.96}$$

with $\omega_\pm = -i\beta \pm \nu$, and $\nu = (\omega_0^2 - \beta^2)^{1/2}$. This yields

$$\mathcal{L}'(\omega_\pm) = \mp(\omega_+ - \omega_-) = \mp 2\nu \quad . \tag{2.97}$$

Then according to equation 2.95,

$$\begin{aligned} G(s) &= \left\{ \frac{e^{-i\omega_+ s}}{i\mathcal{L}'(\omega_+)} + \frac{e^{-i\omega_- s}}{i\mathcal{L}'(\omega_-)} \right\} \Theta(s) \\ &= \left\{ \frac{e^{-\beta s} \, e^{-i\nu s}}{-2i\nu} + \frac{e^{-\beta s} \, e^{i\nu s}}{2i\nu} \right\} \Theta(s) = \nu^{-1} \, e^{-\beta s} \, \sin(\nu s) \, \Theta(s) \quad , \end{aligned} \tag{2.98}$$

exactly as before.

2.4.3 Kramers-Krönig relations

Suppose $\hat{\chi}(\omega) \equiv \hat{G}(\omega)$ is analytic in the UHP[2]. Then for all ν, we must have

$$\int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \, \frac{\hat{\chi}(\nu)}{\nu - \omega + i\epsilon} = 0 \quad , \tag{2.99}$$

where ϵ is a positive infinitesimal. The reason is simple: just close the contour in the UHP, assuming $\hat{\chi}(\omega)$ vanishes sufficiently rapidly that Jordan's lemma can be applied. Clearly this is an extremely weak restriction on $\hat{\chi}(\omega)$, given the fact that the denominator already causes the integrand to vanish as $|\omega|^{-1}$.

Let us examine the function

$$\frac{1}{\nu - \omega + i\epsilon} = \frac{\nu - \omega}{(\nu - \omega)^2 + \epsilon^2} - \frac{i\epsilon}{(\nu - \omega)^2 + \epsilon^2} \quad . \tag{2.100}$$

which we have separated into real and imaginary parts. Under an integral sign, the first term, in the limit $\epsilon \to 0$, is equivalent to taking a *principal part* of the integral. That is, for any function $F(\nu)$ which is regular at $\nu = \omega$,

$$\lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \, \frac{\nu - \omega}{(\nu - \omega)^2 + \epsilon^2} \, F(\nu) \equiv \mathcal{P}\int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \, \frac{F(\nu)}{\nu - \omega} \quad . \tag{2.101}$$

The principal part symbol $\mathcal{P}$ means that the singularity at $\nu = \omega$ is elided, either by smoothing out the function $1/(\nu - \epsilon)$ as above, or by simply cutting out a region of integration of width ϵ on either side of $\nu = \omega$.

The imaginary part is more interesting. Let us write

$$h(u) \equiv \frac{\epsilon}{u^2 + \epsilon^2} \quad . \tag{2.102}$$

For $|u| \gg \epsilon$, $h(u) \simeq \epsilon/u^2$, which vanishes as $\epsilon \to 0$. For $u = 0$, $h(0) = 1/\epsilon$ which diverges as $\epsilon \to 0$. Thus, $h(u)$ has a huge peak at $u = 0$ and rapidly decays to 0 as one moves off the peak in either direction a distance greater that ϵ. Finally, note that

$$\int_{-\infty}^{\infty} du \, h(u) = \pi \quad , \tag{2.103}$$

a result which itself is easy to show using contour integration. Putting it all together, this tells us that

$$\lim_{\epsilon \to 0} \frac{\epsilon}{u^2 + \epsilon^2} = \pi\delta(u) \quad . \tag{2.104}$$

Thus, for positive infinitesimal ϵ,

$$\frac{1}{u \pm i\epsilon} = \mathcal{P}\,\frac{1}{u} \mp i\pi\delta(u) \quad , \tag{2.105}$$

[2] In this section, we use the notation $\hat{\chi}(\omega)$ for the susceptibility, rather than $\hat{G}(\omega)$

a most useful result.

We now return to our initial result 2.99, and we separate $\hat{\chi}(\omega)$ into real and imaginary parts:

$$\hat{\chi}(\omega) = \hat{\chi}'(\omega) + i\hat{\chi}''(\omega) \quad . \tag{2.106}$$

(In this equation, the primes do not indicate differentiation with respect to argument.) We therefore have, for every real value of ω,

$$0 = \int\limits_{-\infty}^{\infty} \frac{d\nu}{2\pi} \Big[\chi'(\nu) + i\chi''(\nu)\Big] \Big[\mathcal{P}\,\frac{1}{\nu - \omega} - i\pi\delta(\nu - \omega)\Big] \quad . \tag{2.107}$$

Taking the real and imaginary parts of this equation, we derive the *Kramers-Krönig relations*:

$$\chi'(\omega) = +\mathcal{P}\int\limits_{-\infty}^{\infty} \frac{d\nu}{\pi}\,\frac{\hat{\chi}''(\nu)}{\nu - \omega} \qquad , \qquad \chi''(\omega) = -\mathcal{P}\int\limits_{-\infty}^{\infty} \frac{d\nu}{\pi}\,\frac{\hat{\chi}'(\nu)}{\nu - \omega} \quad . \tag{2.108}$$

2.4.4 Laplace transforms

Consider a function $F(t)$ defined on nonnegative real numbers $t \geq 0$. The Laplace transform of $F(t)$ is defined to be

$$\check{F}(z) = \int\limits_0^{\infty} dt\, F(t)\, e^{-zt} \quad , \tag{2.109}$$

where in general z is complex. The inverse transform is given by

$$F(t) = \int\limits_{c-i\infty}^{c+i\infty} \frac{dz}{2\pi i}\, \check{F}(z)\, e^{zt} \quad , \tag{2.110}$$

where c is such that the integration lies to the right of any singularities of $\check{F}(z)$ in the complex z-plane. The Laplace transform is particularly useful in cases where we specify initial conditions, and where the inhomogeneous term vanishes for sufficiently small values of t; here we have taken $F(t) = 0$ for $t < 0$.

Note that the Laplace transform of $\dot{F}(t)$ is given by

$$\int\limits_0^{\infty} dt\, \frac{dF(t)}{dt}\, e^{-zt} = z\,\check{F}(z) - F(0) \quad , \tag{2.111}$$

which is easily confirmed via integration by parts. Thus, if $F^{(k)}(t) \equiv d^k F(t)/dt^n$, we have

$$\begin{aligned} \check{F}^{(k)}(z) &= z\,\check{F}^{(k-1)}(z) - F^{(k-1)}(0) \\ &= z^k \check{F}(z) - z^{k-1} F(0) - \cdots - z\,F^{(k-2)}(0) - F^{(k-1)}(0) \\ &= z^k \check{F}(z) - \sum_{p=0}^{n-1} z^p\, F^{(k-1-p)}(0) \quad . \end{aligned} \tag{2.112}$$

Thus, the Laplace transform of the n^{th} order linear, autonomous, inhomogeneous ODE in eqn. 2.76,

$$\frac{d^n x}{dt^n} + a_{n-1}\,\frac{d^{n-1}x}{dt^{n-1}} + \ldots + a_1\,\frac{dx}{dt} + a_0\,x = f(t) \quad , \tag{2.113}$$

is given by

$$Q(z)\,\check{x}(z) = \check{f}(z) + \sum_{p=0}^{n-1} C_p\,z^p \quad , \tag{2.114}$$

with

$$Q(z) = \sum_{k=0}^{n} a_k\,z^k \tag{2.115}$$

and $a_n \equiv 1$, and

$$C_p = \sum_{k=p+1}^{n} a_k\,x^{(k-1-p)}(0) \quad , \tag{2.116}$$

which encodes the n initial conditions on $x^{(l)}(0)$ for $l \in \{0,\ldots,n-1\}$. Explicitly,

$$\begin{aligned}
C_{n-1} &= x(0) \\
C_{n-2} &= x^{(1)}(0) + a_{n-1}\,x(0) \\
C_{n-3} &= x^{(2)}(0) + a_{n-1}\,x^{(1)}(0) + a_{n-2}\,x(0) \\
&\;\;\vdots \\
C_1 &= x^{(n-2)}(0) + a_{n-1}\,x^{(n-3)}(0) + \ldots + a_2\,x(0) \\
C_0 &= x^{(n-1)}(0) + a_{n-1}\,x^{(n-2)}(0) + \ldots + a_2\,x^{(1)}(0) + a_1\,x(0) \quad .
\end{aligned} \tag{2.117}$$

Thus, the solution for $x(t)$ is

$$x(t) = \int\limits_{c-i\infty}^{c+i\infty} \frac{dz}{2\pi i}\,\frac{\check{f}(z) + R(z)}{Q(z)}\,e^{zt} \quad , \tag{2.118}$$

where

$$R(z) = \sum_{p=0}^{n-1} C_p\,z^p \tag{2.119}$$

and where the contour lies to the right of all singularities of the integrand. By the Fundamental Theorem of Algebra, $Q(z)$ may be factorized uniquely as

$$Q(z) = \prod_{j=1}^{n} (z - z_j) \quad . \tag{2.120}$$

Examples

Consider the equation

$$\dot{u} + \gamma u = f(t) \quad . \tag{2.121}$$

Thus $n = 1$ and $a_0 = \gamma$. Let $f(t) = b\,\sin(\omega t)\,\Theta(t)$. Then

$$\check{f}(z) = b\int\limits_0^\infty dt\,\sin(\omega t)\,e^{-\gamma t} = \frac{b\,\omega}{z^2 + \omega^2} \quad . \tag{2.122}$$

From the previous analysis we have

$$Q(z) = z + \gamma \qquad , \qquad R(z) = x(0) \quad . \tag{2.123}$$

Thus,

$$\begin{aligned} x(t) &= \int\limits_{c-i\infty}^{c+i\infty} \frac{dz}{2\pi i}\,\frac{1}{z+\gamma}\bigg(\frac{b\,\omega}{z^2+\omega^2} + x(0)\bigg)e^{zt} \\ &= \bigg(x(0) + \frac{b\omega}{\gamma^2+\omega^2}\bigg)\,e^{-\gamma t} + \frac{b}{\omega^2+\gamma^2}\,\Big(\gamma\,\sin(\omega t) - \omega\,\cos(\omega t)\Big) \quad . \end{aligned} \tag{2.124}$$

The poles of the integrand are at $z = -\gamma$ and $z = \pm i\omega$, so we chose $c > 0$ and closed the integration contour in the left half plane. Note how the solution satisfies the initial conditions at $t = 0$.

As a second example, consider the equation

$$\ddot{x} + (\alpha+\beta)\dot{x} + \alpha\beta x = f(t) \quad , \tag{2.125}$$

whence $n = 2$, $a_1 = \alpha + \beta$ and $a_2 = \alpha\beta$. We next identify

$$Q(z) = (z+\alpha)(z+\beta) \tag{2.126}$$

and $C_0 = \dot{x}(0) + (\alpha+\beta)\,x(0)$, $C_1 = z\,x(0)$. Thus

$$R(z) = \dot{x}(0) + (\alpha+\beta+z)\,x(0) \quad . \tag{2.127}$$

We choose $f(t) = r\,e^{-\gamma t}\,\Theta(t)$, with α, β, and γ all real and positive. The Laplace transform of $f(t)$ is easily obtained as

$$\check{f}(z) = r\int\limits_0^\infty dt\,e^{-\gamma t}\,e^{-zt} = \frac{r}{z+\gamma} \quad . \tag{2.128}$$

We then have

$$\begin{aligned} x(t) &= \int\limits_{c-i\infty}^{c+i\infty} \frac{dz}{2\pi i}\,\bigg(\frac{r}{z+\gamma} + \dot{x}(0) + (\alpha+\beta+z)\,x(0)\bigg)\,\frac{e^{zt}}{(z+\alpha)(z+\beta)} \\ &= -\frac{r}{(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)}\Big((\beta-\gamma)\,e^{-\alpha t} + (\gamma-\alpha)\,e^{-\beta t} + (\alpha-\beta)\,e^{-\gamma t}\Big) \\ &\qquad\qquad + \bigg(\frac{\beta\,e^{-\alpha t} - \alpha\,e^{-\beta t}}{\beta-\alpha}\bigg)\,x(0) - \bigg(\frac{e^{-\beta t} - e^{-\alpha t}}{\beta-\alpha}\bigg)\,\dot{x}(0) \quad . \end{aligned} \tag{2.129}$$

Note again that the initial conditions for $x(0)$ and $\dot{x}(0)$ are satisfied in the $t \to 0$ limit.

Chapter 3

Systems of Particles

3.1 Work-energy theorem

Consider a system of many particles, with positions $\boldsymbol{r}_i$ and velocities $\dot{\boldsymbol{r}}_i$. The kinetic energy of this system is

$$T = \sum_i T_i = \sum_i \tfrac{1}{2} m_i \dot{\boldsymbol{r}}_i^2 \quad . \tag{3.1}$$

Now let's consider how the kinetic energy of the system changes in time. Assuming each m_i is time-independent, we have

$$\frac{dT_i}{dt} = m_i \, \dot{\boldsymbol{r}}_i \cdot \ddot{\boldsymbol{r}}_i \quad . \tag{3.2}$$

Here, we've used the relation

$$\frac{d}{dt}\left(\boldsymbol{A}^2\right) = 2\,\boldsymbol{A} \cdot \frac{d\boldsymbol{A}}{dt} \quad . \tag{3.3}$$

We now invoke Newton's 2nd Law, $m_i \ddot{\boldsymbol{r}}_i = \boldsymbol{F}_i$, to write eqn. 3.2 as $\dot{T}_i = \boldsymbol{F}_i \cdot \dot{\boldsymbol{r}}_i$. We integrate this equation from time t_{A} to t_{B} :

$$T_i^{\mathsf{B}} - T_i^{\mathsf{A}} = \int\limits_{t_{\mathsf{A}}}^{t_{\mathsf{B}}} dt\, \frac{dT_i}{dt} = \int\limits_{t_{\mathsf{A}}}^{t_{\mathsf{B}}} dt\, \boldsymbol{F}_i \cdot \dot{\boldsymbol{r}}_i \equiv \sum_i W_i^{\mathsf{A}\to\mathsf{B}} \quad , \tag{3.4}$$

where $W_i^{\mathsf{A}\to\mathsf{B}}$ is the total *work done* on particle i during its motion from state A to state B, Clearly the total kinetic energy is $T = \sum_i T_i$ and the total work done on all particles is $W^{\mathsf{A}\to\mathsf{B}} = \sum_i W_i^{\mathsf{A}\to\mathsf{B}}$. Eqn. 3.4 is known as the *work-energy theorem*. It says that *In the evolution of a mechanical system, the change in total kinetic energy is equal to the total work done:* $T^{\mathsf{B}} - T^{\mathsf{A}} = W^{\mathsf{A}\to\mathsf{B}}$.

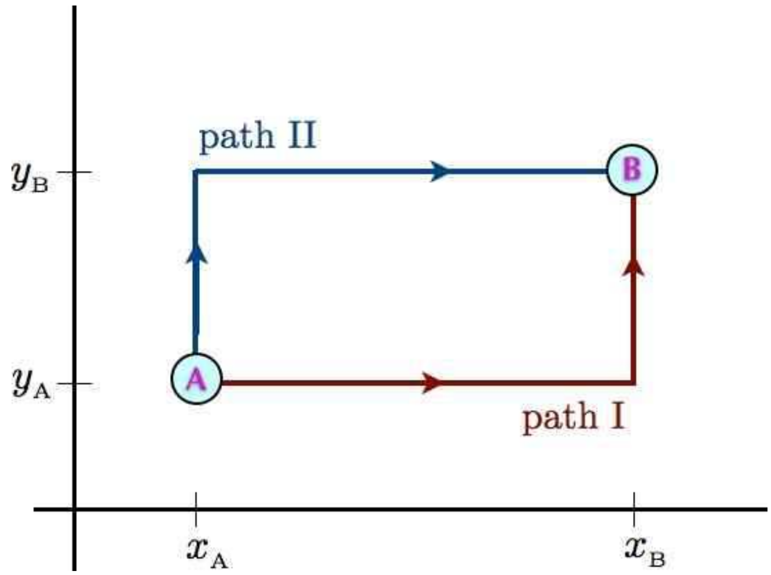

Figure 3.1: Two paths joining points A and B.

3.1.1 Conservative and nonconservative forces

For the sake of simplicity, consider a single particle with kinetic energy $T = \frac{1}{2}m\dot{\boldsymbol{r}}^2$. The work done on the particle during its mechanical evolution is

$$W^{\mathsf{A}\to\mathsf{B}} = \int\limits_{t_\mathsf{A}}^{t_\mathsf{B}} dt\, \boldsymbol{F}\cdot\boldsymbol{v} \quad , \tag{3.5}$$

where $\boldsymbol{v} = \dot{\boldsymbol{r}}$. This is the most general expression for the work done. If the force $\boldsymbol{F}$ depends only on the particle's position $\boldsymbol{r}$, we may write $d\boldsymbol{r} = \boldsymbol{v}\,dt$, and then

$$W^{\mathsf{A}\to\mathsf{B}} = \int\limits_{\boldsymbol{r}_\mathsf{A}}^{\boldsymbol{r}_\mathsf{B}} d\boldsymbol{r}\cdot\boldsymbol{F}(\boldsymbol{r}) \quad . \tag{3.6}$$

Consider now the force

$$\boldsymbol{F}(\boldsymbol{r}) = K_1\, y\, \hat{\boldsymbol{x}} + K_2\, x\, \hat{\boldsymbol{y}} \quad , \tag{3.7}$$

where $K_{1,2}$ are constants. Let's evaluate the work done along each of the two paths in fig. 3.1:

$$\begin{aligned}
W^{(\mathrm{I})} &= K_1 \int\limits_{x_\mathsf{A}}^{x_\mathsf{B}} dx\, y_\mathsf{A} + K_2 \int\limits_{y_\mathsf{A}}^{y_\mathsf{B}} dy\, x_\mathsf{B} = K_1\, y_\mathsf{A}\,(x_\mathsf{B} - x_\mathsf{A}) + K_2\, x_\mathsf{B}\,(y_\mathsf{B} - y_\mathsf{A}) \\
W^{(\mathrm{II})} &= K_1 \int\limits_{x_\mathsf{A}}^{x_\mathsf{B}} dx\, y_\mathsf{B} + K_2 \int\limits_{y_\mathsf{A}}^{y_\mathsf{B}} dy\, x_\mathsf{A} = K_1\, y_\mathsf{B}\,(x_\mathsf{B} - x_\mathsf{A}) + K_2\, x_\mathsf{A}\,(y_\mathsf{B} - y_\mathsf{A}) \quad .
\end{aligned} \tag{3.8}$$

Note that in general $W^{(\mathrm{I})} \neq W^{(\mathrm{II})}$. Thus, if we start at point A, the kinetic energy at point B will depend on the path taken, since the work done is path-dependent.

The difference between the work done along the two paths is

$$W^{(\mathrm{I}-)} - W^{(\mathrm{II})} = (K_2 - K_1)\,(x_{\mathsf{B}} - x_{\mathsf{A}})\,(y_{\mathsf{B}} - y_{\mathsf{A}}) \quad . \tag{3.9}$$

Thus, we see that if $K_1 = K_2$, the work is the same for the two paths. In fact, if $K_1 = K_2$, the work would be path-independent, and would depend only on the endpoints. This is true for *any* path, and not just piecewise linear paths of the type depicted in fig. 3.1. The reason for this is Stokes' theorem:

$$\oint\limits_{\partial\mathcal{C}} d\boldsymbol{\ell}\cdot\boldsymbol{F} = \int\limits_{\mathcal{C}} dS\,\hat{\boldsymbol{n}}\cdot\boldsymbol{\nabla}\times\boldsymbol{F} \quad . \tag{3.10}$$

Here, $\mathcal{C}$ is a connected region in three-dimensional space, $\partial\mathcal{C}$ is mathematical notation for the boundary of $\mathcal{C}$, which is a closed path[1], dS is the scalar differential area element, $\hat{\boldsymbol{n}}$ is the unit normal to that differential area element, and $\boldsymbol{\nabla}\times\boldsymbol{F}$ is the curl of $\boldsymbol{F}$:

$$\begin{aligned}\boldsymbol{\nabla}\times\boldsymbol{F} &= \det\begin{pmatrix}\hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}}\\ \partial_x & \partial_y & \partial_z\\ F_x & F_y & F_z\end{pmatrix}\\ &= \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{\boldsymbol{x}} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{\boldsymbol{y}} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{\boldsymbol{z}} \quad .\end{aligned} \tag{3.11}$$

For the force under consideration, $\boldsymbol{F}(\boldsymbol{r}) = K_1\,y\,\hat{\boldsymbol{x}} + K_2\,x\,\hat{\boldsymbol{y}}$, the curl is

$$\boldsymbol{\nabla}\times\boldsymbol{F} = (K_2 - K_1)\,\hat{\boldsymbol{z}} \quad , \tag{3.12}$$

which is a constant. The RHS of eqn. 3.10 is then simply proportional to the area enclosed by $\mathcal{C}$. When we compute the work difference in eqn. 3.9, we evaluate the integral $\oint_{\mathcal{C}} d\boldsymbol{\ell}\cdot\boldsymbol{F}$ along the path $\gamma_{\mathrm{II}}^{-1}\circ\gamma_{\mathrm{I}}$, which is to say path I followed by the inverse of path II. In this case, $\hat{\boldsymbol{n}} = \hat{\boldsymbol{z}}$ and the integral of $\hat{\boldsymbol{n}}\cdot\boldsymbol{\nabla}\times\boldsymbol{F}$ over the rectangle $\mathcal{C}$ is given by the RHS of eqn. 3.9.

When $\boldsymbol{\nabla}\times\boldsymbol{F} = 0$ everywhere in space, we can always write $\boldsymbol{F} = -\boldsymbol{\nabla}U$, where $U(\boldsymbol{r})$ is the *potential energy*. Such forces are called *conservative forces* because the *total energy* of the system, $E = T + U$, is then conserved during its motion. We can see this by evaluating the work done,

$$W^{\mathsf{A}\to\mathsf{B}} = \int\limits_{\boldsymbol{r}_{\mathsf{A}}}^{\boldsymbol{r}_{\mathsf{B}}} d\boldsymbol{r}\cdot\boldsymbol{F}(\boldsymbol{r}) = -\int\limits_{\boldsymbol{r}_{\mathsf{A}}}^{\boldsymbol{r}_{\mathsf{B}}} d\boldsymbol{r}\cdot\boldsymbol{\nabla}U = U(\boldsymbol{r}_{\mathsf{A}}) - U(\boldsymbol{r}_{\mathsf{B}}) \quad . \tag{3.13}$$

The work-energy theorem then gives

$$T^{\mathsf{B}} - T^{\mathsf{A}} = U(\boldsymbol{r}_{\mathsf{A}}) - U(\boldsymbol{r}_{\mathsf{B}}) \quad , \tag{3.14}$$

which says

$$E^{\mathsf{B}} = T^{\mathsf{B}} + U(\boldsymbol{r}_{\mathsf{B}}) = T^{\mathsf{A}} + U(\boldsymbol{r}_{\mathsf{A}}) = E^{\mathsf{A}} \quad . \tag{3.15}$$

Thus, the total energy $E = T + U$ is conserved.

[1] If $\mathcal{C}$ is multiply connected, then $\partial\mathcal{C}$ is a set of closed paths. For example, if $\mathcal{C}$ is an annulus, $\partial\mathcal{C}$ is two circles, corresponding to the inner and outer boundaries of the annulus.

3.1.2 Integrating $\boldsymbol{F} = -\boldsymbol{\nabla} U$

If $\boldsymbol{\nabla} \times \boldsymbol{F} = 0$, we can compute $U(\boldsymbol{r})$ by integrating, *viz.*

$$U(\boldsymbol{r}) = U(\boldsymbol{0}) - \int\limits_{\boldsymbol{0}}^{\boldsymbol{r}} d\boldsymbol{r}' \cdot \boldsymbol{F}(\boldsymbol{r}') \quad . \tag{3.16}$$

The integral does not depend on the path chosen connecting $\boldsymbol{0}$ and $\boldsymbol{r}$. For example, we can take

$$U(x,y,z) = U(0,0,0) \; - \int\limits_{(0,0,0)}^{(x,0,0)} dx' \, F_x(x',0,0) \; - \int\limits_{(x,0,0)}^{(x,y,0)} dy' \, F_y(x,y',0) \; - \int\limits_{(z,y,0)}^{(x,y,z)} dz' \, F_z(x,y,z') \quad . \tag{3.17}$$

The constant $U(0,0,0)$ is arbitrary and impossible to determine from $\boldsymbol{F}$ alone.

As an example, consider the force

$$\boldsymbol{F}(\boldsymbol{r}) = -ky \, \hat{\boldsymbol{x}} - kx \, \hat{\boldsymbol{y}} - 4bz^3 \, \hat{\boldsymbol{z}} \quad , \tag{3.18}$$

where k and b are constants. We have

$$\begin{aligned} \big(\boldsymbol{\nabla} \times \boldsymbol{F}\big)_x &= \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) = 0 \\ \big(\boldsymbol{\nabla} \times \boldsymbol{F}\big)_y &= \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) = 0 \\ \big(\boldsymbol{\nabla} \times \boldsymbol{F}\big)_z &= \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) = 0 \quad , \end{aligned} \tag{3.19}$$

so $\boldsymbol{\nabla} \times \boldsymbol{F} = 0$ and $\boldsymbol{F}$ must be expressible as $\boldsymbol{F} = -\boldsymbol{\nabla} U$. Integrating using eqn. 3.17, we have

$$\begin{aligned} U(x,y,z) &= U(0,0,0) \; + \int\limits_{(0,0,0)}^{(x,0,0)} dx' \, k \cdot 0 \; + \int\limits_{(x,0,0)}^{(x,y,0)} dy' \, kxy' \; + \int\limits_{(z,y,0)}^{(x,y,z)} dz' \, 4bz'^3 \\ &= U(0,0,0) + kxy + bz^4 \quad . \end{aligned} \tag{3.20}$$

Another approach is to integrate the partial differential equation $\boldsymbol{\nabla} U = -\boldsymbol{F}$. This is in fact three equations, and we shall need all of them to obtain the correct answer. We start with the $\hat{\boldsymbol{x}}$-component,

$$\frac{\partial U}{\partial x} = ky \quad . \tag{3.21}$$

Integrating, we obtain

$$U(x,y,z) = kxy + f(y,z) \quad , \tag{3.22}$$

where $f(y,z)$ is at this point an *arbitrary function* of y and z. The important thing is that it has no x-dependence, so $\partial f/\partial x = 0$. Next, we have

$$\frac{\partial U}{\partial y} = kx \quad \Longrightarrow \quad U(x,y,z) = kxy + g(x,z) \quad . \tag{3.23}$$

Finally, the z-component integrates to yield

$$\frac{\partial U}{\partial z} = 4bz^3 \quad \Longrightarrow \quad U(x,y,z) = bz^4 + h(x,y) \quad . \tag{3.24}$$

We now equate the first two expressions:

$$kxy + f(y,z) = kxy + g(x,z) \quad . \tag{3.25}$$

Subtracting kxy from each side, we obtain the equation $f(y,z) = g(x,z)$. Since the LHS is independent of x and the RHS is independent of y, we must have

$$f(y,z) = g(x,z) = q(z) \quad , \tag{3.26}$$

where $q(z)$ is some unknown function of z. But now we invoke the final equation, to obtain

$$bz^4 + h(x,y) = kxy + q(z) \quad . \tag{3.27}$$

The only possible solution is $h(x,y) = C + kxy$ and $q(z) = C + bz^4$, where C is a constant. Therefore,

$$U(x,y,z) = C + kxy + bz^4 \quad . \tag{3.28}$$

Note that it would be *very wrong* to integrate $\partial U/\partial x = ky$ and obtain $U(x,y,z) = kxy + C'$, where C' is a constant. As we've seen, the 'constant of integration' we obtain upon integrating this first order PDE is in fact a *function* of y and z. The fact that $f(y,z)$ carries no explicit x dependence means that $\partial f/\partial x = 0$, so by construction $U = kxy + f(y,z)$ is a solution to the PDE $\partial U/\partial x = ky$, for any arbitrary function $f(y,z)$.

3.2 Conservative forces in many-particle systems

3.2.1 Kinetic and potential energies

The kinetic and potential energies are given by

$$\begin{aligned} T &= \sum_i \tfrac{1}{2} m_i \dot{\boldsymbol{r}}_i^2 \\ U &= \sum_i V(\boldsymbol{r}_i) + \sum_{i<j} v\big(|\boldsymbol{r}_i - \boldsymbol{r}_j|\big) \quad . \end{aligned} \tag{3.29}$$

Here, $V(\boldsymbol{r})$ is the *external* (or one-body) potential, and $v(\boldsymbol{r} - \boldsymbol{r}')$ is the *interparticle* potential, which we assume to be central, depending only on the distance between any pair of particles. The equations of motion are

$$m_i \ddot{\boldsymbol{r}}_i = \boldsymbol{F}_i^{(\mathsf{ext})} + \boldsymbol{F}_i^{(\mathsf{int})} \quad , \tag{3.30}$$

with

$$\begin{aligned} \boldsymbol{F}_i^{(\text{ext})} &= -\frac{\partial V(\boldsymbol{r}_i)}{\partial \boldsymbol{r}_i} \\ \boldsymbol{F}_i^{(\text{int})} &= -\sum_j \frac{\partial v\big(|\boldsymbol{r}_i - \boldsymbol{r}_j|\big)}{\boldsymbol{r}_i} \equiv \sum_j \boldsymbol{F}_{ij}^{(\text{int})} \quad . \end{aligned} \tag{3.31}$$

Here, $\boldsymbol{F}_{ij}^{(\text{int})}$ is the force exerted on particle i by particle j:

$$\boldsymbol{F}_{ij}^{(\text{int})} = -\frac{\partial v\big(|\boldsymbol{r}_i - \boldsymbol{r}_j|\big)}{\partial \boldsymbol{r}_i} = -\frac{\boldsymbol{r}_i - \boldsymbol{r}_j}{|\boldsymbol{r}_i - \boldsymbol{r}_j|}\, v'\big(|\boldsymbol{r}_i - \boldsymbol{r}_j|\big) \quad . \tag{3.32}$$

Note that $\boldsymbol{F}_{ij}^{(\text{int})} = -\boldsymbol{F}_{ji}^{(\text{int})}$, otherwise known as Newton's Third Law. It is convenient to abbreviate $\boldsymbol{r}_{ij} \equiv \boldsymbol{r}_i - \boldsymbol{r}_j$, in which case we may write the interparticle force as

$$\boldsymbol{F}_{ij}^{(\text{int})} = -\hat{\boldsymbol{r}}_{ij}\, v'\big(r_{ij}\big) \quad . \tag{3.33}$$

3.2.2 Linear and angular momentum

Consider now the total momentum of the system, $\boldsymbol{P} = \sum_i \boldsymbol{p}_i$. Its rate of change is

$$\frac{d\boldsymbol{P}}{dt} = \sum_i \dot{\boldsymbol{p}}_i = \sum_i \boldsymbol{F}_i^{(\text{ext})} + \overbrace{\sum_{i\neq j} \boldsymbol{F}_{ij}^{(\text{int})}}^{\boldsymbol{F}_{ij}^{(\text{int})} + \boldsymbol{F}_{ji}^{(\text{int})} = 0} = \boldsymbol{F}_{\text{tot}}^{(\text{ext})} \quad , \tag{3.34}$$

since the sum over all internal forces cancels as a result of Newton's Third Law. We write

$$\begin{aligned} \boldsymbol{P} &= \sum_i m_i \dot{\boldsymbol{r}}_i = M\dot{\boldsymbol{R}} \\ M &= \sum_i m_i \quad \text{(total mass)} \\ \boldsymbol{R} &= \frac{\sum_i m_i\, \boldsymbol{r}_i}{\sum_i m_i} \quad \text{(center-of-mass)} \quad . \end{aligned} \tag{3.35}$$

Next, consider the total angular momentum,

$$\boldsymbol{L} = \sum_i \boldsymbol{r}_i \times \boldsymbol{p}_i = \sum_i m_i \boldsymbol{r}_i \times \dot{\boldsymbol{r}}_i \quad . \tag{3.36}$$

The rate of change of $\boldsymbol{L}$ is then

$$\begin{aligned} \frac{d\boldsymbol{L}}{dt} &= \sum_i \big\{ m_i \dot{\boldsymbol{r}}_i \times \dot{\boldsymbol{r}}_i + m_i \boldsymbol{r}_i \times \ddot{\boldsymbol{r}}_i \big\} \\ &= \sum_i \boldsymbol{r}_i \times \boldsymbol{F}_i^{(\text{ext})} + \sum_{i\neq j} \boldsymbol{r}_i \times \boldsymbol{F}_{ij}^{(\text{int})} \\ &= \sum_i \boldsymbol{r}_i \times \boldsymbol{F}_i^{(\text{ext})} + \overbrace{\tfrac{1}{2} \sum_{i\neq j} (\boldsymbol{r}_i - \boldsymbol{r}_j) \times \boldsymbol{F}_{ij}^{(\text{int})}}^{\boldsymbol{r}_{ij} \times \boldsymbol{F}_{ij}^{(\text{int})} = 0} = \boldsymbol{N}_{\text{tot}}^{(\text{ext})} \quad . \end{aligned} \tag{3.37}$$

Finally, it is useful to establish the result

$$T = \tfrac{1}{2}\sum_i m_i\,\dot{\boldsymbol{r}}_i^2 = \tfrac{1}{2}M\dot{\boldsymbol{R}}^2 + \tfrac{1}{2}\sum_i m_i\big(\dot{\boldsymbol{r}}_i - \dot{\boldsymbol{R}}\big)^2 \quad , \tag{3.38}$$

which says that the kinetic energy may be written as a sum of two terms, those being the kinetic energy of the center-of-mass motion, and the kinetic energy of the particles relative to the center-of-mass.

Recall the "work-energy theorem" for conservative systems,

$$\begin{aligned} 0 &= \int\limits_{\text{initial}}^{\text{final}} dE = \int\limits_{\text{initial}}^{\text{final}} dT + \int\limits_{\text{initial}}^{\text{final}} dU \\ &= T^{\mathsf{B}} - T^{\mathsf{A}} - \sum_i \int d\boldsymbol{r}_i \cdot \boldsymbol{F}_i \quad , \end{aligned} \tag{3.39}$$

which is to say

$$\Delta T = T^{\mathsf{B}} - T^{\mathsf{A}} = \sum_i \int d\boldsymbol{r}_i \cdot \boldsymbol{F}_i = -\Delta U \quad . \tag{3.40}$$

In other words, the total energy $E = T + U$ is conserved:

$$E = \sum_i \tfrac{1}{2} m_i \dot{\boldsymbol{r}}_i^2 + \sum_i V(\boldsymbol{r}_i) + \sum_{i<j} v\big(|\boldsymbol{r}_i - \boldsymbol{r}_j|\big) \quad . \tag{3.41}$$

Note that for continuous systems, we replace sums by integrals over a mass distribution, *viz.*

$$\sum_i m_i\,\phi(\boldsymbol{r}_i) \longrightarrow \int d^3\!r\,\rho(\boldsymbol{r})\,\phi(\boldsymbol{r}) \quad , \tag{3.42}$$

where $\rho(\boldsymbol{r})$ is the mass density, and $\phi(\boldsymbol{r})$ is any function.

3.3 Scaling of Solutions for Homogeneous Potentials

3.3.1 Euler's theorem for homogeneous functions

In certain cases of interest, the potential is a homogeneous function of the coordinates. This means

$$U\big(\lambda\,\boldsymbol{r}_1,\ldots,\lambda\,\boldsymbol{r}_N\big) = \lambda^k\,U\big(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_N\big) \quad . \tag{3.43}$$

Here, k is the *degree of homogeneity* of U. Familiar examples include gravity,

$$U\big(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_N\big) = -G\sum_{i<j}\frac{m_i\,m_j}{|\boldsymbol{r}_i - \boldsymbol{r}_j|} \quad ; \quad k = -1 \quad , \tag{3.44}$$

and the harmonic oscillator,

$$U\big(q_1,\ldots,q_n\big) = \tfrac{1}{2}\sum_{\sigma,\sigma'} V_{\sigma\sigma'}\, q_\sigma\, q_{\sigma'} \quad ; \quad k=+2 \quad . \tag{3.45}$$

The sum of two homogeneous functions is itself homogeneous only if the component functions themselves are of the same degree of homogeneity. Homogeneous functions obey a special result known as *Euler's Theorem*, which we now prove. Suppose a multivariable function $H(x_1,\ldots,x_n)$ is homogeneous:

$$H(\lambda\, x_1,\ldots,\lambda\, x_n) = \lambda^k\, H(x_1,\ldots,x_n) \quad . \tag{3.46}$$

Then

$$\frac{d}{d\lambda}\bigg|_{\lambda=1} H\big(\lambda\, x_1,\ldots,\lambda\, x_n\big) = \sum_{i=1}^n x_i\, \frac{\partial H}{\partial x_i} = kH \quad . \tag{3.47}$$

3.3.2 Scaled equations of motion

Now suppose the we rescale distances and times, defining

$$\boldsymbol{r}_i = \alpha\, \tilde{\boldsymbol{r}}_i \qquad , \qquad t = \beta\, \tilde{t} \quad . \tag{3.48}$$

Then

$$\frac{d\boldsymbol{r}_i}{dt} = \frac{\alpha}{\beta}\,\frac{d\tilde{\boldsymbol{r}}_i}{d\tilde{t}} \qquad , \qquad \frac{d^2\boldsymbol{r}_i}{dt^2} = \frac{\alpha}{\beta^2}\,\frac{d^2\tilde{\boldsymbol{r}}_i}{d\tilde{t}^2} \quad . \tag{3.49}$$

The force $\boldsymbol{F}_i$ is given by

$$\begin{aligned} \boldsymbol{F}_i &= -\frac{\partial}{\partial \boldsymbol{r}_i}\, U\big(\boldsymbol{r}_1,\ldots,\boldsymbol{r}_N\big) \\ &= -\frac{\partial}{\partial(\alpha\tilde{\boldsymbol{r}}_i)}\, \alpha^k\, U\big(\tilde{\boldsymbol{r}}_1,\ldots,\tilde{\boldsymbol{r}}_N\big) \equiv \alpha^{k-1}\, \widetilde{\boldsymbol{F}}_i \quad , \end{aligned} \tag{3.50}$$

where $\widetilde{\boldsymbol{F}}_i = \partial U\big(\tilde{\boldsymbol{r}}_1,\ldots,\tilde{\boldsymbol{r}}_N\big)/\partial\tilde{\boldsymbol{r}}_i$. Thus, Newton's 2nd Law says

$$\frac{\alpha}{\beta^2}\, m_i\, \frac{d^2\tilde{\boldsymbol{r}}_i}{d\tilde{t}^2} = \alpha^{k-1}\, \widetilde{\boldsymbol{F}}_i \quad . \tag{3.51}$$

If we choose β such that

$$\frac{\alpha}{\beta^2} = \alpha^{k-1} \quad \Rightarrow \quad \beta = \alpha^{1-\frac{1}{2}k} \quad , \tag{3.52}$$

then the equation of motion is invariant under the rescaling transformation, *i.e.*

$$m_i\, \frac{d^2\tilde{\boldsymbol{r}}_i}{d\tilde{t}^2} = \widetilde{\boldsymbol{F}}_i \quad . \tag{3.53}$$

This means that if $\big\{\boldsymbol{r}_i(t)\big\}$ is a solution to the equations of motion, then so is $\big\{\alpha\, \boldsymbol{r}_i(\beta t)\big\}$. This gives us an entire one-parameter family of solutions, for all real positive α. with $\beta = \alpha^{1-\frac{1}{2}k}$.

We see that if $\boldsymbol{r}_i(t)$ is periodic with period T, then $\boldsymbol{r}_i(t;\alpha)$ is periodic with period $T' = \alpha^{1-\frac{1}{2}k}\,T$. Furthermore, if L is a length scale associated with an orbit $\boldsymbol{r}_i(t)$, such as the distance of closest approach, then we have the following relation between the ratios of the time and length scales:

$$\left(\frac{T'}{T}\right) = \left(\frac{L'}{L}\right)^{1-\frac{1}{2}k} \quad . \tag{3.54}$$

Velocities, energies and angular momenta scale accordingly Thus

$$[v] = \frac{L}{T} \quad \Rightarrow \quad \frac{v'}{v} = \frac{L'}{L}\bigg/\frac{T'}{T} = \alpha^{\frac{1}{2}k} \tag{3.55}$$

and

$$[E] = \frac{ML^2}{T^2} \quad \Rightarrow \quad \frac{E'}{E} = \left(\frac{L'}{L}\right)^2\bigg/\left(\frac{T'}{T}\right)^2 = \alpha^k \tag{3.56}$$

and

$$[\boldsymbol{L}] = \frac{ML^2}{T} \quad \Rightarrow \quad \frac{|\boldsymbol{L}'|}{|\boldsymbol{L}|} = \left(\frac{L'}{L}\right)^2\bigg/\frac{T'}{T} = \alpha^{(1+\frac{1}{2}k)} \quad . \tag{3.57}$$

As examples, consider:

(i) *Harmonic Oscillator* : Here $k = 2$ and therefore

$$q_\sigma(t) \longrightarrow q_\sigma(t;\alpha) = \alpha\, q_\sigma(t) \quad . \tag{3.58}$$

Thus, rescaling lengths alone gives another solution.

(ii) *Kepler Problem* : This is gravity, for which $k = -1$. Thus,

$$\boldsymbol{r}(t) \longrightarrow \boldsymbol{r}(t;\alpha) = \alpha\, \boldsymbol{r}\big(\alpha^{-3/2}\, t\big) \quad . \tag{3.59}$$

Thus, $r^3 \propto t^2$, *i.e.*

$$\left(\frac{L'}{L}\right)^3 = \left(\frac{T'}{T}\right)^2 \quad , \tag{3.60}$$

also known as Kepler's Third Law.

3.4 Appendix : Curvilinear Orthogonal Coordinates

The standard cartesian coordinates are $\{x_1, \ldots, x_d\}$, where d is the dimension of space. Consider a different set of coordinates, $\{q_1, \ldots, q_d\}$, which are related to the original coordinates x_μ via the d equations

$$q_\mu = q_\mu\big(x_1, \ldots, x_d\big) \quad . \tag{3.61}$$

In general these are nonlinear equations.

Let $\hat{\boldsymbol{e}}^0_i = \hat{\boldsymbol{x}}_i$ be the Cartesian set of orthonormal unit vectors, and define $\hat{\boldsymbol{e}}_\mu$ to be the unit vector perpendicular to the surface $dq_\mu = 0$. A differential change in position can now be described in both coordinate systems:

$$d\boldsymbol{s} = \sum_{i=1}^{d} \hat{\boldsymbol{e}}^0_i \, dx_i = \sum_{\mu=1}^{d} \hat{\boldsymbol{e}}_\mu \, h_\mu(q) \, dq_\mu \quad , \tag{3.62}$$

where each $h_\mu(q)$ is an as yet unknown function of all the components q_ν. Finding the coefficient of dq_μ then gives

$$h_\mu(q) \, \hat{\boldsymbol{e}}_\mu = \sum_{i=1}^{d} \frac{\partial x_i}{\partial q_\mu} \, \hat{\boldsymbol{e}}^0_i \qquad \Rightarrow \qquad \hat{\boldsymbol{e}}_\mu = \sum_{i=1}^{d} M_{\mu\, i} \, \hat{\boldsymbol{e}}^0_i \quad , \tag{3.63}$$

where

$$M_{\mu i}(q) = \frac{1}{h_\mu(q)} \, \frac{\partial x_i}{\partial q_\mu} \quad . \tag{3.64}$$

The dot product of unit vectors in the new coordinate system is then

$$\hat{\boldsymbol{e}}_\mu \cdot \hat{\boldsymbol{e}}_\nu = \big(M M^{\rm t}\big)_{\mu\nu} = \frac{1}{h_\mu(q) \, h_\nu(q)} \sum_{i=1}^{d} \frac{\partial x_i}{\partial q_\mu} \, \frac{\partial x_i}{\partial q_\nu} \quad . \tag{3.65}$$

The condition that the new basis be orthonormal is then

$$\sum_{i=1}^{d} \frac{\partial x_i}{\partial q_\mu} \, \frac{\partial x_i}{\partial q_\nu} = h^2_\mu(q) \, \delta_{\mu\nu} \quad . \tag{3.66}$$

This gives us the relation

$$h_\mu(q) = \sqrt{\sum_{i=1}^{d} \bigg(\frac{\partial x_i}{\partial q_\mu}\bigg)^2} \quad . \tag{3.67}$$

Note that

$$(d\boldsymbol{s})^2 = \sum_{\mu=1}^{d} h^2_\mu(q) \, (dq_\mu)^2 \quad . \tag{3.68}$$

For general coordinate systems, which are not necessarily orthogonal, we have

$$(d\boldsymbol{s})^2 = \sum_{\mu,\nu=1}^{d} g_{\mu\nu}(q) \, dq_\mu \, dq_\nu \quad , \tag{3.69}$$

where $g_{\mu\nu}(q)$ is a real, symmetric, positive definite matrix called the *metric tensor*.

3.4.1 Example : spherical coordinates

Consider spherical coordinates (ρ, θ, ϕ):

$$x = \rho \, \sin\theta \, \cos\phi \quad , \quad y = \rho \, \sin\theta \, \sin\phi \quad , \quad z = \rho \, \cos\theta \quad . \tag{3.70}$$

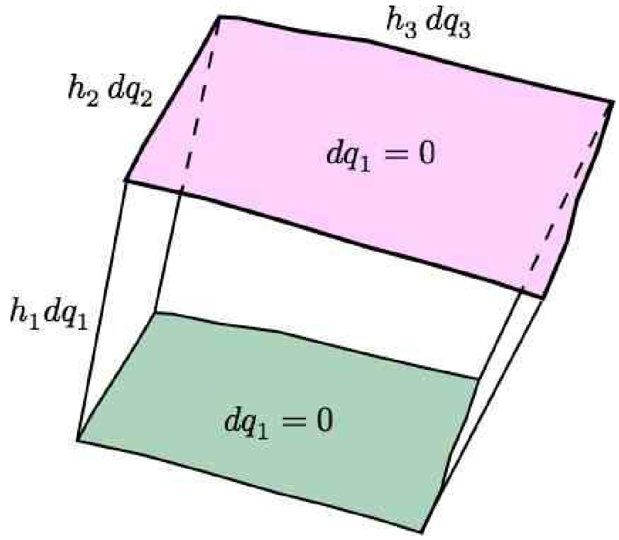

Figure 3.2: Volume element Ω for computing divergences.

It is now a simple matter to derive the results

$$h_\rho^2 = 1 \quad , \quad h_\theta^2 = \rho^2 \quad , \quad h_\phi^2 = \rho^2\, \sin^2\!\theta \quad . \tag{3.71}$$

Thus,

$$d\boldsymbol{s} = \hat{\boldsymbol{\rho}}\, d\rho + \rho\, \hat{\boldsymbol{\theta}}\, d\theta + \rho\, \sin\theta\, \hat{\boldsymbol{\phi}}\, d\phi \quad . \tag{3.72}$$

3.4.2 Vector calculus : grad, div, curl

Here we restrict our attention to $d = 3$. The gradient $\boldsymbol{\nabla} U$ of a function $U(q)$ is defined by

$$\begin{aligned} dU &= \frac{\partial U}{\partial q_1}\, dq_1 + \frac{\partial U}{\partial q_2}\, dq_2 + \frac{\partial U}{\partial q_3}\, dq_3 \\ &\equiv \boldsymbol{\nabla} U \cdot d\boldsymbol{s} \quad . \end{aligned} \tag{3.73}$$

Thus,

$$\boldsymbol{\nabla} = \frac{\hat{\boldsymbol{e}}_1}{h_1(q)}\, \frac{\partial}{\partial q_1} + \frac{\hat{\boldsymbol{e}}_2}{h_2(q)}\, \frac{\partial}{\partial q_2} + \frac{\hat{\boldsymbol{e}}_3}{h_3(q)}\, \frac{\partial}{\partial q_3} \quad . \tag{3.74}$$

For the divergence, we use the divergence theorem, and we appeal to fig. 15.12:

$$\int\limits_{\Omega} dV\, \boldsymbol{\nabla} \cdot \boldsymbol{A} = \int\limits_{\partial\Omega} dS\, \hat{\boldsymbol{n}} \cdot \boldsymbol{A} \quad , \tag{3.75}$$

where Ω is a region of three-dimensional space and $\partial\Omega$ is its closed two-dimensional boundary. The LHS of this equation is

$$\text{LHS} = \boldsymbol{\nabla} \cdot \boldsymbol{A} \cdot (h_1\, dq_1)\, (h_2\, dq_2)\, (h_3\, dq_3) \quad . \tag{3.76}$$

The RHS is

$$\begin{aligned}\text{RHS} &= A_1\, h_2\, h_3 \Big|_{q_1}^{q_1+dq_1} dq_2\, dq_3 + A_2\, h_1\, h_3 \Big|_{q_2}^{q_2+dq_2} dq_1\, dq_3 + A_3\, h_1\, h_2 \Big|_{q_3}^{q_1+dq_3} dq_1\, dq_2 \\ &= \left[\frac{\partial}{\partial q_1}\big(A_1\, h_2\, h_3\big) + \frac{\partial}{\partial q_2}\big(A_2\, h_1\, h_3\big) + \frac{\partial}{\partial q_3}\big(A_3\, h_1\, h_2\big)\right] dq_1\, dq_2\, dq_3 \quad .\end{aligned} \tag{3.77}$$

We therefore conclude

$$\boldsymbol{\nabla}\cdot\boldsymbol{A} = \frac{1}{h_1\, h_2\, h_3}\left[\frac{\partial}{\partial q_1}\big(A_1\, h_2\, h_3\big) + \frac{\partial}{\partial q_2}\big(A_2\, h_1\, h_3\big) + \frac{\partial}{\partial q_3}\big(A_3\, h_1\, h_2\big)\right] \quad . \tag{3.78}$$

To obtain the curl $\boldsymbol{\nabla}\times\boldsymbol{A}$, we use Stokes' theorem again,

$$\int\limits_{\Sigma} dS\,\hat{\boldsymbol{n}}\cdot\boldsymbol{\nabla}\times\boldsymbol{A} = \oint\limits_{\partial\Sigma} d\boldsymbol{\ell}\cdot\boldsymbol{A} \quad , \tag{3.79}$$

where Σ is a two-dimensional region of space and $\partial\Sigma$ is its one-dimensional boundary. Now consider a differential surface element satisfying $dq_1 = 0$, *i.e.* a rectangle of side lengths $h_2\, dq_2$ and $h_3\, dq_3$. The LHS of the above equation is

$$\text{LHS} = \hat{\boldsymbol{e}}_1\cdot\boldsymbol{\nabla}\times\boldsymbol{A}\,(h_2\, dq_2)\,(h_3\, dq_3) \quad . \tag{3.80}$$

The RHS is

$$\begin{aligned}\text{RHS} &= A_3\, h_3 \Big|_{q_2}^{q_2+dq_2} dq_3 - A_2\, h_2 \Big|_{q_3}^{q_3+dq_3} dq_2 \\ &= \left[\frac{\partial}{\partial q_2}\big(A_3\, h_3\big) - \frac{\partial}{\partial q_3}\big(A_2\, h_2\big)\right] dq_2\, dq_3 \quad .\end{aligned} \tag{3.81}$$

Therefore

$$(\boldsymbol{\nabla}\times\boldsymbol{A})_1 = \frac{1}{h_2\, h_3}\left(\frac{\partial(h_3\, A_3)}{\partial q_2} - \frac{\partial(h_2\, A_2)}{\partial q_3}\right) \quad . \tag{3.82}$$

This is one component of the full result

$$\boldsymbol{\nabla}\times\boldsymbol{A} = \frac{1}{h_1\, h_2\, h_3}\,\text{det}\begin{pmatrix} h_1\,\hat{\boldsymbol{e}}_1 & h_2\,\hat{\boldsymbol{e}}_2 & h_3\,\hat{\boldsymbol{e}}_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1\, A_1 & h_2\, A_2 & h_3\, A_3 \end{pmatrix} \quad . \tag{3.83}$$

The Laplacian of a scalar function U is given by

$$\begin{aligned}\nabla^2 U &= \boldsymbol{\nabla}\cdot\boldsymbol{\nabla} U \\ &= \frac{1}{h_1\, h_2\, h_3}\left\{\frac{\partial}{\partial q_1}\left(\frac{h_2\, h_3}{h_1}\,\frac{\partial U}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_1\, h_3}{h_2}\,\frac{\partial U}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1\, h_2}{h_3}\,\frac{\partial U}{\partial q_3}\right)\right\} \quad .\end{aligned} \tag{3.84}$$

Rectangular coordinates

In *rectangular* coordinates (x, y, z), we have

$$h_x = h_y = h_z = 1 \quad . \tag{3.85}$$

Thus

$$d\boldsymbol{s} = \hat{\boldsymbol{x}}\,dx + \hat{\boldsymbol{y}}\,dy + \hat{\boldsymbol{z}}\,dz \tag{3.86}$$

and the velocity squared is

$$\dot{\boldsymbol{s}}^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 \quad . \tag{3.87}$$

The gradient is

$$\boldsymbol{\nabla} U = \hat{\boldsymbol{x}}\,\frac{\partial U}{\partial x} + \hat{\boldsymbol{y}}\,\frac{\partial U}{\partial y} + \hat{\boldsymbol{z}}\,\frac{\partial U}{\partial z} \quad . \tag{3.88}$$

The divergence is

$$\boldsymbol{\nabla} \cdot \boldsymbol{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \quad . \tag{3.89}$$

The curl is

$$\boldsymbol{\nabla} \times \boldsymbol{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\boldsymbol{x}} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{\boldsymbol{y}} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{\boldsymbol{z}} \quad . \tag{3.90}$$

The Laplacian is

$$\nabla^2 U = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} \quad . \tag{3.91}$$

Cylindrical coordinates

In *cylindrical* coordinates (ρ, ϕ, z), we have

$$\hat{\boldsymbol{\rho}} = \hat{\boldsymbol{x}}\,\cos\phi + \hat{\boldsymbol{y}}\,\sin\phi \quad , \quad \hat{\boldsymbol{x}} = \hat{\boldsymbol{\rho}}\,\cos\phi - \hat{\boldsymbol{\phi}}\,\sin\phi \quad , \quad d\hat{\boldsymbol{\rho}} = \hat{\boldsymbol{\phi}}\,d\phi \tag{3.92}$$

and

$$\hat{\boldsymbol{\phi}} = -\hat{\boldsymbol{x}}\,\sin\phi + \hat{\boldsymbol{y}}\,\cos\phi \quad , \quad \hat{\boldsymbol{y}} = \hat{\boldsymbol{\rho}}\,\sin\phi + \hat{\boldsymbol{\phi}}\,\cos\phi \quad , \quad d\hat{\boldsymbol{\phi}} = -\hat{\boldsymbol{\rho}}\,d\phi \quad . \tag{3.93}$$

The metric is given in terms of

$$h_\rho = 1 \quad , \quad h_\phi = \rho \quad , \quad h_z = 1 \quad . \tag{3.94}$$

Thus

$$d\boldsymbol{s} = \hat{\boldsymbol{\rho}}\,d\rho + \hat{\boldsymbol{\phi}}\,\rho\,d\phi + \hat{\boldsymbol{z}}\,dz \tag{3.95}$$

and the velocity squared is

$$\dot{\boldsymbol{s}}^2 = \dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2 \quad . \tag{3.96}$$

The gradient is

$$\boldsymbol{\nabla} U = \hat{\boldsymbol{\rho}}\,\frac{\partial U}{\partial \rho} + \frac{\hat{\boldsymbol{\phi}}}{\rho}\,\frac{\partial U}{\partial \phi} + \hat{\boldsymbol{z}}\,\frac{\partial U}{\partial z} \quad . \tag{3.97}$$

The divergence is

$$\boldsymbol{\nabla}\cdot\boldsymbol{A} = \frac{1}{\rho}\,\frac{\partial(\rho\,A_\rho)}{\partial\rho} + \frac{1}{\rho}\,\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z} \quad . \tag{3.98}$$

The curl is

$$\boldsymbol{\nabla}\times\boldsymbol{A} = \left(\frac{1}{\rho}\,\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right)\hat{\boldsymbol{\rho}} + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right)\hat{\boldsymbol{\phi}} + \left(\frac{1}{\rho}\,\frac{\partial(\rho A_\phi)}{\partial\rho} - \frac{1}{\rho}\,\frac{\partial A_\rho}{\partial\phi}\right)\hat{\boldsymbol{z}} \quad . \tag{3.99}$$

The Laplacian is

$$\nabla^2 U = \frac{1}{\rho}\,\frac{\partial}{\partial\rho}\left(\rho\,\frac{\partial U}{\partial\rho}\right) + \frac{1}{\rho^2}\,\frac{\partial^2 U}{\partial\phi^2} + \frac{\partial^2 U}{\partial z^2} \quad . \tag{3.100}$$

Spherical coordinates

In *spherical* coordinates (r,θ,ϕ), we have

$$\begin{aligned}
\hat{\boldsymbol{r}} &= \hat{\boldsymbol{x}}\sin\theta\cos\phi + \hat{\boldsymbol{y}}\sin\theta\sin\phi + \hat{\boldsymbol{z}}\sin\theta \\
\hat{\boldsymbol{\theta}} &= \hat{\boldsymbol{x}}\cos\theta\cos\phi + \hat{\boldsymbol{y}}\cos\theta\sin\phi - \hat{\boldsymbol{z}}\cos\theta \\
\hat{\boldsymbol{\phi}} &= -\hat{\boldsymbol{x}}\sin\phi + \hat{\boldsymbol{y}}\cos\phi \quad ,
\end{aligned} \tag{3.101}$$

for which

$$\hat{\boldsymbol{r}}\times\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}} \quad , \quad \hat{\boldsymbol{\theta}}\times\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{r}} \quad , \quad \hat{\boldsymbol{\phi}}\times\hat{\boldsymbol{r}} = \hat{\boldsymbol{\theta}} \quad . \tag{3.102}$$

The inverse is

$$\begin{aligned}
\hat{\boldsymbol{x}} &= \hat{\boldsymbol{r}}\sin\theta\cos\phi + \hat{\boldsymbol{\theta}}\cos\theta\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi \\
\hat{\boldsymbol{y}} &= \hat{\boldsymbol{r}}\sin\theta\sin\phi + \hat{\boldsymbol{\theta}}\cos\theta\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi \\
\hat{\boldsymbol{z}} &= \hat{\boldsymbol{r}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta \quad .
\end{aligned} \tag{3.103}$$

The differential relations are

$$\begin{aligned}
d\hat{\boldsymbol{r}} &= \hat{\boldsymbol{\theta}}\,d\theta + \sin\theta\,\hat{\boldsymbol{\phi}}\,d\phi \\
d\hat{\boldsymbol{\theta}} &= -\hat{\boldsymbol{r}}\,d\theta + \cos\theta\,\hat{\boldsymbol{\phi}}\,d\phi \\
d\hat{\boldsymbol{\phi}} &= -\big(\sin\theta\,\hat{\boldsymbol{r}} + \cos\theta\,\hat{\boldsymbol{\theta}}\big)\,d\phi
\end{aligned} \tag{3.104}$$

The metric is given in terms of

$$h_r = 1 \quad , \quad h_\theta = r \quad , \quad h_\phi = r\,\sin\theta \quad . \tag{3.105}$$

Thus

$$d\boldsymbol{s} = \hat{\boldsymbol{r}}\,dr + \hat{\boldsymbol{\theta}}\,r\,d\theta + \hat{\boldsymbol{\phi}}\,r\sin\theta\,d\phi \tag{3.106}$$

and the velocity squared is

$$\dot{\boldsymbol{s}}^2 = \dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\!\theta\,\dot{\phi}^2 \quad . \tag{3.107}$$

The gradient is

$$\boldsymbol{\nabla} U = \hat{\boldsymbol{r}}\,\frac{\partial U}{\partial r} + \frac{\hat{\boldsymbol{\theta}}}{r}\,\frac{\partial U}{\partial \theta} + \frac{\hat{\boldsymbol{\phi}}}{r\sin\theta}\,\frac{\partial U}{\partial \phi} \quad . \tag{3.108}$$

The divergence is

$$\boldsymbol{\nabla}\cdot\boldsymbol{A} = \frac{1}{r^2}\,\frac{\partial(r^2 A_r)}{\partial r} + \frac{1}{r\sin\theta}\,\frac{\partial(\sin\theta\, A_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\,\frac{\partial A_\phi}{\partial\phi} \quad . \tag{3.109}$$

The curl is

$$\begin{aligned}\boldsymbol{\nabla}\times\boldsymbol{A} = \frac{1}{r\sin\theta}&\left(\frac{\partial(\sin\theta\, A_\phi)}{\partial\theta} - \frac{\partial A_\theta}{\partial\phi}\right)\hat{\boldsymbol{r}} + \frac{1}{r}\left(\frac{1}{\sin\theta}\,\frac{\partial A_r}{\partial\phi} - \frac{\partial(rA_\phi)}{\partial r}\right)\hat{\boldsymbol{\theta}} \\ &+ \frac{1}{r}\left(\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right)\hat{\boldsymbol{\phi}} \quad .\end{aligned} \tag{3.110}$$

The Laplacian is

$$\nabla^2 U = \frac{1}{r^2}\,\frac{\partial}{\partial r}\left(r^2\,\frac{\partial U}{\partial r}\right) + \frac{1}{r^2\sin\theta}\,\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial U}{\partial\theta}\right) + \frac{1}{r^2\sin^2\!\theta}\,\frac{\partial^2 U}{\partial\phi^2} \quad . \tag{3.111}$$

Kinetic energy

Note the form of the kinetic energy of a point particle:

$$\begin{aligned} T = \tfrac{1}{2}m\left(\frac{d\boldsymbol{s}}{dt}\right)^2 &= \tfrac{1}{2}m\big(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\big) && \text{(3D Cartesian)} \\ &= \tfrac{1}{2}m\big(\dot{\rho}^2 + \rho^2\dot{\phi}^2\big) && \text{(2D polar)} \\ &= \tfrac{1}{2}m\big(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2\big) && \text{(3D cylindrical)} \\ &= \tfrac{1}{2}m\big(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\!\theta\,\dot{\phi}^2\big) && \text{(3D polar)} \quad . \end{aligned} \tag{3.112}$$

Chapter 4

Lagrangian Mechanics

4.1 Snell's Law

Warm-up problem: You are standing at point (x_1, y_1) on the beach and you want to get to a point (x_2, y_2) in the water, a few meters offshore. The interface between the beach and the water lies at $x = 0$. What path results in the shortest travel time? It is not a straight line! This is because your speed v_1 on the sand is greater than your speed v_2 in the water. The optimal path actually consists of two line segments, as shown in fig. 4.1. Let the path pass through the point $(0, y)$ on the interface. Then the time T is a function of y:

$$T(y) = \frac{1}{v_1} \sqrt{x_1^2 + (y - y_1)^2} + \frac{1}{v_2} \sqrt{x_2^2 + (y_2 - y)^2} \quad . \tag{4.1}$$

To find the minimum time, we set

$$\begin{aligned} \frac{dT}{dy} = 0 &= \frac{1}{v_1} \frac{y - y_1}{\sqrt{x_1^2 + (y - y_1)^2}} - \frac{1}{v_2} \frac{y_2 - y}{\sqrt{x_2^2 + (y_2 - y)^2}} \\ &= \frac{\sin\theta_1}{v_1} - \frac{\sin\theta_2}{v_2} \quad . \end{aligned} \tag{4.2}$$

Thus, the optimal path satisfies

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2} \quad , \tag{4.3}$$

which is known as *Snell's Law*. Snell's Law is familiar from optics, where the speed of light in a polarizable medium is written $v = c/n$, where n is the index of refraction. Thus $n_1 \sin\theta_1 = n_2 \sin\theta_2$. If there are several interfaces, Snell's law holds at each one, so that

$$n_i \sin\theta_i = n_{i+1} \sin\theta_{i+1} \qquad \Leftrightarrow \qquad \frac{\sin\theta_i}{v_i} = \frac{\sin\theta_{i+1}}{v_{i+1}} \quad , \tag{4.4}$$

at the interface between media i and $i + 1$.

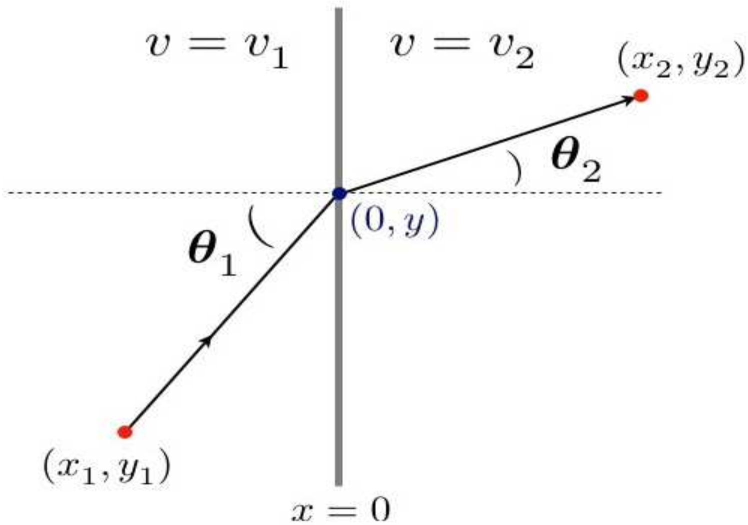

Figure 4.1: The shortest path between (x_1, y_1) and (x_2, y_2) is not a straight line, but rather two successive line segments of different slope.

In the limit where the number of slabs goes to infinity but their thickness is infinitesimal, we can regard n and θ as functions of a continuous variable x. One then has

$$\frac{\sin\theta(x)}{v(x)} = \frac{\sin\theta(x+dx)}{v(x+dx)} \quad , \tag{4.5}$$

which tells us that

$$\frac{d}{dx}\left(\frac{\sin\theta}{v}\right) = 0 \quad . \tag{4.6}$$

On a differential scale, trigonometry tells us that

$$\sin\theta(x) = \frac{dy}{\sqrt{dx^2 + dy^2}} = \frac{y'}{\sqrt{1+y'^2}} \quad , \tag{4.7}$$

and therefore eqn. 4.6 yields

$$\begin{aligned} \frac{d}{dx}\left(\frac{y'}{v\sqrt{1+y'^2}}\right) &= \frac{y''}{v\sqrt{1+y'^2}} - \frac{y'^2 y''}{v\big(1+y'^2\big)^{3/2}} - \frac{v' y'}{v^2\sqrt{1+y'^2}} \\ &= \frac{1}{v\big(1+y'^2\big)^{3/2}}\left(y'' - \frac{v'}{v}\big(1+y'^2\big)y'\right) = 0 \quad . \end{aligned} \tag{4.8}$$

Thus we arrive at the homogeneous second order nonlinear ODE,

$$y'' - \frac{v'}{v}\big(1+y'^2\big)y' = 0 \quad , \tag{4.9}$$

This is a differential equation that $y(x)$ must satisfy if the *functional*

$$T\big[y(x)\big] = \int \frac{ds}{v} = \int\limits_{x_1}^{x_2} dx\, \frac{\sqrt{1+y'^2}}{v(x)} \tag{4.10}$$

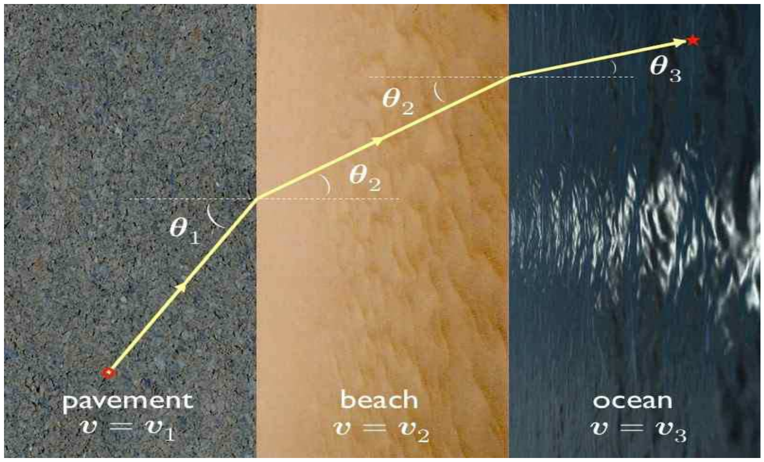

Figure 4.2: The path of shortest length is composed of three line segments. The relation between the angles at each interface is governed by Snell's Law.

is to be minimized. The solution of eqn. 4.9 will require two initial conditions, such as $y(x_0) = y_0$ and $y'(x_0) = y'_0$, or perhaps two boundary conditions $y(x_0) = y_0$ and $y(x_1) = y_1$. Indeed from eqn. 4.6 we can already integrate once, yielding

$$\frac{\sin\theta}{v} = \frac{y'}{v\sqrt{1+y'^2}} = P \quad , \tag{4.11}$$

where P is a constant. Thus, we arrive at a first order ODE, which after isolating y' may be written as

$$\frac{dy}{dx} = \pm\frac{P\,v(x)}{\sqrt{1-P^2v^2(x)}} \quad , \tag{4.12}$$

and for which we must supply one initial/boundary condition.

4.2 The Calculus of Variations

4.2.1 Functions and functionals

A *function* is a mathematical object which takes a real (or complex) variable, or several such variables, and returns a real (or complex) number. A *functional* is a mathematical object which takes an entire function and returns a number. In the case at hand, we have

$$T\big[y(x)\big] = \int\limits_{x_1}^{x_2} dx\, L(y,y',x) \quad , \tag{4.13}$$

where the function $L(y,y',x)$ is given by

$$L(y,y',x) = \frac{1}{v(x)}\sqrt{1+y'^2} \quad . \tag{4.14}$$

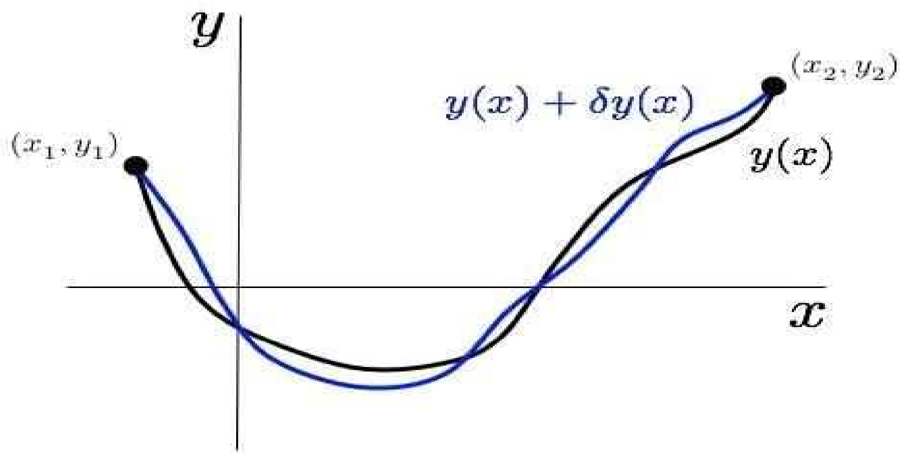

Figure 4.3: A path $y(x)$ and its variation $y(x) + \delta y(x)$.

Here $v(x)$ is a given function characterizing the medium, and $y(x)$ is the path whose time is to be evaluated.

In ordinary calculus, we extremize a function $f(x)$ by demanding that f not change to lowest order when we change $x \to x + dx$:

$$f(x + dx) = f(x) + f'(x)\, dx + \tfrac{1}{2}\, f''(x)\, (dx)^2 + \ldots \quad . \tag{4.15}$$

We say that $x = x^*$ is an extremum when $f'(x^*) = 0$.

For a functional, the first *functional variation* is obtained by sending $y(x) \to y(x) + \delta y(x)$, and extracting the variation in the functional to order δy. Thus, we compute

$$\begin{aligned} T\big[y(x) + \delta y(x)\big] &= \int\limits_{x_1}^{x_2} dx\, L(y + \delta y, y' + \delta y', x) \\ &= \int\limits_{x_1}^{x_2} dx \left\{ L + \frac{\partial L}{\partial y}\, \delta y + \frac{\partial L}{\partial y'}\, \delta y' + \mathcal{O}\big((\delta y)^2\big) \right\} \\ &= T\big[y(x)\big] + \int\limits_{x_1}^{x_2} dx \left\{ \frac{\partial L}{\partial y}\, \delta y + \frac{\partial L}{\partial y'}\, \frac{d}{dx}\, \delta y \right\} \\ &= T\big[y(x)\big] + \int\limits_{x_1}^{x_2} dx \left[\frac{\partial L}{\partial y} - \frac{d}{dx} \bigg(\frac{\partial L}{\partial y'} \bigg) \right] \delta y + \frac{\partial L}{\partial y'}\, \delta y \, \bigg|_{x_1}^{x_2} \quad . \end{aligned} \tag{4.16}$$

Now one very important thing about the variation $\delta y(x)$ is that it must vanish at the endpoints: $\delta y(x_1) = \delta y(x_2) = 0$. This is because the space of functions under consideration satisfy fixed boundary conditions $y(x_1) = y_1$ and $y(x_2) = y_2$. Thus, the last term in the above equation vanishes, and we have

$$\delta T = \int\limits_{x_1}^{x_2} dx \left[\frac{\partial L}{\partial y} - \frac{d}{dx} \bigg(\frac{\partial L}{\partial y'} \bigg) \right] \delta y \quad . \tag{4.17}$$

We say that the first functional derivative of T with respect to $y(x)$ is

$$\frac{\delta T}{\delta y(x)} = \left[\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right)\right]_x \quad , \tag{4.18}$$

where the subscript indicates that the expression inside the square brackets is to be evaluated at x. The functional $T\big[y(x)\big]$ is *extremized* when its first functional derivative vanishes, which results in a differential equation for $y(x)$,

$$\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = 0 \quad , \tag{4.19}$$

known as the *Euler-Lagrange* equation.

$L(y, y', x)$ independent of y

Suppose $L(y, y', x)$ is independent of y. Then from the Euler-Lagrange equations we have that

$$P \equiv \frac{\partial L}{\partial y'} \tag{4.20}$$

is a constant. In classical mechanics, this will turn out to be a *generalized momentum*. For $L = v^{-1}\sqrt{1 + y'^2}$ we have

$$P = \frac{y'}{v\sqrt{1 + y'^2}} \quad . \tag{4.21}$$

Setting $dP/dx = 0$, we recover the second order ODE of eqn. 4.9. Solving for y',

$$\frac{dy}{dx} = \pm \frac{v(x)}{\sqrt{v_0^2 - v^2(x)}} \quad , \tag{4.22}$$

where $v_0 = 1/P$.

$L(y, y', x)$ independent of x

When $L(y, y', x)$ is independent of x, we can again integrate the Euler-Lagrange equation. Consider the quantity

$$H = y' \, \frac{\partial L}{\partial y'} - L \quad . \tag{4.23}$$

Then

$$\begin{aligned} \frac{dH}{dx} = \frac{d}{dx}\left[y' \, \frac{\partial L}{\partial y'} - L\right] &= y'' \, \frac{\partial L}{\partial y'} + y' \, \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) - \frac{\partial L}{\partial y'} \, y'' - \frac{\partial L}{\partial y} y' - \frac{\partial L}{\partial x} \\ &= y' \left[\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) - \frac{\partial L}{\partial y}\right] - \frac{\partial L}{\partial x} = -\frac{\partial L}{\partial x} \quad , \end{aligned} \tag{4.24}$$

where we have used the Euler-Lagrange equations to write $\frac{d}{dx}\big(\frac{\partial L}{\partial y'}\big) = \frac{\partial L}{\partial y}$. So if $\partial L/\partial x = 0$, we have $dH/dx = 0$, *i.e.* H is a constant.

4.2.2 Functional Taylor series

In general, we may expand a functional $F[y+\delta y]$ in a *functional Taylor series*,

$$\begin{aligned}F[y+\delta y] = F[y] + \int dx_1\, K_1(x_1)\,\delta y(x_1) + \tfrac{1}{2!}\int dx_1 \int dx_2\, K_2(x_1,x_2)\,\delta y(x_1)\,\delta y(x_2)\\ + \tfrac{1}{3!}\int dx_1 \int dx_2 \int dx_3\, K_3(x_1,x_2,x_3)\,\delta y(x_1)\,\delta y(x_2)\,\delta y(x_3) + \ldots\end{aligned} \tag{4.25}$$

and we write

$$K_n(x_1,\ldots,x_n) \equiv \frac{\delta^n F}{\delta y(x_1)\cdots\delta y(x_n)} \tag{4.26}$$

for the $n^{\rm th}$ functional derivative.

4.2.3 Examples

Here we present three useful examples of variational calculus as applied to problems in mathematics and physics.

Example 1 : minimal surface of revolution

Consider a surface formed by rotating the function $y(x)$ about the x-axis. The area is then

$$A\big[y(x)\big] = \int\limits_{x_1}^{x_2} dx\; 2\pi y\,\sqrt{1+\bigg(\frac{dy}{dx}\bigg)^2} \quad , \tag{4.27}$$

and is a functional of the curve $y(x)$. Thus we can define $L(y,y') = 2\pi y\sqrt{1+y'^2}$ and make the identification $y(x) \leftrightarrow q(t)$. Since $L(y,y',x)$ is independent of x, we have

$$H = y'\,\frac{\partial L}{\partial y'} - L \quad \Rightarrow \quad \frac{dH}{dx} = -\frac{\partial L}{\partial x} \quad , \tag{4.28}$$

and when L has no explicit x-dependence, H is conserved. One finds

$$H = 2\pi y \cdot \frac{y'^2}{\sqrt{1+y'^2}} - 2\pi y\,\sqrt{1+y'^2} = -\frac{2\pi y}{\sqrt{1+y'^2}} \quad . \tag{4.29}$$

Solving for y',

$$\frac{dy}{dx} = \pm\sqrt{\bigg(\frac{2\pi y}{H}\bigg)^2 - 1} \quad , \tag{4.30}$$

which may be integrated with the substitution $y = \frac{H}{2\pi}\cosh u$, yielding

$$y(x) = b\,\cosh\!\bigg(\frac{x-a}{b}\bigg) \quad , \tag{4.31}$$

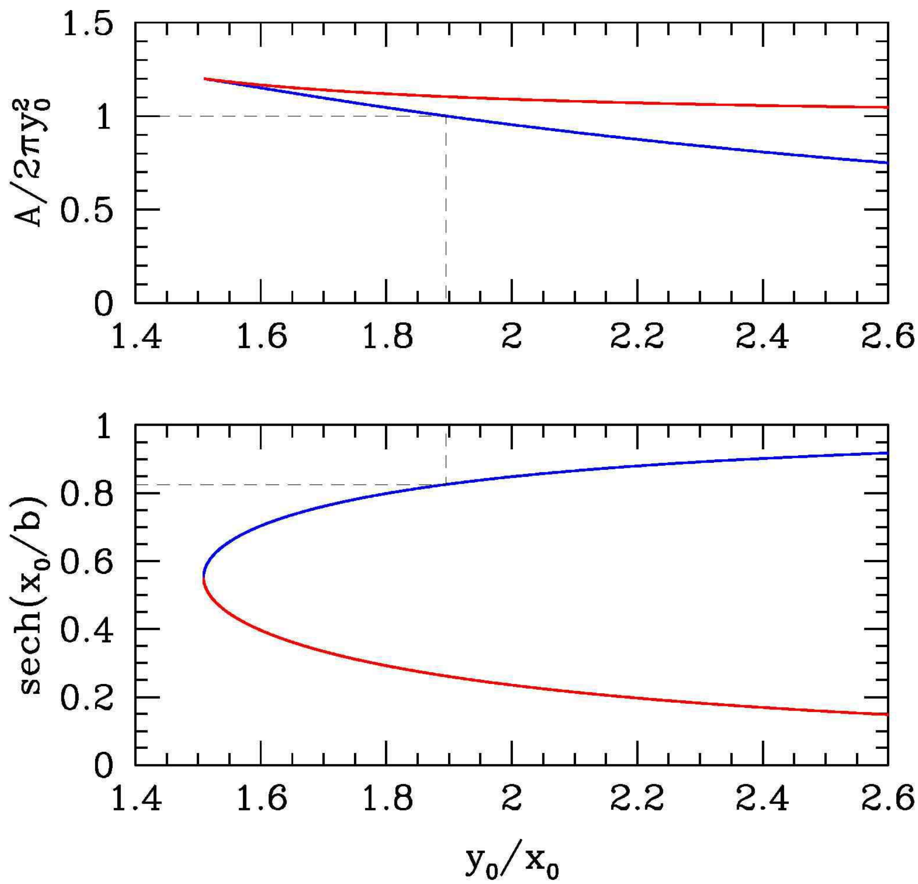

Figure 4.4: Minimal surface solution, with $y(x) = b\cosh(x/b)$ and $y(x_0) = y_0$. Top panel: $A/2\pi y_0^2$ *vs.* y_0/x_0. The discontinuous configuration is shown by the dashed black line. Bottom panel: $\mathrm{sech}(x_0/b)$ *vs.* y_0/x_0. The blue curve corresponds to a global minimum of $A[y(x)]$, and the red curve to a local minimum or saddle point.

where a and $b = \frac{H}{2\pi}$ are constants of integration. Note there are two such constants, as the original equation was second order. This shape is called a *catenary*. As we shall later find, it is also the shape of a uniformly dense rope hanging between two supports, under the influence of gravity. To fix the constants a and b, we invoke the boundary conditions $y(x_1) = y_1$ and $y(x_2) = y_2$.

Consider the case where $-x_1 = x_2 \equiv x_0$ and $y_1 = y_2 \equiv y_0$. Then clearly $a = 0$, and we have

$$y_0 = b\,\cosh\left(\frac{x_0}{b}\right) \quad \Rightarrow \quad \gamma = \kappa^{-1}\,\cosh\kappa \quad , \tag{4.32}$$

with $\gamma \equiv y_0/x_0$ and $\kappa \equiv x_0/b$. One finds that for any $\gamma > 1.5089$ there are two solutions, one of which is a global minimum and one of which is a local minimum or saddle of $A[y(x)]$. The solution with the smaller value of κ (*i.e.* the larger value of $\mathrm{sech}\,\kappa$) yields the smaller value of A, as shown in fig. 4.4. Note that

$$\frac{y}{y_0} = \frac{\cosh(x/b)}{\cosh(x_0/b)} \quad , \tag{4.33}$$

so $y(x=0) = y_0\,\mathrm{sech}(x_0/b)$.

When extremizing functions that are defined over a finite or semi-infinite interval, one must take care to evaluate the function at the boundary, for it may be that the boundary yields a global extremum even though the derivative may not vanish there. Similarly, when extremizing functionals, one must investigate the functions at the boundary of function space. In this case, such a function would be the discontinuous solution, with

$$y(x) = \begin{cases} y_1 & \text{if } x = x_1 \\ 0 & \text{if } x_1 < x < x_2 \\ y_2 & \text{if } x = x_2 \end{cases} \quad . \tag{4.34}$$

This solution corresponds to a surface consisting of two discs of radii y_1 and y_2, joined by an infinitesimally thin thread. The area functional evaluated for this particular $y(x)$ is clearly $A = \pi(y_1^2 + y_2^2)$. In fig. 4.4, we plot $A/2\pi y_0^2$ *versus* the parameter $\gamma = y_0/x_0$. For $\gamma > \gamma_c \approx 1.564$, one of the catenary solutions is the global minimum. For $\gamma < \gamma_c$, the minimum area is achieved by the discontinuous solution.

Note that the functional derivative,

$$K_1(x) = \frac{\delta A}{\delta y(x)} = \frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = \frac{2\pi\big(1 + y'^2 - yy''\big)}{(1 + y'^2)^{3/2}} \quad , \tag{4.35}$$

indeed vanishes for the catenary solutions, but does not vanish for the discontinuous solution, where $K_1(x) = 2\pi$ throughout the interval $(-x_0, x_0)$. Since $y = 0$ on this interval, y cannot be decreased. The fact that $K_1(x) > 0$ means that increasing y will result in an increase in A, so the boundary value for A, which is $2\pi y_0^2$, is indeed a local minimum.

We furthermore see in fig. 4.4 that for $\gamma < \gamma_* \approx 1.5089$ the local minimum and saddle are no longer present. This is the familiar saddle-node bifurcation, here in function space. Thus, for $\gamma \in [0, \gamma_*)$ there are no extrema of $A[y(x)]$, and the minimum area occurs for the discontinuous $y(x)$ lying at the boundary of function space. For $\gamma \in (\gamma_*, \gamma_c)$, two extrema exist, one of which is a local minimum and the other a saddle point. Still, the area is minimized for the discontinuous solution. For $\gamma \in (\gamma_c, \infty)$, the local minimum is the global minimum, and has smaller area than for the discontinuous solution.

Example 2 : geodesic on a surface of revolution

We use cylindrical coordinates (ρ, ϕ, z) on the surface $z = z(\rho)$. Thus,

$$\begin{aligned} ds^2 &= d\rho^2 + \rho^2\, d\phi^2 + dx^2 \\ &= \Big\{1 + \big[z'(\rho)\big]^2\Big\}\, d\rho + \rho^2\, d\phi^2 \quad , \end{aligned} \tag{4.36}$$

and the distance functional $D\big[\phi(\rho)\big]$ is

$$D\big[\phi(\rho)\big] = \int\limits_{\rho_1}^{\rho_2} d\rho\, L(\phi, \phi', \rho) \quad , \tag{4.37}$$

where

$$L(\phi, \phi', \rho) = \sqrt{1 + z'^2(\rho) + \rho^2\, \phi'^2(\rho)} \quad . \tag{4.38}$$

The Euler-Lagrange equation is

$$\frac{\partial L}{\partial \phi} - \frac{d}{d\rho}\left(\frac{\partial L}{\partial \phi'}\right) = 0 \quad \Rightarrow \quad \frac{\partial L}{\partial \phi'} = \text{const.} \tag{4.39}$$

Thus,

$$\frac{\partial L}{\partial \phi'} = \frac{\rho^2\, \phi'}{\sqrt{1 + z'^2 + \rho^2\, \phi'^2}} = a \quad , \tag{4.40}$$

where a is a constant. Solving for ϕ', we obtain

$$d\phi = \frac{a\sqrt{1 + \big[z'(\rho)\big]^2}}{\rho\sqrt{\rho^2 - a^2}}\, d\rho \quad , \tag{4.41}$$

which we must integrate to find $\phi(\rho)$, subject to boundary conditions $\phi(\rho_i) = \phi_i$, with $i = 1, 2$.

On a cone, $z(\rho) = \lambda\rho$, and we have

$$d\phi = a\sqrt{1 + \lambda^2}\, \frac{d\rho}{\rho\sqrt{\rho^2 - a^2}} = \sqrt{1 + \lambda^2}\, d\tan^{-1}\sqrt{\frac{\rho^2}{a^2} - 1} \quad , \tag{4.42}$$

which yields

$$\phi(\rho) = \beta + \sqrt{1 + \lambda^2}\ \tan^{-1}\sqrt{\frac{\rho^2}{a^2} - 1} \quad , \tag{4.43}$$

which is equivalent to

$$\rho\,\cos\left(\frac{\phi - \beta}{\sqrt{1 + \lambda^2}}\right) = a \quad . \tag{4.44}$$

The constants β and a are determined from $\phi(\rho_i) = \phi_i$.

Example 3 : brachistochrone

Problem: find the path between (x_1, y_1) and (x_2, y_2) which a particle sliding frictionlessly and under constant gravitational acceleration will traverse in the shortest time. To solve this we first must invoke some elementary mechanics. Assuming the particle is released from (x_1, y_1) at rest, energy conservation says

$$\tfrac{1}{2}mv^2 + mgy = mgy_1 \quad . \tag{4.45}$$

Then the time, which is a functional of the curve $y(x)$, is

$$T\big[y(x)\big] =\equiv \int\limits_{x_1}^{x_2} dx\, L(y, y', x) = \int\limits_{x_1}^{x_2} \frac{ds}{v} = \frac{1}{\sqrt{2g}} \int\limits_{x_1}^{x_2} dx\, \sqrt{\frac{1 + y'^2}{y_1 - y}} \quad , \tag{4.46}$$

with

$$L(y, y', x) = \sqrt{\frac{1 + y'^2}{2g(y_1 - y)}} \quad . \tag{4.47}$$

Since L is independent of x, eqn. 4.24, we have that

$$H = y' \frac{\partial L}{\partial y'} - L = -\Big[2g\,(y_1 - y)\,\big(1 + y'^2\big)\Big]^{-1/2} \tag{4.48}$$

is conserved. This yields

$$dx = -\sqrt{\frac{y_1 - y}{2a - y_1 + y}}\,dy \quad , \tag{4.49}$$

with $a = (4gH^2)^{-1}$. This may be integrated parametrically, writing

$$y_1 - y = 2a\sin^2\!\big(\tfrac{1}{2}\theta\big) \quad \Rightarrow \quad dx = 2a\sin^2\!\big(\tfrac{1}{2}\theta\big)\,d\theta \quad , \tag{4.50}$$

which results in the parametric equations

$$\begin{aligned} x - x_1 &= a\,\big(\theta - \sin\theta\big) \\ y - y_1 &= -a\,(1 - \cos\theta) \quad . \end{aligned} \tag{4.51}$$

This curve is known as a *cycloid.*

4.2.4 Ocean waves

Surface waves in fluids propagate with a definite relation between their angular frequency ω and their wavevector $k = 2\pi/\lambda$, where λ is the wavelength. The *dispersion relation* is a function $\omega = \omega(k)$. The *group velocity* of the waves is then $v(k) = d\omega/dk$.

In a fluid with a flat bottom at depth h, the dispersion relation turns out to be

$$\omega(k) = \sqrt{gk\,\tanh kh} \approx \begin{cases} \sqrt{gh}\,k & \text{shallow } (kh \ll 1) \\ \\ \sqrt{gk} & \text{deep } (kh \gg 1) \quad . \end{cases} \tag{4.52}$$

Suppose we are in the shallow case, where the wavelength λ is significantly greater than the depth h of the fluid. This is the case for ocean waves which break at the shore. The phase velocity and group velocity are then identical, and equal to $v(h) = \sqrt{gh}$. The waves propagate more slowly as they approach the shore.

Let us choose the following coordinate system: x represents the distance parallel to the shoreline, y the distance perpendicular to the shore (which lies at $y = 0$), and $h(y)$ is the depth profile of the bottom. We assume $h(y)$ to be a slowly varying function of y which satisfies $h(0) = 0$. Suppose a disturbance in the ocean at position (x_2, y_2) propagates until it reaches the shore at $(x_1, y_1 = 0)$. The time of propagation is

$$T\big[y(x)\big] = \int \frac{ds}{v} = \int\limits_{x_1}^{x_2} \!dx\,\sqrt{\frac{1 + y'^2}{g\,h(y)}} \quad . \tag{4.53}$$

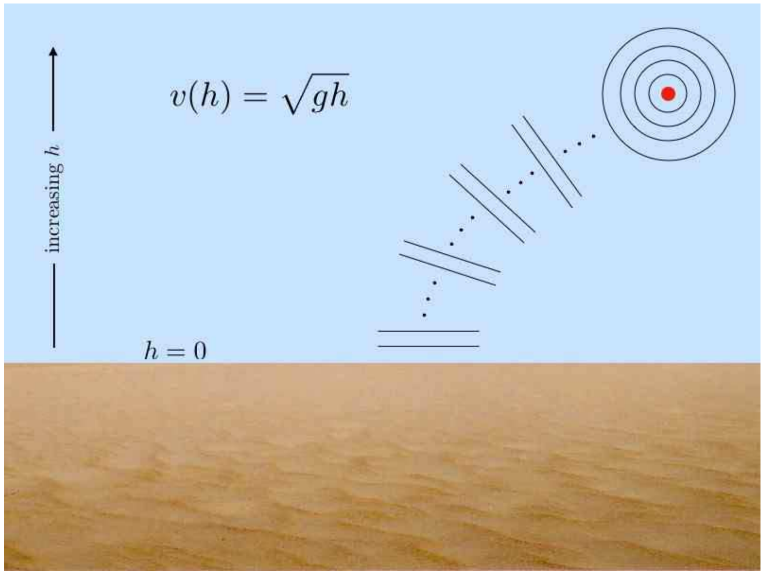

Figure 4.5: For shallow water waves, $v = \sqrt{gh}$. To minimize the propagation time from a source to the shore, the waves break parallel to the shoreline.

We thus identify the integrand

$$L(y, y', x) = \sqrt{\frac{1 + y'^2}{g\,h(y)}} \quad . \tag{4.54}$$

As with the brachistochrone problem, to which this bears an obvious resemblance, L is cyclic in the independent variable x, hence

$$H = y'\,\frac{\partial L}{\partial y'} - L = -\Big[g\,h(y)\,\big(1 + y'^2\big)\Big]^{-1/2} \tag{4.55}$$

is constant. Solving for $y'(x)$, we have

$$\tan\theta = \frac{dy}{dx} = \sqrt{\frac{a}{h(y)} - 1} \quad , \tag{4.56}$$

where $a = (gH)^{-1}$ is a constant, and where θ is the local slope of the function $y(x)$. Thus, we conclude that near $y = 0$, where $h(y) \to 0$, the waves come in *parallel to the shoreline.* If $h(y) = \alpha y$ has a linear profile, the solution is again a cycloid, with

$$x(\theta) = b\,(\theta - \sin\theta) \qquad , \qquad y(\theta) = b\,(1 - \cos\theta) \quad , \tag{4.57}$$

where $b = 2a/\alpha$ and where the shore lies at $\theta = 0$. Expanding in a Taylor series in θ for small θ, we may eliminate θ and obtain $y(x)$ as

$$y(x) = \big(\tfrac{9}{2}\big)^{1/3}\, b^{1/3}\, x^{2/3} + \ldots \quad . \tag{4.58}$$

A *tsunami* is a shallow water wave that propagates in deep water. This requires $\lambda > h$, as we've seen, which means the disturbance must have a very long spatial extent out in the open ocean, where $h \sim$

10 km. An undersea earthquake is the only possible source; the characteristic length of earthquake fault lines can be hundreds of kilometers. If we take $h = 10\,\mathrm{km}$, we obtain $v = \sqrt{gh} \approx 310\,\mathrm{m/s}$ or $1100\,\mathrm{km/hr}$. At these speeds, a tsunami can cross the Pacific Ocean in less than a day.

As the wave approaches the shore, it must slow down, since $v = \sqrt{gh}$ is diminishing. But energy is conserved, which means that the amplitude must concomitantly rise. In extreme cases, the water level rise at shore may be 20 meters or more.

4.2.5 More on functionals

We remarked in section 4.2.1 that a function f is an animal which gets fed a real number x and excretes a real number $f(x)$. We say f maps the reals to the reals, or $f\colon \mathbb{R} \to \mathbb{R}$. Of course we also have functions $g\colon \mathbb{C} \to \mathbb{C}$ which eat and excrete complex numbers, multivariable functions $h\colon \mathbb{R}^N \to \mathbb{R}$ which eat N-tuples of numbers and excrete a single number, *etc.*

A *functional* $F[f(x)]$ eats entire functions (!) and excretes numbers. That is,

$$F\colon \Big\{ f(x) \;\big|\; x \in \mathbb{R} \Big\} \to \mathbb{R} \tag{4.59}$$

We may write $F\colon C(\mathbb{R}) \to \mathbb{R}$, where $C(\mathbb{R})$ is the *space of continuous functions*[1]. This says that F operates on the set of real-valued functions of a single real variable, yielding a real number. Some examples:

$$\begin{aligned}
F[f(x)] &= \tfrac{1}{2}\int\limits_{-\infty}^{\infty} dx\, \big[f(x)\big]^2 \\
F[f(x)] &= \tfrac{1}{2}\int\limits_{-\infty}^{\infty} dx \int\limits_{-\infty}^{\infty} dx'\, K(x,x')\, f(x)\, f(x') \\
F[f(x)] &= \tfrac{1}{2}\int\limits_{-\infty}^{\infty} dx\, \bigg[A\, f^2(x) + B \bigg(\frac{df}{dx}\bigg)^{\!2}\, \bigg] \quad .
\end{aligned} \tag{4.60}$$

In classical mechanics, the action S is a functional of the path $q(t)$:

$$S[q(t)] = \int\limits_{t_a}^{t_b} dt\, \Big\{ \tfrac{1}{2} m \dot{q}^2 - U(q) \Big\} \quad . \tag{4.61}$$

We can also have functionals which feed on functions of more than one independent variable, such as

$$S[y(x,t)] = \int\limits_{t_a}^{t_b} dt \int\limits_{x_a}^{x_b} dx\, \bigg\{ \tfrac{1}{2}\mu \bigg(\frac{\partial y}{\partial t}\bigg)^{\!2} - \tfrac{1}{2}\tau \bigg(\frac{\partial y}{\partial x}\bigg)^{\!2} \bigg\} \quad , \tag{4.62}$$

[1]The notation $C^\infty(\mathbb{R})$ indicates the space of continuous *smooth* (*i.e.* infinitely differentiable) real functions of a real variable.

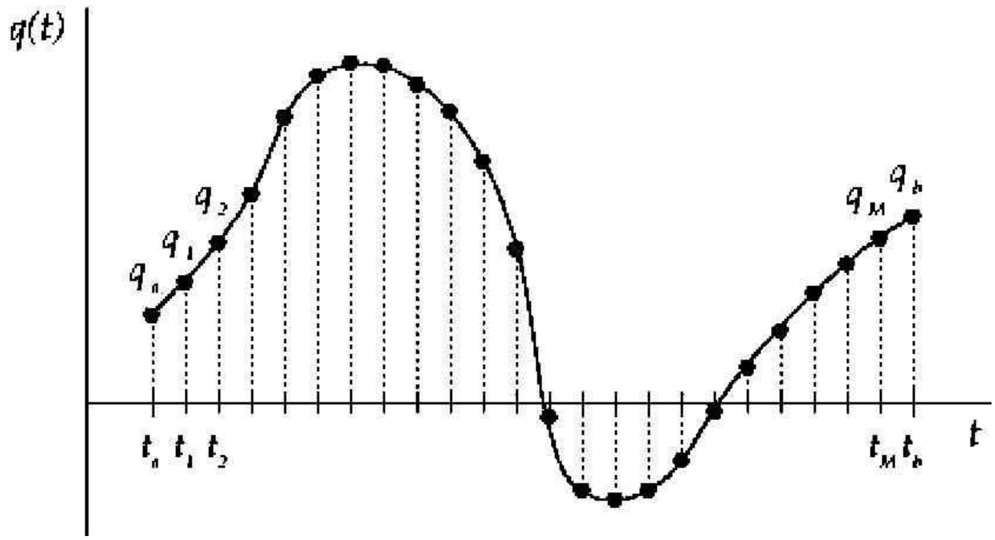

Figure 4.6: A functional $S[q(t)]$ is the continuum limit of a function of a large number of variables, $S(q_1, \ldots, q_M)$.

which happens to be the functional for a string of mass density μ under uniform tension τ. Another example comes from electrodynamics:

$$S[A^\mu(\boldsymbol{x}, t)] = -\int dt \int d^3x \left\{ \frac{1}{16\pi} F_{\mu\nu}\, F^{\mu\nu} + \frac{1}{c} j_\mu\, A^\mu \right\} \quad , \tag{4.63}$$

which is a functional of the four fields $\{A^0, A^1, A^2, A^3\}$, where $A^0 = c\phi$. These are the components of the 4-potential, each of which is itself a function of four independent variables (x^0, x^1, x^2, x^3), with $x^0 = ct$. The field strength tensor is written in terms of derivatives of the A^μ: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, where we use a metric $g_{\mu\nu} = \mathrm{diag}(+, -, -, -)$ to raise and lower indices. The 4-potential couples linearly to the source term J_μ, which is the electric 4-current $(c\rho, \boldsymbol{J})$.

We extremize functions by sending the independent variable x to $x + dx$ and demanding that the variation $df = 0$ to first order in dx. That is,

$$f(x + dx) = f(x) + f'(x)\, dx + \tfrac{1}{2} f''(x)(dx)^2 + \ldots \quad , \tag{4.64}$$

whence $df = f'(x)\, dx + \mathcal{O}\big((dx)^2\big)$ and thus

$$f'(x^*) = 0 \quad \Longleftrightarrow \quad x^* \text{ an extremum.} \tag{4.65}$$

We extremize *functionals* by sending

$$f(x) \to f(x) + \delta f(x) \tag{4.66}$$

and demanding that the variation δF in the functional $F[f(x)]$ vanish to first order in $\delta f(x)$. The variation $\delta f(x)$ must sometimes satisfy certain boundary conditions. For example, if $F[f(x)]$ only operates on functions which vanish at a pair of endpoints, *i.e.* $f(x_a) = f(x_b) = 0$, then when we extremize the functional F we must do so *within the space of allowed functions.* Thus, we would in this case require $\delta f(x_a) = \delta f(x_b) = 0$. We may expand the functional $F[f + \delta f]$ in a *functional Taylor series*,

$$\begin{aligned} F[f+\delta f] = F[f] &+ \int dx_1 \, K_1(x_1)\, \delta f(x_1) + \tfrac{1}{2!} \int dx_1 \int dx_2 \, K_2(x_1,x_2)\, \delta f(x_1)\, \delta f(x_2) \\ &+ \tfrac{1}{3!} \int dx_1 \int dx_2 \int dx_3 \, K_3(x_1,x_2,x_3)\, \delta f(x_1)\, \delta f(x_2)\, \delta f(x_3) + \ldots \end{aligned} \tag{4.67}$$

and we write

$$K_n(x_1, \ldots, x_n) \equiv \frac{\delta^n F}{\delta f(x_1) \cdots \delta f(x_n)} \quad . \tag{4.68}$$

In a more general case, $F = F\big[\{f_i(\boldsymbol{x})\}\big]$ is a functional of several functions, each of which is a function of several independent variables[2]. We then write

$$\begin{aligned} F[\{f_i+\delta f_i\}] = F[\{f_i\}] &+ \int d\boldsymbol{x}_1 \, K_1^i(\boldsymbol{x}_1)\, \delta f_i(\boldsymbol{x}_1) + \frac{1}{2!} \int d\boldsymbol{x}_1 \int d\boldsymbol{x}_2 \, K_2^{ij}(\boldsymbol{x}_1,\boldsymbol{x}_2)\, \delta f_i(\boldsymbol{x}_1)\, \delta f_j(\boldsymbol{x}_2) \\ &+ \frac{1}{3!} \int d\boldsymbol{x}_1 \int d\boldsymbol{x}_2 \int d\boldsymbol{x}_3 \; K_3^{ijk}(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3)\, \delta f_i(\boldsymbol{x}_1)\, \delta f_j(\boldsymbol{x}_2)\, \delta f_k(\boldsymbol{x}_3) + \ldots \quad , \end{aligned} \tag{4.69}$$

with

$$K_n^{i_1 i_2 \cdots i_n}(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n) = \frac{\delta^n F}{\delta f_{i_1}(\boldsymbol{x}_1)\, \delta f_{i_2}(\boldsymbol{x}_2)\, \delta f_{i_n}(\boldsymbol{x}_n)} \quad . \tag{4.70}$$

Another way to compute functional derivatives is to send

$$f(x) \to f(x) + \epsilon_1 \, \delta(x - x_1) + \ldots + \epsilon_n \, \delta(x - x_n) \tag{4.71}$$

and then differentiate n times with respect to ϵ_1 through ϵ_n. That is,

$$\frac{\delta^n F}{\delta f(x_1) \cdots \delta f(x_n)} = \frac{\partial^n}{\partial \epsilon_1 \cdots \partial \epsilon_n} \bigg|_{\epsilon_1=\epsilon_2=\cdots \epsilon_n=0} F\big[f(x) + \epsilon_1 \, \delta(x-x_1) + \ldots + \epsilon_n \, \delta(x - x_n)\big] \quad . \tag{4.72}$$

Let's see how this works. As an example, we'll take the action functional from classical mechanics,

$$S[q(t)] = \int\limits_{t_a}^{t_b} dt \, \Big\{ \tfrac{1}{2} m \dot{q}^2 - U(q) \Big\} \quad . \tag{4.73}$$

To compute the first functional derivative, we replace the function $q(t)$ with $q(t) + \epsilon \, \delta(t - t_1)$, and expand in powers of ϵ:

$$\begin{aligned} S\big[q(t) + \epsilon \delta(t-t_1)\big] &= S[q(t)] + \epsilon \int\limits_{t_a}^{t_b} dt \, \Big\{ m \dot{q} \, \delta'(t-t_1) - U'(q) \, \delta(t-t_1) \Big\} \\ &= -\epsilon \, \Big\{ m \ddot{q}(t_1) + U'\big(q(t_1)\big) \Big\} \quad , \end{aligned} \tag{4.74}$$

[2]It may be also be that different functions depend on a different number of independent variables. *E.g.* $F = F[f(x),\, g(x,y),\, h(x,y,z)]$.

hence

$$\frac{\delta S}{\delta q(t)} = -\Big\{ m\ddot{q}(t) + U'\big(q(t)\big)\Big\} \tag{4.75}$$

and setting the first functional derivative to zero yields Newton's Second Law, $m\ddot{q} = -U'(q)$, for all $t \in [t_a, t_b]$. Note that we have used the result

$$\int\limits_{-\infty}^{\infty} dt\, \delta'(t-t_1)\, h(t) = -h'(t_1) \quad , \tag{4.76}$$

which is easily established upon integration by parts.

To compute the second functional derivative, we replace

$$q(t) \to q(t) + \epsilon_1\, \delta(t-t_1) + \epsilon_2\, \delta(t-t_2) \tag{4.77}$$

and extract the term of order $\epsilon_1\,\epsilon_2$ in the double Taylor expansion. One finds this term to be

$$\epsilon_1\,\epsilon_2 \int\limits_{t_a}^{t_b} dt\, \Big\{ m\, \delta'(t-t_1)\, \delta'(t-t_2) - U''(q)\, \delta(t-t_1)\, \delta(t-t_2)\Big\} \quad . \tag{4.78}$$

Note that we needn't bother with terms proportional to ϵ_1^2 or ϵ_2^2 since the recipe is to differentiate once with respect to each of ϵ_1 and ϵ_2 and then to set $\epsilon_1 = \epsilon_2 = 0$. This procedure uniquely selects the term proportional to $\epsilon_1\,\epsilon_2$, and yields

$$\frac{\delta^2 S}{\delta q(t_1)\, \delta q(t_2)} = -\Big\{ m\, \delta''(t_1 - t_2) + U''\big(q(t_1)\big)\, \delta(t_1 - t_2)\Big\} \quad . \tag{4.79}$$

In multivariable calculus, the stability of an extremum is assessed by computing the matrix of second derivatives at the extremal point, known as the Hessian matrix. One has

$$\frac{\partial f}{\partial x_i}\bigg|_{\boldsymbol{x}^*} = 0 \quad \forall\, i \qquad ; \qquad H_{ij} = \frac{\partial^2 f}{\partial x_i\, \partial x_j}\bigg|_{\boldsymbol{x}^*} \quad . \tag{4.80}$$

The eigenvalues of the Hessian H_{ij} determine the stability of the extremum. Since H_{ij} is a symmetric matrix, its eigenvectors η^α may be chosen to be orthogonal. The associated eigenvalues λ_α, defined by the equation

$$H_{ij}\, \eta^\alpha_j = \lambda_\alpha\, \eta^\alpha_i \quad , \tag{4.81}$$

are the respective curvatures in the directions η^α, where $\alpha \in \{1, \ldots, n\}$ where n is the number of variables. The extremum is a local minimum if all the eigenvalues λ_α are positive, a maximum if all are negative, and otherwise is a saddle point. Near a saddle point, there are some directions in which the function increases and some in which it decreases.

In the case of functionals, the second functional derivative $K_2(x_1, x_2)$ defines an eigenvalue problem for $\delta f(x)$:

$$\int\limits_{x_a}^{x_b} dx_2\, K_2(x_1, x_2)\, \delta f(x_2) = \lambda\, \delta f(x_1) \quad . \tag{4.82}$$

In general there are an infinite number of solutions to this equation which form a basis in function space, subject to appropriate boundary conditions at x_a and x_b. For example, in the case of the action functional from classical mechanics, the above eigenvalue equation becomes a differential equation,

$$-\left\{m\frac{d^2}{dt^2} + U''\big(q^*(t)\big)\right\}\delta q(t) = \lambda\,\delta q(t) \quad , \tag{4.83}$$

where $q^*(t)$ is the solution to the Euler-Lagrange equations. As with the case of ordinary multivariable functions, the functional extremum is a local minimum (in function space) if every eigenvalue λ_α is positive, a local maximum if every eigenvalue is negative, and a saddle point otherwise.

Consider the simple harmonic oscillator, for which $U(q) = \frac{1}{2}\,m\omega_0^2\,q^2$. Then $U''\big(q^*(t)\big) = m\,\omega_0^2$; note that we don't even need to know the solution $q^*(t)$ to obtain the second functional derivative in this special case. The eigenvectors obey $m(\delta\ddot{q} + \omega_0^2\,\delta q) = -\lambda\,\delta q$, hence

$$\delta q(t) = A\cos\left(\sqrt{\omega_0^2 + (\lambda/m)}\;t + \varphi\right) \quad , \tag{4.84}$$

where A and φ are constants. Demanding $\delta q(t_a) = \delta q(t_b) = 0$ requires

$$\sqrt{\omega_0^2 + (\lambda/m)}\,\big(t_b - t_a\big) = n\pi \quad , \tag{4.85}$$

where n is an integer. Thus, the eigenfunctions are

$$\delta q_n(t) = A\sin\left(n\pi\cdot\frac{t-t_a}{t_b-t_a}\right) \quad , \tag{4.86}$$

and the eigenvalues are

$$\lambda_n = m\Big(\frac{n\pi}{T}\Big)^2 - m\omega_0^2 \quad , \tag{4.87}$$

where $T = t_b - t_a$. Thus, so long as $T > \pi/\omega_0$, there is at least one negative eigenvalue. Indeed, for $\frac{n\pi}{\omega_0} < T < \frac{(n+1)\pi}{\omega_0}$ there will be n negative eigenvalues. This means the action is generally not a minimum, but rather lies at a *saddle point* in the (infinite-dimensional) function space.

To test this explicitly, consider a harmonic oscillator with the boundary conditions $q(0) = 0$ and $q(T) = Q$. The equations of motion, $\ddot{q} + \omega_0^2\,q = 0$, along with the boundary conditions, determine the motion,

$$q^*(t) = \frac{Q\,\sin(\omega_0 t)}{\sin(\omega_0 T)} \quad . \tag{4.88}$$

The action for this path is then

$$\begin{aligned} S[q^*(t)] &= \int_0^T\! dt\,\Big(\tfrac{1}{2}m\dot{q}^{*2} - \tfrac{1}{2}m\omega_0^2\,q^{*2}\Big) \\ &= \frac{m\,\omega_0^2\,Q^2}{2\sin^2\!\omega_0 T}\int_0^T\! dt\,\Big(\cos^2\!\omega_0 t - \sin^2\!\omega_0 t\Big) = \tfrac{1}{2}m\omega_0\,Q^2\ \mathrm{ctn}\,(\omega_0 T) \quad . \end{aligned} \tag{4.89}$$

Next consider the path $q(t) = Q\,t/T$ which satisfies the boundary conditions but does not satisfy the equations of motion (it proceeds with constant velocity). One finds the action for this path is

$$S[q(t)] = \tfrac{1}{2}m\omega_0\,Q^2\left(\frac{1}{\omega_0 T} - \tfrac{1}{3}\omega_0 T\right) \quad . \tag{4.90}$$

Thus, provided $\omega_0 T \neq n\pi$, in the limit $T \to \infty$ we find that the constant velocity path has lower action.

Finally, consider the general mechanical action,

$$S\big[q(t)\big] = \int\limits_{t_a}^{t_b} dt\, L(q, \dot{q}, t) \quad . \tag{4.91}$$

We now evaluate the first few terms in the functional Taylor series:

$$\begin{aligned} S\big[q^*(t) + \delta q(t)\big] = \int\limits_{t_a}^{t_b} dt\, \Bigg\{ & L(q^*, \dot{q}^*, t) + \frac{\partial L}{\partial q_i}\bigg|_{q^*} \delta q_i + \frac{\partial L}{\partial \dot{q}_i}\bigg|_{q^*} \delta \dot{q}_i \\ & + \frac{1}{2}\,\frac{\partial^2 L}{\partial q_i\, \partial q_j}\bigg|_{q^*} \delta q_i\, \delta q_j + \frac{\partial^2 L}{\partial q_i\, \partial \dot{q}_j}\bigg|_{q^*} \delta q_i\, \delta \dot{q}_j + \frac{1}{2}\,\frac{\partial^2 L}{\partial \dot{q}_i\, \partial \dot{q}_j}\bigg|_{q^*} \delta \dot{q}_i\, \delta \dot{q}_j + \ldots \Bigg\} \quad . \end{aligned} \tag{4.92}$$

To identify the functional derivatives, we integrate by parts. Let $\Phi_{\ldots}(t)$ be an arbitrary function of time. Then

$$\int\limits_{t_a}^{t_b} dt\, \Phi_i(t)\, \delta \dot{q}_i(t) = -\int\limits_{t_a}^{t_b} dt\, \dot{\Phi}_i(t)\, \delta q_i(t) \tag{4.93}$$

and

$$\begin{aligned} \int\limits_{t_a}^{t_b} dt\; \Phi_{ij}(t)\; \delta q_i(t)\, \delta \dot{q}_j(t) &= \int\limits_{t_a}^{t_b} dt \int\limits_{t_a}^{t_b} dt'\; \Phi_{ij}(t)\, \delta(t - t')\, \frac{d}{dt'}\, \delta q_i(t)\, \delta q_j(t') \\ &= \int\limits_{t_a}^{t_b} dt \int\limits_{t_a}^{t_b} dt'\; \Phi_{ij}(t))\, \delta'(t - t')\, \delta q_i(t)\, \delta q_j(t') \quad , \end{aligned} \tag{4.94}$$

and

$$\begin{aligned} \int\limits_{t_a}^{t_b} dt\; \Phi_{ij}(t)\, d\dot{q}_i(t)\, \delta \dot{q}_j(t) &= \int\limits_{t_a}^{t_b} dt \int\limits_{t_a}^{t_b} dt'\; \Phi_{ij}(t)\, \delta(t - t')\, \frac{d}{dt}\, \frac{d}{dt'}\, \delta q_i(t)\, \delta q_j(t') \\ &= -\int\limits_{t_a}^{t_b} dt \int\limits_{t_a}^{t_b} dt' \Big[\dot{\Phi}_{ij}(t)\, \delta'(t - t') + \Phi_{ij}(t)\, \delta''(t - t') \Big]\, \delta q_i(t)\, \delta q_j(t') \quad . \end{aligned} \tag{4.95}$$

Thus, the first two functional derivatives are given by

$$\frac{\delta S}{\delta q_i(t)} = \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i}\right)\right]_{q^*(t)} \tag{4.96}$$

and

$$\begin{aligned}\frac{\delta^2 S}{\delta q_i(t)\, \delta q_j(t')} = \Bigg\{ \frac{\partial^2 L}{\partial q_i\, \partial q_j}\bigg|_{q^*(t)} \delta(t-t') - \frac{\partial^2 L}{\partial \dot q_i\, \partial \dot q_j}\bigg|_{q^*(t)} \delta''(t-t') \\ + \left[2\, \frac{\partial^2 L}{\partial q_i\, \partial \dot q_j} - \frac{d}{dt}\left(\frac{\partial^2 L}{\partial \dot q_i\, \partial \dot q_j}\right)\right]_{q^*(t)} \delta'(t-t') \Bigg\} \quad .\end{aligned} \tag{4.97}$$

4.3 Lagrangian Mechanics

4.3.1 Generalized coordinates

A set of *generalized coordinates* $q_1, \ldots, q_n$ completely describes the positions of all particles in a mechanical system. In a system with $d_{\rm f}$ degrees of freedom and k constraints, $n = d_{\rm f} - k$ independent generalized coordinates are needed to completely specify all the positions. A constraint is a relation among coordinates, such as $x^2 + y^2 + z^2 = a^2$ for a particle moving on a sphere of radius a. In this case, $d_{\rm f} = 3$ and $k = 1$. In this case, we could eliminate z in favor of x and y, *i.e.* by writing $z = \pm\sqrt{a^2 - x^2 - y^2}$, or we could choose as coordinates the polar and azimuthal angles θ and ϕ.

For the moment we will assume that $n = d_{\rm f} - k$, and that the generalized coordinates are independent, satisfying no additional constraints among them. Later on we will learn how to deal with any remaining constraints among the $\{q_1, \ldots, q_n\}$.

The generalized coordinates may have units of length, or angle, or perhaps something totally different. In the theory of small oscillations, the normal coordinates are conventionally chosen to have units of $({\rm mass})^{1/2} \times ({\rm length})$. However, once a choice of generalized coordinate is made, with a concomitant set of units, the units of the conjugate momentum and force are determined:

$$\big[p_\sigma\big] = \frac{ML^2}{T} \cdot \frac{1}{\big[q_\sigma\big]} \quad , \quad \big[F_\sigma\big] = \frac{ML^2}{T^2} \cdot \frac{1}{\big[q_\sigma\big]} \quad , \tag{4.98}$$

where $\big[A\big]$ means 'the units of A', and where M, L, and T stand for mass, length, and time, respectively. Thus, if q_σ has dimensions of length, then p_σ has dimensions of momentum and F_σ has dimensions of force. If q_σ is dimensionless, as is the case for an angle, p_σ has dimensions of angular momentum (ML^2/T) and F_σ has dimensions of torque (ML^2/T^2).

4.3.2 Hamilton's principle

The equations of motion of classical mechanics are embodied in a variational principle, called *Hamilton's principle.* Hamilton's principle states that the motion of a system is such that the *action functional*

$$S\big[q(t)\big] = \int_{t_1}^{t_2} dt\, L(q, \dot{q}, t) \tag{4.99}$$

is an extremum, *i.e.* $\delta S = 0$. Here, $q = \{q_1, \ldots, q_n\}$ is a complete set of *generalized coordinates* for our mechanical system, and

$$L = T - U \tag{4.100}$$

is the *Lagrangian*, where T is the kinetic energy and U is the potential energy. Setting the first variation of the action to zero gives the Euler-Lagrange equations,

$$\frac{d}{dt}\overbrace{\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right)}^{\text{momentum } p_\sigma} = \overbrace{\frac{\partial L}{\partial q_\sigma}}^{\text{force } F_\sigma} \quad . \tag{4.101}$$

Thus, we have the familiar $\dot{p}_\sigma = F_\sigma$, also known as Newton's second law. Note, however, that the $\{q_\sigma\}$ are *generalized coordinates*, so p_σ may not have dimensions of momentum, nor F_σ of force. For example, if the generalized coordinate in question is an angle ϕ, then the corresponding generalized momentum is the angular momentum about the axis of ϕ's rotation, and the generalized force is the torque.

4.3.3 Invariance of the equations of motion

Suppose

$$\tilde{L}(q, \dot{q}, t) = L(q, \dot{q}, t) + \frac{d}{dt} G(q, t) \quad . \tag{4.102}$$

Then

$$\tilde{S}[q(t)] = S[q(t)] + G(q_b, t_b) - G(q_a, t_a) \quad . \tag{4.103}$$

Since the difference $\tilde{S} - S$ is a function only of the endpoint values $\{q_a, q_b\}$, their variations are identical: $\delta\tilde{S} = \delta S$. This means that L and $\tilde{L}$ result in the same equations of motion. Thus, the equations of motion are invariant under a shift of L by a total time derivative of a function of coordinates and time.

4.3.4 Remarks on the order of the equations of motion

The equations of motion are second order in time. This follows from the fact that $L = L(q, \dot{q}, t)$. Using the chain rule,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right) = \frac{\partial^2 L}{\partial \dot{q}_\sigma\, \partial \dot{q}_{\sigma'}}\, \ddot{q}_{\sigma'} + \frac{\partial^2 L}{\partial \dot{q}_\sigma\, \partial q_{\sigma'}}\, \dot{q}_{\sigma'} + \frac{\partial^2 L}{\partial \dot{q}_\sigma\, \partial t} \quad . \tag{4.104}$$

That the equations are second order in time can be regarded as an empirical fact. It follows, as we have just seen, from the fact that L depends on q and on $\dot{q}$, but on no higher time derivative terms. Suppose the Lagrangian did depend on the generalized accelerations $\ddot{q}$ as well. What would the equations of motion look like? Taking the variation of S,

$$\delta \int_{t_a}^{t_b} dt\, L(q,\dot{q},\ddot{q},t) = \left[\frac{\partial L}{\partial \dot{q}_\sigma}\, \delta q_\sigma + \frac{\partial L}{\partial \ddot{q}_\sigma}\, \delta \dot{q}_\sigma - \frac{d}{dt}\bigg(\frac{\partial L}{\partial \ddot{q}_\sigma}\bigg)\, \delta q_\sigma \right]_{t_a}^{t_b} + \int_{t_a}^{t_b} dt \left\{ \frac{\partial L}{\partial q_\sigma} - \frac{d}{dt}\bigg(\frac{\partial L}{\partial \dot{q}_\sigma}\bigg) + \frac{d^2}{dt^2}\bigg(\frac{\partial L}{\partial \ddot{q}_\sigma}\bigg) \right\} \delta q_\sigma \quad . \tag{4.105}$$

The boundary term vanishes if we require $\delta q_\sigma(t_a) = \delta q_\sigma(t_b) = \delta \dot{q}_\sigma(t_a) = \delta \dot{q}_\sigma(t_b) = 0 \;\forall\, \sigma$. The equations of motion would then be *fourth order* in time.

4.3.5 Lagrangian for a free particle

For a free particle, we can use Cartesian coordinates for each particle as our system of generalized coordinates. For a single particle, the Lagrangian $L(\boldsymbol{x}, \boldsymbol{v}, t)$ must be a function solely of $\boldsymbol{v}^2$. This is because homogeneity with respect to space and time preclude any dependence of L on $\boldsymbol{x}$ or on t, and isotropy of space means L must depend on $\boldsymbol{v}^2$. We next invoke Galilean relativity, which says that the equations of motion are invariant under transformation to a reference frame moving with constant velocity. Let $\boldsymbol{V}$ be the velocity of the new reference frame $\mathcal{K}'$ relative to our initial reference frame $\mathcal{K}$. Then $\boldsymbol{x}' = \boldsymbol{x} - \boldsymbol{V} t$, and $\boldsymbol{v}' = \boldsymbol{v} - \boldsymbol{V}$. In order that the equations of motion be invariant under the change in reference frame, we demand

$$L'(\boldsymbol{v}') = L(\boldsymbol{v}) + \frac{d}{dt}\, G(\boldsymbol{x}, t) \quad . \tag{4.106}$$

The only possibility is $L = \frac{1}{2} m \boldsymbol{v}^2$, where the constant m is the mass of the particle. Note:

$$L' = \tfrac{1}{2} m (\boldsymbol{v} - \boldsymbol{V})^2 = \tfrac{1}{2} m \boldsymbol{v}^2 + \frac{d}{dt}\Big(\tfrac{1}{2} m \boldsymbol{V}^2\, t - m \boldsymbol{V} \cdot \boldsymbol{x}\Big) = L + \frac{dG}{dt} \quad . \tag{4.107}$$

For N interacting particles,

$$L = \tfrac{1}{2} \sum_{a=1}^{N} m_a \left(\frac{d\boldsymbol{x}_a}{dt} \right)^2 - U\big(\{\boldsymbol{x}_a\}, \{\dot{\boldsymbol{x}}_a\}\big) \quad . \tag{4.108}$$

Here, U is the *potential energy*. Generally, U is of the form

$$U = \sum_a U_1(\boldsymbol{x}_a) + \sum_{a<a'} v(\boldsymbol{x}_a - \boldsymbol{x}_{a'}) \quad , \tag{4.109}$$

however, as we shall see, velocity-dependent potentials appear in the case of charged particles interacting with electromagnetic fields. In general, though,

$$L = T - U \quad , \tag{4.110}$$

where T is the kinetic energy, and U is the potential energy.

4.3.6 Conserved quantities

A conserved quantity $\Lambda(q,\dot{q},t)$ is one which does not vary throughout the motion of the system. This means

$$\frac{d\Lambda}{dt}\bigg|_{q=q(t)} = 0 \quad . \tag{4.111}$$

We shall discuss conserved quantities in detail in the chapter on Noether's Theorem, which follows.

Momentum conservation

The simplest case of a conserved quantity occurs when the Lagrangian does not explicitly depend on one or more of the generalized coordinates, *i.e.* when

$$F_\sigma = \frac{\partial L}{\partial q_\sigma} = 0 \quad . \tag{4.112}$$

We then say that L is *cyclic* in the coordinate q_σ. In this case, the Euler-Lagrange equations $\dot{p}_\sigma = F_\sigma$ say that the conjugate momentum p_σ is conserved. Consider, for example, the motion of a particle of mass m near the surface of the earth. Let (x,y) be coordinates parallel to the surface and z the height. We then have

$$\begin{aligned} T &= \tfrac{1}{2}m\big(\dot{x}^2+\dot{y}^2+\dot{z}^2\big) \\ U &= mgz \\ L = T-U &= \tfrac{1}{2}m\big(\dot{x}^2+\dot{y}^2+\dot{z}^2\big) - mgz \quad . \end{aligned} \tag{4.113}$$

Since

$$F_x = \frac{\partial L}{\partial x} = 0 \quad \text{and} \quad F_y = \frac{\partial L}{\partial y} = 0 \quad , \tag{4.114}$$

we have that p_x and p_y are conserved, with

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \quad , \quad p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y} \quad . \tag{4.115}$$

These first order equations can be integrated to yield

$$x(t) = x(0) + \frac{p_x}{m}\,t \quad , \quad y(t) = y(0) + \frac{p_y}{m}\,t \quad . \tag{4.116}$$

The z equation is of course

$$\dot{p}_z = m\ddot{z} = -mg = F_z \quad , \tag{4.117}$$

with solution

$$z(t) = z(0) + \dot{z}(0)\,t - \tfrac{1}{2}gt^2 \quad . \tag{4.118}$$

As another example, consider a particle moving in the (x,y) plane under the influence of a potential $U(x,y) = U\big(\sqrt{x^2+y^2}\big)$ which depends only on the particle's distance from the origin $\rho = \sqrt{x^2+y^2}$. The Lagrangian, expressed in two-dimensional polar coordinates (ρ,ϕ), is

$$L = \tfrac{1}{2}m\big(\dot{\rho}^2 + \rho^2\dot{\phi}^2\big) - U(\rho) \quad . \tag{4.119}$$

We see that L is cyclic in the angle ϕ, hence

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m\rho^2 \dot{\phi} \tag{4.120}$$

is conserved. p_ϕ is the angular momentum of the particle about the $\hat{\boldsymbol{z}}$ axis. In the language of the calculus of variations, momentum conservation is what follows when the integrand of a functional is independent of the *independent variable*.

Energy conservation

When the integrand of a functional is independent of the *dependent* variable, another conservation law follows. For Lagrangian mechanics, consider the expression

$$H(q,\dot{q},t) = \sum_{\sigma=1}^{n} p_\sigma\, \dot{q}_\sigma - L \quad . \tag{4.121}$$

Now we take the total time derivative of H:

$$\frac{dH}{dt} = \sum_{\sigma=1}^{n} \left\{ p_\sigma\, \ddot{q}_\sigma + \dot{p}_\sigma\, \dot{q}_\sigma - \frac{\partial L}{\partial q_\sigma}\, \dot{q}_\sigma - \frac{\partial L}{\partial \dot{q}_\sigma}\, \ddot{q}_\sigma \right\} - \frac{\partial L}{\partial t} \quad . \tag{4.122}$$

We evaluate $\dot{H}$ along the motion of the system, which entails that the terms in the curly brackets above cancel for each σ:

$$p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma} \quad , \quad \dot{p}_\sigma = \frac{\partial L}{\partial q_\sigma} \quad . \tag{4.123}$$

Thus, we find

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t} \quad , \tag{4.124}$$

which means that H is conserved *whenever the Lagrangian contains no explicit time dependence*. For a Lagrangian of the form

$$L = \sum_a \tfrac{1}{2}\, m_a \dot{\boldsymbol{r}}_a^2 - U(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N) \quad , \tag{4.125}$$

we have that $\boldsymbol{p}_a = m_a\, \dot{\boldsymbol{r}}_a$, and

$$H = T + U = \sum_a \tfrac{1}{2}\, m_a \dot{\boldsymbol{r}}_a^2 + U(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N) \quad . \tag{4.126}$$

However, it is not always the case that $H = T + U$ is the total energy, as we shall further on below.

4.3.7 Choosing generalized coordinates

Any choice of generalized coordinates will yield an equivalent set of equations of motion. However, some choices result in an apparently simpler set than others. This is often true with respect to the form of the potential energy. Additionally, certain constraints that may be present are more amenable to treatment using a particular set of generalized coordinates.

The kinetic energy T is always simple to write in Cartesian coordinates, and it is good practice, at least when one is first learning the method, to write T in Cartesian coordinates and then convert to generalized coordinates. In Cartesian coordinates, the kinetic energy of a single particle of mass m is

$$T = \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) \quad . \tag{4.127}$$

If the motion is two-dimensional, and confined to the plane $z = \text{const.}$, one of course has $T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right)$.

Two other commonly used coordinate systems are the cylindrical and spherical systems. In cylindrical coordinates (ρ, ϕ, z), ρ is the radial coordinate in the (x, y) plane and ϕ is the azimuthal angle:

$$x = \rho\,\cos\phi \quad , \quad y = \rho\,\sin\phi \quad , \quad \dot{x} = \cos\phi\,\dot{\rho} - \rho\sin\phi\,\dot{\phi} \qquad \dot{y} = \sin\phi\,\dot{\rho} + \rho\cos\phi\,\dot{\phi} \quad , \tag{4.128}$$

and the third, orthogonal coordinate is of course z. The kinetic energy is

$$T = \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{x}^2\right) = \tfrac{1}{2}m\left(\dot{\rho}^2 + \rho^2\,\dot{\phi}^2 + \dot{z}^2\right) \quad . \tag{4.129}$$

When the motion is confined to a plane with $z = \text{const.}$, this coordinate system is often referred to as 'two-dimensional polar' coordinates.

In spherical coordinates (r, θ, ϕ), r is the radius, θ is the polar angle, and ϕ is the azimuthal angle. On the globe, θ would be the 'colatitude', which is $\theta = \frac{\pi}{2} - \lambda$, where λ is the latitude. *I.e.* $\theta = 0$ at the north pole. In spherical polar coordinates,

$$x = r\,\sin\theta\,\cos\phi \qquad \dot{x} = \sin\theta\,\cos\phi\,\dot{r} + r\,\cos\theta\,\cos\phi\,\dot{\theta} - r\,\sin\theta\sin\phi\,\dot{\phi} \tag{4.130}$$

$$y = r\,\sin\theta\,\sin\phi \qquad \dot{y} = \sin\theta\,\sin\phi\,\dot{r} + r\,\cos\theta\,\sin\phi\,\dot{\theta} + r\,\sin\theta\cos\phi\,\dot{\phi} \tag{4.131}$$

$$z = r\,\cos\theta \qquad \dot{z} = \cos\theta\,\dot{r} - r\,\sin\theta\,\dot{\theta} \quad . \tag{4.132}$$

The kinetic energy is

$$T = \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) = \tfrac{1}{2}m\left(\dot{r}^2 + r^2\,\dot{\theta}^2 + r^2\,\sin^2\!\theta\,\dot{\phi}^2\right) \quad . \tag{4.133}$$

4.4 How to Solve Mechanics Problems

Here are some simple steps you can follow toward obtaining the equations of motion:

1. Choose a set of generalized coordinates $\{q_1, \ldots, q_n\}$.

2. Find the kinetic energy $T(q, \dot{q}, t)$, the potential energy $U(q, t)$, and the Lagrangian $L(q, \dot{q}, t) = T - U$. It is often helpful to first write the kinetic energy in Cartesian coordinates for each particle before converting to generalized coordinates.

3. Find the canonical momenta $p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma}$ and the generalized forces $F_\sigma = \frac{\partial L}{\partial q_\sigma}$.

4. Identify any conserved quantities (more about this later).
5. Evaluate the time derivatives $\dot{p}_\sigma$ and write the equations of motion $\dot{p}_\sigma = F_\sigma$. Be careful to differentiate properly, using the chain rule and the Leibniz rule where appropriate.
6. Integrate the equations of motion to obtain $\{q_\sigma(t)\}$ (easier said than done).

We not consider several examples:

4.4.1 One-dimensional motion

For a one-dimensional mechanical system with potential energy $U(x)$,

$$L = T - U = \tfrac{1}{2}m\dot{x}^2 - U(x) \quad . \tag{4.134}$$

The canonical momentum is

$$p = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \tag{4.135}$$

and the equation of motion is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x} \quad \Rightarrow \quad m\ddot{x} = -U'(x) \quad , \tag{4.136}$$

which is of course $F = ma$.

Note that we can multiply the equation of motion by $\dot{x}$ to get

$$0 = \dot{x}\,\Big\{m\ddot{x} + U'(x)\Big\} = \frac{d}{dt}\Big\{\tfrac{1}{2}m\dot{x}^2 + U(x)\Big\} = \frac{dE}{dt} \quad , \tag{4.137}$$

where $E = T + U$.

4.4.2 Central force in two dimensions

Consider next a particle of mass m moving in two dimensions under the influence of a potential $U(\rho)$ which is a function of the distance from the origin $\rho = \sqrt{x^2 + y^2}$. Clearly cylindrical (2d polar) coordinates are called for:

$$L = \tfrac{1}{2}m\left(\dot{\rho}^2 + \rho^2\,\dot{\phi}^2\right) - U(\rho) \quad . \tag{4.138}$$

The equations of motion are

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\rho}}\right) = \frac{\partial L}{\partial \rho} \quad &\Rightarrow \quad m\ddot{\rho} = m\rho\,\dot{\phi}^2 - U'(\rho) \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{\partial L}{\partial \phi} \quad &\Rightarrow \quad \frac{d}{dt}\left(m\rho^2\,\dot{\phi}\right) = 0 \quad . \end{aligned} \tag{4.139}$$

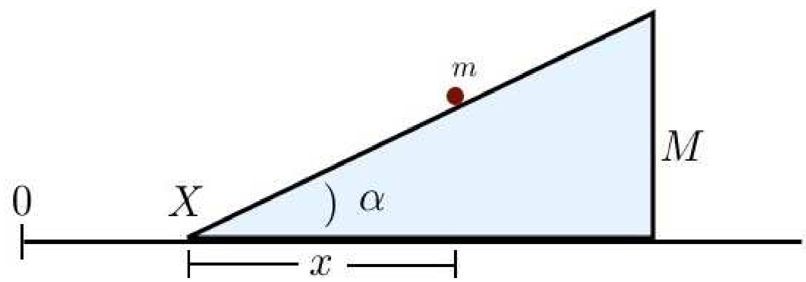

Figure 4.7: A wedge of mass M and opening angle α slides frictionlessly along a horizontal surface, while a small object of mass m slides frictionlessly along the wedge.

Note that the canonical momentum conjugate to ϕ, which is to say the angular momentum, is conserved:

$$p_\phi = m\rho^2\,\dot\phi = \text{const.} \tag{4.140}$$

We can use this to eliminate $\dot\phi$ from the first Euler-Lagrange equation, obtaining

$$m\ddot\rho = \frac{p_\phi^2}{m\rho^3} - U'(\rho) \quad . \tag{4.141}$$

We can also write the total energy as

$$\begin{aligned} E &= \tfrac{1}{2}m\left(\dot\rho^2 + \rho^2\,\dot\phi^2\right) + U(\rho) \\ &= \tfrac{1}{2}m\dot\rho^2 + \frac{p_\phi^2}{2m\rho^2} + U(\rho) \quad , \end{aligned} \tag{4.142}$$

from which it may be shown that E is also a constant:

$$\frac{dE}{dt} = \Big(m\ddot\rho - \frac{p_\phi^2}{m\rho^3} + U'(\rho)\Big)\,\dot\rho = 0 \quad . \tag{4.143}$$

We shall discuss this case in much greater detail in the coming weeks.

4.4.3 A sliding point mass on a sliding wedge

Consider the situation depicted in fig. 4.7, in which a point object of mass m slides frictionlessly along a wedge of opening angle α. The wedge itself slides frictionlessly along a horizontal surface, and its mass is M. We choose as generalized coordinates the horizontal position X of the left corner of the wedge, and the horizontal distance x from the left corner to the sliding point mass. The vertical coordinate of the sliding mass is then $y = x\,\tan\alpha$, where the horizontal surface lies at $y = 0$. With these generalized coordinates, the kinetic energy is

$$\begin{aligned} T &= \tfrac{1}{2}M\dot X^2 + \tfrac{1}{2}m\,(\dot X + \dot x)^2 + \tfrac{1}{2}m\dot y^2 \\ &= \tfrac{1}{2}(M+m)\dot X^2 + m\dot X\dot x + \tfrac{1}{2}m\,(1+\tan^2\!\alpha)\,\dot x^2 \quad . \end{aligned} \tag{4.144}$$

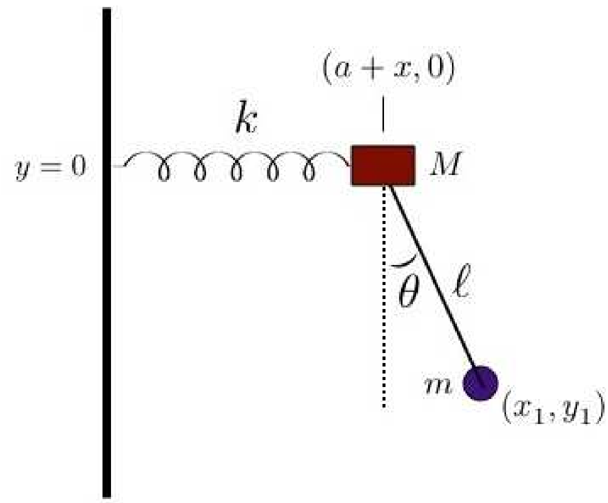

Figure 4.8: The spring–pendulum system.

The potential energy is simply

$$U = mgy = mg\,x\,\tan\alpha \quad . \tag{4.145}$$

Thus, the Lagrangian is

$$L = \tfrac{1}{2}(M+m)\dot{X}^2 + m\dot{X}\dot{x} + \tfrac{1}{2}m\,(1+\tan^2\!\alpha)\,\dot{x}^2 - mg\,x\,\tan\alpha \quad , \tag{4.146}$$

and the equations of motion are

$$\begin{aligned} \frac{d}{dt}\bigg(\frac{\partial L}{\partial \dot{X}}\bigg) &= \frac{\partial L}{\partial X} \quad\Rightarrow\quad (M+m)\ddot{X} + m\ddot{x} = 0 \\ \frac{d}{dt}\bigg(\frac{\partial L}{\partial \dot{x}}\bigg) &= \frac{\partial L}{\partial x} \quad\Rightarrow\quad m\ddot{X} + m\,(1+\tan^2\!\alpha)\,\ddot{x} = -mg\tan\alpha \quad . \end{aligned} \tag{4.147}$$

At this point we can use the first of these equations to write

$$\ddot{X} = -\frac{m}{M+m}\,\ddot{x} \quad . \tag{4.148}$$

Substituting this into the second equation, we obtain the constant accelerations

$$\ddot{x} = -\frac{(M+m)g\,\sin\alpha\,\cos\alpha}{M+m\sin^2\!\alpha} \quad , \quad \ddot{X} = \frac{mg\,\sin\alpha\,\cos\alpha}{M+m\sin^2\!\alpha} \quad . \tag{4.149}$$

4.4.4 A pendulum attached to a mass on a spring

Consider next the system depicted in fig. 4.8 in which a mass M moves horizontally while attached to a spring of spring constant k. Hanging from this mass is a pendulum of arm length ℓ and bob mass m.

A convenient set of generalized coordinates is (x, θ), where x is the displacement of the mass M relative to the equilibrium extension a of the spring, and θ is the angle the pendulum arm makes with respect to the vertical. Let the Cartesian coordinates of the pendulum bob be (x_1, y_1). Then

$$x_1 = a + x + \ell\,\sin\theta \quad , \quad y_1 = -l\cos\theta \quad . \tag{4.150}$$

The kinetic energy is

$$\begin{aligned} T &= \tfrac{1}{2}M\dot{x}^2 + \tfrac{1}{2}m\,(\dot{x}^2 + \dot{y}^2) \\ &= \tfrac{1}{2}M\dot{x}^2 + \tfrac{1}{2}m\left[(\dot{x} + \ell\cos\theta\,\dot{\theta})^2 + (\ell\sin\theta\,\dot{\theta})^2\right] \\ &= \tfrac{1}{2}(M+m)\,\dot{x}^2 + \tfrac{1}{2}m\ell^2\,\dot{\theta}^2 + m\ell\cos\theta\,\dot{x}\,\dot{\theta} \quad , \end{aligned} \tag{4.151}$$

and the potential energy is

$$\begin{aligned} U &= \tfrac{1}{2}kx^2 + mgy_1 \\ &= \tfrac{1}{2}kx^2 - mg\ell\cos\theta \quad . \end{aligned} \tag{4.152}$$

Thus,

$$L = \tfrac{1}{2}(M+m)\,\dot{x}^2 + \tfrac{1}{2}m\ell^2\,\dot{\theta}^2 + m\ell\cos\theta\,\dot{x}\,\dot{\theta} - \tfrac{1}{2}kx^2 + mg\ell\cos\theta \quad . \tag{4.153}$$

The canonical momenta are

$$\begin{aligned} p_x &= \frac{\partial L}{\partial \dot{x}} = (M+m)\,\dot{x} + m\ell\cos\theta\,\dot{\theta} \\ p_\theta &= \frac{\partial L}{\partial \dot{\theta}} = m\ell\cos\theta\,\dot{x} + m\ell^2\,\dot{\theta} \quad , \end{aligned} \tag{4.154}$$

and the canonical forces are

$$\begin{aligned} F_x &= \frac{\partial L}{\partial x} = -kx \\ F_\theta &= \frac{\partial L}{\partial \theta} = -m\ell\sin\theta\,\dot{x}\,\dot{\theta} - mg\ell\,\sin\theta \quad . \end{aligned} \tag{4.155}$$

The equations of motion then yield

$$\begin{aligned} (M+m)\,\ddot{x} + m\ell\,\cos\theta\,\ddot{\theta} - m\ell\,\sin\theta\,\dot{\theta}^2 &= -kx \\ m\ell\,\cos\theta\,\ddot{x} + m\ell^2\,\ddot{\theta} &= -mg\ell\,\sin\theta \quad . \end{aligned} \tag{4.156}$$

Small Oscillations : If we assume both x and θ are small, we may write $\sin\theta \approx \theta$ and $\cos\theta \approx 1$, in which case the equations of motion may be linearized to

$$\begin{aligned} (M+m)\,\ddot{x} + m\ell\,\ddot{\theta} + kx &= 0 \\ m\ell\,\ddot{x} + m\ell^2\,\ddot{\theta} + mg\ell\,\theta &= 0 \quad . \end{aligned} \tag{4.157}$$

If we define

$$u \equiv \frac{x}{\ell} \quad , \quad \alpha \equiv \frac{m}{M} \quad , \quad \omega_0^2 \equiv \frac{k}{M} \quad , \quad \omega_1^2 \equiv \frac{g}{\ell} \quad , \tag{4.158}$$

then may be linearized to

$$\begin{aligned} (1+\alpha)\,\ddot{u} + \alpha\,\ddot{\theta} + \omega_0^2\,u &= 0 \\ \ddot{u} + \ddot{\theta} + \omega_1^2\,\theta &= 0 \quad . \end{aligned} \tag{4.159}$$

We can solve by writing

$$\begin{pmatrix} u(t) \\ \theta(t) \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} e^{-i\omega t} \quad , \tag{4.160}$$

in which case

$$\begin{pmatrix} \omega_0^2 - (1+\alpha)\,\omega^2 & -\alpha\,\omega^2 \\ -\omega^2 & \omega_1^2 - \omega^2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad . \tag{4.161}$$

In order to have a nontrivial solution (*i.e.* without $a = b = 0$), the determinant of the above 2×2 matrix must vanish. This gives a condition on ω^2, with solutions

$$\omega_\pm^2 = \tfrac{1}{2}\big[\,\omega_0^2 + (1+\alpha)\,\omega_1^2\,\big] \pm \tfrac{1}{2}\sqrt{\big[\,\omega_0^2 - (1+\alpha)\omega_1^2\,\big]^2 + 4\alpha\,\omega_0^2\,\omega_1^2} \quad . \tag{4.162}$$

4.4.5 The double pendulum

As yet another example of the generalized coordinate approach to Lagrangian dynamics, consider the double pendulum system, sketched in fig. 4.9. We choose as generalized coordinates the two angles θ_1 and θ_2. In order to evaluate the Lagrangian, we must obtain the kinetic and potential energies in terms of the generalized coordinates $\{\theta_1, \theta_2\}$ and their corresponding velocities $\{\dot{\theta}_1, \dot{\theta}_2\}$.

In Cartesian coordinates,

$$\begin{aligned} T &= \tfrac{1}{2}m_1\,(\dot{x}_1^2 + \dot{y}_1^2) + \tfrac{1}{2}m_2\,(\dot{x}_2^2 + \dot{y}_2^2) \\ U &= m_1\,g\,y_1 + m_2\,g\,y_2 \quad . \end{aligned} \tag{4.163}$$

We therefore express the Cartesian coordinates $\{x_1, y_1, x_2, y_2\}$ in terms of the generalized coordinates $\{\theta_1, \theta_2\}$:

$$\begin{aligned} x_1 &= +\ell_1\,\sin\theta_1 \quad , & x_2 &= \ell_1\,\sin\theta_1 + \ell_2\,\sin\theta_2 \\ y_1 &= -\ell_1\,\cos\theta_1 \quad , & y_2 &= -\ell_1\,\cos\theta_1 - \ell_2\,\cos\theta_2 \quad . \end{aligned} \tag{4.164}$$

Thus, the velocities are

$$\begin{aligned} \dot{x}_1 &= \ell_1\,\dot{\theta}_1\,\cos\theta_1 \quad , & \dot{x}_2 &= \ell_1\,\dot{\theta}_1\,\cos\theta_1 + \ell_2\,\dot{\theta}_2\,\cos\theta_2 \\ \dot{y}_1 &= \ell_1\,\dot{\theta}_1\,\sin\theta_1 \quad , & \dot{y}_2 &= \ell_1\,\dot{\theta}_1\,\sin\theta_1 + \ell_2\,\dot{\theta}_2\,\sin\theta_2 \quad . \end{aligned} \tag{4.165}$$

Thus,

$$\begin{aligned} T &= \tfrac{1}{2}m_1\,\ell_1^2\,\dot{\theta}_1^2 + \tfrac{1}{2}m_2\Big\{\ell_1^2\,\dot{\theta}_1^2 + 2\ell_1\,\ell_2\,\cos(\theta_1 - \theta_2)\,\dot{\theta}_1\,\dot{\theta}_2 + \ell_2^2\,\dot{\theta}_2^2\Big\} \\ U &= -m_1\,g\,\ell_1\,\cos\theta_1 - m_2\,g\,\ell_1\,\cos\theta_1 - m_2\,g\,\ell_2\,\cos\theta_2 \quad , \end{aligned} \tag{4.166}$$

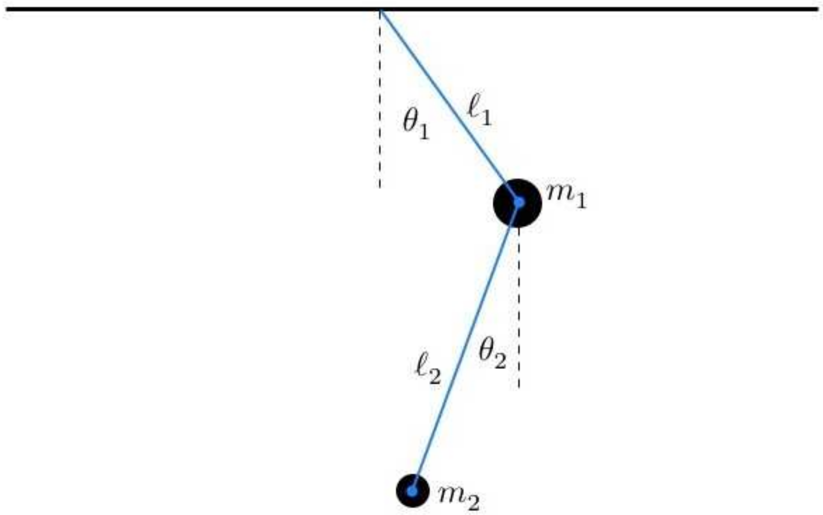

Figure 4.9: The double pendulum, with generalized coordinates θ_1 and θ_2. All motion is confined to a single plane.

and

$$\begin{aligned} L = T - U = \tfrac{1}{2}(m_1 + m_2)\,\ell_1^2\,\dot\theta_1^2 + m_2\,\ell_1\,\ell_2\,\cos(\theta_1 - \theta_2)\,\dot\theta_1\,\dot\theta_2 + \tfrac{1}{2} m_2\,\ell_2^2\,\dot\theta_2^2 \\ + (m_1 + m_2)\,g\,\ell_1\,\cos\theta_1 + m_2\,g\,\ell_2\,\cos\theta_2 \quad . \end{aligned} \tag{4.167}$$

The generalized (canonical) momenta are

$$\begin{aligned} p_1 &= \frac{\partial L}{\partial\dot\theta_1} = (m_1 + m_2)\,\ell_1^2\,\dot\theta_1 + m_2\,\ell_1\,\ell_2\,\cos(\theta_1 - \theta_2)\,\dot\theta_2 \\ p_2 &= \frac{\partial L}{\partial\dot\theta_2} = m_2\,\ell_1\,\ell_2\,\cos(\theta_1 - \theta_2)\,\dot\theta_1 + m_2\,\ell_2^2\,\dot\theta_2 \quad , \end{aligned} \tag{4.168}$$

and the equations of motion are

$$\begin{aligned} \dot p_1 &= (m_1 + m_2)\,\ell_1^2\,\ddot\theta_1 + m_2\,\ell_1\,\ell_2\,\cos(\theta_1 - \theta_2)\,\ddot\theta_2 - m_2\,\ell_1\,\ell_2\,\sin(\theta_1 - \theta_2)\,(\dot\theta_1 - \dot\theta_2)\,\dot\theta_2 \\ &= -(m_1 + m_2)\,g\,\ell_1\,\sin\theta_1 - m_2\,\ell_1\,\ell_2\,\sin(\theta_1 - \theta_2)\,\dot\theta_1\,\dot\theta_2 = \frac{\partial L}{\partial\theta_1} \end{aligned} \tag{4.169}$$

and

$$\begin{aligned} \dot p_2 &= m_2\,\ell_1\,\ell_2\,\cos(\theta_1 - \theta_2)\,\ddot\theta_1 - m_2\,\ell_1\,\ell_2\,\sin(\theta_1 - \theta_2)\,(\dot\theta_1 - \dot\theta_2)\,\dot\theta_1 + m_2\,\ell_2^2\,\ddot\theta_2 \\ &= -m_2\,g\,\ell_2\,\sin\theta_2 + m_2\,\ell_1\,\ell_2\,\sin(\theta_1 - \theta_2)\,\dot\theta_1\,\dot\theta_2 = \frac{\partial L}{\partial\theta_2} \quad . \end{aligned} \tag{4.170}$$

We therefore find

$$\begin{aligned} \ell_1\,\ddot\theta_1 + \frac{m_2\,\ell_2}{m_1+m_2}\,\cos(\theta_1-\theta_2)\,\ddot\theta_2 + \frac{m_2\,\ell_2}{m_1+m_2}\,\sin(\theta_1-\theta_2)\,\dot\theta_2^2 + g\,\sin\theta_1 = 0 \\ \ell_1\,\cos(\theta_1-\theta_2)\,\ddot\theta_1 + \ell_2\,\ddot\theta_2 - \ell_1\,\sin(\theta_1-\theta_2)\,\dot\theta_1^2 + g\,\sin\theta_2 = 0 \quad . \end{aligned} \tag{4.171}$$

Small Oscillations : The equations of motion are coupled, nonlinear second order ODEs. When the system is close to equilibrium, the amplitudes of the motion are small, and we may expand in powers of the θ_1 and θ_2. The linearized equations of motion are then

$$\begin{aligned} \ddot\theta_1 + \alpha\beta\,\ddot\theta_2 + \omega_0^2\,\theta_1 = 0 \\ \ddot\theta_1 + \beta\,\ddot\theta_2 + \omega_0^2\,\theta_2 = 0 \quad , \end{aligned} \tag{4.172}$$

where we have defined

$$\alpha \equiv \frac{m_2}{m_1+m_2} \quad , \quad \beta \equiv \frac{\ell_2}{\ell_1} \quad , \quad \omega_0^2 \equiv \frac{g}{\ell_1} \quad . \tag{4.173}$$

We can solve this coupled set of equations by a nifty trick. Let's take a linear combination of the first equation plus an undetermined coefficient, r, times the second:

$$(1+r)\,\ddot\theta_1 + (\alpha+r)\beta\,\ddot\theta_2 + \omega_0^2\,(\theta_1 + r\,\theta_2) = 0 \quad . \tag{4.174}$$

We now demand that the ratio of the coefficients of θ_2 and θ_1 is the same as the ratio of the coefficients of $\ddot\theta_2$ and $\ddot\theta_1$:

$$\frac{(\alpha+r)\,\beta}{1+r} = r \quad \Rightarrow \quad r_\pm = \tfrac{1}{2}(\beta-1) \pm \tfrac{1}{2}\sqrt{(1-\beta)^2 + 4\alpha\beta} \tag{4.175}$$

When $r = r_\pm$, the equation of motion may be written

$$\frac{d^2}{dt^2}\,\big(\theta_1 + r_\pm\,\theta_2\big) = -\frac{\omega_0^2}{1+r_\pm}\,\big(\theta_1 + r_\pm\,\theta_2\big) \tag{4.176}$$

and defining the (unnormalized) *normal modes* $\xi_\pm \equiv \big(\theta_1 + r_\pm\,\theta_2\big)$ we find $\ddot\xi_\pm + \omega_\pm^2\,\xi_\pm = 0$ with

$$\omega_\pm = \frac{\omega_0}{\sqrt{1+r_\pm}} \quad . \tag{4.177}$$

Thus, by switching to the normal coordinates, we have decoupled the equations of motion, and identified the two *normal frequencies of oscillation*. We shall have much more to say about small oscillations in chapter 6.

For example, with $\ell_1 = \ell_2 = \ell$ and $m_1 = m_2 = m$, we have $\alpha = \frac{1}{2}$, and $\beta = 1$, in which case

$$r_\pm = \pm\tfrac{1}{\sqrt{2}} \quad , \quad \xi_\pm = \theta_1 \pm \tfrac{1}{\sqrt{2}}\,\theta_2 \quad , \quad \omega_\pm = \sqrt{2\mp\sqrt{2}}\,\sqrt{\frac{g}{\ell}} \quad . \tag{4.178}$$

Note that the oscillation frequency for the 'in-phase' mode ξ_+ is low, and that for the 'out of phase' mode ξ_- is high.

4.4.6 The thingy

Four massless rods of length L are hinged together at their ends to form a rhombus. A particle of mass M is attached to each vertex. The opposite corners are joined by springs of spring constant k. In the square configuration, the strings are unstretched. The motion is confined to a plane, and the particles move only along the diagonals of the rhombus. Introduce suitable generalized coordinates and find the Lagrangian of the system. Deduce the equations of motion and find the frequency of small oscillations about equilibrium.

Solution

The rhombus is depicted in figure 4.10. Let a be the equilibrium length of the springs; clearly $b = \frac{1}{\sqrt{2}}\,a$. Let ϕ be half of one of the opening angles, as shown. Then the masses are located at $(\pm X, 0)$ and $(0, \pm Y)$, with $X = \frac{1}{\sqrt{2}}\,a\cos\phi$ and $Y = \frac{1}{\sqrt{2}}\,a\sin\phi$. The spring extensions are $\delta X = 2X - a$ and $\delta Y = 2Y - a$. The

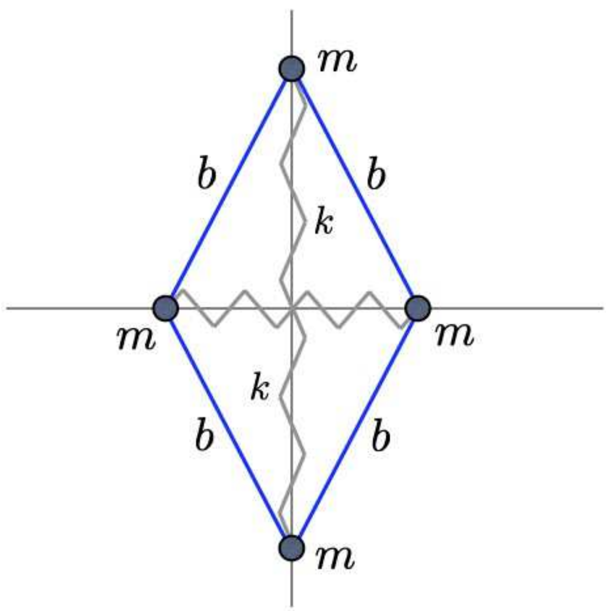

Figure 4.10: The thingy: a rhombus with opening angles 2ϕ and $\pi - 2\phi$.

kinetic and potential energies are therefore

$$T = M\big(\dot{X}^2 + \dot{Y}^2\big) = \tfrac{1}{2}Ma^2\,\dot{\phi}^2 \tag{4.179}$$

and

$$\begin{aligned} U &= \tfrac{1}{2}k\big(\delta X\big)^2 + \tfrac{1}{2}k\big(\delta Y\big)^2 \\ &= \tfrac{1}{2}ka^2\left\{\big(\sqrt{2}\,\cos\phi - 1\big)^2 + \big(\sqrt{2}\,\sin\phi - 1\big)^2\right\} \\ &= \tfrac{1}{2}ka^2\left\{3 - 2\sqrt{2}(\cos\phi + \sin\phi)\right\} \quad . \end{aligned} \tag{4.180}$$

Note that minimizing $U(\phi)$ gives $\sin\phi = \cos\phi$, *i.e.* $\phi_{\rm eq} = \frac{\pi}{4}$. The Lagrangian is then

$$L = T - U = \tfrac{1}{2} M a^2\, \dot\phi^2 + \sqrt{2}\, k a^2 \big(\cos\phi + \sin\phi\big) + \text{const.} \tag{4.181}$$

The equations of motion are

$$\frac{d}{dt}\frac{\partial L}{\partial\dot\phi} = \frac{\partial L}{\partial\phi} \qquad \Rightarrow \qquad M a^2\, \ddot\phi = \sqrt{2}\, k a^2\, (\cos\phi - \sin\phi) \tag{4.182}$$

It's always smart to expand about equilibrium, so let's write $\phi = \frac{\pi}{4} + \delta$, which leads to

$$\ddot\delta + \omega_0^2 \, \sin\delta = 0 \quad , \tag{4.183}$$

with $\omega_0 = \sqrt{2k/M}$. This is the equation of a pendulum! Linearizing gives $\ddot\delta + \omega_0^2\,\delta = 0$, so the small oscillation frequency is just ω_0.

4.5 The Virial Theorem

The virial theorem is a statement about the time-averaged motion of a mechanical system. Define the *virial*,

$$G(q,p) = \sum_\sigma p_\sigma \, q_\sigma \quad . \tag{4.184}$$

Then

$$\begin{aligned} \frac{dG}{dt} &= \sum_\sigma \big(\dot p_\sigma\, q_\sigma + p_\sigma\, \dot q_\sigma\big) \\ &= \sum_\sigma q_\sigma\, F_\sigma + \sum_\sigma \dot q_\sigma\, \frac{\partial L}{\partial \dot q_\sigma} \quad . \end{aligned} \tag{4.185}$$

Now suppose that $T = \frac{1}{2}\sum_{\sigma,\sigma'} {\rm T}_{\sigma\sigma'}(q)\, \dot q_\sigma\, \dot q_{\sigma'}$ is homogeneous of degree $k=2$ in $\dot q$, and that U is homogeneous of degree zero in $\dot q$. Then

$$\sum_\sigma \dot q_\sigma\, \frac{\partial L}{\partial \dot q_\sigma} = \sum_\sigma \dot q_\sigma\, \frac{\partial T}{\partial \dot q_\sigma} = 2\,T, \tag{4.186}$$

which follows from Euler's theorem on homogeneous functions.

Now consider the time average of $\dot G$ over a period τ:

$$\Big\langle \frac{dG}{dt} \Big\rangle = \frac{1}{\tau}\int\limits_0^\tau \! dt\, \frac{dG}{dt} = \frac{1}{\tau}\Big[G(\tau) - G(0)\Big] \quad . \tag{4.187}$$

If $G(t)$ is bounded, then in the limit $\tau\to\infty$ we must have $\langle \dot G\rangle = 0$. Any bounded motion, such as the orbit of the earth around the Sun, will result in $\langle \dot G\rangle_{\tau\to\infty} = 0$. But then

$$\Big\langle \frac{dG}{dt} \Big\rangle = 2\,\langle T\rangle + \Big\langle \sum_\sigma q_\sigma\, F_\sigma \Big\rangle = 0 \quad , \tag{4.188}$$

which implies

$$\langle T \rangle = -\frac{1}{2} \Big\langle \sum_\sigma q_\sigma \, F_\sigma \Big\rangle = \Big\langle \frac{1}{2} \sum_i \boldsymbol{x}_i \cdot \boldsymbol{\nabla}_i \, U\big(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N\big) \Big\rangle = \tfrac{1}{2} \, k \, \langle U \rangle \quad , \tag{4.189}$$

where above equation pertains to homogeneous potentials of degree k in the Cartesian coordinates[3]. Finally, since $T + U = E$ is conserved, we have

$$\langle T \rangle = \frac{k\,E}{k+2} \quad , \quad \langle U \rangle = \frac{2\,E}{k+2} \quad . \tag{4.190}$$

4.6 Noether's Theorem

4.6.1 Continuous symmetry implies conserved charges

Consider a particle moving in two dimensions under the influence of an external potential $U(r)$. The potential is a function only of the magnitude of the vector $\boldsymbol{r}$. The Lagrangian is then

$$L = T - U = \tfrac{1}{2} m \left(\dot{r}^2 + r^2 \, \dot{\phi}^2 \right) - U(r) \quad , \tag{4.191}$$

where we have chosen generalized coordinates (r, ϕ). The momentum conjugate to ϕ is $p_\phi = m \, r^2 \, \dot{\phi}$. The generalized force F_ϕ clearly vanishes, since L does not depend on the coordinate ϕ. (One says that L is 'cyclic' in ϕ.) Thus, although $r = r(t)$ and $\phi = \phi(t)$ will in general be time-dependent, the combination $p_\phi = m \, r^2 \, \dot{\phi}$ is constant. This is the conserved angular momentum about the $\hat{\boldsymbol{z}}$ axis.

If instead the particle moved in a potential $U(y)$, independent of x, then writing

$$L = \tfrac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 \right) - U(y) \quad , \tag{4.192}$$

we have that the momentum $p_x = \partial L / \partial \dot{x} = m\dot{x}$ is conserved, because the generalized force $F_x = \partial L / \partial x = 0$ vanishes. This situation pertains in a uniform gravitational field, with $U(x, y) = mgy$, independent of x. The horizontal component of momentum is conserved.

In general, whenever the system exhibits a *continuous symmetry*, there is an associated *conserved charge*. (The terminology 'charge' is from field theory.) Indeed, this is a rigorous result, known as *Noether's Theorem*. Consider a one-parameter family of transformations,

$$q_\sigma \longrightarrow \tilde{q}_\sigma(q, \zeta) \quad , \tag{4.193}$$

where ζ is the continuous parameter. Suppose further (without loss of generality) that at $\zeta = 0$ this transformation is the identity, *i.e.* $\tilde{q}_\sigma(q, 0) = q_\sigma$. The transformation may be nonlinear in the generalized

[3]Note that $-\sum_\sigma q_\sigma F_\sigma = -\sum_\sigma q_\sigma \left(\partial L / \partial q_\sigma \right) \neq \sum_\sigma q_\sigma \left(\partial U / \partial q_\sigma \right)$ in general because $T = \frac{1}{2} \sum_{\sigma\sigma'} \mathrm{T}_{\sigma\sigma'}(q) \, \dot{q}_\sigma \, \dot{q}_{\sigma'}$, and so the inequality holds whenever $\mathrm{T}_{\sigma\sigma'}(q)$ is q-dependent. In a Cartesian coordinate system, however, we have $T = \frac{1}{2} \sum_j m_j \dot{\boldsymbol{x}}_j^2$ and therefore eqn. 4.189 holds

coordinates. Suppose further that the Lagrangian L is invariant under the replacement $q \to \tilde{q}$. Then we must have

$$\begin{aligned}0 = \frac{d}{d\zeta}\bigg|_{\zeta=0} L(\tilde{q},\dot{\tilde{q}},t) &= \sum_{\sigma=1}^{n} \left\{ \frac{\partial L}{\partial q_\sigma}\,\frac{\partial \tilde{q}_\sigma}{\partial \zeta}\bigg|_{\zeta=0} + \frac{\partial L}{\partial \dot{q}_\sigma}\,\frac{\partial \dot{\tilde{q}}_\sigma}{\partial \zeta}\bigg|_{\zeta=0} \right\} \\ &= \sum_{\sigma=1}^{n} \left\{ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right) \frac{\partial \tilde{q}_\sigma}{\partial \zeta}\bigg|_{\zeta=0} + \frac{\partial L}{\partial \dot{q}_\sigma}\,\frac{d}{dt}\left(\frac{\partial \tilde{q}_\sigma}{\partial \zeta}\right)_{\zeta=0} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\,\frac{\partial \tilde{q}_\sigma}{\partial \zeta}\right)_{\zeta=0} \right\} \quad .\end{aligned} \tag{4.194}$$

Thus, there is an associated conserved charge

$$\Lambda = \sum_{\sigma=1}^{n} \frac{\partial L}{\partial \dot{q}_\sigma}\,\frac{\partial \tilde{q}_\sigma}{\partial \zeta}\bigg|_{\zeta=0} \quad . \tag{4.195}$$

4.6.2 Examples of one-parameter families of transformations

Consider the Lagrangian

$$L = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) - U\big(\sqrt{x^2+y^2}\big) \quad . \tag{4.196}$$

In two-dimensional polar coordinates, we have

$$L = \tfrac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\phi}^2) - U(\rho) \quad , \tag{4.197}$$

and we may now define

$$\tilde{\rho}(\zeta) = \rho \qquad , \qquad \tilde{\phi}(\zeta) = \phi + \zeta \quad . \tag{4.198}$$

Note that $\tilde{\rho}(0) = \rho$ and $\tilde{\phi}(0) = \phi$, *i.e.* the transformation is the identity when $\zeta = 0$. We now have

$$\Lambda = \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma}\,\frac{\partial \tilde{q}_\sigma}{\partial \zeta}\bigg|_{\zeta=0} = \frac{\partial L}{\partial \dot{\rho}}\,\frac{\partial \tilde{\rho}}{\partial \zeta}\bigg|_{\zeta=0} + \frac{\partial L}{\partial \dot{\phi}}\,\frac{\partial \tilde{\phi}}{\partial \zeta}\bigg|_{\zeta=0} = m\rho^2\dot{\phi} \quad . \tag{4.199}$$

Another way to derive the same result which is somewhat instructive is to work out the transformation in Cartesian coordinates. We then have

$$\begin{aligned}\tilde{x}(\zeta) &= x\,\cos\zeta - y\,\sin\zeta \\ \tilde{y}(\zeta) &= x\,\sin\zeta + y\,\cos\zeta \quad .\end{aligned} \tag{4.200}$$

Thus,

$$\frac{\partial \tilde{x}}{\partial \zeta} = -\tilde{y} \quad , \quad \frac{\partial \tilde{y}}{\partial \zeta} = \tilde{x} \tag{4.201}$$

and

$$\Lambda = \frac{\partial L}{\partial \dot{x}}\,\frac{\partial \tilde{x}}{\partial \zeta}\bigg|_{\zeta=0} + \frac{\partial L}{\partial \dot{y}}\,\frac{\partial \tilde{y}}{\partial \zeta}\bigg|_{\zeta=0} = m(x\dot{y} - y\dot{x}) \quad . \tag{4.202}$$

But

$$m(x\dot{y} - y\dot{x}) = m\hat{\boldsymbol{z}} \cdot \boldsymbol{\rho} \times \dot{\boldsymbol{\rho}} = m\rho^2\dot{\phi} \quad . \tag{4.203}$$

As another example, consider the potential

$$U(\rho,\phi,z)=V(\rho,a\phi+z) \quad , \tag{4.204}$$

where (ρ,ϕ,z) are cylindrical coordinates for a particle of mass m, and where a is a constant with dimensions of length. The Lagrangian is

$$\tfrac{1}{2}m\big(\dot\rho^2+\rho^2\dot\phi^2+\dot z^2\big)-V(\rho,a\phi+z) \quad . \tag{4.205}$$

This model possesses a helical symmetry, with a one-parameter family

$$\tilde\rho(\zeta)=\rho \qquad , \qquad \tilde\phi(\zeta)=\phi+\zeta \qquad , \qquad \tilde z(\zeta)=z-\zeta a \quad . \tag{4.206}$$

Note that

$$a\tilde\phi+\tilde z=a\phi+z \quad , \tag{4.207}$$

so the potential energy, and the Lagrangian as well, is invariant under this one-parameter family of transformations. The conserved charge for this symmetry is

$$\Lambda=\frac{\partial L}{\partial\dot\rho}\,\frac{\partial\tilde\rho}{\partial\zeta}\bigg|_{\zeta=0}+\frac{\partial L}{\partial\dot\phi}\,\frac{\partial\tilde\phi}{\partial\zeta}\bigg|_{\zeta=0}+\frac{\partial L}{\partial\dot z}\,\frac{\partial\tilde z}{\partial\zeta}\bigg|_{\zeta=0}=m\rho^2\dot\phi-ma\dot z \quad . \tag{4.208}$$

We can check explicitly that Λ is conserved, using the equations of motion

$$\begin{aligned}\frac{d}{dt}\bigg(\frac{\partial L}{\partial\dot\phi}\bigg)&=\frac{d}{dt}\big(m\rho^2\dot\phi\big)=\frac{\partial L}{\partial\phi}=-a\frac{\partial V}{\partial z}\\ \frac{d}{dt}\bigg(\frac{\partial L}{\partial\dot z}\bigg)&=\frac{d}{dt}(m\dot z)=\frac{\partial L}{\partial z}=-\frac{\partial V}{\partial z} \quad .\end{aligned} \tag{4.209}$$

Thus,

$$\dot\Lambda=\frac{d}{dt}\big(m\rho^2\dot\phi\big)-a\,\frac{d}{dt}(m\dot z)=0 \quad . \tag{4.210}$$

4.6.3 Conservation of linear and angular momentum

Suppose that the Lagrangian of a mechanical system is invariant under a uniform translation of all particles in the $\hat{\boldsymbol n}$ direction. Then our one-parameter family of transformations is given by

$$\tilde{\boldsymbol x}_a=\boldsymbol x_a+\zeta\,\hat{\boldsymbol n} \quad , \tag{4.211}$$

and the associated conserved Noether charge is

$$\Lambda=\sum_a\frac{\partial L}{\partial\dot{\boldsymbol x}_a}\cdot\hat{\boldsymbol n}=\hat{\boldsymbol n}\cdot\boldsymbol P \quad , \tag{4.212}$$

where $\boldsymbol P=\sum_a\boldsymbol p_a$ is the *total momentum* of the system.

If the Lagrangian of a mechanical system is invariant under rotations about an axis $\hat{\boldsymbol{n}}$, then

$$\begin{aligned}\tilde{\boldsymbol{x}}_a &= R(\zeta,\hat{\boldsymbol{n}})\,\boldsymbol{x}_a \\ &= \boldsymbol{x}_a + \zeta\,\hat{\boldsymbol{n}}\times\boldsymbol{x}_a + \mathcal{O}(\zeta^2) \quad ,\end{aligned} \tag{4.213}$$

where we have expanded the rotation matrix $R(\zeta,\hat{\boldsymbol{n}})$ in powers of ζ. The conserved Noether charge associated with this symmetry is

$$\Lambda = \sum_a \frac{\partial L}{\partial\dot{\boldsymbol{x}}_a}\cdot\hat{\boldsymbol{n}}\times\boldsymbol{x}_a = \hat{\boldsymbol{n}}\cdot\sum_a \boldsymbol{x}_a\times\boldsymbol{p}_a = \hat{\boldsymbol{n}}\cdot\boldsymbol{L} \quad , \tag{4.214}$$

where $\boldsymbol{L}$ is the *total angular momentum* of the system.

4.6.4 Invariance of L *vs.* invariance of S

Observant readers might object that demanding invariance of L is too strict. We should instead be demanding invariance of the action S[4]. Suppose S is invariant under

$$t \to \tilde{t}(q,t,\zeta) \qquad , \qquad q_\sigma(t) \to \tilde{q}_\sigma(q,t,\zeta) \quad . \tag{4.215}$$

Then invariance of S means

$$S = \int\limits_{t_a}^{t_b}\! dt\, L(q,\dot{q},t) = \int\limits_{\tilde{t}_a}^{\tilde{t}_b}\! dt\, L(\tilde{q},\dot{\tilde{q}},t) \quad . \tag{4.216}$$

Note that t is a dummy variable of integration, so it doesn't matter whether we call it t or $\tilde{t}$. The endpoints of the integral, however, do change under the transformation. Now consider an infinitesimal transformation, for which $\delta t = \tilde{t} - t$ and $\delta q = \tilde{q}(\tilde{t}) - q(t)$ are both small. Thus,

$$S = \int\limits_{t_a}^{t_b}\! dt\, L(q,\dot{q},t) = \int\limits_{t_a+\delta t_a}^{t_b+\delta t_b}\! dt\,\Big\{ L(q,\dot{q},t) + \frac{\partial L}{\partial q_\sigma}\,\bar{\delta} q_\sigma + \frac{\partial L}{\partial \dot{q}_\sigma}\,\bar{\delta}\dot{q}_\sigma + \ldots\Big\} \quad , \tag{4.217}$$

where

$$\begin{aligned}\bar{\delta} q_\sigma(t) &\equiv \tilde{q}_\sigma(t) - q_\sigma(t) \\ &= \tilde{q}_\sigma\big(\tilde{t}\big) - \tilde{q}_\sigma\big(\tilde{t}\big) + \tilde{q}_\sigma(t) - q_\sigma(t) \\ &= \delta q_\sigma - \dot{q}_\sigma\,\delta t + \mathcal{O}(\delta q\,\delta t)\end{aligned} \tag{4.218}$$

Subtracting eqn. 4.217 from eqn. 4.216, we obtain

$$\begin{aligned} 0 &= L_b\,\delta t_b - L_a\,\delta t_a + \frac{\partial L}{\partial\dot{q}_\sigma}\bigg|_b \bar{\delta} q_{\sigma,b} - \frac{\partial L}{\partial\dot{q}_\sigma}\bigg|_a \bar{\delta} q_{\sigma,a} + \int\limits_{t_a+\delta t_a}^{t_b+\delta t_b}\! dt\,\bigg\{\frac{\partial L}{\partial q_\sigma} - \frac{d}{dt}\bigg(\frac{\partial L}{\partial\dot{q}_\sigma}\bigg)\bigg\}\,\bar{\delta} q_\sigma(t) \\ &= \int\limits_{t_a}^{t_b}\! dt\,\frac{d}{dt}\bigg\{\bigg(L - \frac{\partial L}{\partial\dot{q}_\sigma}\,\dot{q}_\sigma\bigg)\,\delta t + \frac{\partial L}{\partial\dot{q}_\sigma}\,\delta q_\sigma\bigg\} \quad ,\end{aligned} \tag{4.219}$$

[4]Indeed, we should be demanding that S only change by a function of the endpoint values.

where $L_{a,b}$ is $L(q,\dot{q},t)$ evaluated at $t = t_{a,b}$. Thus, if $\zeta \equiv \delta\zeta$ is infinitesimal, and

$$\delta t = A(q,t)\,\delta\zeta \qquad , \qquad \delta q_\sigma = B_\sigma(q,t)\,\delta\zeta \quad , \tag{4.220}$$

then the conserved charge is

$$\begin{aligned} \Lambda &= \left(L - \frac{\partial L}{\partial \dot{q}_\sigma}\,\dot{q}_\sigma\right) A(q,t) + \frac{\partial L}{\partial \dot{q}_\sigma}\,B_\sigma(q,t) \\ &= -\,H(q,p,t)\,A(q,t) + p_\sigma\,B_\sigma(q,t) \quad . \end{aligned} \tag{4.221}$$

Thus, when $A = 0$, we recover our earlier results, obtained by assuming invariance of L. Note that conservation of H follows from time translation invariance: $t \to t + \zeta$, for which $A = 1$ and $B_\sigma = 0$. Here we have written

$$H = p_\sigma\,\dot{q}_\sigma - L \quad , \tag{4.222}$$

and expressed it in terms of the momenta p_σ, the coordinates q_σ, and time t. H is called the *Hamiltonian*.

4.7 The Hamiltonian

4.7.1 From Lagrangian to Hamiltonian

The Lagrangian is a function of generalized coordinates, velocities, and time. The canonical momentum conjugate to the generalized coordinate q_σ is

$$p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma} \quad . \tag{4.223}$$

The Hamiltonian is a function of coordinates, *momenta*, and time. It is defined as the Legendre transform[5] of L:

$$H(q,p,t) = \sum_\sigma p_\sigma\,\dot{q}_\sigma - L \quad . \tag{4.224}$$

Let's examine the differential of H:

$$\begin{aligned} dH &= \sum_\sigma \left(\dot{q}_\sigma\,dp_\sigma + p_\sigma\,d\dot{q}_\sigma - \frac{\partial L}{\partial q_\sigma}\,dq_\sigma - \frac{\partial L}{\partial \dot{q}_\sigma}\,d\dot{q}_\sigma\right) - \frac{\partial L}{\partial t}\,dt \\ &= \sum_\sigma \left(\dot{q}_\sigma\,dp_\sigma - \frac{\partial L}{\partial q_\sigma}\,dq_\sigma\right) - \frac{\partial L}{\partial t}\,dt \quad , \end{aligned} \tag{4.225}$$

where we have invoked the definition of p_σ to cancel the coefficients of $d\dot{q}_\sigma$. Since $\dot{p}_\sigma = \partial L/\partial q_\sigma$, we have *Hamilton's equations of motion*,

$$\dot{q}_\sigma = \frac{\partial H}{\partial p_\sigma} \quad , \quad \dot{p}_\sigma = -\frac{\partial H}{\partial q_\sigma} \quad . \tag{4.226}$$

[5]See the appendix in §4.12 for more on Legendre transformations.

Thus, we can write

$$dH = \sum_\sigma \Big(\dot q_\sigma \, dp_\sigma - \dot p_\sigma \, dq_\sigma \Big) - \frac{\partial L}{\partial t}\, dt \quad . \tag{4.227}$$

Dividing by dt, we obtain

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t} \quad , \tag{4.228}$$

which says that the Hamiltonian is *conserved* (*i.e.* it does not change with time) whenever there is no *explicit* time dependence to L.

Example #1 : For a simple $d = 1$ system with $L = \frac{1}{2} m \dot x^2 - U(x)$, we have $p = m\dot x$ and

$$H = p\,\dot x - L = \tfrac{1}{2} m\dot x^2 + U(x) = \frac{p^2}{2m} + U(x) \quad . \tag{4.229}$$

Example #2 : Consider now the mass point – wedge system analyzed above, with

$$L = \tfrac{1}{2}(M+m)\dot X^2 + m\dot X\dot x + \tfrac{1}{2} m \sec^2\!\alpha\, \dot x^2 - mg\tan(\alpha)\, x \quad , \tag{4.230}$$

The canonical momenta are

$$\begin{aligned} P &= \frac{\partial L}{\partial \dot X} = (M+m)\,\dot X + m\dot x \\ p &= \frac{\partial L}{\partial \dot x} = m\dot X + m\sec^2\!\alpha\,\dot x \quad . \end{aligned} \tag{4.231}$$

The Hamiltonian is given by

$$\begin{aligned} H &= P\,\dot X + p\,\dot x - L \\ &= \tfrac{1}{2}(M+m)\dot X^2 + m\dot X\dot x + \tfrac{1}{2} m\sec^2\!\alpha\,\dot x^2 + mg\tan(\alpha)\,x \\ &= \frac{1}{2}\begin{pmatrix} \dot X & \dot x \end{pmatrix} \overbrace{\begin{pmatrix} M+m & m \\ m & m\sec^2\!\alpha \end{pmatrix}}^{A} \begin{pmatrix} \dot X \\ \dot x \end{pmatrix} + mg\tan(\alpha)\,x \quad . \end{aligned} \tag{4.232}$$

However, this is not quite H, since $H = H(X, x, P, p, t)$ must be expressed in terms of the coordinates and the *momenta* and not the coordinates and velocities. So we must eliminate $\dot X$ and $\dot x$ in favor of P and p. We do this by inverting the relations

$$\begin{pmatrix} P \\ p \end{pmatrix} = \begin{pmatrix} M+m & m \\ m & m\,\sec^2\!\alpha \end{pmatrix} \begin{pmatrix} \dot X \\ \dot x \end{pmatrix} = A \begin{pmatrix} \dot X \\ \dot x \end{pmatrix} \tag{4.233}$$

to obtain

$$\begin{pmatrix} \dot X \\ \dot x \end{pmatrix} = \frac{1}{M\sec^2\!\alpha + m\tan^2\!\alpha} \begin{pmatrix} \sec^2\!\alpha & -1 \\ -1 & \frac{M}{m} + 1 \end{pmatrix} \begin{pmatrix} P \\ p \end{pmatrix} = A^{-1} \begin{pmatrix} P \\ p \end{pmatrix} \quad . \tag{4.234}$$

Substituting into 4.232, we obtain

$$\begin{aligned} H &= \frac{1}{2}\begin{pmatrix} P & p \end{pmatrix} A^{-1} \begin{pmatrix} P \\ p \end{pmatrix} + mg\tan(\alpha)\, x \\ &= \frac{P^2}{2(M+m\sin^2\!\alpha)} - \frac{2Pp\cos^2\!\alpha}{2(M+m\sin^2\!\alpha)} + \frac{\left(\frac{M}{m}+1\right)p^2\cos^2\!\alpha}{2(M+m\sin^2\!\alpha)} + mg\tan(\alpha)x \quad . \end{aligned} \tag{4.235}$$

Notice that $\dot{P} = 0$ since $\frac{\partial L}{\partial X} = 0$. P is the total horizontal momentum of the system (wedge plus particle) and it is conserved. As a sanity check, consider the limit $M \to \infty$ with P and p finite. The wedge then has infinite inertia and remains fixed. Accordingly, we find

$$H(X,x,P,p,t)\big|_{M\to\infty} = \frac{p^2\cos^2\!\alpha}{2m} + mg\tan(\alpha)\, x \quad . \tag{4.236}$$

4.7.2 Is $H = T + U$?

The most general form of the kinetic energy is

$$\begin{aligned} T &= T_2 + T_1 + T_0 \\ &= \tfrac{1}{2} T_2^{\sigma\sigma'}(q,t)\,\dot{q}_\sigma\,\dot{q}_{\sigma'} + T_1^\sigma(q,t)\,\dot{q}_\sigma + T_0(q,t) \quad , \end{aligned} \tag{4.237}$$

where $T_n(q,\dot{q},t)$ is homogeneous of degree n in the velocities[6]. We assume a potential energy of the form

$$\begin{aligned} U &= U_1 + U_0 \\ &= U_1^\sigma(q,t)\,\dot{q}_\sigma + U_0(q,t) \quad , \end{aligned} \tag{4.238}$$

which allows for velocity-dependent forces, as we have with charged particles moving in an electromagnetic field. The Lagrangian is then

$$L = T - U = \tfrac{1}{2} T_2^{\sigma\sigma'}(q,t)\,\dot{q}_\sigma\,\dot{q}_{\sigma'} + T_1^\sigma(q,t)\,\dot{q}_\sigma + T_0(q,t) - U_1^\sigma(q,t)\,\dot{q}_\sigma - U_0(q,t) \quad . \tag{4.239}$$

The canonical momentum conjugate to q_σ is

$$p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma} = T_2^{\sigma\sigma'}\,\dot{q}_{\sigma'} + T_1^\sigma(q,t) - U_1^\sigma(q,t) \tag{4.240}$$

which is inverted to give

$$\dot{q}_\sigma = \left(T_2^{-1}\right)^{\sigma\sigma'}\left(p_{\sigma'} - T_1^{\sigma'} + U_1^{\sigma'}\right) \quad . \tag{4.241}$$

The Hamiltonian is then

$$\begin{aligned} H &= p_\sigma\,\dot{q}_\sigma - L \\ &= \tfrac{1}{2}\left(T_2^{-1}\right)^{\sigma\sigma'}\left(p_\sigma - T_1^\sigma + U_1^\sigma\right)\left(p_{\sigma'} - T_1^{\sigma'} + U_1^{\sigma'}\right) - T_0 + U_0 \\ &= T_2 - T_0 + U_0 \quad . \end{aligned} \tag{4.242}$$

[6]A homogeneous function of degree k satisfies $f(\lambda x_1,\ldots,\lambda x_n) = \lambda^k f(x_1,\ldots,x_n)$. It is then easy to prove *Euler's theorem*, $\sum_{i=1}^n x_i \frac{\partial f}{\partial x_i} = kf$.

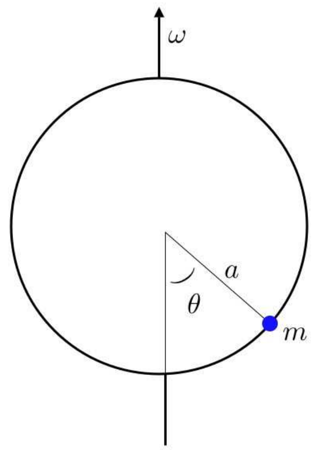

Figure 4.11: A bead of mass m on a rotating hoop of radius a.

If T_0, T_1, and U_1 vanish, *i.e.* if $T(q, \dot{q}, t)$ is a homogeneous function of degree two in the generalized velocities, and $U(q, t)$ is velocity-independent, then $H = T + U$. But if T_0 or T_1 is nonzero, or the potential is velocity-dependent, then $H \neq T + U$.

4.7.3 Example: a bead on a rotating hoop

Consider a bead of mass m constrained to move along a hoop of radius a. The hoop is further constrained to rotate with angular velocity $\dot{\phi} = \omega$ about the $\hat{\boldsymbol{z}}$-axis, as shown in fig. 4.11.

The most convenient set of generalized coordinates is spherical polar (r, θ, ϕ), in which case

$$\begin{aligned} T &= \tfrac{1}{2} m \big(\dot{r}^2 + r^2\,\dot{\theta}^2 + r^2 \sin^2\theta\,\dot{\phi}^2\big) \\ &= \tfrac{1}{2} m a^2 \big(\dot{\theta}^2 + \omega^2 \sin^2\theta\big) \quad . \end{aligned} \tag{4.243}$$

Thus, $T_2 = \frac{1}{2} m a^2 \dot{\theta}^2$ and $T_0 = \frac{1}{2} m a^2 \omega^2 \sin^2\theta$. The potential energy is $U(\theta) = mga(1 - \cos\theta)$. The momentum conjugate to θ is $p_\theta = m a^2 \dot{\theta}$, and thus

$$\begin{aligned} H(\theta, p_\theta) &= T_2 - T_0 + U \\ &= \tfrac{1}{2} m a^2 \dot{\theta}^2 - \tfrac{1}{2} m a^2 \omega^2 \sin^2\theta + mga(1 - \cos\theta) \\ &= \frac{p_\theta^2}{2ma^2} - \tfrac{1}{2} m a^2 \omega^2 \sin^2\theta + mga(1 - \cos\theta) \quad . \end{aligned} \tag{4.244}$$

For this problem, we can define the *effective potential*

$$\begin{aligned} U_{\text{eff}}(\theta) \equiv U - T_0 &= mga(1 - \cos\theta) - \tfrac{1}{2} m a^2 \omega^2 \sin^2\theta \\ &= mga\Big(1 - \cos\theta - \frac{\omega^2}{2\omega_0^2}\,\sin^2\theta\Big) \quad , \end{aligned} \tag{4.245}$$

where $\omega_0^2 \equiv g/a$. The Lagrangian may then be written

$$L = \tfrac{1}{2} m a^2 \dot{\theta}^2 - U_{\text{eff}}(\theta) \quad , \tag{4.246}$$

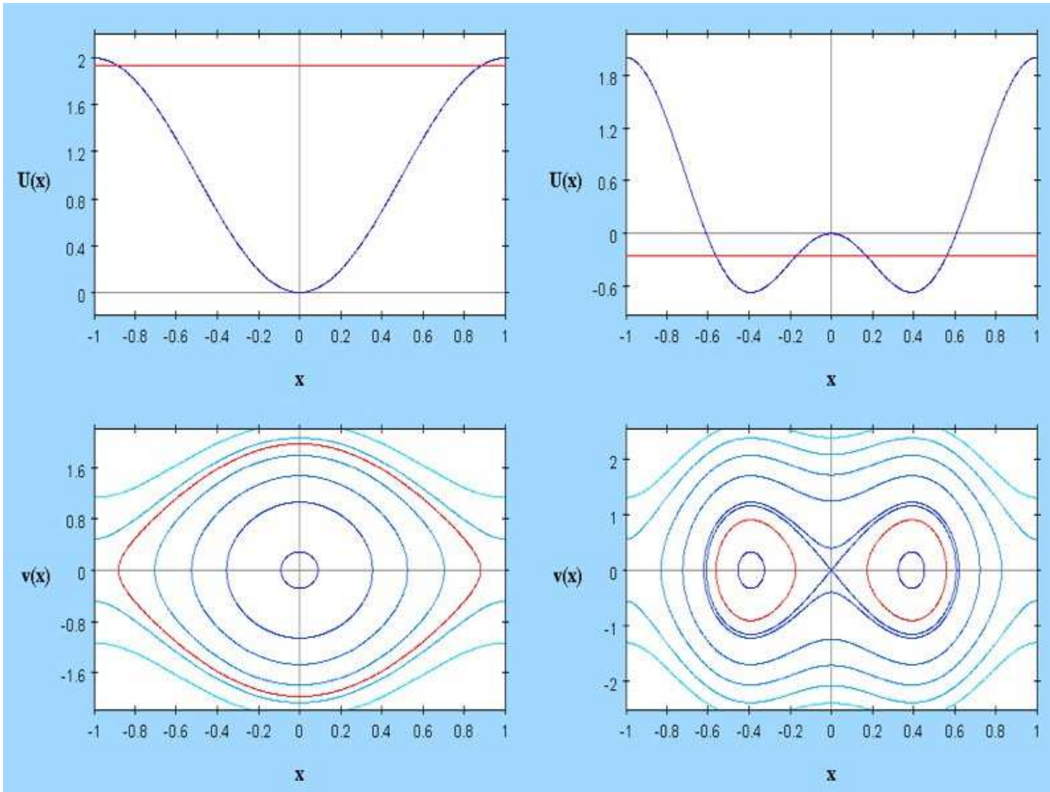

Figure 4.12: The effective potential $U_{\rm eff}(\theta) = mga\big[1 - \cos\theta - \frac{\omega^2}{2\omega_0^2}\,\sin^2\theta\big]$. (The dimensionless potential $\tilde U_{\rm eff}(x) = U_{\rm eff}/mga$ is shown, where $x = \theta/\pi$.) Left panels: $\omega = \frac{1}{2}\sqrt{3}\,\omega_0$. Right panels: $\omega = \sqrt{3}\,\omega_0$.

and thus the equations of motion are

$$ma^2\ddot\theta = -\frac{\partial U_{\rm eff}}{\partial\theta} \quad . \tag{4.247}$$

Equilibrium is achieved when $U'_{\rm eff}(\theta) = 0$, which gives

$$\frac{\partial U_{\rm eff}}{\partial\theta} = mga\sin\theta\left\{1 - \frac{\omega^2}{\omega_0^2}\cos\theta\right\} = 0 \quad , \tag{4.248}$$

i.e. $\theta^* = 0$, $\theta^* = \pi$, or $\theta^* = \pm\cos^{-1}(\omega_0^2/\omega^2)$, where the last pair of equilibria are present only for $\omega^2 > \omega_0^2$. The stability of these equilibria is assessed by examining the sign of $U''_{\rm eff}(\theta^*)$. We have

$$U''_{\rm eff}(\theta) = mga\left\{\cos\theta - \frac{\omega^2}{\omega_0^2}\big(2\cos^2\theta - 1\big)\right\} \quad . \tag{4.249}$$

Thus,

$$U''_{\text{eff}}(\theta^*) = \begin{cases} +mga\left(1 - \frac{\omega^2}{\omega_0^2}\right) & \text{at } \theta^* = 0 \\ \\ -mga\left(1 + \frac{\omega^2}{\omega_0^2}\right) & \text{at } \theta^* = \pi \\ \\ +mga\left(\frac{\omega^2}{\omega_0^2} - \frac{\omega_0^2}{\omega^2}\right) & \text{at } \theta^* = \pm\cos^{-1}\left(\frac{\omega_0^2}{\omega^2}\right) \quad . \end{cases} \tag{4.250}$$

Thus, $\theta^* = 0$ is stable for $\omega^2 < \omega_0^2$ but becomes unstable when the rotation frequency ω is sufficiently large, *i.e.* when $\omega^2 > \omega_0^2$. In this regime, there are two new equilibria, at $\theta^* = \pm\cos^{-1}(\omega_0^2/\omega^2)$, which are both stable. The equilibrium at $\theta^* = \pi$ is always unstable, independent of the value of ω. The situation is depicted in fig. 4.12.

4.7.4 Charged particle in an electromagnetic field

Consider next the case of a charged particle moving in the presence of an electromagnetic field. The particle's potential energy is

$$U(\boldsymbol{r}, \dot{\boldsymbol{r}}) = q\,\phi(\boldsymbol{r}, t) - \frac{q}{c}\,\boldsymbol{A}(\boldsymbol{r}, t) \cdot \dot{\boldsymbol{r}} \quad , \tag{4.251}$$

which *is* velocity-dependent. The kinetic energy is $T = \frac{1}{2}m\dot{\boldsymbol{r}}^2$, as usual, and $L = T - U$. Here $\phi(\boldsymbol{r}, t)$ is the scalar potential and $\boldsymbol{A}(\boldsymbol{r}, t)$ the vector potential. The electric and magnetic fields are given by

$$\boldsymbol{E} = -\boldsymbol{\nabla}\phi - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} \quad , \quad \boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A} \quad . \tag{4.252}$$

The canonical momenta and forces are

$$\begin{aligned} \boldsymbol{p} &= \frac{\partial L}{\partial \dot{\boldsymbol{r}}} = m\dot{\boldsymbol{r}} + \frac{q}{c}\,\boldsymbol{A} \\ \boldsymbol{F} &= \frac{\partial L}{\partial \boldsymbol{r}} = -q\boldsymbol{\nabla}\phi + \frac{q}{c}\,\boldsymbol{\nabla}\big(\boldsymbol{A} \cdot \dot{\boldsymbol{r}}\big) \quad . \end{aligned} \tag{4.253}$$

The Euler-Lagrange equations are

$$\dot{\boldsymbol{p}} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{r}}}\right) = \frac{\partial L}{\partial \boldsymbol{r}} = \boldsymbol{F} \tag{4.254}$$

which is to say

$$m\ddot{\boldsymbol{r}} + \frac{q}{c}\frac{d\boldsymbol{A}}{dt} = -q\,\boldsymbol{\nabla}\phi + \frac{q}{c}\,\boldsymbol{\nabla}(\boldsymbol{A} \cdot \dot{\boldsymbol{r}}) \quad , \tag{4.255}$$

or, in component notation,

$$m\ddot{x}_i + \frac{q}{c}\overbrace{\left(\frac{\partial A_i}{\partial x_j}\,\dot{x}_j + \frac{\partial A_i}{\partial t}\right)}^{dA_i/dt} = -q\,\frac{\partial \phi}{\partial x_i} + \frac{q}{c}\,\frac{\partial A_j}{\partial x_i}\,\dot{x}_j \quad . \tag{4.256}$$

Here we are using the Einstein convention of summing over repeated indices. Thus,

$$m\ddot{x}_i = -q\,\frac{\partial\phi}{\partial x_i} - \frac{q}{c}\,\frac{\partial A_i}{\partial t} + \frac{q}{c}\left(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}\right)\dot{x}_j \quad . \tag{4.257}$$

It is convenient to express the cross product in terms of the completely antisymmetric tensor of rank three, ϵ_{ijk} :

$$B_k = \epsilon_{klm}\,\frac{\partial A_m}{\partial x_l} \quad , \tag{4.258}$$

and using the result

$$\epsilon_{kij}\,\epsilon_{klm} = \delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl} \quad , \tag{4.259}$$

we have

$$\epsilon_{kij}\,B_k = \frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j} \quad . \tag{4.260}$$

and therefore

$$m\ddot{x}_i = -q\,\frac{\partial\phi}{\partial x_i} - \frac{q}{c}\,\frac{\partial A_i}{\partial t} + \frac{q}{c}\,\epsilon_{ijk}\,\dot{x}_j\,B_k \quad . \tag{4.261}$$

In vector notation, using $\epsilon_{ijk}\,v_j\,B_k = (\boldsymbol{v}\times\boldsymbol{B})_i$, we have

$$m\ddot{\boldsymbol{r}} = q\boldsymbol{E} + \frac{q}{c}\,\dot{\boldsymbol{r}}\times\boldsymbol{B} \quad , \tag{4.262}$$

which is, of course, the Lorentz force law.

Next, we compute the Hamiltonian:

$$\begin{aligned} H(\boldsymbol{r},\boldsymbol{p},t) &= \boldsymbol{p}\cdot\dot{\boldsymbol{r}} - L \\ &= m\dot{\boldsymbol{r}}^2 + \frac{q}{c}\,\boldsymbol{A}\cdot\dot{\boldsymbol{r}} - \tfrac{1}{2}m\dot{\boldsymbol{r}}^2 - \frac{q}{c}\,\boldsymbol{A}\cdot\dot{\boldsymbol{r}} + q\phi \\ &= \tfrac{1}{2}m\dot{\boldsymbol{r}}^2 + q\phi \\ &= \frac{1}{2m}\Big(\boldsymbol{p} - \frac{q}{c}\,\boldsymbol{A}(\boldsymbol{r},t)\Big)^2 + q\phi(\boldsymbol{r},t) \quad . \end{aligned} \tag{4.263}$$

If $\boldsymbol{A}$ and ϕ are time-independent, then $dH/dt = -\partial L/\partial t = 0$ and $H(\boldsymbol{r},\boldsymbol{p})$ is conserved.

4.8 Motion in Rapidly Oscillating Fields

4.8.1 Slow and fast dynamics

Consider a free particle moving under the influence of an oscillating force $F(t) = F_0\,\cos(\omega t)$. Newton's second law is then $m\ddot{q} = F\cos\omega t$, the solution to which is

$$q(t) = a + b\,t - \frac{F_0\,\cos\omega t}{m\omega^2} \quad . \tag{4.264}$$

where $q_{\rm h}(t) \equiv a + b\,t$ is the solution to the homogeneous (unforced) equation of motion. Note that the amplitude of the response $q - q_{\rm h}$ goes as ω^{-2} and is therefore small when ω is large.

Now consider a general $n = 1$ system, with

$$H(q,p,t) = H^0(q,p) + \widetilde{V}(q)\cos(\omega t) \quad , \tag{4.265}$$

where we will assume $\widetilde{V}(q)$ is small. We also assume that ω is much greater than any natural oscillation frequency associated with H_0. We separate the motion $q(t)$ and $p(t)$ into slow and fast components:

$$\begin{aligned} q(t) &= Q(t) + \zeta(t) \\ p(t) &= P(t) + \pi(t) \quad , \end{aligned} \tag{4.266}$$

where $\zeta(t)$ and $\pi(t)$ oscillate with the driving frequency ω. Since ζ and π will be small, we expand Hamilton's equations in these quantities:

$$\begin{aligned} \dot{Q} + \dot{\zeta} &= \frac{\partial H^0}{\partial P} + \frac{\partial^2 H^0}{\partial P^2}\,\pi + \frac{\partial^2 H^0}{\partial Q\,\partial P}\,\zeta + \frac{1}{2}\,\frac{\partial^3 H^0}{\partial Q^2\,\partial P}\,\zeta^2 + \frac{\partial^3 H^0}{\partial Q\,\partial P^2}\,\zeta\pi + \frac{1}{2}\,\frac{\partial^3 H^0}{\partial P^3}\,\pi^2 + \ldots \\ \dot{P} + \dot{\pi} &= -\frac{\partial H^0}{\partial Q} - \frac{\partial^2 H^0}{\partial Q^2}\,\zeta - \frac{\partial^2 H^0}{\partial Q\,\partial P}\,\pi - \frac{1}{2}\,\frac{\partial^3 H^0}{\partial Q^3}\,\zeta^2 - \frac{\partial^3 H^0}{\partial Q^2\,\partial P}\,\zeta\pi - \frac{1}{2}\,\frac{\partial^3 H^0}{\partial Q\,\partial P^2}\,\pi^2 \\ &\qquad - \frac{\partial\widetilde{V}}{\partial Q}\,\cos(\omega t) - \frac{\partial^2\widetilde{V}}{\partial Q^2}\,\zeta\cos(\omega t) - \ldots \quad . \end{aligned} \tag{4.267}$$

We now average over the fast degrees of freedom to obtain an equation of motion for the slow variables Q and P, which we here carry to lowest nontrivial order in averages of fluctuating quantities:

$$\begin{aligned} \dot{Q} &= H^0_P + \tfrac{1}{2} H^0_{QQP}\,\langle\zeta^2\rangle + H^0_{QPP}\,\langle\zeta\pi\rangle + \tfrac{1}{2} H^0_{PPP}\,\langle\pi^2\rangle \\ \dot{P} &= -H^0_Q - \tfrac{1}{2} H^0_{QQQ}\,\langle\zeta^2\rangle - H^0_{QQP}\,\langle\zeta\pi\rangle - \tfrac{1}{2} H^0_{QPP}\,\langle\pi^2\rangle - \widetilde{V}_{QQ}\,\langle\zeta\cos\omega t\rangle \quad , \end{aligned} \tag{4.268}$$

where we now adopt the shorthand notation $H^0_{QQP} = \frac{\partial^3 H^0}{\partial^2 Q\,\partial P}$, *etc.* The fast degrees of freedom obey

$$\begin{aligned} \dot{\zeta} &= H^0_{QP}\,\zeta + H^0_{PP}\,\pi \\ \dot{\pi} &= -H^0_{QQ}\,\zeta - H^0_{QP}\,\pi - \widetilde{V}_Q\,\cos(\omega t) \quad . \end{aligned} \tag{4.269}$$

We can solve these by replacing $\widetilde{V}_Q\,\cos\omega t$ with $\widetilde{V}_Q\,e^{-i\omega t}$, and writing $\zeta(t) = \zeta_0\,e^{-i\omega t}$ and $\pi(t) = \pi_0\,e^{-i\omega t}$, resulting in

$$\begin{pmatrix} H^0_{QP} + i\omega & H^0_{PP} \\ -H^0_{QQ} & -H^0_{QP} + i\omega \end{pmatrix} \begin{pmatrix} \zeta_0 \\ \pi_0 \end{pmatrix} = \begin{pmatrix} 0 \\ \widetilde{V}_Q \end{pmatrix} \quad . \tag{4.270}$$

We now invert the matrix to obtain ζ_0 and π_0, then take the real part, which yields

$$\begin{aligned} \zeta(t) &= \omega^{-2}\,H^0_{PP}\,\widetilde{V}_Q\,\cos\omega t + \mathcal{O}\big(\omega^{-4}\big) \\ \pi(t) &= -\omega^{-2}\,H^0_{QP}\,\widetilde{V}_Q\,\cos\omega t - \omega^{-1}\,\widetilde{V}_Q\,\sin\omega t + \mathcal{O}\big(\omega^{-3}\big) \quad . \end{aligned} \tag{4.271}$$

Invoking $\langle\cos^2(\omega t)\rangle = \langle\sin^2(\omega t)\rangle = \frac{1}{2}$ and $\langle\cos(\omega t)\sin(\omega t)\rangle = 0$, we substitute into eqns. 4.268 to obtain

$$\begin{aligned}\dot{Q} &= H^0_P + \tfrac{1}{4}\,\omega^{-2}\,H^0_{PPP}\,\widetilde{V}^2_Q + \mathcal{O}(\omega^{-4})\\ \dot{P} &= -H^0_Q - \tfrac{1}{4}\,\omega^{-2}\,H^0_{QPP}\,\widetilde{V}^2_Q - \tfrac{1}{2}\,\omega^{-2} H^0_{PP}\,\widetilde{V}_Q\,\widetilde{V}_{QQ} + \mathcal{O}(\omega^{-4}) \quad .\end{aligned} \tag{4.272}$$

These equations may be written compactly as

$$\dot{Q} = \frac{\partial K}{\partial P} \quad , \quad \dot{P} = -\frac{\partial K}{\partial Q} \quad , \tag{4.273}$$

where

$$K(Q,P) = H^0(Q,P) + \frac{1}{4\omega^2}\,\frac{\partial^2 H^0}{\partial P^2}\bigg(\frac{\partial \widetilde{V}}{\partial Q}\bigg)^2 + \ldots \quad . \tag{4.274}$$

4.8.2 Example : pendulum with oscillating support

Consider a pendulum with a vertically oscillating point of support. The coordinates of the pendulum bob are

$$x = \ell\,\sin\theta \qquad , \qquad y = a(t) - \ell\,\cos\theta \quad . \tag{4.275}$$

The Lagrangian is easily obtained:

$$\begin{aligned}L &= \tfrac{1}{2}m\ell^2\,\dot\theta^2 + m\ell\dot{a}\,\dot\theta\sin\theta + mg\ell\cos\theta + \tfrac{1}{2}m\dot{a}^2 - mga\\ &= \tfrac{1}{2}m\ell^2\,\dot\theta^2 + m(g+\ddot{a})\ell\cos\theta + \overbrace{\tfrac{1}{2}m\dot{a}^2 - mga - \frac{d}{dt}\big(m\ell\dot{a}\,\cos\theta\big)}^{\text{these may be dropped}} \quad .\end{aligned} \tag{4.276}$$

Thus we may take the Lagrangian to be

$$L = \tfrac{1}{2}m\ell^2\,\dot\theta^2 + m(g+\ddot{a})\,\ell\cos\theta \quad , \tag{4.277}$$

from which we derive the Hamiltonian

$$\begin{aligned}H(\theta,p_\theta) &= \frac{p^2_\theta}{2m\ell^2} - mg\ell\cos\theta - m\ell\ddot{a}\,\cos\theta\\ &= H_0(\theta,p_\theta,t) + \widetilde{V}(\theta)\,\sin\omega t \quad .\end{aligned} \tag{4.278}$$

We have assumed $a(t) = a_0\sin\omega t$, so

$$\widetilde{V}(\theta) = m\ell a_0\,\omega^2\cos\theta \quad . \tag{4.279}$$

Writing $\theta \equiv \Theta + \zeta$ and $p_\theta \equiv L + \pi$, the effective Hamiltonian, per eqn. 4.274, is

$$K(\Theta,L) = \frac{L^2}{2m\ell^2} - mg\ell\cos\Theta + \tfrac{1}{4}m\,a_0^2\,\omega^2\sin^2\!\Theta \quad . \tag{4.280}$$

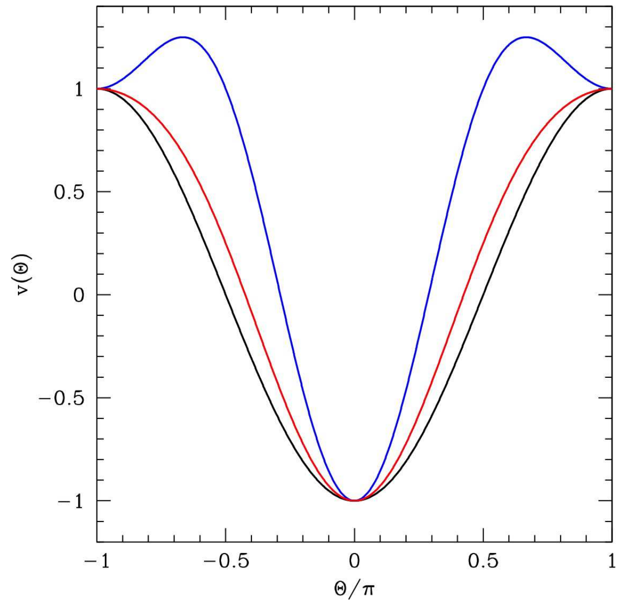

Figure 4.13: Dimensionless potential $v(\Theta)$ for $r = 0$ (black curve), $r = 0.5$ (red), and $r = 2$ (blue).

Let's define the dimensionless parameter $r \equiv \omega^2 a_0^2 / 2g\ell$. The slow variable Θ then executes motion in the *effective potential* $V_{\rm eff}(\Theta) = mg\ell\, v(\Theta)$, with

$$v(\Theta) = -\cos\Theta + \frac{r}{2}\sin^2\!\Theta \quad . \tag{4.281}$$

Differentiating, we find that $V_{\rm eff}(\Theta)$ is stationary when

$$v'(\Theta) = 0 \quad \Rightarrow \quad r\sin\Theta\,\cos\Theta = -\sin\Theta \quad . \tag{4.282}$$

Thus, $\Theta = 0$ and $\Theta = \pi$, where $\sin\Theta = 0$, are equilibria. When $r > 1$ (note $r > 0$ always), there are two new solutions, given by the roots of $\cos\Theta = -r^{-1}$.

To assess stability of these equilibria, we compute the second derivative:

$$v''(\Theta) = \cos\Theta + r\,\cos 2\Theta \quad . \tag{4.283}$$

From this, we see that $\Theta = 0$ is stable, *i.e.* $v''(\Theta = 0) > 0$, always, but $\Theta = \pi$ is stable for $r > 1$ and unstable for $r < 1$. When $r > 1$, two new solutions appear, at $\cos\Theta = -r^{-1}$, for which

$$v''\big(\cos^{-1}(-1/r)\big) = r^{-1} - r \quad , \tag{4.284}$$

which is always negative since $r > 1$ in order for these equilibria to exist. The situation is sketched in fig. 4.13, showing $v(\Theta)$ for three representative values of the parameter r. For $r < 1$, the equilibrium at $\Theta = \pi$ is unstable, but as r increases, a subcritical pitchfork bifurcation is encountered at $r = 1$, and $\Theta = \pi$ becomes stable, while the outlying $\Theta = \cos^{-1}(-1/r)$ solutions are unstable.

4.9 Field Theory: Systems with Several Independent Variables

4.9.1 Equations of motion and Noether's theorem

Suppose $\phi_a(x)$ depends on several independent variables: $x = \{x^1, x^2, \ldots, x^n\}$. Furthermore, suppose

$$S\big[\{\phi_a(x)\}\big] = \int\limits_\Omega d^n\!x\, \mathcal{L}(\phi_a\, \partial_\mu \phi_a, x) \quad , \tag{4.285}$$

i.e. the *Lagrangian density* $\mathcal{L}$ is a function of the fields ϕ_a; their partial derivatives $\partial\phi_a/\partial x^\mu$, and possibly the independent variables x^μ as well. Here Ω is a region in $\mathbb{R}^n$. In dynamical field theories, we write $x = (x^0, x^1, \ldots, x^d)$ where d is the dimension of space and $x^0 = ct$, where t is time and c is a constant with dimensions of speed. In such cases $n = d + 1$ and we can identify $x^0 \equiv x^n$.

Then the first variation of S is

$$\begin{aligned} \delta S &= \int\limits_\Omega d^n\!x \left\{ \frac{\partial \mathcal{L}}{\partial \phi_a}\, \delta\phi_a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\, \frac{\partial\, \delta\phi_a}{\partial x^\mu} \right\} \\ &= \oint\limits_{\partial\Omega} d\Sigma\, n^\mu\, \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\, \delta\phi_a + \int\limits_\Omega d^n\!x \left\{ \frac{\partial \mathcal{L}}{\partial \phi_a} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right) \right\} \delta\phi_a \quad , \end{aligned} \tag{4.286}$$

where $\partial\Omega$ is the $(n-1)$-dimensional boundary of Ω, $d\Sigma$ is the differential surface area, and n^μ is the unit vector normal to $\partial\Omega$. If we demand $\partial\mathcal{L}/\partial(\partial_\mu\phi_a)\big|_{\partial\Omega} = 0$ or $\delta\phi_a\big|_{\partial\Omega} = 0$, the surface term vanishes, and we conclude

$$\frac{\delta S}{\delta \phi_a(\boldsymbol{x})} = \frac{\partial \mathcal{L}}{\partial \phi_a} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right) \quad . \tag{4.287}$$

Next, consider the one-parameter family of *field* transformations

$$\phi_a(x) \to \widetilde{\phi}_a\big(\boldsymbol{\phi}(x), \zeta\big) \tag{4.288}$$

such that $\widetilde{\phi}_a\big(\boldsymbol{\phi}(x), \zeta = 0\big) = \phi_a(x)$. If the Lagrangian density $\mathcal{L}$ is independent of this transformation, then

$$\begin{aligned} \frac{d\mathcal{L}}{d\zeta}\bigg|_{\zeta=0} &= \frac{\partial \mathcal{L}}{\partial \phi_a}\, \frac{\partial \widetilde{\phi}_a}{\partial \zeta}\bigg|_{\zeta=0} + \sum_{\mu=1}^n \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\, \frac{\partial(\partial_\mu \widetilde{\phi}_a)}{\partial \zeta}\bigg|_{\zeta=0} \\ &= \sum_{\mu=1}^n \left\{ \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right) \frac{\partial \widetilde{\phi}_a}{\partial \zeta}\bigg|_{\zeta=0} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\, \frac{\partial}{\partial x^\mu}\left(\frac{\partial \widetilde{\phi}_a}{\partial \zeta} \right)_{\zeta=0} \right\} \\ &= \sum_{\mu=1}^n \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\, \frac{\partial \widetilde{\phi}_a}{\partial \zeta} \right)_{\zeta=0} \end{aligned} \tag{4.289}$$

We can write this as $\partial_\mu\, J^\mu = 0$, where

$$J^\mu \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)}\, \frac{\partial \widetilde{\phi}_a}{\partial \zeta}\bigg|_{\zeta=0} \quad . \tag{4.290}$$

We call $\Lambda = J^0/c$ the *total charge*. If we assume $\boldsymbol{J} = 0$ at the spatial boundaries of our system, then integrating the conservation law $\partial_\mu J^\mu$ (summation convention) over the spatial region Ω gives

$$\frac{d\Lambda}{dt} = \int\limits_\Omega d^3\!x\, \partial_0\, J^0 = -\int\limits_\Omega d^3\!x\, \boldsymbol{\nabla}\!\cdot\!\boldsymbol{J} = -\oint\limits_{\partial\Omega} d\Sigma\, \hat{\boldsymbol{n}}\cdot\boldsymbol{J} = 0 \quad , \tag{4.291}$$

assuming $\boldsymbol{J} = 0$ at the boundary $\partial\Omega$.

As an example, consider the case of a stretched string of linear mass density ρ and tension τ. The action is a functional of the height $y(x,t)$, where the coordinate along the string, x, and time, t, are the two independent variables. The Lagrangian density is

$$\mathcal{L} = \tfrac{1}{2}\,\rho \left(\frac{\partial y}{\partial t}\right)^{\!2} - \tfrac{1}{2}\,\tau \left(\frac{\partial y}{\partial x}\right)^{\!2} \quad . \tag{4.292}$$

The Euler-Lagrange equations are

$$\begin{aligned} 0 = \frac{\delta S}{\delta y(x,t)} &= -\frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial y'}\right) - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) \\ &= \frac{\partial}{\partial x}\left(\tau\,\frac{\partial y}{\partial x}\right) - \rho\,\frac{\partial^2 y}{\partial t^2} \quad , \end{aligned} \tag{4.293}$$

where $y' = \partial y/\partial x$ and $\dot{y} = \partial y/\partial t$. We've assumed boundary conditions where $\delta y(x_a, t) = \delta y(x_b, t) = \delta y(x, t_a) = \delta y(x, t_b) = 0$. At this point, $\rho(x)$ and $\tau(x)$ may be position-dependent. For constant ρ and τ, we obtain the Helmholtz equation $\rho\ddot{y} = \tau y''$, where $c = (\tau/\rho)^{1/2}$ is the speed of wave propagation.

For practice with the Minkowski notation, we define $x^0 \equiv ct$ and $x^1 \equiv x$ and the two-dimensional space-time coordinate vector is then $x^\mu = (x^0, x^1) = (ct, x)$. The Lagrangian can then be written $\mathcal{L} = \frac{1}{2}\,\tau(\partial_\mu y)(\partial^\mu y)$, where $x_\mu = g_{\mu\nu}\,x^\nu = (ct, -x)$, in which case $\partial_\mu = \partial/\partial x^\mu$ and $\partial^\mu = \partial/\partial x_\mu$. Clearly $\mathcal{L}$ remains invariant under the one-parameter family of transformations $y \to y + \zeta$, and the conserved Noether current is

$$J^\mu = \tau\,\frac{\partial y}{\partial x_\mu} \quad , \tag{4.294}$$

and we have $\partial_\mu J^\mu = 0$, which is equivalent to $\partial^\mu J_\mu = 0$. (Upper indices are called *covariant* while lower ones are *contravariant*.) Current conservation in this system is simply a restatement of the Helmholtz equation.

Maxwell's equations

The Lagrangian density for an electromagnetic field with sources is

$$\mathcal{L} = -\frac{1}{16\pi}\,F_{\mu\nu}\,F^{\mu\nu} - \frac{1}{c}\,j_\mu\,A^\mu \quad . \tag{4.295}$$

The equations of motion are then

$$\frac{\partial \mathcal{L}}{\partial A^\mu} - \frac{\partial}{\partial x^\nu}\left(\frac{\partial \mathcal{L}}{\partial(\partial^\mu A^\nu)}\right) = 0 \quad \Rightarrow \quad \partial_\mu\,F^{\mu\nu} = \frac{4\pi}{c}\,j^\nu \quad , \tag{4.296}$$

which are Maxwell's equations.

Relativistic complex scalar field

As an example, consider the case of a complex scalar field, with Lagrangian density

$$\mathcal{L}(\psi, \psi^*, \partial_\mu\psi, \partial_\mu\psi^*) = \tfrac{1}{2}K\,(\partial_\mu\psi^*)(\partial^\mu\psi) - U\big(\psi^*\psi\big) \quad . \tag{4.297}$$

This is invariant under the transformation $\psi \to e^{i\zeta}\,\psi$, $\psi^* \to e^{-i\zeta}\,\psi^*$. Thus,

$$\frac{\partial\tilde{\psi}}{\partial\zeta} = i\,e^{i\zeta}\,\psi \qquad , \qquad \frac{\partial\tilde{\psi}^*}{\partial\zeta} = -i\,e^{-i\zeta}\,\psi^* \quad , \tag{4.298}$$

and, summing over both ψ and ψ^* fields, we have

$$\begin{aligned} J^\mu &= \frac{\partial\mathcal{L}}{\partial\,(\partial_\mu\psi)}\cdot(i\psi) + \frac{\partial\mathcal{L}}{\partial\,(\partial_\mu\psi^*)}\cdot(-i\psi^*) \\ &= \frac{K}{2i}\big(\psi^*\partial^\mu\psi - \psi\,\partial^\mu\psi^*\big) \quad . \end{aligned} \tag{4.299}$$

The potential, which depends on $|\psi|^2$, is independent of ζ. Hence, this form of conserved 4-current is valid for an entire class of potentials.

4.9.2 Gross-Pitaevskii model

As one final example of a field theory, consider the Gross-Pitaevskii model, with

$$\mathcal{L} = i\hbar\,\psi^*\,\frac{\partial\psi}{\partial t} - \frac{\hbar^2}{2m}\,\boldsymbol{\nabla}\psi^*\cdot\boldsymbol{\nabla}\psi - g\,\big(|\psi|^2 - n_0\big)^2 \quad . \tag{4.300}$$

This describes a nonrelativistic Bose fluid with repulsive short-ranged interactions. Here $\psi(\boldsymbol{x}, t)$ is again a complex scalar field, and ψ^* is its complex conjugate. Using the Leibniz rule, we have

$$\begin{aligned} \delta S[\psi^*,\psi] &= S[\psi^* + \delta\psi^*, \psi + \delta\psi] \\ &= \int\! dt \!\int\! d^d\!x\,\bigg\{ i\hbar\,\psi^*\,\frac{\partial\delta\psi}{\partial t} + i\hbar\,\delta\psi^*\,\frac{\partial\psi}{\partial t} - \frac{\hbar^2}{2m}\,\boldsymbol{\nabla}\psi^*\cdot\boldsymbol{\nabla}\delta\psi - \frac{\hbar^2}{2m}\,\boldsymbol{\nabla}\delta\psi^*\cdot\boldsymbol{\nabla}\psi \\ &\qquad\qquad - 2g\,\big(|\psi|^2 - n_0\big)\,(\psi^*\delta\psi + \psi\delta\psi^*)\bigg\} \\ &= \int\! dt \!\int\! d^d\!x\,\bigg\{ \bigg[-i\hbar\,\frac{\partial\psi^*}{\partial t} + \frac{\hbar^2}{2m}\,\nabla^2\psi^* - 2g\,\big(|\psi|^2 - n_0\big)\,\psi^* \bigg]\,\delta\psi \\ &\qquad\qquad + \bigg[i\hbar\,\frac{\partial\psi}{\partial t} + \frac{\hbar^2}{2m}\,\nabla^2\psi - 2g\,\big(|\psi|^2 - n_0\big)\,\psi \bigg]\,\delta\psi^* \bigg\} \quad , \end{aligned} \tag{4.301}$$

where we have integrated by parts where necessary and discarded the boundary terms. Extremizing $S[\psi^*,\psi]$ therefore results in the *nonlinear Schrödinger equation* (NLSE),

$$i\hbar\,\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\,\nabla^2\psi + 2g\,\big(|\psi|^2 - n_0\big)\,\psi \tag{4.302}$$

as well as its complex conjugate,

$$-\,i\hbar\,\frac{\partial\psi^*}{\partial t} = -\frac{\hbar^2}{2m}\,\nabla^2\psi^* + 2g\,\big(|\psi|^2 - n_0\big)\,\psi^* \quad . \tag{4.303}$$

Note that these equations are indeed the Euler-Lagrange equations:

$$\begin{aligned}\frac{\delta S}{\delta\psi} &= \frac{\partial\mathcal{L}}{\partial\psi} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial\,\partial_\mu\psi}\right)\\[2ex] \frac{\delta S}{\delta\psi^*} &= \frac{\partial\mathcal{L}}{\partial\psi^*} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial\mathcal{L}}{\partial\,\partial_\mu\psi^*}\right) \quad ,\end{aligned} \tag{4.304}$$

with $x^\mu = (t, \boldsymbol{x})$[7]. Plugging in

$$\frac{\partial\mathcal{L}}{\partial\psi} = -2g\,\big(|\psi|^2 - n_0\big)\,\psi^* \quad , \quad \frac{\partial\mathcal{L}}{\partial\,\partial_t\psi} = i\hbar\,\psi^* \quad , \quad \frac{\partial\mathcal{L}}{\partial\,\boldsymbol{\nabla}\psi} = -\frac{\hbar^2}{2m}\,\boldsymbol{\nabla}\psi^* \tag{4.305}$$

and

$$\frac{\partial\mathcal{L}}{\partial\psi^*} = i\hbar\,\psi - 2g\,\big(|\psi|^2 - n_0\big)\,\psi \quad , \quad \frac{\partial\mathcal{L}}{\partial\,\partial_t\psi^*} = 0 \quad , \quad \frac{\partial\mathcal{L}}{\partial\,\boldsymbol{\nabla}\psi^*} = -\frac{\hbar^2}{2m}\,\boldsymbol{\nabla}\psi \quad , \tag{4.306}$$

we recover the NLSE and its conjugate.

The Gross-Pitaevskii model also possesses a $\mathsf{U}(1)$ or $\mathsf{O}(2)$ invariance, *viz.*

$$\psi(\boldsymbol{x},t) \to \widetilde{\psi}(\boldsymbol{x},t) = e^{i\zeta}\,\psi(\boldsymbol{x},t) \quad , \quad \psi^*(\boldsymbol{x},t) \to \widetilde{\psi}^*(\boldsymbol{x},t) = e^{-i\zeta}\,\psi^*(\boldsymbol{x},t) \quad . \tag{4.307}$$

Thus, the conserved Noether current is then a $(d+1)$-dimensional vector with components

$$J^\mu = \frac{\partial\mathcal{L}}{\partial\,\partial_\mu\psi}\,\frac{\partial\widetilde{\psi}}{\partial\zeta}\bigg|_{\zeta=0} + \frac{\partial\mathcal{L}}{\partial\,\partial_\mu\psi^*}\,\frac{\partial\widetilde{\psi}^*}{\partial\zeta}\bigg|_{\zeta=0} \quad . \tag{4.308}$$

In terms of time ($\mu = 0$) and space ($\mu \in \{1,\ldots,d\}$) components, we have

$$\begin{aligned}J^0 &= -\hbar\,|\psi|^2\\[2ex] \boldsymbol{J} &= -\frac{\hbar^2}{2im}\,\big(\psi^*\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\psi^*\big) \quad .\end{aligned} \tag{4.309}$$

Dividing out by $\hbar$, taking $J^0 \equiv -\hbar\rho$ and $\boldsymbol{J} \equiv -\hbar\boldsymbol{j}$, we obtain the continuity equation,

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\!\cdot\!\boldsymbol{j} = 0 \quad , \tag{4.310}$$

where

$$\rho = |\psi|^2 \qquad , \qquad \boldsymbol{j} = \frac{\hbar}{2im}\,\big(\psi^*\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\psi^*\big) \quad . \tag{4.311}$$

are the particle density and the particle current, respectively.

[7] In the nonrelativistic case, there is no utility in defining $x^0 = ct$, so we simply define $x^0 = t$.

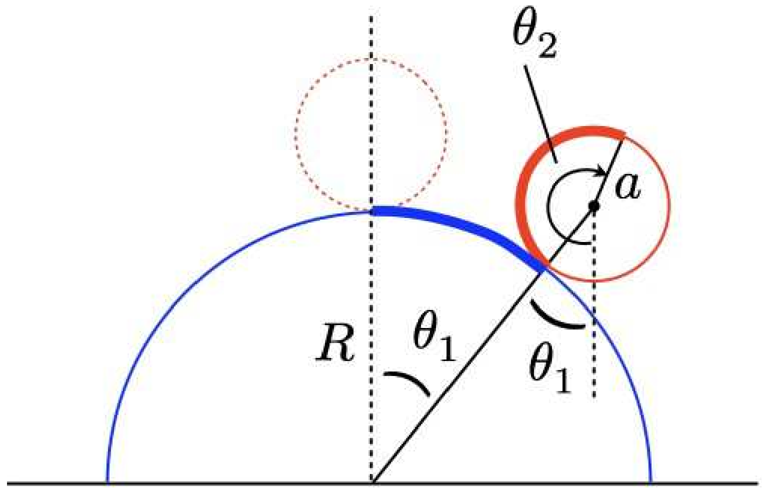

Figure 4.14: A cylinder of radius a rolls along a half-cylinder of radius R. When there is no slippage, the angles θ_1 and θ_2 obey the constraint equation $R\theta_1 = a(\theta_2 - \theta_1)$.

4.10 Constraints: General Theory

4.10.1 Introduction

A mechanical system of N point particles in d dimensions possesses $n = dN$ degrees of freedom[8]. To specify these degrees of freedom, we can choose any independent set of generalized coordinates $\{q_1, \ldots, q_n\}$. Oftentimes, however, not all n coordinates are independent.

Consider, for example, the situation in fig. 4.14, where a cylinder of radius a rolls over a half-cylinder of radius R. If there is no slippage, then the angles θ_1 and θ_2 are not independent, and they obey the *equation of constraint*,

$$R\,\theta_1 = a\,(\theta_2 - \theta_1) \quad . \tag{4.312}$$

In this case, we can easily solve the constraint equation and substitute $\theta_2 = \big(1 + \frac{R}{a}\big)\,\theta_1$. In other cases, though, the equation of constraint might not be so easily solved (*e.g.* it may be nonlinear). How then do we proceed?

4.10.2 Constrained extremization of functions: Lagrange multipliers

Given $F(x_1, \ldots, x_n)$ to be extremized subject to k constraints of the form $G_j(x_1, \ldots, x_n) = 0$ where $j = 1, \ldots, k$, construct

$$F^*\big(x_1, \ldots, x_n; \lambda_1, \ldots, \lambda_k\big) \equiv F(x_1, \ldots, x_n) + \sum_{j=1}^{k} \lambda_j\, G_j(x_1, \ldots, x_n) \tag{4.313}$$

[8]For N rigid bodies, the number of degrees of freedom is $n' = \frac{1}{2}d(d+1)N$, corresponding to d center-of-mass coordinates and $\frac{1}{2}d(d-1)$ angles of orientation for each particle. The dimension of the group of rotations in d dimensions is $\frac{1}{2}d(d-1)$, corresponding to the number of parameters in a general rank-d orthogonal matrix (*i.e.* an element of the group $O(d)$).

which is a function of the $(n+k)$ variables $\{x_1, \ldots, x_n; \lambda_1, \ldots, \lambda_k\}$, where the quantities $\{\lambda_1, \ldots, \lambda_k\}$ are *Lagrange undetermined multipliers*. We now *freely* extremize the extended function F^*:

$$\begin{aligned} dF^* &= \sum_{\sigma=1}^{n} \frac{\partial F^*}{\partial x_\sigma}\, dx_\sigma + \sum_{j=1}^{k} \frac{\partial F^*}{\partial \lambda_j}\, d\lambda_j \\ &= \sum_{\sigma=1}^{n} \left(\frac{\partial F}{\partial x_\sigma} + \sum_{j=1}^{k} \lambda_j\, \frac{\partial G_j}{\partial x_\sigma} \right) dx_\sigma + \sum_{j=1}^{k} G_j\, d\lambda_j = 0 \end{aligned} \tag{4.314}$$

This results in the $(n+k)$ equations

$$\begin{aligned} \frac{\partial F}{\partial x_\sigma} + \sum_{j=1}^{k} \lambda_j\, \frac{\partial G_j}{\partial x_\sigma} &= 0 \qquad (\sigma = 1, \ldots, n) \\ G_j &= 0 \qquad (j = 1, \ldots, k) \quad . \end{aligned} \tag{4.315}$$

The interpretation of all this is as follows. The first n equations in 4.315 can be written in vector form as

$$\boldsymbol{\nabla} F + \sum_{j=1}^{k} \lambda_j\, \boldsymbol{\nabla} G_j = 0 \quad . \tag{4.316}$$

This says that the (n-component) vector $\boldsymbol{\nabla} F$ is linearly dependent upon the k vectors $\boldsymbol{\nabla} G_j$. Thus, any movement in the direction of $\boldsymbol{\nabla} F$ must necessarily entail movement along one or more of the directions $\boldsymbol{\nabla} G_j$. This would require violating the constraints, since movement along $\boldsymbol{\nabla} G_j$ takes us off the level set $G_j = 0$. Were $\boldsymbol{\nabla} F$ linearly *independent* of the set $\{\boldsymbol{\nabla} G_j\}$, this would mean that we could find a differential displacement $d\boldsymbol{x}$ which has finite overlap with $\boldsymbol{\nabla} F$ but zero overlap with each $\boldsymbol{\nabla} G_j$. Thus $\boldsymbol{x} + d\boldsymbol{x}$ would still satisfy $G_j(\boldsymbol{x} + d\boldsymbol{x}) = 0$, but F would change by the finite amount $dF = \boldsymbol{\nabla} F(\boldsymbol{x}) \cdot d\boldsymbol{x}$.

Put another way, when we extremize $F(\boldsymbol{x})$ without constraints, we identify points $\boldsymbol{x} \in \mathbb{R}^n$ where the gradient $\boldsymbol{\nabla} F$ vanishes. However, when we have k constraints of the form $G_j(\boldsymbol{x}) = 0$, the subset

$$\Upsilon = \big\{ \boldsymbol{x} \in \mathbb{R}^n \,|\, G_j(\boldsymbol{x}) = 0 \;\forall\, j \in \{1, \ldots, k\} \big\} \tag{4.317}$$

is a hypersurface of dimension $n-k$. Generically we should not expect any of the solutions to $\boldsymbol{\nabla} F = 0$ to lie within the subspace Υ. Extremizing $F(\boldsymbol{x})$ subject to the k constraints $G_j(\boldsymbol{x}) = 0$ means that we must find the extrema of $F(\boldsymbol{x})$ for $\boldsymbol{x} \in \Upsilon \subset \mathbb{R}^n$. All such extrema satisfy that $\boldsymbol{\nabla} F(\boldsymbol{x})$ is *perpendicular* to the hypersurface Υ, *i.e.* $\boldsymbol{\nabla} F(\boldsymbol{x})$ must lie in the k-dimensional subspace spanned by the vectors $\boldsymbol{\nabla} G_j(\boldsymbol{x})$.

Example : volume of a cylinder

To see how this formalism works in practice, let's extremize the volume $V = \pi a^2 h$ of a cylinder of radius a and height h, subject to the constraint

$$G(a,h) = 2\pi a + \frac{h^2}{b} - \ell = 0 \quad . \tag{4.318}$$

We therefore define

$$V^*(a,h,\lambda) \equiv V(a,h) + \lambda\, G(a,h) \quad , \tag{4.319}$$

and set

$$\frac{\partial V^*}{\partial a} = 2\pi a h + 2\pi\lambda = 0 \tag{4.320}$$

$$\frac{\partial V^*}{\partial h} = \pi a^2 + 2\lambda\,\frac{h}{b} = 0 \tag{4.321}$$

$$\frac{\partial V^*}{\partial \lambda} = 2\pi a + \frac{h^2}{b} - \ell = 0 \quad . \tag{4.322}$$

Solving these three equations simultaneously gives

$$a = \frac{2\ell}{5\pi} \quad , \quad h = \sqrt{\frac{b\ell}{5}} \quad , \quad \lambda = -\frac{2}{5^{3/2}\pi}\, b^{1/2}\,\ell^{3/2} \quad , \quad V^* = \frac{4}{5^{5/2}\,\pi}\,\ell^{5/2}\, b^{1/2} \quad . \tag{4.323}$$

4.10.3 Constraints and variational calculus

Before addressing the subject of constrained dynamical systems, let's consider the issue of constraints in the broader context of variational calculus. Suppose we have a functional

$$F[\boldsymbol{y}(x)] = \int\limits_{x_a}^{x_b} dx\, L(\boldsymbol{y},\boldsymbol{y}',x) \quad , \tag{4.324}$$

which we want to extremize subject to some constraints. Here $\boldsymbol{y}$ stands for an n-component vector of functions $\{y_\sigma(x)\}$. We assume that the endpoint values $y_\sigma(x_a)$ and $y_\sigma(x_b)$ are fixed for each σ. There are two classes of constraints we will consider:

1. *Integral constraints:* These are of the form

   $$\int\limits_{x_a}^{x_b} dx\, N_j(\boldsymbol{y},\boldsymbol{y}',x) = C_j \quad , \tag{4.325}$$

   where j labels the constraint.

2. *Holonomic constraints:* These are of the form

   $$G_j(\boldsymbol{y},x) = 0 \quad . \tag{4.326}$$

The cylinders system in fig. 4.14 provides an example of a holonomic constraint. There, $G(\theta,t) = R\,\theta_1 - a\,(\theta_2-\theta_1) = 0$. As an example of a problem with an integral constraint, suppose we want to know the shape of a hanging rope of fixed length C. This means we minimize the rope's potential energy,

$$U[y(x)] = \rho g \int\limits_{x_a}^{x_b} ds\, y(x) = \rho g \int\limits_{x_a}^{x_b} dx\; y\sqrt{1+y'^2} \quad , \tag{4.327}$$

where ρ is the linear mass density of the rope, subject to the fixed-length constraint

$$C = \int_{s_a}^{s_b} ds = \int_{x_a}^{x_b} dx\, \sqrt{1 + y'^2} \quad . \tag{4.328}$$

Note $ds = \sqrt{dx^2 + dy^2}$ is the differential element of arc length along the rope. To solve problems like these, we again use the method of Lagrange multipliers.

4.10.4 Extremization of functionals : integral constraints

Given a functional

$$F\big[\{y_\sigma(x)\}\big] = \int_{x_a}^{x_b} dx\, L\big(\{y_\sigma\}, \{y'_\sigma\}, x\big) \qquad (\sigma = 1, \ldots, n) \tag{4.329}$$

subject to boundary conditions $\delta y_\sigma(x_a) = \delta y_\sigma(x_b) = 0$ and k constraints of the form

$$\int_{x_a}^{x_b} dx\, N_l\big(\{y_\sigma\}, \{y'_\sigma\}, x\big) = C_l \qquad (l = 1, \ldots, k) \quad , \tag{4.330}$$

construct the extended functional

$$F^*\big[\{y_\sigma(x)\}; \{\lambda_j\}\big] \equiv \int_{x_a}^{x_b} dx \left\{ L\big(\{y_\sigma\}, \{y'_\sigma\}, x\big) + \sum_{l=1}^{k} \lambda_l\, N_l\big(\{y_\sigma\}, \{y'_\sigma\}, x\big) \right\} - \sum_{l=1}^{k} \lambda_l\, C_l \tag{4.331}$$

and freely extremize over $\{y_1, \ldots, y_n; \lambda_1, \ldots, \lambda_k\}$. This results in $(n + k)$ equations

$$\begin{aligned} \frac{\partial L}{\partial y_\sigma} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'_\sigma}\right) + \sum_{l=1}^{k} \lambda_l \left\{ \frac{\partial N_l}{\partial y_\sigma} - \frac{d}{dx}\left(\frac{\partial N_l}{\partial y'_\sigma}\right) \right\} = 0 \qquad (\sigma = 1, \ldots, n) \\ \int_{x_a}^{x_b} dx\, N_l\big(\{y_\sigma\}, \{y'_\sigma\}, x\big) = C_l \qquad (l = 1, \ldots, k) \quad . \end{aligned} \tag{4.332}$$

4.10.5 Extremization of functionals : holonomic constraints

Given a functional

$$F\big[\{y_\sigma(x)\}\big] = \int_{x_a}^{x_b} dx\, L\big(\{y_\sigma\}, \{y'_\sigma\}, x\big) \qquad (\sigma = 1, \ldots, n) \tag{4.333}$$

subject to boundary conditions $\delta y_\sigma(x_a) = \delta y_\sigma(x_b) = 0$ and k constraints of the form

$$G_j\big(\{y_\sigma(x)\}, x\big) = 0 \qquad (j = 1, \ldots, k) \quad , \tag{4.334}$$

construct the extended functional

$$F^*\big[\{y_\sigma(x)\};\{\lambda_j(x)\}\big] \equiv \int\limits_{x_a}^{x_b} dx\,\bigg\{L\big(\{y_\sigma\},\{y'_\sigma\},x\big) + \sum_{j=1}^{k} \lambda_j\, G_j\big(\{y_\sigma\},x\big)\bigg\} \tag{4.335}$$

and freely extremize over the $(n+k)$ *functions* $\{y_1(x),\ldots,y_n(x);\lambda_1(x),\ldots,\lambda_k(x)\}$:

$$\delta F^* = \int\limits_{x_a}^{x_b} dx\,\Bigg\{\sum_{\sigma=1}^{n}\Bigg(\frac{\partial L}{\partial y_\sigma} - \frac{d}{dx}\bigg(\frac{\partial L}{\partial y'_\sigma}\bigg) + \sum_{j=1}^{k}\lambda_j\,\frac{\partial G_j}{\partial y_\sigma}\Bigg)\delta y_\sigma + \sum_{j=1}^{k} G_j\,\delta\lambda_j\Bigg\} = 0 \quad , \tag{4.336}$$

resulting in the $(n+k)$ equations

$$\begin{aligned} \frac{d}{dx}\bigg(\frac{\partial L}{\partial y'_\sigma}\bigg) - \frac{\partial L}{\partial y_\sigma} &= \sum_{j=1}^{k}\lambda_j\,\frac{\partial G_j}{\partial y_\sigma} \qquad (\sigma = 1,\ldots,n) \\ G_j &= 0 \qquad (j = 1,\ldots,k) \quad . \end{aligned} \tag{4.337}$$

4.10.6 Examples of functional extremization with constraints

Hanging rope

We minimize the potential energy functional

$$U\big[y(x)\big] = \rho g \int\limits_{x_1}^{x_2} dx\; y\sqrt{1+y'^2} \quad , \tag{4.338}$$

where ρ is the linear mass density, subject to the constraint of fixed total length,

$$C\big[y(x)\big] = \int\limits_{x_1}^{x_2} dx\,\sqrt{1+y'^2} \quad . \tag{4.339}$$

Thus,

$$U^*\big[y(x),\lambda\big] = U\big[y(x)\big] + \lambda C\big[y(x)\big] = \int\limits_{x_1}^{x_2} dx\; L^*(y,y',x) \quad , \tag{4.340}$$

with

$$L^*(y,y',x) = (\rho g y + \lambda)\,\sqrt{1+y'^2} \quad . \tag{4.341}$$

Since $\partial L^*/\partial x = 0$ we have that

$$H = y'\,\frac{\partial L^*}{\partial y'} - L^* = -\frac{\rho g y + \lambda}{\sqrt{1+y'^2}} \tag{4.342}$$

is constant. Thus,

$$\frac{dy}{dx} = \pm H^{-1} \sqrt{(\rho g y + \lambda)^2 - H^2} \quad , \tag{4.343}$$

with solution

$$y(x) = -\frac{\lambda}{\rho g} + \frac{H}{\rho g} \cosh\!\left(\frac{\rho g}{H}\,(x-a)\right) \quad . \tag{4.344}$$

Here, H, a, and λ are constants to be determined by demanding $y(x_i) = y_i$ $(i = 1, 2)$, and that the total length of the rope is C.

Geodesic on a curved surface

Consider next the problem of a geodesic on a curved surface. Let the equation for the surface be

$$G(x, y, z) = 0 \quad . \tag{4.345}$$

We wish to extremize the distance,

$$D = \int_a^b ds = \int_a^b \sqrt{dx^2 + dy^2 + dz^2} \quad . \tag{4.346}$$

We introduce a parameter t defined on the unit interval: $t \in [0, 1]$, such that $x(0) = x_a$, $x(1) = x_b$, *etc.* Then D may be regarded as a functional, *viz.*

$$D\big[x(t), y(t), z(t)\big] = \int_0^1 dt\, \sqrt{\dot x^2 + \dot y^2 + \dot z^2} \quad . \tag{4.347}$$

We impose the constraint by forming the extended functional, D^*:

$$D^*\big[x(t), y(t), z(t), \lambda(t)\big] \equiv \int_0^1 dt\, \left\{ \sqrt{\dot x^2 + \dot y^2 + \dot z^2} + \lambda\, G(x, y, z) \right\} \quad , \tag{4.348}$$

and we demand that the first functional derivatives of D^* vanish:

$$\begin{aligned}
\frac{\delta D^*}{\delta x(t)} &= -\frac{d}{dt}\left(\frac{\dot x}{\sqrt{\dot x^2 + \dot y^2 + \dot z^2}}\right) + \lambda\,\frac{\partial G}{\partial x} = 0 \\
\frac{\delta D^*}{\delta y(t)} &= -\frac{d}{dt}\left(\frac{\dot y}{\sqrt{\dot x^2 + \dot y^2 + \dot z^2}}\right) + \lambda\,\frac{\partial G}{\partial y} = 0 \\
\frac{\delta D^*}{\delta z(t)} &= -\frac{d}{dt}\left(\frac{\dot z}{\sqrt{\dot x^2 + \dot y^2 + \dot z^2}}\right) + \lambda\,\frac{\partial G}{\partial z} = 0 \\
\frac{\delta D^*}{\delta \lambda(t)} &= G(x, y, z) = 0 \quad .
\end{aligned} \tag{4.349}$$

Thus,

$$\lambda(t) = \frac{v\ddot{x} - \dot{x}\dot{v}}{v^2\,\partial_x G} = \frac{v\ddot{y} - \dot{y}\dot{v}}{v^2\,\partial_y G} = \frac{v\ddot{z} - \dot{z}\dot{v}}{v^2\,\partial_z G} \quad , \tag{4.350}$$

with $v = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$ and $\partial_x \equiv \frac{\partial}{\partial x}$, *etc.* These three equations are supplemented by $G(x, y, z) = 0$, which is the fourth.

4.10.7 Constraints in Lagrangian mechanics

Let us write our system of constraints in the differential form

$$\sum_{\sigma=1}^{n} g_{j\sigma}(q,t)\,dq_\sigma + h_j(q,t)\,dt = 0 \qquad (j = 1, \ldots, k) \quad . \tag{4.351}$$

If the partial derivatives satisfy

$$\frac{\partial g_{j\sigma}}{\partial q_{\sigma'}} = \frac{\partial g_{j\sigma'}}{\partial q_\sigma} \qquad , \qquad \frac{\partial g_{j\sigma}}{\partial t} = \frac{\partial h_j}{\partial q_\sigma} \quad , \tag{4.352}$$

then the k differentials can be integrated to give $dG_j(q,t) = 0$ for each $j \in \{1, \ldots, k\}$, where

$$g_{j\sigma} = \frac{\partial G_j}{\partial q_\sigma} \qquad , \qquad h_j = \frac{\partial G_j}{\partial t} \quad . \tag{4.353}$$

The action functional is

$$S[\{q_\sigma(t)\}] = \int\limits_{t_a}^{t_b} dt\, L\big(\{q_\sigma\}, \{\dot{q}_\sigma\}, t\big) \qquad (\sigma = 1, \ldots, n) \quad , \tag{4.354}$$

subject to boundary conditions $\delta q_\sigma(t_a) = \delta q_\sigma(t_b) = 0$. The first variation of S is given by

$$\delta S = \int\limits_{t_a}^{t_b} dt \sum_{\sigma=1}^{n} \left\{ \frac{\partial L}{\partial q_\sigma} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right) \right\} \delta q_\sigma \quad . \tag{4.355}$$

Since the $\{q_\sigma(t)\}$ are no longer independent, we cannot infer that the term in brackets vanishes for each index σ. What are the constraints on the variations $\delta q_\sigma(t)$? The constraints are expressed in terms of *virtual displacements* which take no time: $\delta t = 0$. Thus,

$$\sum_{\sigma=1}^{n} g_{j\sigma}(q,t)\,\delta q_\sigma(t) = 0 \quad , \tag{4.356}$$

where $j = 1, \ldots, k$ is the constraint index. We may now relax the constraint by introducing k undetermined functions $\lambda_j(t)$, by adding integrals of the above equations with undetermined coefficient functions to δS:

$$\sum_{\sigma=1}^{n} \left\{ \frac{\partial L}{\partial q_\sigma} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right) + \sum_{j=1}^{k} \lambda_j(t)\, g_{j\sigma}(q,t) \right\} \delta q_\sigma(t) = 0 \quad . \tag{4.357}$$

Now we can demand that the term in brackets vanish for all σ. Thus, we obtain a set of $(n+k)$ equations,

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right) - \frac{\partial L}{\partial q_\sigma} &= \sum_{j=1}^{k} \lambda_j(t)\, g_{j\sigma}(q,t) \equiv Q_\sigma \\
\sum_{\sigma=1}^{n} g_{j\sigma}(q,t)\, \dot{q}_\sigma + h_j(q,t) &= 0 \quad ,
\end{aligned}
\tag{4.358}
$$

in $(n+k)$ unknowns $\{q_1, \ldots, q_n, \lambda_1, \ldots, \lambda_k\}$. Here, Q_σ is the *force of constraint conjugate to the generalized coordinate* q_σ. Thus, with

$$
p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma} \quad , \quad F_\sigma = \frac{\partial L}{\partial q_\sigma} \quad , \quad Q_\sigma = \sum_{j=1}^{k} \lambda_j\, g_{j\sigma} \quad , \tag{4.359}
$$

we write Newton's second law as

$$
\dot{p}_\sigma = F_\sigma + Q_\sigma \quad . \tag{4.360}
$$

Note that we can write

$$
\frac{\delta S}{\delta \boldsymbol{q}(t)} = \frac{\partial L}{\partial \boldsymbol{q}} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\right) \tag{4.361}
$$

and that the *instantaneous* constraints may be written

$$
\boldsymbol{g}_j \cdot \delta \boldsymbol{q} = 0 \qquad (j = 1, \ldots, k) \quad . \tag{4.362}
$$

Thus, by demanding

$$
\frac{\delta S}{\delta \boldsymbol{q}(t)} + \sum_{j=1}^{k} \lambda_j\, \boldsymbol{g}_j = 0 \tag{4.363}
$$

we require that the functional derivative be linearly dependent on the k vectors $\boldsymbol{g}_j$.

4.10.8 Constraints and conservation laws

We have seen how invariance of the Lagrangian with respect to a one-parameter family of coordinate transformations results in an associated conserved quantity Λ, and how a lack of explicit time dependence in L results in the conservation of the Hamiltonian H. In deriving both these results, however, we used the equations of motion $\dot{p}_\sigma = F_\sigma$. What happens when we have constraints, in which case $\dot{p}_\sigma = F_\sigma + Q_\sigma$?

Let's begin with the Hamiltonian. We have $H = \dot{q}_\sigma\, p_\sigma - L$, hence

$$
\begin{aligned}
\frac{dH}{dt} &= \overbrace{\left(p_\sigma - \frac{\partial L}{\partial \dot{q}_\sigma}\right)}^{\text{this vanishes}} \ddot{q}_\sigma + \overbrace{\left(\dot{p}_\sigma - \frac{\partial L}{\partial q_\sigma}\right)}^{\text{this is } \dot{Q}_\sigma} \dot{q}_\sigma - \frac{\partial L}{\partial t} \\
&= Q_\sigma\, \dot{q}_\sigma - \frac{\partial L}{\partial t} \quad .
\end{aligned}
\tag{4.364}
$$

We now use

$$Q_\sigma\,\dot{q}_\sigma = \lambda_j\,g_{j\sigma}\,\dot{q}_\sigma = -\lambda_j\,h_j \tag{4.365}$$

to obtain

$$\frac{dH}{dt} = -\lambda_j\,h_j - \frac{\partial L}{\partial t} \quad . \tag{4.366}$$

We therefore conclude that *in a system with constraints of the form $g_{j\sigma}\,\dot{q}_\sigma + h_j = 0$, the Hamiltonian is conserved if each $h_j = 0$ and if L is not explicitly dependent on time.* In the case of holonomic constraints, $h_j = \frac{\partial G_j}{\partial t}$, so H is conserved if neither L nor any of the constraints G_j is explicitly time-dependent.

Next, let us rederive Noether's theorem when constraints are present. We assume a one-parameter family of transformations $q_\sigma \to \tilde{q}_\sigma(\zeta)$ leaves L invariant. Then

$$\begin{aligned} 0 = \frac{dL}{d\zeta} &= \frac{\partial L}{\partial \tilde{q}_\sigma}\,\frac{\partial \tilde{q}_\sigma}{\partial \zeta} + \frac{\partial L}{\partial \dot{\tilde{q}}_\sigma}\,\frac{\partial \dot{\tilde{q}}_\sigma}{\partial \zeta} \\ &= \big(\dot{\tilde{p}}_\sigma - \tilde{Q}_\sigma\big)\,\frac{\partial \tilde{q}_\sigma}{\partial \zeta} + \tilde{p}_\sigma\,\frac{d}{dt}\bigg(\frac{\partial \tilde{q}_\sigma}{\partial \zeta}\bigg) \\ &= \frac{d}{dt}\bigg(\tilde{p}_\sigma\,\frac{\partial \tilde{q}_\sigma}{\partial \zeta}\bigg) - \lambda_j\,\tilde{g}_{j\sigma}\,\frac{\partial \tilde{q}_\sigma}{\partial \zeta} \quad . \end{aligned} \tag{4.367}$$

Now let us write the constraints in differential form as

$$\tilde{g}_{j\sigma}\,d\tilde{q}_\sigma + \tilde{h}_j\,dt + \tilde{k}_j\,d\zeta = 0 \quad . \tag{4.368}$$

We now have

$$\frac{d\Lambda}{dt} = \lambda_j\,\tilde{k}_j \quad , \tag{4.369}$$

which says that *if the constraints are independent of ζ then Λ is conserved.* For holonomic constraints, this means that

$$G_j\big(\tilde{q}(\zeta),t\big) = 0 \quad \Rightarrow \quad \tilde{k}_j = \frac{\partial G_j}{\partial \zeta} = 0 \quad , \tag{4.370}$$

i.e. $G_j(\tilde{q},t)$ has no explicit ζ dependence.

4.11 Constraints: Worked Examples

Here we consider several example problems of constrained dynamics, and work each out in full detail.

4.11.1 One cylinder rolling off another

As an example of the constraint formalism, consider the system in fig. 4.14, where a cylinder of radius a rolls atop a cylinder of radius R. We have two constraints:

$$G_1(r,\theta_1,\theta_2) = r - R - a = 0 \qquad \text{(cylinders in contact)} \tag{4.371}$$

$$G_2(r,\theta_1,\theta_2) = R\,\theta_1 - a\,(\theta_2 - \theta_1) = 0 \qquad \text{(no slipping)} \quad , \tag{4.372}$$

from which we obtain the $g_{j\sigma}$:

$$g_{j\sigma} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & R+a & -a \end{pmatrix} \quad , \tag{4.373}$$

which is to say

$$\begin{aligned} \frac{\partial G_1}{\partial r} &= 1 \quad , \quad \frac{\partial G_1}{\partial \theta_1} = 0 \quad , \quad \frac{\partial G_1}{\partial \theta_2} = 0 \\ \frac{\partial G_2}{\partial r} &= 0 \quad , \quad \frac{\partial G_2}{\partial \theta_1} = R+a \quad , \quad \frac{\partial G_2}{\partial \theta_2} = -a \quad . \end{aligned} \tag{4.374}$$

The Lagrangian is

$$L = T - U = \tfrac{1}{2}M\big(\dot r^2 + r^2\dot\theta_1^2\big) + \tfrac{1}{2}I\dot\theta_2^2 - Mgr\,\cos\theta_1 \quad , \tag{4.375}$$

where M and I are the mass and rotational inertia of the rolling cylinder, respectively. Note that the kinetic energy is a sum of center-of-mass translation $T_{\rm tr} = \frac{1}{2}M\big(\dot r^2 + r^2\,\dot\theta_1^2\big)$ and rotation about the center-of-mass, $T_{\rm rot} = \frac{1}{2}I\,\dot\theta_2^2$. The equations of motion are

$$\begin{aligned} \frac{d}{dt}\bigg(\frac{\partial L}{\partial \dot r}\bigg) - \frac{\partial L}{\partial r} &= M\ddot r - Mr\,\dot\theta_1^2 + Mg\cos\theta_1 = \lambda_1 \equiv Q_r \\ \frac{d}{dt}\bigg(\frac{\partial L}{\partial \dot\theta_1}\bigg) - \frac{\partial L}{\partial \theta_1} &= Mr^2\ddot\theta_1 + 2Mr\dot r\,\dot\theta_1 - Mgr\sin\theta_1 = (R+a)\,\lambda_2 \equiv Q_{\theta_1} \\ \frac{d}{dt}\bigg(\frac{\partial L}{\partial \dot\theta_2}\bigg) - \frac{\partial L}{\partial \theta_2} &= I\ddot\theta_2 = -a\,\lambda_2 \equiv Q_{\theta_2} \quad . \end{aligned} \tag{4.376}$$

To these three equations we add the two constraints, resulting in five equations in the five unknowns $\big\{r, \theta_1, \theta_2, \lambda_1, \lambda_2\big\}$.

We solve by first implementing the constraints, which give $r = (R+a)$ a constant (*i.e.* $\dot r = 0$), and $\dot\theta_2 = \big(1 + \frac{R}{a}\big)\,\dot\theta_1$. Substituting these into the above equations gives

$$-M(R+a)\,\dot\theta_1^2 + Mg\cos\theta_1 = \lambda_1 \tag{4.377}$$

$$M(R+a)^2\ddot\theta_1 - Mg(R+a)\sin\theta_1 = (R+a)\,\lambda_2 \tag{4.378}$$

$$I\bigg(\frac{R+a}{a}\bigg)\ddot\theta_1 = -a\lambda_2 \quad . \tag{4.379}$$

From eqn. 4.379 we obtain

$$\lambda_2 = -\frac{I}{a}\,\ddot\theta_2 = -\frac{R+a}{a^2}\,I\,\ddot\theta_1 \quad , \tag{4.380}$$

which we substitute into eqn. 4.378 to obtain

$$\bigg(M + \frac{I}{a^2}\bigg)(R+a)^2\ddot\theta_1 - Mg(R+a)\sin\theta_1 = 0 \quad . \tag{4.381}$$

Multiplying by $\dot{\theta}_1$, we obtain an exact differential, which may be integrated to yield

$$\tfrac{1}{2}M\left(1+\frac{I}{Ma^2}\right)\dot{\theta}_1^2+\frac{Mg}{R+a}\,\cos\theta_1=\frac{Mg}{R+a}\cos\theta_1^\circ \quad . \tag{4.382}$$

Here, we have assumed that $\dot{\theta}_1=0$ when $\theta_1=\theta_1^\circ$, *i.e.* the rolling cylinder is released from rest at $\theta_1=\theta_1^\circ$. Finally, inserting this result into eqn. 4.377, we obtain the radial force of constraint,

$$Q_r=\frac{Mg}{1+\alpha}\Big\{(3+\alpha)\,\cos\theta_1-2\cos\theta_1^\circ\Big\} \quad , \tag{4.383}$$

where $\alpha=I/Ma^2$ is a dimensionless parameter ($0\le\alpha\le1$). This is the radial component of the normal force between the two cylinders. When Q_r vanishes, the cylinders lose contact – the rolling cylinder flies off. Clearly this occurs at an angle $\theta_1=\theta_1^*$, where

$$\theta_1^*=\cos^{-1}\left(\frac{2\cos\theta_1^\circ}{3+\alpha}\right) \quad . \tag{4.384}$$

The detachment angle θ_1^* is an increasing function of α, which means that larger I delays detachment. This makes good sense, since when I is larger the gain in kinetic energy is split between translational and rotational motion of the rolling cylinder. Note also that $Q_r(\theta_1^\circ)=Mg\cos\theta_1^\circ$ balances the initial radial component of the force of gravity.

Finally, note that the differential equation

$$dt=\left(\frac{R+a}{2g}\right)^{1/2}\frac{d\theta}{\sqrt{\cos\theta_1^\circ-\cos\theta_1}} \tag{4.385}$$

may be integrated to yield $\theta_1(t)$ for $t\in[0,t^*]$, where $\theta_1(t^*)=\theta_1^*$, *i.e.* t^* is the time to detachment.

4.11.2 Frictionless motion along a curve

Consider the situation in fig. 4.15 where a skier moves frictionlessly under the influence of gravity along a general curve $y=h(x)$. The Lagrangian for this problem is

$$L=\tfrac{1}{2}m(\dot{x}^2+\dot{y}^2)-mgy \tag{4.386}$$

and the (holonomic) constraint is

$$G(x,y)=y-h(x)=0 \quad . \tag{4.387}$$

Accordingly, the Euler-Lagrange equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{q}_\sigma}\right)-\frac{\partial L}{\partial q_\sigma}=\lambda\,\frac{\partial G}{\partial q_\sigma} \quad , \tag{4.388}$$

where $q_1=x$ and $q_2=y$. Thus, we obtain

$$\begin{aligned} m\ddot{x}&=-\lambda\,h'(x)=Q_x\\ m\ddot{y}+mg&=\lambda=Q_y \quad . \end{aligned} \tag{4.389}$$

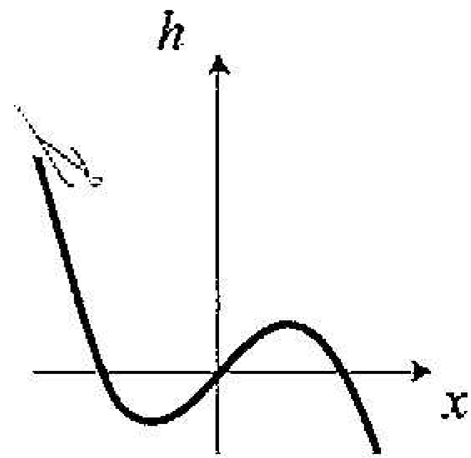

Figure 4.15: Frictionless motion under gravity along a curved surface. The skier flies off the surface when the normal force vanishes.

We eliminate y in favor of x by invoking the constraint. Since we need $\ddot{y}$, we must differentiate the constraint, which gives

$$\dot{y} = h'(x)\,\dot{x} \qquad , \qquad \ddot{y} = h'(x)\,\ddot{x} + h''(x)\,\dot{x}^2 \quad . \tag{4.390}$$

Using the second Euler-Lagrange equation, we then obtain

$$\frac{\lambda}{m} = g + h'(x)\,\ddot{x} + h''(x)\,\dot{x}^2 \quad . \tag{4.391}$$

Finally, we substitute this into the first E-L equation to obtain an equation for x alone:

$$\Big(1 + \big[h'(x)\big]^2\Big)\,\ddot{x} + h'(x)\,h''(x)\,\dot{x}^2 + g\,h'(x) = 0 \quad . \tag{4.392}$$

Had we started by eliminating $y = h(x)$ at the outset, writing

$$L(x,\dot{x}) = \tfrac{1}{2}m\Big(1 + \big[h'(x)\big]^2\Big)\,\dot{x}^2 - mg\,h(x) \quad , \tag{4.393}$$

we would also have obtained this equation of motion.

The skier flies off the curve when the vertical force of constraint $Q_y = \lambda$ starts to become negative, because the curve can only supply a positive normal force. Suppose the skier starts from rest at a height y_0. We may then determine the point x at which the skier detaches from the curve by setting $\lambda(x) = 0$. To do so, we must eliminate $\dot{x}$ and $\ddot{x}$ in terms of x. For $\ddot{x}$, we may use the equation of motion to write

$$\ddot{x} = -\bigg(\frac{gh' + h'\,h''\,\dot{x}^2}{1 + h'^2}\bigg) \quad , \tag{4.394}$$

which allows us to write

$$\lambda = m\bigg(\frac{g + h''\,\dot{x}^2}{1 + h'^2}\bigg) \quad . \tag{4.395}$$

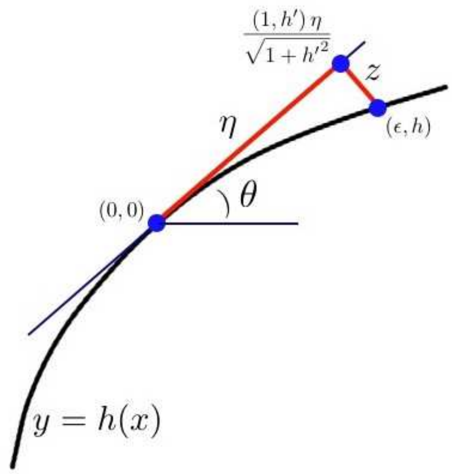

Figure 4.16: Finding the local radius of curvature: $z = \eta^2/2R$.

To eliminate $\dot{x}$, we use conservation of energy,

$$E = mgy_0 = \tfrac{1}{2}m\big(1 + h'^2\big)\,\dot{x}^2 + mgh \quad , \tag{4.396}$$

which fixes

$$\dot{x}^2 = 2g\bigg(\frac{y_0 - h}{1 + h'^2}\bigg) \quad . \tag{4.397}$$

Putting it all together, we have

$$\lambda(x) = \frac{mg}{\big(1 + h'^2\big)^2}\Big\{1 + h'^2 + 2(y_0 - h)\,h''\Big\} \quad . \tag{4.398}$$

The skier detaches from the curve when $\lambda(x) = 0$, *i.e.* when

$$1 + h'^2 + 2(y_0 - h)\,h'' = 0 \quad . \tag{4.399}$$

There is a somewhat easier way of arriving at the same answer. This is to note that the skier must fly off when the local centripetal force equals the gravitational force normal to the curve, *i.e.*

$$\frac{m\,v^2(x)}{R(x)} = mg\,\cos\theta(x) \quad , \tag{4.400}$$

where $R(x)$ is the local radius of curvature. Now $\tan\theta = h'$, so $\cos\theta = \big(1 + h'^2\big)^{-1/2}$. The square of the velocity is $v^2 = \dot{x}^2 + \dot{y}^2 = \big(1 + h'^2\big)\,\dot{x}^2$. What is the local radius of curvature $R(x)$? This can be determined from the following argument, and from the sketch in fig. 4.16. Writing $x = x^* + \epsilon$, we have

$$y = h(x^*) + h'(x^*)\,\epsilon + \tfrac{1}{2}h''(x^*)\,\epsilon^2 + \ldots \quad . \tag{4.401}$$

We now drop a perpendicular segment of length z from the point (x, y) to the line which is tangent to the curve at $\big(x^*, h(x^*)\big)$. According to fig. 4.16, this means

$$\begin{pmatrix} \epsilon \\ y \end{pmatrix} = \eta \cdot \tfrac{1}{\sqrt{1+h'^2}} \begin{pmatrix} 1 \\ h' \end{pmatrix} - z \cdot \tfrac{1}{\sqrt{1+h'^2}} \begin{pmatrix} -h' \\ 1 \end{pmatrix} \quad . \tag{4.402}$$

Thus, we have

$$\begin{aligned} y &= h'\,\epsilon + \tfrac{1}{2}h''\,\epsilon^2 \\ &= h'\left(\frac{\eta + z\,h'}{\sqrt{1+h'^2}}\right) + \tfrac{1}{2}h''\left(\frac{\eta + z\,h'}{\sqrt{1+h'^2}}\right)^2 \\ &= \frac{\eta\,h' + z\,h'^2}{\sqrt{1+h'^2}} + \frac{h''\,\eta^2}{2\big(1+h'^2\big)} + \mathcal{O}(\eta z) = \frac{\eta\,h' - z}{\sqrt{1+h'^2}} \quad , \end{aligned} \tag{4.403}$$

from which we obtain

$$z = -\frac{h''\,\eta^2}{2\big(1+h'^2\big)^{3/2}} + \mathcal{O}(\eta^3) \tag{4.404}$$

and therefore

$$R(x) = -\frac{1}{h''(x)} \cdot \Big(1 + \big[h'(x)\big]^2\Big)^{3/2} \quad . \tag{4.405}$$

Thus, the detachment condition,

$$\frac{mv^2}{R} = -\frac{m\,h''\,\dot{x}^2}{\sqrt{1+h'^2}} = \frac{mg}{\sqrt{1+h'^2}} = mg\,\cos\theta \tag{4.406}$$

reproduces the result from eqn. 4.395.

4.11.3 Disk rolling down an inclined plane

A hoop of mass m and radius R rolls without slipping down an inclined plane. The inclined plane has opening angle α and mass M, and itself slides frictionlessly along a horizontal surface. Find the motion of the system.

Solution : Referring to the sketch in fig. 4.17, the center of the hoop is located at

$$\begin{aligned} x &= X + s\,\cos\alpha - a\,\sin\alpha \\ y &= s\,\sin\alpha + a\,\cos\alpha \quad , \end{aligned} \tag{4.407}$$

where X is the location of the lower left corner of the wedge, and s is the distance along the wedge to the bottom of the hoop. If the hoop rotates through an angle θ, the no-slip condition is $a\,\dot{\theta} + \dot{s} = 0$. Thus,

$$\begin{aligned} L &= \tfrac{1}{2}M\dot{X}^2 + \tfrac{1}{2}m\big(\dot{x}^2 + \dot{y}^2\big) + \tfrac{1}{2}I\dot{\theta}^2 - mgy \\ &= \tfrac{1}{2}\left(m + \frac{I}{a^2}\right)\dot{s}^2 + \tfrac{1}{2}(M+m)\dot{X}^2 + m\cos\alpha\,\dot{X}\,\dot{s} - mgs\,\sin\alpha - mga\,\cos\alpha \quad . \end{aligned} \tag{4.408}$$

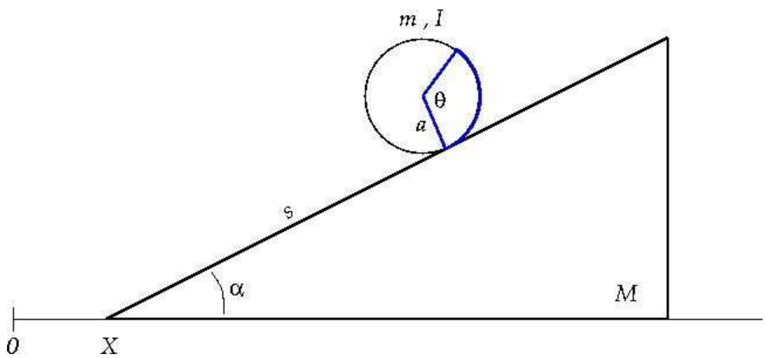

Figure 4.17: A hoop rolling down an inclined plane lying on a frictionless surface.

Since X is cyclic in L, the momentum

$$P_X = (M+m)\dot{X} + m\cos\alpha\,\dot{s} \quad , \tag{4.409}$$

is preserved: $\dot{P}_X = 0$. The second equation of motion, corresponding to the generalized coordinate s, is

$$\left(1 + \frac{I}{ma^2}\right)\ddot{s} + \cos\alpha\,\ddot{X} = -g\,\sin\alpha \quad . \tag{4.410}$$

Using conservation of P_X, we eliminate $\ddot{s}$ in favor of $\ddot{X}$, and immediately obtain

$$\ddot{X} = \frac{g\sin\alpha\cos\alpha}{\left(1+\frac{M}{m}\right)\left(1+\frac{I}{ma^2}\right) - \cos^2\alpha} \equiv a_X \quad . \tag{4.411}$$

The result

$$\ddot{s} = -\frac{g\left(1+\frac{M}{m}\right)\sin\alpha}{\left(1+\frac{M}{m}\right)\left(1+\frac{I}{ma^2}\right) - \cos^2\alpha} \equiv a_s \tag{4.412}$$

follows immediately. Thus,

$$\begin{aligned} X(t) &= X(0) + \dot{X}(0)\,t + \tfrac{1}{2}a_X t^2 \\ s(t) &= s(0) + \dot{s}(0)\,t + \tfrac{1}{2}a_s t^2 \quad . \end{aligned} \tag{4.413}$$

Note that $a_s < 0$ while $a_X > 0$, *i.e.* the hoop rolls down and to the left as the wedge slides to the right. Note that $I = ma^2$ for a hoop; we've computed the answer here for general I.

4.11.4 Pendulum with nonrigid support

A particle of mass m is suspended from a flexible string of length ℓ in a uniform gravitational field. While hanging motionless in equilibrium, it is struck a horizontal blow resulting in an initial angular velocity ω_0. Treating the system as one with *two* degrees of freedom and a constraint, answer the following:

(a) Compute the Lagrangian, the equation of constraint, and the equations of motion.

Solution : The Lagrangian is

$$L = \tfrac{1}{2}m\left(\dot{r}^2 + r^2\,\dot{\theta}^2\right) + mgr\,\cos\theta \quad . \tag{4.414}$$

The constraint is $r = \ell$. The equations of motion are

$$\begin{aligned} m\ddot{r} - mr\,\dot{\theta}^2 - mg\,\cos\theta &= \lambda \\ mr^2\,\ddot{\theta} + 2mr\,\dot{r}\,\dot{\theta} - mg\sin\theta &= 0 \quad . \end{aligned} \tag{4.415}$$

(b) Compute the tension in the string as a function of angle θ.

Solution : Energy is conserved, hence

$$\tfrac{1}{2}m\ell^2\,\dot{\theta}^2 - mg\ell\cos\theta = \tfrac{1}{2}m\ell^2\,\dot{\theta}_0^2 - mg\ell\cos\theta_0 \quad . \tag{4.416}$$

We take $\theta_0 = 0$ and $\dot{\theta}_0 = \omega_0$. Thus,

$$\dot{\theta}^2 = \omega_0^2 - 2\,\Omega^2\left(1 - \cos\theta\right) \quad , \tag{4.417}$$

with $\Omega = \sqrt{g/\ell}$. Substituting this into the equation for λ, we obtain

$$\lambda = mg\left\{2 - 3\cos\theta - \frac{\omega_0^2}{\Omega^2}\right\} \quad . \tag{4.418}$$

(c) Show that if $\omega_0^2 < 2g/\ell$ then the particle's motion is confined below the horizontal and that the tension in the string is always positive (defined such that positive means exerting a pulling force and negative means exerting a pushing force). Note that the difference between a string and a rigid rod is that the string can only pull but the rod can pull or push. Thus, *the string tension must always be positive or else the string goes "slack"*.

Solution : Since $\dot{\theta}^2 \geq 0$, we must have

$$\frac{\omega_0^2}{2\Omega^2} \geq 1 - \cos\theta \quad . \tag{4.419}$$

The condition for slackness is $\lambda = 0$, or

$$\frac{\omega_0^2}{2\Omega^2} = 1 - \tfrac{3}{2}\cos\theta \quad . \tag{4.420}$$

Thus, if $\omega_0^2 < 2\Omega^2$, we have

$$1 > \frac{\omega_0^2}{2\Omega^2} > 1 - \cos\theta > 1 - \tfrac{3}{2}\cos\theta \quad , \tag{4.421}$$

and the string never goes slack. Note the last equality follows from $\cos\theta > 0$. The string rises to a maximum angle

$$\theta_{\text{max}} = \cos^{-1}\left(1 - \frac{\omega_0^2}{2\Omega^2}\right) \quad . \tag{4.422}$$

(d) Show that if $2g/\ell < \omega_0^2 < 5g/\ell$ the particle rises above the horizontal and the string becomes slack (the tension vanishes) at an angle θ^*. Compute θ^*.

Solution : When $\omega^2 > 2\Omega^2$, the string rises above the horizontal and goes slack at an angle

$$\theta^* = \cos^{-1}\left(\tfrac{2}{3} - \frac{\omega_0^2}{3\Omega^2}\right) \quad . \tag{4.423}$$

This solution craps out when the string is still taut at $\theta = \pi$, which means $\omega_0^2 = 5\Omega^2$.

(e) Show that if $\omega_0^2 > 5g/\ell$ the tension is always positive and the particle executes circular motion.

Solution : For $\omega_0^2 > 5\Omega^2$, the string never goes slack. Furthermore, $\dot{\theta}$ never vanishes. Therefore, the pendulum undergoes circular motion, albeit not with constant angular velocity.

4.11.5 Falling ladder

A uniform ladder of length ℓ and mass m has one end on a smooth horizontal floor and the other end against a smooth vertical wall. The ladder is initially at rest and makes an angle θ_0 with respect to the horizontal.

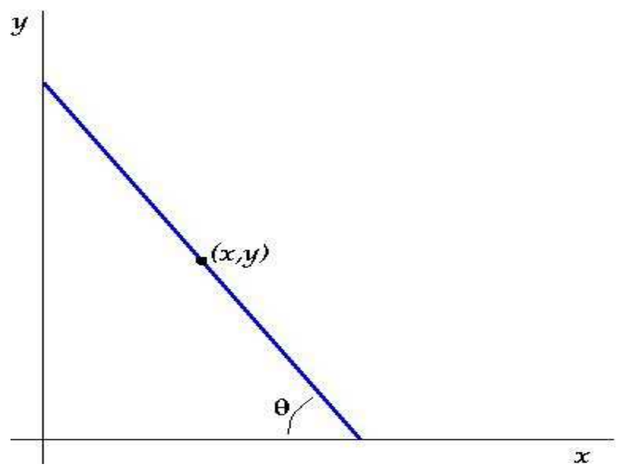

Figure 4.18: A ladder sliding down a wall and across a floor.

(a) Make a convenient choice of generalized coordinates and find the Lagrangian.

Solution : I choose as generalized coordinates the Cartesian coordinates (x, y) of the ladder's center of mass, and the angle θ it makes with respect to the floor. The Lagrangian is then

$$L = \tfrac{1}{2}\, m\, (\dot{x}^2 + \dot{y}^2) + \tfrac{1}{2}\, I\, \dot{\theta}^2 - mgy \quad . \tag{4.424}$$

There are two constraints: one enforcing contact along the wall, and the other enforcing contact along the floor. These are written

$$\begin{aligned} G_1(x, y, \theta) &= x - \tfrac{1}{2}\,\ell\, \cos\theta = 0 \\ G_2(x, y, \theta) &= y - \tfrac{1}{2}\,\ell\, \sin\theta = 0 \quad . \end{aligned} \tag{4.425}$$

(b) Prove that the ladder leaves the wall when its upper end has fallen to a height $\frac{2}{3} L \sin\theta_0$.

Solution : The equations of motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right) - \frac{\partial L}{\partial q_\sigma} = \sum_j \lambda_j \, \frac{\partial G_j}{\partial q_\sigma} \quad . \tag{4.426}$$

Thus, we have

$$\begin{aligned} m\ddot{x} &= \lambda_1 = Q_x \\ m\ddot{y} + mg &= \lambda_2 = Q_y \\ I\,\ddot{\theta} &= \tfrac{1}{2}\ell\big(\lambda_1 \sin\theta - \lambda_2 \cos\theta\big) = Q_\theta \quad . \end{aligned} \tag{4.427}$$

We now implement the constraints to eliminate x and y in terms of θ. We have

$$\begin{aligned} \dot{x} &= -\tfrac{1}{2}\,\ell\,\sin\theta\,\dot{\theta} \qquad , \qquad \ddot{x} = -\tfrac{1}{2}\,\ell\,\cos\theta\,\dot{\theta}^2 - \tfrac{1}{2}\,\ell\,\sin\theta\,\ddot{\theta} \\ \dot{y} &= \tfrac{1}{2}\,\ell\,\cos\theta\,\dot{\theta} \qquad , \qquad \ddot{y} = -\tfrac{1}{2}\,\ell\,\sin\theta\,\dot{\theta}^2 + \tfrac{1}{2}\,\ell\,\cos\theta\,\ddot{\theta} \quad . \end{aligned} \tag{4.428}$$

We can now obtain the forces of constraint in terms of the function $\theta(t)$:

$$\begin{aligned} \lambda_1 &= -\tfrac{1}{2}m\ell\left(\sin\theta\,\ddot{\theta} + \cos\theta\,\dot{\theta}^2\right) \\ \lambda_2 &= +\tfrac{1}{2}m\ell\left(\cos\theta\,\ddot{\theta} - \sin\theta\,\dot{\theta}^2\right) + mg \quad . \end{aligned} \tag{4.429}$$

We substitute these into the last equation of motion to obtain the result

$$I\,\ddot{\theta} = -I_0\,\ddot{\theta} - \tfrac{1}{2}mg\ell\,\cos\theta \quad , \tag{4.430}$$

which is to say $(1+\alpha)\,\ddot{\theta} = -2\omega_0^2\,\cos\theta$, with $I_0 = \frac{1}{4}m\ell^2$, $\alpha \equiv I/I_0$ and $\omega_0 = \sqrt{g/\ell}$. This may be integrated once (multiply by $\dot{\theta}$ to convert to a total derivative) to yield

$$\tfrac{1}{2}(1+\alpha)\,\dot{\theta}^2 + 2\,\omega_0^2\,\sin\theta = 2\,\omega_0^2\,\sin\theta_0 \quad , \tag{4.431}$$

which is of course a statement of energy conservation. This,

$$\dot{\theta}^2 = \frac{4\,\omega_0^2\,(\sin\theta_0 - \sin\theta)}{1+\alpha} \qquad , \qquad \ddot{\theta} = -\frac{2\,\omega_0^2\,\cos\theta}{1+\alpha} \quad . \tag{4.432}$$

We may now obtain $\lambda_1(\theta)$ and $\lambda_2(\theta)$:

$$\begin{aligned} \lambda_1(\theta) &= -\frac{mg}{1+\alpha}\,\big(3\sin\theta - 2\sin\theta_0\big)\,\cos\theta \\ \lambda_2(\theta) &= \frac{mg}{1+\alpha}\,\Big\{\big(3\sin\theta - 2\sin\theta_0\big)\,\sin\theta + \alpha\Big\} \quad . \end{aligned} \tag{4.433}$$

Demanding $\lambda_1(\theta) = 0$ gives the detachment angle $\theta = \theta_{\rm d}$, where

$$\sin\theta_{\rm d} = \tfrac{2}{3}\sin\theta_0 \quad . \tag{4.434}$$

Note that $\lambda_2(\theta_d) = mg\alpha/(1+\alpha) > 0$, so the normal force from the floor is always positive for $\theta > \theta_d$. The time to detachment is

$$T_1(\theta_0) = \int \frac{d\theta}{\dot{\theta}} = \frac{\sqrt{1+\alpha}}{2\,\omega_0} \int\limits_{\theta_d}^{\theta_0} \frac{d\theta}{\sqrt{\sin\theta_0 - \sin\theta}} \quad . \tag{4.435}$$

(c) Show that the subsequent motion can be reduced to quadratures (*i.e.* explicit integrals).

Solution : After the detachment, there is no longer a constraint G_1. The equations of motion are

$$\begin{aligned} m\ddot{x} &= 0 \qquad \text{(conservation of } x\text{-momentum)} \\ m\ddot{y} + m\,g &= \lambda \\ I\,\ddot{\theta} &= -\tfrac{1}{2}\,\ell\,\lambda\,\cos\theta \quad , \end{aligned} \tag{4.436}$$

along with the constraint $y = \frac{1}{2}\,\ell\,\sin\theta$. Eliminating y in favor of θ using the constraint, the second equation yields

$$\lambda = mg - \tfrac{1}{2}m\ell\,\sin\theta\,\dot{\theta}^2 + \tfrac{1}{2}m\ell\,\cos\theta\,\ddot{\theta} \quad . \tag{4.437}$$

Plugging this into the third equation of motion, we find

$$I\,\ddot{\theta} = -2\,I_0\,\omega_0^2\,\cos\theta + I_0\,\sin\theta\,\cos\theta\,\dot{\theta}^2 - I_0\,\cos^2\theta\,\ddot{\theta} \quad . \tag{4.438}$$

Multiplying by $\dot{\theta}$ one again obtains a total time derivative, which is equivalent to rediscovering energy conservation:

$$E = \tfrac{1}{2}\big(I + I_0\,\cos^2\theta\big)\,\dot{\theta}^2 + 2\,I_0\,\omega_0^2\,\sin\theta \quad . \tag{4.439}$$

By continuity with the first phase of the motion, we obtain the initial conditions for this second phase:

$$\theta = \sin^{-1}\big(\tfrac{2}{3}\sin\theta_0\big) \qquad , \qquad \dot{\theta} = -2\,\omega_0\,\sqrt{\frac{\sin\theta_0}{3\,(1+\alpha)}} \quad . \tag{4.440}$$

Thus,

$$\begin{aligned} E &= \tfrac{1}{2}\big(I + I_0 - \tfrac{4}{9}\,I_0\,\sin^2\theta_0\big)\cdot\frac{4\,\omega_0^2\,\sin\theta_0}{3\,(1+\alpha)} + \tfrac{1}{3}\,mg\ell\,\sin\theta_0 \\ &= 2\,I_0\,\omega_0^2\cdot\bigg\{1 + \tfrac{4}{27}\,\frac{\sin^2\theta_0}{1+\alpha}\bigg\}\,\sin\theta_0 \quad . \end{aligned} \tag{4.441}$$

(d) Find an expression for the time $T(\theta_0)$ it takes the ladder to smack against the floor. Note that, expressed in units of the time scale $\sqrt{L/g}$, T is a dimensionless function of θ_0. Numerically integrate this expression and plot T *versus* θ_0.

Solution : The time from detachment to smack is

$$T_2(\theta_0) = \int \frac{d\theta}{\dot{\theta}} = \frac{1}{2\,\omega_0} \int\limits_0^{\theta_d} d\theta\,\sqrt{\frac{1+\alpha\,\cos^2\theta}{\big(1 - \frac{4}{27}\frac{\sin^2\theta_0}{1+\alpha}\big)\sin\theta_0 - \sin\theta}} \quad . \tag{4.442}$$

```
In[37]:= T[x_] := NIntegrate[√((4 / 3) / (x - Sin[y])) , {y, ArcSin[2 x / 3], ArcSin[x] - 10^-9}] / 2

In[38]:= S[x_] := NIntegrate[
           √((1 + (4 / 3) (Cos[y]) ^2) / ((1 - (x / 3) ^2) x - Sin[y])) , {y, 0, ArcSin[2 x / 3]}] / 2

In[39]:= Q[x_] := T[x] + S[x]

In[43]:= Plot[Q[x], {x, 0, 1}]
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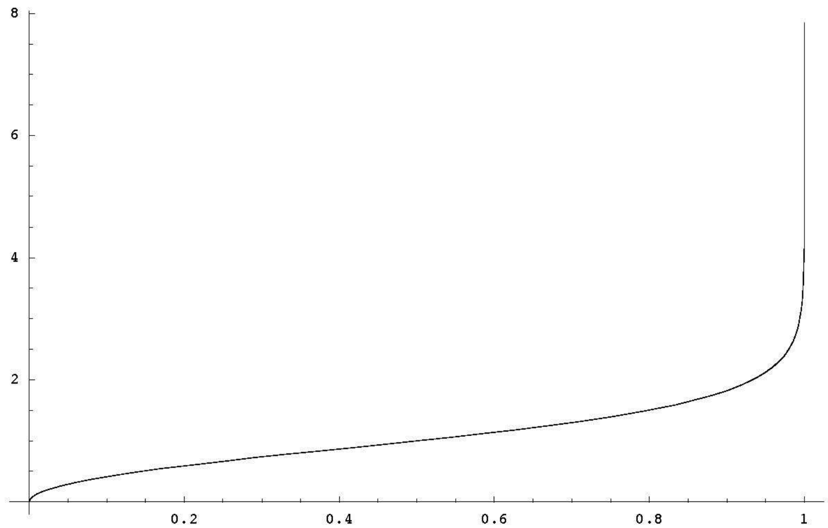


Figure 4.19: Plot of time to fall for the slipping ladder. Here $x = \sin\theta_0$.

The total time is then $T(\theta_0) = T_1(\theta_0) + T_2(\theta_0)$. For a uniformly dense ladder, $I = \frac{1}{12}\,m\ell^2 = \frac{1}{3}\,I_0$, and therefore $\alpha = \frac{1}{3}$.

(e) What is the horizontal velocity of the ladder at long times?

Solution : From the moment of detachment, and thereafter,

$$\dot{x} = -\tfrac{1}{2}\,\ell\,\sin\theta\,\dot{\theta} = \sqrt{\frac{4\,g\,\ell}{27\,(1+\alpha)}}\;\sin^{3/2}\!\theta_0 \quad . \tag{4.443}$$

(f) Describe in words the motion of the ladder subsequent to it slapping against the floor.

Solution : Only a fraction of the ladder's initial potential energy is converted into kinetic energy of horizontal motion. The rest is converted into kinetic energy of vertical motion and of rotation. The slapping of the ladder against the floor is an elastic collision. After the collision, the ladder must rise again, and continue to rise and fall *ad infinitum*, as it slides along with constant horizontal velocity.

4.11.6 Point mass inside rolling hoop

Consider the point mass m inside the hoop of radius R, depicted in fig. 4.20. We choose as generalized coordinates the Cartesian coordinates (X, Y) of the center of the hoop, the Cartesian coordinates (x, y) for the point mass, the angle ϕ through which the hoop turns, and the angle θ which the point mass makes with respect to the vertical. These six coordinates are not all independent. Indeed, there are only two independent coordinates for this system, which can be taken to be θ and ϕ. Thus, there are *four* constraints:

$$\begin{aligned} X - R\phi \equiv G_1 &= 0 \\ Y - R \equiv G_2 &= 0 \\ x - X - R\sin\theta \equiv G_3 &= 0 \\ y - Y + R\cos\theta \equiv G_4 &= 0 \quad . \end{aligned} \tag{4.444}$$

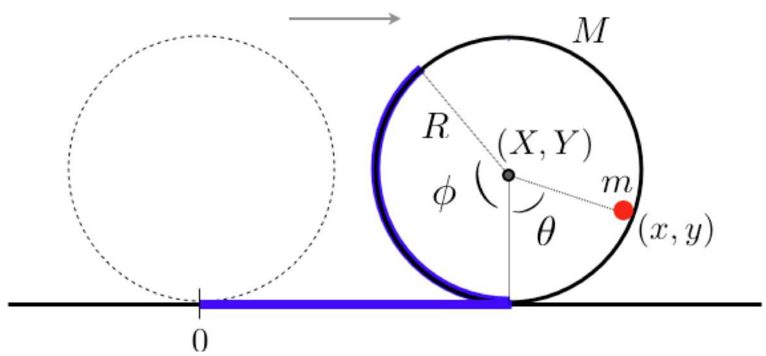


Figure 4.20: A point mass m inside a hoop of mass M, radius R, and moment of inertia I.

The kinetic and potential energies are easily expressed in terms of the Cartesian coordinates, aside from the energy of rotation of the hoop about its CM, which is expressed in terms of $\dot{\phi}$:

$$\begin{aligned} T &= \tfrac{1}{2}M(\dot{X}^2 + \dot{Y}^2) + \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \tfrac{1}{2}I\,\dot{\phi}^2 \\ U &= MgY + mgy \quad . \end{aligned} \tag{4.445}$$

The moment of inertia of the hoop about its CM is $I = MR^2$, but we could imagine a situation in which I were different. For example, we could instead place the point mass inside a very short cylinder with two solid end caps, in which case $I = \frac{1}{2}MR^2$. The Lagrangian is then

$$L = \tfrac{1}{2}M(\dot{X}^2 + \dot{Y}^2) + \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \tfrac{1}{2}I\,\dot{\phi}^2 - MgY - mgy \quad . \tag{4.446}$$

Note that L as written is completely independent of θ and $\dot{\theta}$!

Continuous symmetry

Note that there is a continuous symmetry to L which is satisfied by all the constraints, under

$$\tilde{X}(\zeta) = X + \zeta \quad , \quad \tilde{Y}(\zeta) = Y \quad , \quad \tilde{x}(\zeta) = x + \zeta \quad , \quad \tilde{y}(\zeta) = y \quad , \quad \tilde{\phi}(\zeta) = \phi + \frac{\zeta}{R} \quad , \quad \tilde{\theta}(\zeta) = \theta \quad . \tag{4.447}$$

Thus, according to Noether's theorem, there is a conserved quantity

$$\Lambda = \frac{\partial L}{\partial \dot{X}} + \frac{\partial L}{\partial \dot{x}} + \frac{1}{R}\frac{\partial L}{\partial \dot{\phi}} = M\dot{X} + m\dot{x} + \frac{I}{R}\,\dot{\phi} \quad . \tag{4.448}$$

This means $\dot{\Lambda} = 0$. This reflects the overall conservation of momentum in the x-direction.

Energy conservation

Since neither L nor any of the constraints are explicitly time-dependent, the Hamiltonian is conserved. And since T is homogeneous of degree two in the generalized velocities, we have $H = E = T + U$:

$$E = \tfrac{1}{2}M(\dot{X}^2 + \dot{Y}^2) + \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \tfrac{1}{2}I\,\dot{\phi}^2 + MgY + mgy \quad . \tag{4.449}$$

Equations of motion

We have $n = 6$ generalized coordinates and $k = 4$ constraints. Thus, there are four undetermined multipliers $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ used to impose the constraints. This makes for ten unknowns: X, Y, x, y, ϕ, θ, λ_1, λ_2, λ_3, and λ_4. Accordingly, we have ten equations: six equations of motion plus the four equations of constraint. The equations of motion are obtained from

$$\frac{d}{dt}\bigg(\frac{\partial L}{\partial \dot{q}_\sigma}\bigg) = \frac{\partial L}{\partial q_\sigma} + \sum_{j=1}^{k} \lambda_j\,\frac{\partial G_j}{\partial q_\sigma} \quad . \tag{4.450}$$

Taking each generalized coordinate in turn, the equations of motion are thus

$$\begin{aligned}
M\ddot{X} &= \lambda_1 - \lambda_3 \\
M\ddot{Y} &= -Mg + \lambda_2 - \lambda_4 \\
m\ddot{x} &= \lambda_3 \\
m\ddot{y} &= -mg + \lambda_4 \\
I\,\ddot{\phi} &= -R\,\lambda_1 \\
0 &= -R\cos\theta\,\lambda_3 - R\sin\theta\,\lambda_4 \quad .
\end{aligned} \tag{4.451}$$

Along with the four constraint equations, these determine the motion of the system. Note that the last of the equations of motion, for the generalized coordinate $q_\sigma = \theta$, says that $Q_\theta = 0$, which means that the force of constraint on the point mass is radial. Were the point mass replaced by a rolling object, there would be an angular component to this constraint in order that there be no slippage.

Implementation of constraints

We now use the constraint equations to eliminate X, Y, x, and y in terms of θ and ϕ:

$$X = R\phi \quad , \quad Y = R \quad , \quad x = R\phi + R\sin\theta \quad , \quad y = R(1 - \cos\theta) \quad . \tag{4.452}$$

We also need the derivatives:

$$\dot{x} = R\,\dot{\phi} + R\cos\theta\,\dot{\theta} \quad , \quad \ddot{x} = R\,\ddot{\phi} + R\cos\theta\,\ddot{\theta} - R\sin\theta\,\dot{\theta}^2 \quad , \tag{4.453}$$

and

$$\dot{y} = R\sin\theta\,\dot{\theta} \quad , \quad \ddot{y} = R\sin\theta\,\ddot{\theta} + R\cos\theta\,\dot{\theta}^2 \quad , \tag{4.454}$$

as well as

$$\dot{X} = R\,\dot{\phi} \quad , \quad \ddot{X} = R\,\ddot{\phi} \quad , \quad \dot{Y} = 0 \quad , \quad \ddot{Y} = 0 \quad . \tag{4.455}$$

We now may write the conserved charge as

$$\Lambda = \frac{1}{R}(I + MR^2 + mR^2)\,\dot{\phi} + mR\cos\theta\,\dot{\theta} \quad . \tag{4.456}$$

This, in turn, allows us to eliminate $\dot{\phi}$ in terms of $\dot{\theta}$ and the constant Λ:

$$\dot{\phi} = \frac{\gamma}{1+\gamma}\left(\frac{\Lambda}{mR} - \dot{\theta}\cos\theta\right) \quad , \tag{4.457}$$

where $\gamma = mR^2/(I + MR^2)$.

The energy is then

$$\begin{aligned} E &= \tfrac{1}{2}(I + MR^2)\,\dot{\phi}^2 + \tfrac{1}{2}m\big(R^2\,\dot{\phi}^2 + R^2\,\dot{\theta}^2 + 2R^2\cos\theta\,\dot{\phi}\,\dot{\theta}\big) + MgR + mgR(1-\cos\theta) \\ &= \tfrac{1}{2}mR^2\left\{\left(\frac{1+\gamma\sin^2\!\theta}{1+\gamma}\right)\dot{\theta}^2 + \frac{2g}{R}\,(1-\cos\theta) + \frac{\gamma}{1+\gamma}\left(\frac{\Lambda}{mR}\right)^2 + \frac{2Mg}{mR}\right\} \quad . \end{aligned} \tag{4.458}$$

The last two terms inside the big bracket are constant, so we can write this as

$$\left(\frac{1+\gamma\sin^2\!\theta}{1+\gamma}\right)\dot{\theta}^2 + \frac{2g}{R}\,(1-\cos\theta) = \frac{4gk}{R} \quad . \tag{4.459}$$

Here, k is a dimensionless measure of the energy of the system, after subtracting the aforementioned constants. If $k > 1$, then $\dot{\theta}^2 > 0$ for all θ, which would result in 'loop-the-loop' motion of the point mass inside the hoop – provided, that is, the normal force of the hoop doesn't vanish and the point mass doesn't detach from the hoop's surface.

Equation motion for $\theta(t)$

The equation of motion for θ obtained by eliminating all other variables from the original set of ten equations is the same as $\dot{E} = 0$, and may be written

$$\left(\frac{1+\gamma\sin^2\!\theta}{1+\gamma}\right)\ddot{\theta} + \left(\frac{\gamma\sin\theta\cos\theta}{1+\gamma}\right)\dot{\theta}^2 = -\frac{g}{R} \quad . \tag{4.460}$$

We can use this to write $\ddot{\theta}$ in terms of $\dot{\theta}^2$, and, after invoking eqn. 4.459, in terms of θ itself. We find

$$\begin{aligned} \dot{\theta}^2 &= \frac{4g}{R}\cdot\left(\frac{1+\gamma}{1+\gamma\sin^2\!\theta}\right)\big(k - \sin^2\tfrac{1}{2}\theta\big) \\ \ddot{\theta} &= -\frac{g}{R}\cdot\frac{(1+\gamma)\sin\theta}{\big(1+\gamma\sin^2\!\theta\big)^2}\left[4\gamma\,\big(k - \sin^2\tfrac{1}{2}\theta\big)\cos\theta + 1 + \gamma\sin^2\!\theta\right] \quad . \end{aligned} \tag{4.461}$$

Forces of constraint

We can solve for the λ_j, and thus obtain the forces of constraint

$$\begin{aligned}\lambda_3 = m\ddot{x} &= mR\,\ddot{\phi} + mR\cos\theta\,\ddot{\theta} - mR\sin\theta\,\dot{\theta}^2 \\ &= \frac{mR}{1+\gamma}\Big[\ddot{\theta}\,\cos\theta - \dot{\theta}^2\sin\theta\Big]\end{aligned} \tag{4.462}$$

and

$$\begin{aligned}\lambda_4 = m\ddot{y} + mg &= mg + mR\sin\theta\,\ddot{\theta} + mR\cos\theta\,\dot{\theta}^2 \\ &= mR\Big[\ddot{\theta}\,\sin\theta + \dot{\theta}^2\sin\theta + \frac{g}{R}\Big]\end{aligned} \tag{4.463}$$

and

$$\begin{aligned}\lambda_1 &= -\frac{I}{R}\,\ddot{\phi} = \frac{(1+\gamma)I}{mR^2}\,\lambda_3 \\ \lambda_2 &= (M+m)g + m\ddot{y} = \lambda_4 + Mg \quad .\end{aligned} \tag{4.464}$$

One can check that $\lambda_3\cos\theta + \lambda_4\sin\theta = 0$.

The condition that the normal force of the hoop on the point mass vanish is $\lambda_3 = 0$, which entails $\lambda_4 = 0$. This gives

$$-\,(1+\gamma\sin^2\!\theta)\,\cos\theta = 4(1+\gamma)\,\big(k - \sin^2\!\tfrac{1}{2}\theta\big) \quad . \tag{4.465}$$

Note that this requires $\cos\theta < 0$, *i.e.* the point of detachment lies above the horizontal diameter of the hoop. Clearly if k is sufficiently large, the equality cannot be satisfied, and the point mass executes a periodic 'loop-the-loop' motion. In particular, setting $\theta = \pi$, we find that

$$k_{\rm c} = 1 + \frac{1}{4(1+\gamma)} \quad . \tag{4.466}$$

If $k > k_{\rm c}$, then there is periodic 'loop-the-loop' motion. If $k < k_{\rm c}$, then the point mass may detach at a critical angle θ^*, but only if the motion allows for $\cos\theta < 0$. From the energy conservation equation, we have that the maximum value of θ achieved occurs when $\dot{\theta} = 0$, which means

$$\cos\theta_{\rm max} = 1 - 2k \quad . \tag{4.467}$$

If $\frac{1}{2} < k < k_{\rm c}$, then, we have the possibility of detachment. This means the energy must be large enough but not too large.

4.12 Appendix: Legendre Transformations

A *convex function* of a single variable $f(x)$ is one for which $f''(x) > 0$ everywhere. The *Legendre transform* of a convex function $f(x)$ is a function $g(p)$ defined as follows. Let p be a real number, and consider the

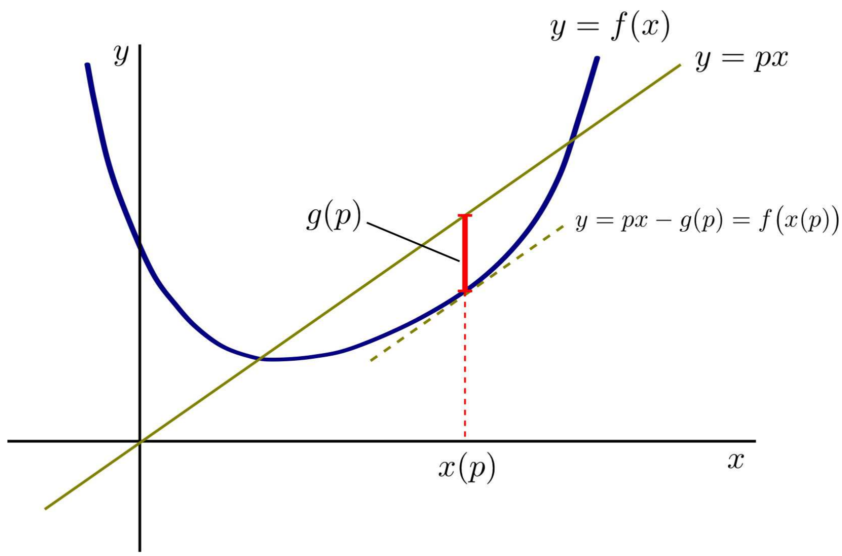


Figure 4.21: Construction for the Legendre transformation of a function $f(x)$.

line $y = px$, as shown in fig. 4.21. We define the point $x(p)$ as the value of x for which the difference $F(x,p) = px - f(x)$ is greatest. Then define $g(p) = F\big(x(p),p\big)$.[9] The value $x(p)$ is unique if $f(x)$ is convex, since $x(p)$ is determined by the equation

$$f'\big(x(p)\big) = p \quad . \tag{4.468}$$

Note that from $p = f'\big(x(p)\big)$ we have, according to the chain rule,

$$1 = \frac{d}{dp}\, f'\big(x(p)\big) = f''\big(x(p)\big)\, x'(p) \qquad \Longrightarrow \qquad x'(p) = \Big[f''\big(x(p)\big)\Big]^{-1} \quad . \tag{4.469}$$

From this, we can prove that $g(p)$ is itself convex:

$$\begin{aligned} g'(p) &= \frac{d}{dp}\Big[p\, x(p) - f\big(x(p)\big)\Big] \\ &= p\, x'(p) + x(p) - f'\big(x(p)\big)\, x'(p) = x(p) \quad , \end{aligned} \tag{4.470}$$

hence

$$g''(p) = x'(p) = \Big[f''\big(x(p)\big)\Big]^{-1} > 0 \quad . \tag{4.471}$$

In higher dimensions, the generalization of the definition $f''(x) > 0$ is that a function $F(x_1, \ldots, x_n)$ is convex if the matrix of second derivatives, called the *Hessian*,

$$H_{ij}(\boldsymbol{x}) = \frac{\partial^2 F}{\partial x_i\, \partial x_j} \tag{4.472}$$

[9]Note that $g(p)$ may be a negative number, if the line $y = px$ lies everywhere below $f(x)$.

is positive definite. That is, all the eigenvalues of $H_{ij}(\boldsymbol{x})$ must be positive for every $\boldsymbol{x}$. We then define the Legendre transform $\boldsymbol{G}(\boldsymbol{p})$ as

$$\boldsymbol{G}(\boldsymbol{p}) = \boldsymbol{p}\cdot\boldsymbol{x} - F(\boldsymbol{x}) \tag{4.473}$$

where $\boldsymbol{p} = \boldsymbol{\nabla} F$. Note that

$$dG = \boldsymbol{x}\cdot d\boldsymbol{p} + \boldsymbol{p}\cdot d\boldsymbol{x} - \boldsymbol{\nabla} F\cdot d\boldsymbol{x} = \boldsymbol{x}\cdot d\boldsymbol{p} \quad , \tag{4.474}$$

which establishes that G is a function of $\boldsymbol{p}$ and that

$$\frac{\partial G}{\partial p_j} = x_j \quad . \tag{4.475}$$

Note also that the Legendre transformation is *self dual*, which is to say that the Legendre transform of $G(\boldsymbol{p})$ is $F(\boldsymbol{x})$: $F \to G \to F$ under successive Legendre transformations.

We can also define a *partial Legendre transformation* as follows. Consider a function of q variables $F(\boldsymbol{x},\boldsymbol{y})$, where $\boldsymbol{x} = \{x_1,\ldots,x_m\}$ and $\boldsymbol{y} = \{y_1,\ldots,y_n\}$, with $q = m+n$. Define $\boldsymbol{p} = \{p_1,\ldots,p_m\}$, and

$$G(\boldsymbol{p},\boldsymbol{y}) = \boldsymbol{p}\cdot\boldsymbol{x} - F(\boldsymbol{x},\boldsymbol{y}) \quad , \tag{4.476}$$

where

$$p_a = \frac{\partial F}{\partial x_a} \quad , \quad a\in\{1,\ldots,m\} \quad . \tag{4.477}$$

These equations are then to be inverted to yield

$$x_a = x_a(\boldsymbol{p},\boldsymbol{y}) = \frac{\partial G}{\partial p_a} \quad . \tag{4.478}$$

Note that

$$p_a = \frac{\partial F}{\partial x_a}\big(\boldsymbol{x}(\boldsymbol{p},\boldsymbol{y}),\boldsymbol{y}\big) \quad . \tag{4.479}$$

Thus, from the chain rule,

$$\delta_{ab} = \frac{\partial p_a}{\partial p_b} = \frac{\partial^2 F}{\partial x_a\,\partial x_c}\,\frac{\partial x_c}{\partial p_b} = \frac{\partial^2 F}{\partial x_a\,\partial x_c}\,\frac{\partial^2 G}{\partial p_c\,\partial p_b} \quad , \tag{4.480}$$

which says

$$\frac{\partial^2 G}{\partial p_a\,\partial p_b} = \frac{\partial x_a}{\partial p_b} = \mathsf{K}^{-1}_{ab} \quad , \tag{4.481}$$

where the $m\times m$ partial Hessian is

$$\frac{\partial^2 F}{\partial x_a\,\partial x_b} = \frac{\partial p_a}{\partial x_b} = \mathsf{K}_{ab} \quad . \tag{4.482}$$

Note that $\mathsf{K}_{ab} = \mathsf{K}_{ba}$ is symmetric. And with respect to the $\boldsymbol{y}$ coordinates,

$$\frac{\partial^2 G}{\partial y_\mu\,\partial y_\nu} = -\frac{\partial^2 F}{\partial y_\mu\,\partial y_\nu} = -\mathsf{L}_{\mu\nu} \quad , \tag{4.483}$$

where

$$\mathsf{L}_{\mu\nu} = \frac{\partial^2 F}{\partial y_\mu \, \partial y_\nu} \tag{4.484}$$

is the partial Hessian in the $\boldsymbol{y}$ coordinates. Now it is easy to see that if the full $q \times q$ Hessian matrix H_{ij} is positive definite, then any submatrix such as K_{ab} or $\mathsf{L}_{\mu\nu}$ must also be positive definite. In this case, the partial Legendre transform is convex in $\{p_1, \ldots, p_m\}$ and concave in $\{y_1, \ldots, y_n\}$.

Chapter 5

Central Forces and Orbital Mechanics

5.1 Reduction to a one-body problem

Consider two particles interacting via a potential $U(\boldsymbol{r}_1, \boldsymbol{r}_2) = U\big(|\boldsymbol{r}_1 - \boldsymbol{r}_2|\big)$. Such a potential, which depends only on the relative distance between the particles, is called a *central* potential. The Lagrangian of this system is then

$$L = T - U = \tfrac{1}{2} m_1 \dot{\boldsymbol{r}}_1^2 + \tfrac{1}{2} m_2 \dot{\boldsymbol{r}}_2^2 - U\big(|\boldsymbol{r}_1 - \boldsymbol{r}_2|\big) \quad . \tag{5.1}$$

5.1.1 Center-of-mass (CM) and relative coordinates

The two-body central force problem may always be reduced to two independent one-body problems, by transforming to center-of-mass ($\boldsymbol{R}$) and relative ($\boldsymbol{r}$) coordinates (see fig. 5.1), *viz.*

$$\boldsymbol{R} = \frac{m_1 \boldsymbol{r}_1 + m_2 \boldsymbol{r}_2}{m_1 + m_2} \qquad\qquad \boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2 \tag{5.2}$$

$$\boldsymbol{r}_1 = \boldsymbol{R} + \frac{m_2}{m_1 + m_2}\,\boldsymbol{r} \qquad\qquad \boldsymbol{r}_2 = \boldsymbol{R} - \frac{m_1}{m_1 + m_2}\,\boldsymbol{r} \tag{5.3}$$

We then have

$$\begin{aligned} L &= \tfrac{1}{2} m_1 \dot{\boldsymbol{r}}_1{}^2 + \tfrac{1}{2} m_2 \dot{\boldsymbol{r}}_2{}^2 - U\big(|\boldsymbol{r}_1 - \boldsymbol{r}_2|\big) \\ &= \tfrac{1}{2} M \dot{\boldsymbol{R}}^2 + \tfrac{1}{2} \mu \dot{\boldsymbol{r}}^2 - U(r) \quad . \end{aligned} \tag{5.4}$$

where

$$\begin{aligned} M &= m_1 + m_2 \qquad \text{(total mass)} \\ \mu &= \frac{m_1 m_2}{m_1 + m_2} \qquad \text{(reduced mass)} \quad . \end{aligned} \tag{5.5}$$

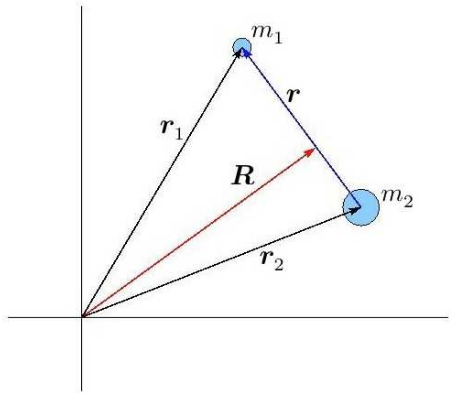


Figure 5.1: Center-of-mass ($\boldsymbol{R}$) and relative ($\boldsymbol{r}$) coordinates.

We may thus write $L = L_{\text{CM}} + L_{\text{rel}}$, where $L_{\text{CM}} = \frac{1}{2}M\dot{\boldsymbol{R}}^2$ and

$$L_{\text{rel}} = \tfrac{1}{2}\mu\dot{\boldsymbol{r}}^2 - U(r) = \tfrac{1}{2}\mu\big(\dot{r}^2 + r^2\dot{\phi}^2\big) - U(r) \quad . \tag{5.6}$$

2D polar coordinates

Recall that in 2D polar coordinates,

$$\begin{aligned} \hat{\boldsymbol{r}} &= \cos\phi\,\hat{\boldsymbol{x}} + \sin\phi\,\hat{\boldsymbol{y}} \\ \hat{\boldsymbol{\phi}} &= -\sin\phi\,\hat{\boldsymbol{x}} + \cos\phi\,\hat{\boldsymbol{y}} \quad , \end{aligned} \tag{5.7}$$

whence $\hat{\boldsymbol{r}} \times \hat{\boldsymbol{\phi}} = \hat{\boldsymbol{z}}$ *et cyc.* The differentials are then $d\hat{\boldsymbol{r}} = d\phi\,\hat{\boldsymbol{\phi}}$ and $d\hat{\boldsymbol{\phi}} = -d\phi\,\hat{\boldsymbol{r}}$, and so

$$\dot{\boldsymbol{r}} = \frac{d}{dt}(r\,\hat{\boldsymbol{r}}) = \dot{r}\,\hat{\boldsymbol{r}} + r\,\dot{\hat{\boldsymbol{r}}} = \dot{r}\,\hat{\boldsymbol{r}} + r\dot{\phi}\,\hat{\boldsymbol{\phi}} \quad . \tag{5.8}$$

Squaring, we obtain $\dot{\boldsymbol{r}}^2 = \dot{r}^2 + r^2\dot{\phi}^2$. Note also that

$$\frac{\partial U(r)}{\partial r^\alpha} = U'(r)\,\frac{\partial r}{\partial r^\alpha} = \frac{r^\alpha}{r}\,U'(r) \quad , \tag{5.9}$$

since

$$r\,dr = \tfrac{1}{2}d(r^2) = \tfrac{1}{2}d(x^2+y^2+z^2) = x\,dx + y\,dy + z\,dz \qquad \Rightarrow \qquad \frac{\partial r}{\partial r^\alpha} = \frac{r^\alpha}{r} = \hat{r}^\alpha \quad . \tag{5.10}$$

Thus, $\boldsymbol{F}(\boldsymbol{r}) = -\boldsymbol{\nabla}U(r) = -U'(r)\,\hat{\boldsymbol{r}}$.

5.1.2 Solution to the CM problem

We have $\partial L/\partial\boldsymbol{R} = 0$, which gives $\dot{\boldsymbol{R}} = 0$ and hence

$$\boldsymbol{R}(t) = \boldsymbol{R}(0) + \dot{\boldsymbol{R}}(0)\,t \quad . \tag{5.11}$$

Thus, the CM problem is trivial. The center-of-mass moves at constant velocity.

5.1.3 Solution to the relative coordinate problem

`Angular momentum conservation:` We have that $\boldsymbol{\ell} = \boldsymbol{r} \times \boldsymbol{p} = \mu \boldsymbol{r} \times \dot{\boldsymbol{r}}$ is a constant of the motion. This means that the motion $\boldsymbol{r}(t)$ is confined to a plane perpendicular to $\boldsymbol{\ell}$. It is convenient to adopt two-dimensional polar coordinates (r, ϕ). The magnitude of $\boldsymbol{\ell}$ is

$$\ell = \mu r^2 \dot{\phi} = 2\mu \dot{\mathcal{A}} \tag{5.12}$$

where $d\mathcal{A} = \frac{1}{2} r^2 d\phi$ is the differential element of area subtended relative to the force center. *The relative coordinate vector for a central force problem subtends equal areas in equal times.* This is known as *Kepler's Second Law.*

`Energy conservation:` The equation of motion for the relative coordinate is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{r}}}\right) = \frac{\partial L}{\partial \boldsymbol{r}} \quad \Rightarrow \quad \mu \ddot{\boldsymbol{r}} = -\frac{\partial U}{\partial \boldsymbol{r}} \quad . \tag{5.13}$$

Taking the dot product with $\dot{\boldsymbol{r}}$, we have

$$\begin{aligned} 0 &= \mu \ddot{\boldsymbol{r}} \cdot \dot{\boldsymbol{r}} + \frac{\partial U}{\partial \boldsymbol{r}} \cdot \dot{\boldsymbol{r}} \\ &= \frac{d}{dt}\Big\{ \tfrac{1}{2}\mu \dot{\boldsymbol{r}}^2 + U(r) \Big\} = \frac{dE}{dt} \quad . \end{aligned} \tag{5.14}$$

Thus, the relative coordinate contribution to the total energy is itself conserved. The total energy is of course $E_{\text{tot}} = E + \frac{1}{2} M \dot{\boldsymbol{R}}^2$.

Since $\boldsymbol{\ell}$ is conserved, and since $\boldsymbol{r} \cdot \boldsymbol{\ell} = 0$, all motion is confined to a plane perpendicular to $\boldsymbol{\ell}$. Choosing coordinates such that $\hat{\boldsymbol{z}} = \hat{\boldsymbol{\ell}}$, we have

$$\begin{aligned} E &= \tfrac{1}{2}\mu \dot{\boldsymbol{r}}^2 + U(r) = \tfrac{1}{2}\mu \dot{r}^2 + \frac{\ell^2}{2\mu r^2} + U(r) \\ &= \tfrac{1}{2}\mu \dot{r}^2 + U_{\text{eff}}(r) \\ U_{\text{eff}}(r) &= \frac{\ell^2}{2\mu r^2} + U(r) \quad . \end{aligned} \tag{5.15}$$

`Integration of the Equations of Motion, Step I:` The second order equation for $r(t)$ is

$$\frac{dE}{dt} = 0 \quad \Rightarrow \quad \mu \ddot{r} = \frac{\ell^2}{\mu r^3} - \frac{dU(r)}{dr} = -\frac{dU_{\text{eff}}(r)}{dr} \quad . \tag{5.16}$$

However, conservation of energy reduces this to a first order equation, via

$$\dot{r} = \pm \sqrt{\frac{2}{\mu}\Big(E - U_{\text{eff}}(r)\Big)} \quad \Rightarrow \quad dt = \pm \frac{\sqrt{\frac{\mu}{2}}\, dr}{\sqrt{E - \frac{\ell^2}{2\mu r^2} - U(r)}} \quad . \tag{5.17}$$

This gives $t(r)$, which must be inverted to obtain $r(t)$. In principle this is possible. Note that a constant of integration also appears at this stage – call it $r_0 = r(t=0)$.

`Integration of the Equations of Motion, Step II:` After finding $r(t)$ one can integrate to find $\phi(t)$ using the conservation of ℓ:

$$\dot{\phi} = \frac{\ell}{\mu r^2} \quad \Rightarrow \quad d\phi = \frac{\ell}{\mu r^2(t)}\, dt \quad . \tag{5.18}$$

This gives $\phi(t)$, and introduces another constant of integration – call it $\phi_0 = \phi(t=0)$.

`Pause to Reflect on the Number of Constants:` Confined to the plane perpendicular to $\boldsymbol{\ell}$, the relative coordinate vector has two degrees of freedom. The equations of motion are second order in time, leading to *four* constants of integration. Our four constants are E, ℓ, r_0, and ϕ_0.

The original problem involves two particles, hence six positions and six velocities, making for 12 initial conditions. Six constants are associated with the CM system: $\boldsymbol{R}(0)$ and $\dot{\boldsymbol{R}}(0)$. The six remaining constants associated with the relative coordinate system are $\boldsymbol{\ell}$ (three components), E, r_0, and ϕ_0.

`Geometric Equation of the Orbit:` From $\ell = \mu r^2 \dot{\phi}$, we have

$$\frac{d}{dt} = \frac{\ell}{\mu r^2} \frac{d}{d\phi} \quad , \tag{5.19}$$

leading to

$$\frac{d^2 r}{d\phi^2} - \frac{2}{r}\left(\frac{dr}{d\phi}\right)^2 = \frac{\mu r^4}{\ell^2}\, F(r) + r \tag{5.20}$$

where $F(r) = -dU(r)/dr$ is the magnitude of the central force. This second order equation may be reduced to a first order one using energy conservation:

$$\begin{aligned} E &= \tfrac{1}{2}\mu \dot{r}^2 + U_{\text{eff}}(r) \\ &= \frac{\ell^2}{2\mu r^4}\left(\frac{dr}{d\phi}\right)^2 + U_{\text{eff}}(r) \quad . \end{aligned} \tag{5.21}$$

Thus,

$$d\phi = \pm \frac{\ell}{\sqrt{2\mu}} \cdot \frac{dr}{r^2\sqrt{E - U_{\text{eff}}(r)}} \quad , \tag{5.22}$$

which can be integrated to yield $\phi(r)$, and then inverted to yield $r(\phi)$. Note that only one integration need be performed to obtain the geometric shape of the orbit, while two integrations – one for $r(t)$ and one for $\phi(t)$ – must be performed to obtain the full motion of the system.

It is sometimes convenient to rewrite Eqn. 5.20 in terms of the variable $s = 1/r$:

$$\frac{d^2 s}{d\phi^2} + s = -\frac{\mu}{\ell^2 s^2}\, F\big(s^{-1}\big) \quad . \tag{5.23}$$

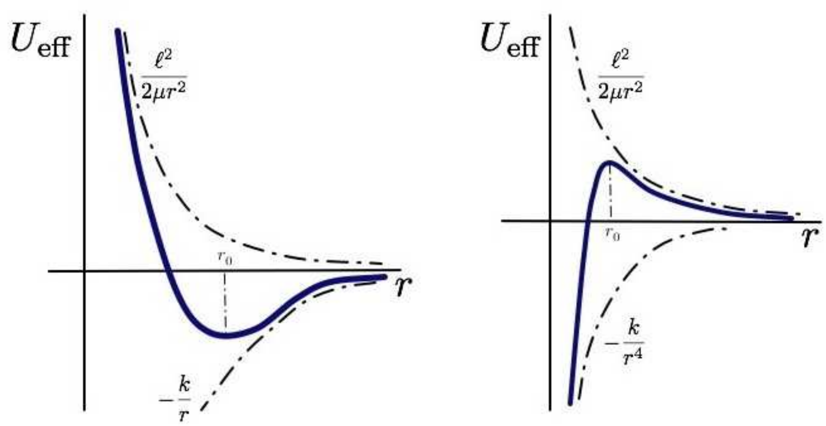


Figure 5.2: Stable and unstable circular orbits. Left panel: $U(r) = -k/r$ produces a stable circular orbit. Right panel: $U(r) = -k/r^4$ produces an unstable circular orbit.

As an example, suppose the geometric orbit is $r(\phi) = k\, e^{\alpha\phi}$, known as a logarithmic spiral. What is the force? We invoke (5.20), with $s''(\phi) = \alpha^2\, s$, yielding

$$F\big(s^{-1}\big) = -(1+\alpha^2)\,\frac{\ell^2}{\mu}\, s^3 \qquad \Rightarrow \qquad F(r) = -\frac{C}{r^3} \tag{5.24}$$

with

$$\alpha^2 = \frac{\mu C}{\ell^2} - 1 \quad . \tag{5.25}$$

The general solution for $s(\phi)$ for this force law is

$$s(\phi) = \begin{cases} A\cosh(\alpha\phi) + B\sinh(-\alpha\phi) & \text{if } \ell^2 > \mu C \\ \\ A'\,\cos\big(|\alpha|\phi\big) + B'\,\sin\big(|\alpha|\phi\big) & \text{if } \ell^2 < \mu C \quad . \end{cases} \tag{5.26}$$

The logarithmic spiral shape is a special case of the first kind of orbit.

5.2 Almost Circular Orbits

A circular orbit with $r(t) = r_0$ satisfies $\ddot{r} = 0$, which means that $U'_{\text{eff}}(r_0) = 0$, which says that $F(r_0) = -\ell^2/\mu r_0^3$. This is negative, indicating that a circular orbit is possible only if the force is attractive over some range of distances. Since $\dot{r} = 0$ as well, we must also have $E = U_{\text{eff}}(r_0)$. An almost circular orbit has $r(t) = r_0 + \eta(t)$, where $|\eta/r_0| \ll 1$. To lowest order in η, one derives the equations

$$\frac{d^2\eta}{dt^2} = -\omega^2\,\eta \qquad , \qquad \omega^2 = \frac{1}{\mu}\, U''_{\text{eff}}(r_0) \quad . \tag{5.27}$$

If $\omega^2 > 0$, the circular orbit is *stable* and the perturbation oscillates harmonically. If $\omega^2 < 0$, the circular orbit is *unstable* and the perturbation grows exponentially. For the geometric shape of the perturbed orbit, we write $r = r_0 + \eta$, and from (5.20) we obtain

$$\frac{d^2\eta}{d\phi^2} = \left(\frac{\mu r_0^4}{\ell^2}\, F'(r_0) - 3\right)\eta = -\beta^2\,\eta \quad , \tag{5.28}$$

with

$$\beta^2 = 3 + \frac{d\log F(r)}{d\log r}\bigg|_{r_0} \quad . \tag{5.29}$$

The solution here is

$$\eta(\phi) = \eta_0\, \cos\beta(\phi - \delta_0) \quad , \tag{5.30}$$

where η_0 and δ_0 are initial conditions. Setting $\eta = \eta_0$, we obtain the sequence of ϕ values

$$\phi_n = \delta_0 + \frac{2\pi n}{\beta} \quad , \tag{5.31}$$

at which $\eta(\phi)$ is a local maximum, *i.e.* at *apoapsis*, where $r = r_0 + \eta_0$. Setting $\eta = -\eta_0$ (*i.e.* $r = r_0 - \eta_0$) is the condition for closest approach, *i.e. periapsis*. The condition for periapsis is thus $\phi = \phi_n + \pi\beta^{-1}$. The difference,

$$\Delta\phi = \phi_{n+1} - \phi_n = \frac{2\pi}{\beta} \quad , \tag{5.32}$$

and so $\Delta\phi - 2\pi$ is the amount by which the apsides (*i.e.* periapsis and apoapsis) *precess* during each cycle. If $\beta > 1$, the apsides advance, *i.e.* it takes less than a complete revolution $\Delta\phi = 2\pi$ between successive periapses. If $\beta < 1$, the apsides retreat, and it takes longer than a complete revolution between successive periapses. The situation is depicted in fig. 5.3 for the case $\beta = 1.1$. Below, we will exhibit a soluble model in which the precessing orbit may be determined exactly. Finally, note that if $\beta = p/q$ is a rational number, then the orbit is *closed*, *i.e.* it eventually retraces itself, after every q revolutions.

As an example, let $F(r) = -kr^{-\alpha}$. Solving for a circular orbit, we write

$$U'_{\text{eff}}(r) = \frac{k}{r^\alpha} - \frac{\ell^2}{\mu r^3} = 0 \quad , \tag{5.33}$$

which has a solution only for $k > 0$, corresponding to an attractive potential. We then find

$$r_0 = \left(\frac{\ell^2}{\mu k}\right)^{1/(3-\alpha)} \quad , \tag{5.34}$$

and $\beta^2 = 3 - \alpha$. The shape of the perturbed orbits follows from $\eta'' = -\beta^2\,\eta$. Thus, while circular orbits exist whenever $k > 0$, small perturbations about these orbits are stable only for $\beta^2 > 0$, *i.e.* for $\alpha < 3$. One then has $\eta(\phi) = A\cos\beta(\phi - \delta_0)$. The perturbed orbits are closed, at least to lowest order in η, for $\alpha = 3 - (p/q)^2$, *i.e.* for $\beta = p/q$. The situation is depicted in fig. 5.2, for the potentials $U(r) = -k/r$ ($\alpha = 2$) and $U(r) = -k/r^4$ ($\alpha = 5$).

5.3 Precession in a Soluble Model

Let's start with the answer and work backwards. Consider the geometrical orbit,

$$r(\phi) = \frac{r_0}{1 - \varepsilon \cos \beta\phi} \quad . \tag{5.35}$$

Our interest is in bound orbits, for which $0 \le \varepsilon < 1$ (see fig. 5.3). What sort of potential gives rise to this orbit? Writing $s = 1/r$ as before, we have

$$s(\phi) = s_0 \, (1 - \varepsilon \cos \beta\phi) \quad . \tag{5.36}$$

Substituting into (5.23), we have

$$-\frac{\mu}{\ell^2 s^2} \, F\big(s^{-1}\big) = \frac{d^2 s}{d\phi^2} + s = \beta^2 \, s_0 \, \varepsilon \cos \beta\phi + s = (1 - \beta^2) \, s + \beta^2 \, s_0 \quad , \tag{5.37}$$

from which we conclude

$$F(r) = -\frac{k}{r^2} + \frac{C}{r^3} \quad , \tag{5.38}$$

with

$$k = \beta^2 s_0 \, \frac{\ell^2}{\mu} \qquad , \qquad C = (\beta^2 - 1) \, \frac{\ell^2}{\mu} \quad . \tag{5.39}$$

The corresponding potential is

$$U(r) = -\frac{k}{r} + \frac{C}{2r^2} + U_\infty \quad , \tag{5.40}$$

where U_∞ is an arbitrary constant, conveniently set to zero. If μ and C are given, we have

$$r_0 = \frac{\ell^2}{\mu k} + \frac{C}{k} \qquad , \qquad \beta = \sqrt{1 + \frac{\mu C}{\ell^2}} \quad . \tag{5.41}$$

When $C = 0$, these expressions recapitulate those from the Kepler problem. Note that when $\ell^2 + \mu C < 0$ that the effective potential is monotonically increasing as a function of r. In this case, the angular momentum barrier is overwhelmed by the (attractive, $C < 0$) inverse square part of the potential, and $U_{\rm eff}(r)$ is monotonically increasing. The orbit then passes through the force center. It is a useful exercise to derive the total energy for the orbit,

$$E = (\varepsilon^2 - 1) \, \frac{\mu k^2}{2(\ell^2 + \mu C)} \qquad \Longleftrightarrow \qquad \varepsilon^2 = 1 + \frac{2E(\ell^2 + \mu C)}{\mu k^2} \quad . \tag{5.42}$$

5.4 The Kepler Problem: $U(r) = -k/r$

5.4.1 Geometric shape of orbits

The force is $F(r) = -kr^{-2}$, hence the equation for the geometric shape of the orbit is

$$\frac{d^2 s}{d\phi^2} + s = -\frac{\mu}{\ell^2 s^2} \, F(s^{-1}) = \frac{\mu k}{\ell^2} \quad , \tag{5.43}$$

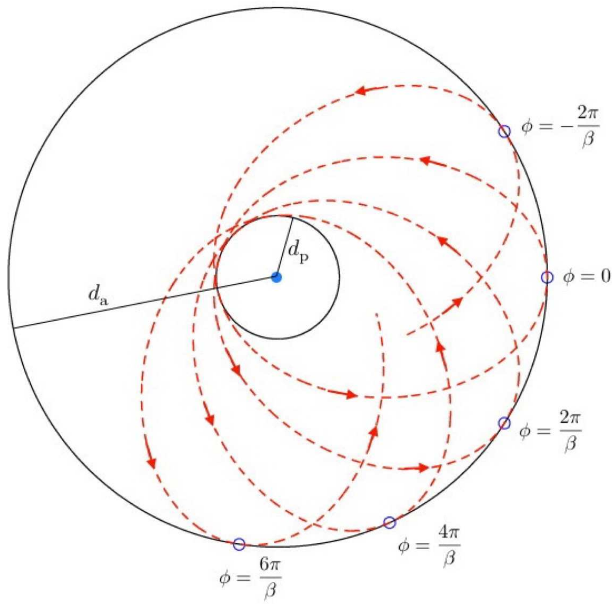


Figure 5.3: Precession in a soluble model, with geometric orbit $r(\phi) = r_0/(1 - \varepsilon \cos \beta\phi)$, shown here with $\beta = 1.1$. Periapsis and apoapsis advance by an angle $2\pi(1 - \beta^{-1})$ per cycle.

with $s = 1/r$. Thus, the most general solution is

$$s(\phi) = s_0 + C \cos(\phi - \phi_0) \quad , \tag{5.44}$$

where C and ϕ_0 are constants. Thus,

$$r(\phi) = \frac{r_0}{1 + \varepsilon \cos(\phi - \phi_0)} \quad , \tag{5.45}$$

where $r_0 = \ell^2/\mu k$ and where we have defined a new constant $\varepsilon \equiv C r_0$. The closest approach of the two bodies occurs when their relative distance r is minimized. This occurs for $\phi = \phi_0 + 2\pi n$, where $r(\phi_0) = r_0/(1 + \varepsilon)$, corresponding to periapsis. The furthest separation occurs for $\phi = \phi_0 + (2n + 1)\pi$, where $r(\phi_0 + \pi) = r_0/(1 - \varepsilon)$, corresponding to apoapsis.

5.4.2 Laplace-Runge-Lenz vector

Consider the *Laplace-Runge-Lenz vector*,

$$\boldsymbol{A} = \boldsymbol{p} \times \boldsymbol{\ell} - \mu k \, \hat{\boldsymbol{r}} \quad , \tag{5.46}$$

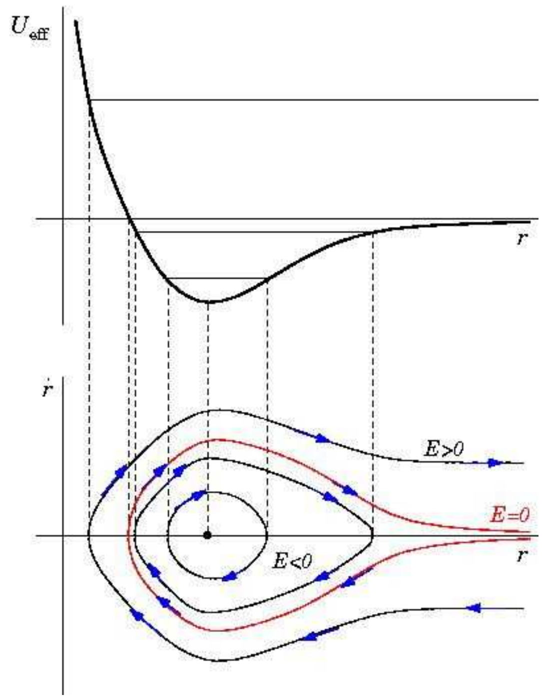


Figure 5.4: The effective potential for the Kepler problem, and associated phase curves. The orbits are geometrically described as conic sections: hyperbolae ($E > 0$), parabolae ($E = 0$), ellipses ($E_{\rm min} < E < 0$), and circles ($E = E_{\rm min}$).

where $\hat{\boldsymbol{r}} = \boldsymbol{r}/|\boldsymbol{r}|$ is the unit vector pointing in the direction of $\boldsymbol{r}$. We may now show that $\boldsymbol{A}$ is conserved:

$$\begin{aligned}\frac{d\boldsymbol{A}}{dt} = \frac{d}{dt}\Big\{\boldsymbol{p}\times\boldsymbol{\ell} - \mu k\frac{\boldsymbol{r}}{r}\Big\} &= \dot{\boldsymbol{p}}\times\boldsymbol{\ell} + \boldsymbol{p}\times\dot{\boldsymbol{\ell}} - \mu k\,\frac{r\dot{\boldsymbol{r}} - \boldsymbol{r}\dot{r}}{r^2} \\ &= -\frac{k\boldsymbol{r}}{r^3}\times(\mu\boldsymbol{r}\times\dot{\boldsymbol{r}}) - \mu k\,\frac{\dot{\boldsymbol{r}}}{r} + \mu k\,\frac{\dot{r}\boldsymbol{r}}{r^2} \\ &= -\mu k\,\frac{\boldsymbol{r}(\boldsymbol{r}\cdot\dot{\boldsymbol{r}})}{r^3} + \mu k\,\frac{\dot{\boldsymbol{r}}(\boldsymbol{r}\cdot\boldsymbol{r})}{r^3} - \mu k\,\frac{\dot{\boldsymbol{r}}}{r} + \mu k\,\frac{\dot{r}\boldsymbol{r}}{r^2} = 0 \quad .\end{aligned} \tag{5.47}$$

So $\boldsymbol{A}$ is a conserved vector which clearly lies in the plane of the motion. $\boldsymbol{A}$ points toward periapsis, *i.e.* toward the point of closest approach to the force center.

Let's assume periapsis occurs at $\phi = \phi_0$. Then

$$\boldsymbol{A}\cdot\boldsymbol{r} = Ar\cos(\phi-\phi_0) = \ell^2 - \mu k r \tag{5.48}$$

giving

$$r(\phi) = \frac{\ell^2}{\mu k + A\cos(\phi-\phi_0)} = \frac{a\,|1-\varepsilon^2|}{1+\varepsilon\cos(\phi-\phi_0)} \quad , \tag{5.49}$$

where

$$\varepsilon = \frac{A}{\mu k} \qquad , \qquad a\,|1-\varepsilon^2| = \frac{\ell^2}{\mu k} \quad . \tag{5.50}$$

The orbit is a *conic section* with eccentricity ε. Squaring $\boldsymbol{A}$, one finds

$$\begin{aligned}\boldsymbol{A}^2 &= (\boldsymbol{p}\times\boldsymbol{\ell})^2 - 2\mu k\,\hat{\boldsymbol{r}}\cdot\boldsymbol{p}\times\boldsymbol{\ell} + \mu^2k^2 \\ &= p^2\ell^2 - 2\mu\ell^2\,\frac{k}{r} + \mu^2k^2 \\ &= 2\mu\ell^2\bigg(\frac{p^2}{2\mu} - \frac{k}{r} + \frac{\mu k^2}{2\ell^2}\bigg) = 2\mu\ell^2\bigg(E + \frac{\mu k^2}{2\ell^2}\bigg)\quad,\end{aligned}\tag{5.51}$$

and thus

$$\varepsilon^2 = \frac{A^2}{\mu^2k^2} = 1 + \frac{2E\ell^2}{\mu k^2}\qquad,\qquad a = \frac{1}{|1-\varepsilon^2|}\cdot\frac{\ell^2}{\mu k} = \frac{k}{2\,|E|}\qquad.\tag{5.52}$$

Note that for circular orbits $\boldsymbol{A}=0$. Furthermore, by squaring the equation $\mu k\,\hat{\boldsymbol{r}} = \boldsymbol{p}\times\boldsymbol{\ell} - \boldsymbol{A}$, we obtain the relation

$$\mu^2k^2 = \boldsymbol{A}^2 + \ell^2\,\boldsymbol{p}^2 + 2\boldsymbol{\ell}\cdot\boldsymbol{p}\times\boldsymbol{A} = \big(\ell\,\boldsymbol{p} - \hat{\boldsymbol{z}}\times\boldsymbol{A}\big)^2\quad,\tag{5.53}$$

where we have taken $\boldsymbol{\ell} = \ell\hat{\boldsymbol{z}}$. This says that the momentum vector $\boldsymbol{p}$ always lies on a circle of radius $\mu k/\ell$ centered at the momentum value $\hat{\boldsymbol{z}}\times\boldsymbol{A}/\ell$.

Aside : hidden symmetry in the Kepler problem

The fact that the Laplace-Runge-Lenz vector is conserved and lies in the plane of the motion entails that bound Keplerian orbits, which are ellipsoidal, do not precess. Remarkably, this feature is also responsible for the degeneracy of the energy spectrum of hydrogenic atoms in quantum mechanics. The energy eigenvalues $E_{n,l} = -Ze^2/2na_{\rm B}$, where $+Ze$ is the nuclear charge and $a_{\rm B} = \hbar^2/me^2 = 0.529\,\text{Å}$ is the Bohr radius, are dependent only on the principal quantum number $n \in \{1,2,\ldots\}$ and not on the angular momentum quantum number $l \in \{0,1,\ldots,n-1\}$, nor the angular momentum and spin polarizations $m_l \in \{-l,\ldots,+l\}$ and $m_s = \pm\frac{1}{2}$. While one might expect the symmetry of the relative coordinate problem to be SO(3), corresponding to the isotropy of three-dimensional space, in fact the symmetry group is SO(4)[1].

5.4.3 Kepler orbits are conic sections

There are four classes of conic sections:

- *Circle:* $\varepsilon = 0$, $E = -\mu k^2/2\ell^2$, radius $a = \ell^2/\mu k$. The force center lies at the center of circle.
- *Ellipse:* $0 < \varepsilon < 1$, $-\mu k^2/2\ell^2 < E < 0$, semimajor axis $a = -k/2E$, semiminor axis $b = a\sqrt{1-\varepsilon^2}$. The force center is at one of the foci.

[1]Defining the scaled Laplace-Runge-Lenz vector $d_i = A_i/\sqrt{2m|E|}$, the Poisson brackets, which we will discuss in ch. 16, are given by $\{\ell_i,\ell_j\} = \varepsilon_{ijk}\,\ell_k$, $\{d_i,\ell_j\} = \varepsilon_{ijk}\,d_k$, and $\{d_i,d_j\} = -\,\text{sgn}\,(E)\,\varepsilon_{ijk}\,d_k$. When $E<0$ (bound orbits), this corresponds to the Lie algebra so(4), while for $E>0$ (unbound orbits) this corresponds to the Lie algebra so(3,1). In quantum mechanics, the Poisson bracket becomes the commutator, *viz.* $\{A,B\}\to[\hat{A},\hat{B}]/i\hbar$, where $\hat{A}$ and $\hat{B}$ are operators. The LRL and angular momentum vectors are conserved because $\{d_i,H\} = \{\ell_i,H\} = 0$, where H is the Hamiltonian.

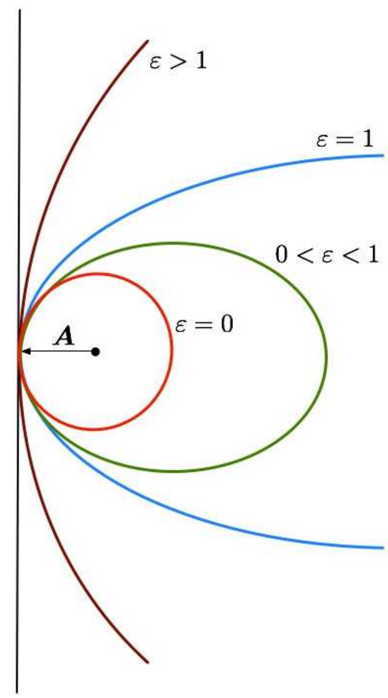


Figure 5.5: Keplerian orbits are conic sections, classified according to eccentricity: hyperbola ($\varepsilon > 1$), parabola ($\varepsilon = 1$), ellipse ($0 < \varepsilon < 1$), and circle ($\varepsilon = 0$). The Laplace-Runge-Lenz vector, $\boldsymbol{A}$, points toward periapsis, but its length $A = \mu k \varepsilon$ vanishes for circular orbits.

- *Parabola:* $\varepsilon = 1$, $E = 0$, force center is the focus.
- *Hyperbola:* $\varepsilon > 1$, $E > 0$, force center is closest focus (attractive) or farthest focus (repulsive).

To see that the Keplerian orbits are indeed conic sections, consider the ellipse of fig. 5.6. The law of cosines gives

$$\rho^2 = r^2 + 4f^2 - 4rf \cos\phi \quad , \tag{5.54}$$

where $f = \varepsilon a$ is the focal distance. Now for any point on an ellipse, the sum of the distances to the left and right foci is a constant, and taking $\phi = 0$ we see that this constant is $2a$. Thus, $\rho = 2a - r$, and we have

$$(2a - r)^2 = 4a^2 - 4ar + r^2 = r^2 + 4\varepsilon^2 a^2 - 4\varepsilon r \cos\phi \quad , \tag{5.55}$$

which says

$$r(\phi) = \frac{a\,(1 - \varepsilon^2)}{1 - \varepsilon \cos\phi} \quad , \tag{5.56}$$

corresponding to periapsis at $\phi = \phi_0 = \pi$. We therefore conclude that

$$r_0 = \frac{\ell^2}{\mu k} = a\,(1 - \varepsilon^2) \quad . \tag{5.57}$$

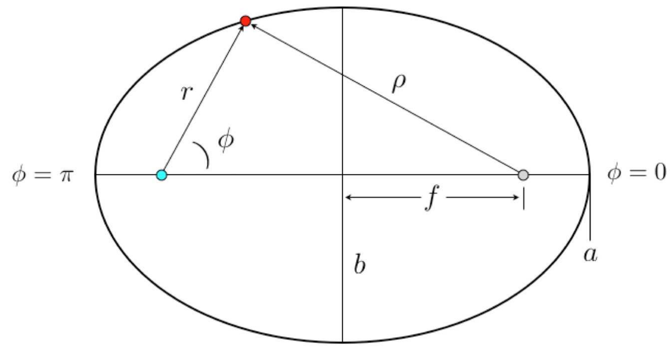


Figure 5.6: The Keplerian ellipse, with the force center at the left focus. The focal distance is $f = \varepsilon a$, where a is the semimajor axis length. The length of the semiminor axis is $b = \sqrt{1-\varepsilon^2}\,a$.

Next let us examine the energy,

$$\begin{aligned} E &= \overbrace{\tfrac{1}{2}\mu\bigg(\frac{\ell}{\mu r^2}\,\frac{dr}{d\phi}\bigg)^2}^{\frac{1}{2}\mu\dot{r}^2} + \overbrace{\frac{\ell^2}{2\mu r^2} - \frac{k}{r}}^{U_{\rm eff}(r)} \\ &= \frac{\ell^2}{2\mu}\bigg(\frac{ds}{d\phi}\bigg)^2 + \frac{\ell^2}{2\mu}\,s^2 - ks \quad , \end{aligned} \tag{5.58}$$

with

$$s = \frac{1}{r} = \frac{\mu k}{\ell^2}\Big(1 - \varepsilon\cos\phi\Big) \quad . \tag{5.59}$$

Thus,

$$\frac{ds}{d\phi} = \frac{\mu k}{\ell^2}\,\varepsilon\,\sin\phi \quad , \tag{5.60}$$

and

$$\begin{aligned} \bigg(\frac{ds}{d\phi}\bigg)^2 &= \frac{\mu^2 k^2}{\ell^4}\,\varepsilon^2\,\sin^2\!\phi \\ &= \frac{\mu^2 k^2 \varepsilon^2}{\ell^4} - \bigg(\frac{\mu k}{\ell^2} - s\bigg)^2 = -s^2 + \frac{2\mu k}{\ell^2}\,s + \big(\varepsilon^2 - 1\big)\,\frac{\mu^2 k^2}{\ell^4} \quad . \end{aligned} \tag{5.61}$$

Substituting this into eqn. 5.58, we obtain

$$E = \big(\varepsilon^2 - 1\big)\,\frac{\mu k^2}{2\ell^2} \quad . \tag{5.62}$$

For the hyperbolic orbit, depicted in fig. 5.7, we have $r - \rho = \mp 2a$, depending on whether we are on the attractive or repulsive branch, respectively. We then have

$$(r \pm 2a)^2 = 4a^2 \pm 4ar + r^2 = r^2 + 4\varepsilon^2 a^2 - 4\varepsilon r\cos\phi \quad , \tag{5.63}$$

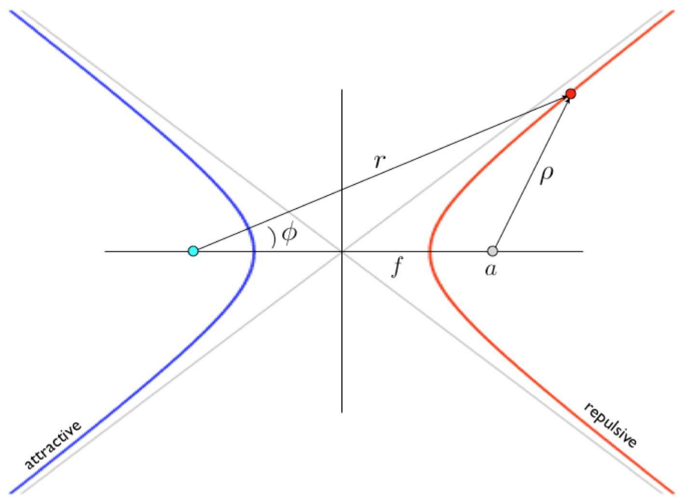


Figure 5.7: The Keplerian hyperbolae, with the force center at the left focus. The left (blue) branch corresponds to an attractive potential, while the right (red) branch corresponds to a repulsive potential. The equations of these branches are $r = \rho = \mp 2a$, where the top sign corresponds to the left branch and the bottom sign to the right branch.

from which we obtain

$$r(\phi) = \frac{a\,(\varepsilon^2 - 1)}{\pm 1 + \varepsilon \cos\phi} \quad . \tag{5.64}$$

Note that $r(\pm\phi_\infty) = \infty$, where $\phi_\infty = \cos^{-1}(\mp 1/\varepsilon)$.

5.4.4 Period of bound Kepler orbits

From $\ell = \mu r^2 \dot{\phi} = 2\mu \dot{\mathcal{A}}$, the period is $\tau = 2\mu\mathcal{A}/\ell$, where $\mathcal{A} = \pi a^2 \sqrt{1-\varepsilon^2}$ is the area enclosed by the orbit. This gives

$$\tau = 2\pi \left(\frac{\mu a^3}{k}\right)^{1/2} = 2\pi \left(\frac{a^3}{GM}\right)^{1/2} \tag{5.65}$$

as well as

$$\frac{a^3}{\tau^2} = \frac{GM}{4\pi^2} \quad , \tag{5.66}$$

where $k = Gm_1m_2$ and $M = m_1 + m_2$ is the total mass. For planetary orbits, $m_1 = M_\odot$ is the solar mass and $m_2 = m_{\rm p}$ is the planetary mass. We then have

$$\frac{a^3}{\tau^2} = \left(1 + \frac{m_{\rm p}}{M_\odot}\right)\frac{GM_\odot}{4\pi^2} \approx \frac{GM_\odot}{4\pi^2} \quad , \tag{5.67}$$

which is to an excellent approximation independent of the planetary mass. (Note that $m_{\rm p}/M_\odot \approx 10^{-3}$ even for Jupiter.) This analysis also holds, *mutatis mutandis*, for the case of satellites orbiting the earth, and indeed in any case where the masses are grossly disproportionate in magnitude.

5.4.5 Escape velocity

The threshold for escape from a gravitational potential occurs at $E = 0$. Since $E = T + U$ is conserved, we determine the *escape velocity* for a body a distance r from the force center by setting

$$E = 0 = \tfrac{1}{2}\mu v_{\rm esc}^2(t) - \frac{G m_1 m_2}{r} \quad \Rightarrow \quad v_{\rm esc}(r) = \sqrt{\frac{2GM}{r}} \quad . \tag{5.68}$$

with $M = m_1 + m_2$. For an object on earth's surface, $v_{\rm esc} = \sqrt{2gR_{\rm E}} = 11.2\,{\rm km/s}$, assuming the object is much less massive than the earth itself.

5.4.6 Satellites and spacecraft

A satellite in a circular orbit a distance h above the earth's surface has an orbital period

$$\tau = \frac{2\pi}{\sqrt{GM_{\rm E}}}\,(R_{\rm E} + h)^{3/2} \quad , \tag{5.69}$$

where we take $m_{\rm satellite} \ll M_{\rm E}$. For low earth orbit (LEO), $h \ll R_{\rm E} = 6.37 \times 10^6\,{\rm m}$, in which case $\tau_{\rm LEO} = 2\pi\sqrt{R_{\rm E}/g} = 1.4\,{\rm hr}$.

Consider a weather satellite in an elliptical orbit whose closest approach to the earth (perigee) is 200 km above the earth's surface and whose farthest distance (apogee) is 7200 km above the earth's surface. What is the satellite's orbital period? From fig. 5.6, we see that

$$\begin{aligned} d_{\rm apogee} &= R_{\rm E} + 7200\,{\rm km} = 13571\,{\rm km} \\ d_{\rm perigee} &= R_{\rm E} + 200\,{\rm km} = 6971\,{\rm km} \\ a &= \tfrac{1}{2}(d_{\rm apogee} + d_{\rm perigee}) = 10071\,{\rm km} \quad . \end{aligned} \tag{5.70}$$

We then have

$$\tau = \left(\frac{a}{R_{\rm E}}\right)^{3/2} \tau_{\rm LEO} \approx 2.65\,{\rm hr} \quad . \tag{5.71}$$

What happens if a spacecraft in orbit about the earth fires its rockets? Clearly the energy and angular momentum of the orbit will change, and this means the shape will change. If the rockets are fired (in the direction of motion) at perigee, then perigee itself is unchanged, because $\boldsymbol{v} \cdot \boldsymbol{r} = 0$ is left unchanged at this point. However, E is increased, hence the eccentricity $\varepsilon = \sqrt{1 + \frac{2E\ell^2}{\mu k^2}}$ increases. This is the most efficient way of boosting a satellite into an orbit with higher eccentricity. Conversely, and somewhat paradoxically, when a satellite in LEO loses energy due to frictional drag of the atmosphere, the energy E decreases. Initially, because the drag is weak and the atmosphere is isotropic, the orbit remains circular. Since E decreases, $\langle T \rangle = -E$ must *increase*, which means that the frictional forces cause the satellite to speed up!

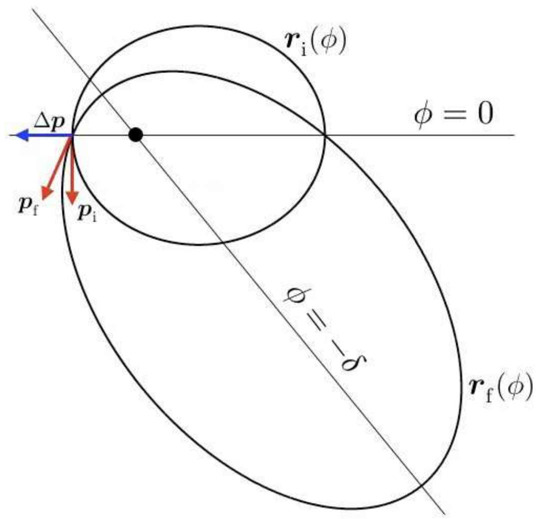


Figure 5.8: At perigee of an elliptical orbit $r_{\rm i}(\phi)$, a radial impulse $\Delta\boldsymbol{p}$ is applied. The shape of the resulting orbit $r_{\rm f}(\phi)$ is shown.

5.4.7 Two examples of orbital mechanics

- Problem #1: At perigee of an elliptical Keplerian orbit, a satellite receives an impulse $\Delta\boldsymbol{p} = p_0\hat{\boldsymbol{r}}$. Describe the resulting orbit.

- ◦ Solution #1: Since the impulse is radial, the angular momentum $\boldsymbol{\ell} = \boldsymbol{r}\times\boldsymbol{p}$ is unchanged. The energy, however, does change, with $\Delta E = p_0^2/2\mu$. Thus, using $\varepsilon^2 = 1 + \frac{2E\ell^2}{\mu k^2}$, we have

$$\varepsilon_{\rm f}^2 = 1 + \frac{2E_{\rm f}\ell^2}{\mu k^2} = \varepsilon_{\rm i}^2 + \left(\frac{\ell p_0}{\mu k}\right)^2 \quad . \tag{5.72}$$

The new semimajor axis length is

$$a_{\rm f} = \frac{\ell^2/\mu k}{1-\varepsilon_{\rm f}^2} = a_{\rm i}\cdot\frac{1-\varepsilon_{\rm i}^2}{1-\varepsilon_{\rm f}^2} = \frac{a_{\rm i}}{1-(a_{\rm i}p_0^2/\mu k)} \quad . \tag{5.73}$$

The shape of the final orbit must also be a Keplerian ellipse, described by

$$r_{\rm f}(\phi) = \frac{\ell^2}{\mu k}\cdot\frac{1}{1-\varepsilon_{\rm f}\cos(\phi+\delta)} \quad , \tag{5.74}$$

where the phase shift δ is determined by setting

$$r_{\rm i}(\pi) = r_{\rm f}(\pi) = \frac{\ell^2}{\mu k}\cdot\frac{1}{1+\varepsilon_{\rm i}} \quad . \tag{5.75}$$

Solving for δ, we obtain

$$\delta = \cos^{-1}\!\big(\varepsilon_{\rm i}/\varepsilon_{\rm f}\big) \quad . \tag{5.76}$$

The situation is depicted in fig. 5.8.

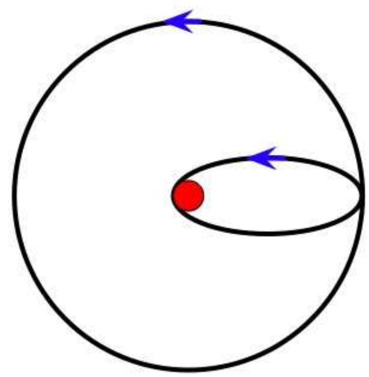


Figure 5.9: The larger circular orbit represents the orbit of the earth. The elliptical orbit represents that for an object orbiting the Sun with distance at perihelion equal to the Sun's radius.

- Problem #2: Which is more energy efficient – to send nuclear waste outside the solar system, or to send it into the Sun? Neglect the eccentricity of earth's orbit.

◦ Solution #2: Escape velocity for the solar system is $v_{\rm esc,\odot}(r) = \sqrt{2GM_\odot/r}$. At a distance of the earth's orbital radius $a_{\rm E} = 149.6 \times 10^6$ km, we have $v_{\rm esc,\odot}(a_{\rm E}) = \sqrt{2}\,v_{\rm E}$, where $v_{\rm E}$ is earth's orbital speed[2], $v_{\rm E} = \sqrt{GM_\odot/a_{\rm E}} = 2\pi a_{\rm E}/\tau_{\rm E} = 29.9\,{\rm km/s}$. The rocket is launched from earth, and clearly the most energy efficient launch will be one in the direction of the earth's motion, in which case the velocity after escape from earth must be $u = \big(\sqrt{2}-1\big)v_{\rm E} = 12.4\,{\rm km/s}$. The speed just above the earth's atmosphere must then be $\tilde u$, where

$$\tfrac{1}{2}m\tilde u^2 - \frac{GM_{\rm E}m}{R_{\rm E}} = \tfrac{1}{2}mu^2 \quad , \tag{5.77}$$

or, in other words,

$$\tilde u^2 = u^2 + v_{\rm esc,E}^2 \quad . \tag{5.78}$$

We find $\tilde u = 16.7\,{\rm km/s}$. This is the speed of the rocket *relative to the earth* once it has escaped earth's gravitational pull.

The second method is to place the trash ship in an elliptical orbit whose perihelion is the Sun's radius, $R_\odot = 6.98 \times 10^8$ m, and whose aphelion is $a_{\rm E}$. Invoking the general equation for the shape of the Keplerian orbit $r(\phi) = (\ell^2/\mu k)/(1 - \varepsilon\cos\phi)$, we then solve the two equations

$$\begin{aligned} r(\phi=\pi) = R_\odot &= \frac{1}{1+\varepsilon}\cdot\frac{\ell^2}{\mu k} \\ r(\phi=0) = a_{\rm E} &= \frac{1}{1-\varepsilon}\cdot\frac{\ell^2}{\mu k} \quad . \end{aligned} \tag{5.79}$$

We thereby obtain

$$\varepsilon = \frac{a_{\rm E} - R_\odot}{a_{\rm E} + R_\odot} = 0.991 \quad , \tag{5.80}$$

[2]Assuming a circular orbit, we equate the centrifugal and gravitational forces: $M_{\rm E}v_{\rm E}^2/a_{\rm E} = GM_\odot M_{\rm E}/a_{\rm E}^2$. This yields earth's orbital speed $v_{\rm E} = \sqrt{GM_\odot/a_{\rm E}}$.

which is a very eccentric ellipse, and

$$\begin{aligned}\frac{\ell^2}{\mu k} &= \frac{a_{\rm E}^2\, v^2}{G(M_\odot + m)} \approx \frac{a_{\rm E}\, v^2}{v_{\rm E}^2} \\ &= (1-\varepsilon)\, a_{\rm E} = \frac{2a_{\rm E}R_\odot}{a_{\rm E}+R_\odot} \quad .\end{aligned} \tag{5.81}$$

Hence,

$$v^2 = \frac{2R_\odot}{a_{\rm E}+R_\odot}\, v_{\rm E}^2 \quad , \tag{5.82}$$

and the necessary velocity relative to earth is

$$u = \left(\sqrt{\frac{2R_\odot}{a_{\rm E}+R_\odot}} - 1\right) v_{\rm E} \approx -0.904\, v_{\rm E} \quad , \tag{5.83}$$

i.e. $u = -27.0\,{\rm km/s}$. Launch is in the opposite direction from the earth's orbital motion, and from $\tilde u^2 = u^2 + v_{\rm esc,E}^2$ we find $\tilde u = -29.2\,{\rm km/s}$, which is larger (in magnitude) than in the first scenario. Thus, it is cheaper to ship the trash out of the solar system than to send it crashing into the Sun, by a factor $\tilde u_{\rm I}^2/\tilde u_{\rm II}^2 = 0.327$.

5.5 Mission to Neptune

Four earth-launched spacecraft have escaped the solar system: *Pioneer 10* (launch 3/3/72), *Pioneer 11* (launch 4/6/73), *Voyager 1* (launch 9/5/77), and *Voyager 2* (launch 8/20/77)[3]. The latter two are still functioning, and each are moving away from the Sun at a velocity of roughly 3.5 AU/yr.

As the first objects of earthly origin to leave our solar system, both *Pioneer* spacecraft featured a graphic message in the form of a 6" x 9" gold anodized plaque affixed to the spacecrafts' frame. This plaque was designed in part by the late astronomer and popular science writer Carl Sagan. The humorist Dave Barry, in an essay entitled *Bring Back Carl's Plaque*, remarks,

> *It's all well and good for Carl Sagan to talk about how neat it would be to get in touch with the aliens, but I bet he'd change his mind pronto if they actually started oozing under his front door. I bet he'd be whapping at them with his golf clubs just like the rest of us.*
>
> *But the really bad part is what they put on the plaque. I mean, if we're going to have a plaque, it ought to at least show the aliens what we're really like, right? Maybe a picture of people eating cheeseburgers and watching "The Dukes of Hazzard." Then if aliens found it, they'd say, "Ah. Just plain folks."*
>
> *But no. Carl came up with this incredible science-fair-wimp plaque that features drawings of – you are not going to believe this – a hydrogen atom and naked people. To represent the entire Earth! This is crazy! Walk the streets of any town on this planet, and the two things you will almost never see are hydrogen atoms and naked people.*

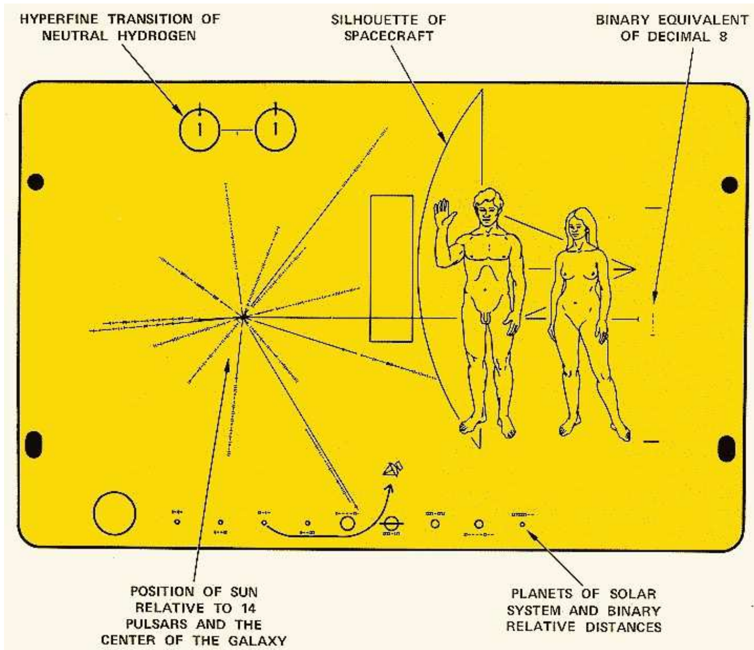


Figure 5.10: The unforgivably dorky *Pioneer 10* and *Pioneer 11* plaque.

During August, 1989, *Voyager 2* investigated the planet Neptune. A direct trip to Neptune along a Keplerian ellipse with $r_{\rm p} = a_{\rm E} = 1\,{\rm AU}$ and $r_{\rm a} = a_{\rm N} = 30.06\,{\rm AU}$ would take 30.6 years. To see this, note that $r_{\rm p} = a\,(1-\varepsilon)$ and $r_{\rm a} = a\,(1+\varepsilon)$ yield

$$a = \tfrac{1}{2}\big(a_{\rm E} + a_{\rm N}\big) = 15.53\,{\rm AU} \quad , \quad \varepsilon = \frac{a_{\rm N} - a_{\rm E}}{a_{\rm N} + a_{\rm E}} = 0.9356 \quad . \tag{5.84}$$

Thus,

$$\tau = \tfrac{1}{2}\,\tau_{\rm E} \cdot \left(\frac{a}{a_{\rm E}}\right)^{3/2} = 30.6\,{\rm yr} \quad . \tag{5.85}$$

The energy cost per kilogram of such a mission is computed as follows. Let the speed of the probe after its escape from earth be $v_{\rm p} = \lambda v_{\rm E}$, and the speed just above the atmosphere (*i.e.* neglecting atmospheric friction) is v_0. For the most efficient launch possible, the probe is shot in the direction of earth's instantaneous motion about the Sun. Then we must have

$$\tfrac{1}{2} m\, v_0^2 - \frac{G M_{\rm E} m}{R_{\rm E}} = \tfrac{1}{2} m\,(\lambda-1)^2\, v_{\rm E}^2 \quad , \tag{5.86}$$

since the speed of the probe in the frame of the earth is $v_{\rm p} - v_{\rm E} = (\lambda - 1)\, v_{\rm E}$. Thus,

$$\begin{aligned} \frac{E}{m} &= \tfrac{1}{2} v_0^2 = \Big[\tfrac{1}{2}(\lambda-1)^2 + h\Big]\, v_{\rm E}^2 \\ v_{\rm E}^2 &= \frac{G M_\odot}{a_{\rm E}} = 6.24 \times 10^7\, R_{\rm J}/{\rm kg} \quad , \end{aligned} \tag{5.87}$$

[3] There is a very nice discussion in the Barger and Olsson book on 'Grand Tours of the Outer Planets'. Here I reconstruct and extend their discussion.

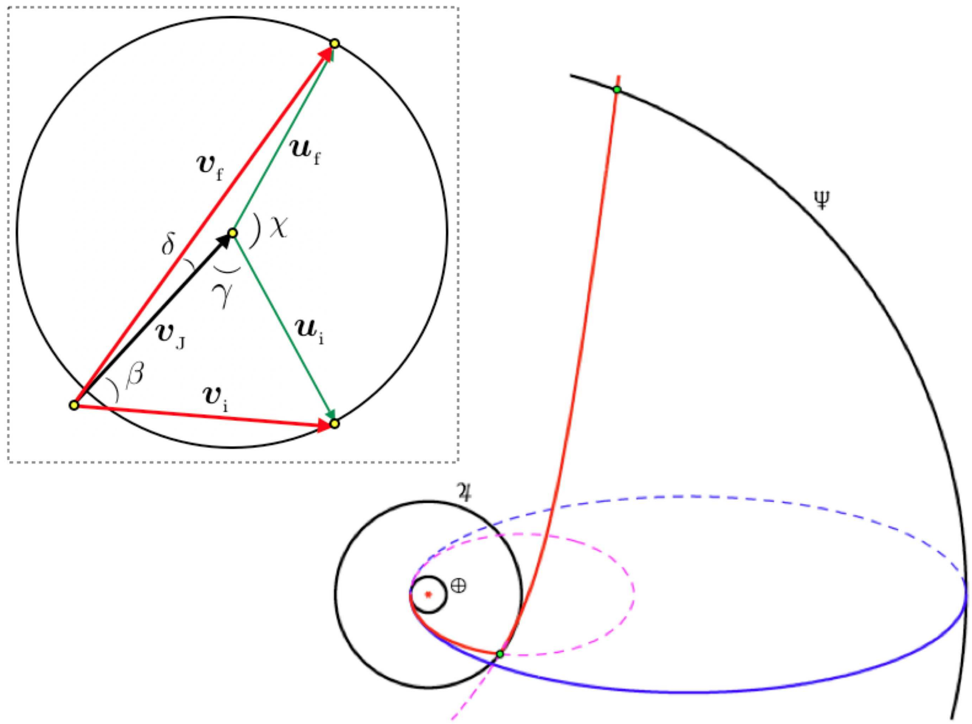


Figure 5.11: Mission to Neptune. The figure at the lower right shows the orbits of Earth, Jupiter, and Neptune in black. The cheapest (in terms of energy) direct flight to Neptune, shown in blue, would take 30.6 years. By swinging past the planet Jupiter, the satellite can pick up great speed and with even less energy the mission time can be cut to 8.5 years (red curve). The inset in the upper left shows the scattering event with Jupiter.

where

$$h \equiv \frac{M_{\rm E}}{M_\odot} \cdot \frac{a_{\rm E}}{R_{\rm E}} = 7.050 \times 10^{-2} \quad . \tag{5.88}$$

Therefore, a convenient dimensionless measure of the energy is

$$\eta \equiv \frac{2E}{mv_{\rm E}^2} = \frac{v_0^2}{v_{\rm E}^2} = (\lambda - 1)^2 + 2h \quad . \tag{5.89}$$

As we shall derive below, a direct mission to Neptune requires

$$\lambda \geq \sqrt{\frac{2a_{\rm N}}{a_{\rm N} + a_{\rm E}}} = 1.3913 \quad , \tag{5.90}$$

which is close to the criterion for escape from the solar system, $\lambda_{\rm esc} = \sqrt{2}$. Note that about 52% of the energy is expended after the probe escapes the Earth's pull, and 48% is expended in liberating the probe from Earth itself.

This mission can be done much more economically by taking advantage of a Jupiter flyby, as shown in fig. 5.11. The idea of a flyby is to steal some of Jupiter's momentum and then fly away very fast before Jupiter realizes and gets angry. The CM frame of the probe-Jupiter system is of course the rest frame of Jupiter, and in this frame conservation of energy means that the final velocity $\boldsymbol{u}_{\rm f}$ is of the same magnitude as the initial velocity $\boldsymbol{u}_{\rm i}$. However, in the frame of the Sun, the initial and final velocities are $\boldsymbol{v}_{\rm J} + \boldsymbol{u}_{\rm i}$ and $\boldsymbol{v}_{\rm J} + \boldsymbol{u}_{\rm f}$, respectively, where $\boldsymbol{v}_{\rm J}$ is the velocity of Jupiter in the rest frame of the Sun. If, as shown in the inset to fig. 5.11, $\boldsymbol{u}_{\rm f}$ is roughly parallel to $\boldsymbol{v}_{\rm J}$, the probe's velocity in the Sun's frame will be enhanced. Thus, the motion of the probe is broken up into three segments:

I : Earth to Jupiter
II : Scatter off Jupiter's gravitational pull
III : Jupiter to Neptune

We now analyze each of these segments in detail. In so doing, it is useful to recall that the general form of a Keplerian orbit is

$$r(\phi) = \frac{d}{1 - \varepsilon\cos\phi} \quad , \quad d = \frac{\ell^2}{\mu k} = \big|\varepsilon^2 - 1\big|\, a \quad . \tag{5.91}$$

The energy is $E = (\varepsilon^2 - 1)\,\mu k^2/2\ell^2$, with $k = GMm$, where M is the mass of either the Sun or a planet. In either case, M dominates, and $\mu = Mm/(M+m) \simeq m$ to extremely high accuracy. The time for the trajectory to pass from $\phi = \phi_1$ to $\phi = \phi_2$ is

$$T = \int\! dt = \int\limits_{\phi_1}^{\phi_2} \frac{d\phi}{\dot\phi} = \frac{\mu}{\ell}\int\limits_{\phi_1}^{\phi_2} d\phi\, r^2(\phi) = \frac{\ell^3}{\mu k^2}\int\limits_{\phi_1}^{\phi_2} \frac{d\phi}{\big[1 - \varepsilon\cos\phi\big]^2} \quad . \tag{5.92}$$

For reference,

$$\begin{array}{lll} a_{\rm E} = 1\,{\rm AU} & a_{\rm J} = 5.20\,{\rm AU} & a_{\rm N} = 30.06\,{\rm AU} \\ M_{\rm E} = 5.972 \times 10^{24}\,{\rm kg} & M_{\rm J} = 1.900 \times 10^{27}\,{\rm kg} & M_{\odot} = 1.989 \times 10^{30}\,{\rm kg} \end{array}$$

with $1\,{\rm AU} = 1.496 \times 10^8$ km. Here $a_{\rm E,J,N}$ and $M_{\rm E,J,\odot}$ are the orbital radii and masses of Earth, Jupiter, and Neptune, and the Sun. The last thing we need to know is the radius of Jupiter,

$$R_{\rm J} = 9.558 \times 10^{-4}\,{\rm AU} \quad .$$

We need $R_{\rm J}$ because the distance of closest approach to Jupiter, or *perijove*, must be $R_{\rm J}$ or greater, or else the probe crashes into Jupiter!

5.5.1 Earth to Jupiter (Phase I)

The probe's velocity at perihelion is $v_{\rm p} = \lambda v_{\rm E}$. The angular momentum is $\ell = \mu a_{\rm E} \cdot \lambda v_{\rm E}$, whence

$$d = \frac{(a_{\rm E}\lambda v_{\rm E})^2}{GM_\odot} = \lambda^2\, a_{\rm E} \quad . \tag{5.93}$$

From $r(\pi) = a_{\rm E}$, we obtain

$$\varepsilon_{\rm I} = \lambda^2 - 1 \quad . \tag{5.94}$$

This orbit will intersect the orbit of Jupiter if $r_a \geq a_J$, which means

$$\frac{d}{1-\varepsilon_I} \geq a_J \quad \Rightarrow \quad \lambda \geq \sqrt{\frac{2a_J}{a_J + a_E}} = 1.2952 \quad . \tag{5.95}$$

If this inequality holds, then intersection of Jupiter's orbit will occur for

$$\phi_J = 2\pi - \cos^{-1}\left(\frac{a_J - \lambda^2 a_E}{(\lambda^2 - 1)\, a_J}\right) \quad . \tag{5.96}$$

Finally, the time for this portion of the trajectory is

$$\tau_{EJ} = \tau_E \cdot \lambda^3 \int\limits_{\pi}^{\phi_J} \frac{d\phi}{2\pi} \frac{1}{\big[1 - (\lambda^2 - 1)\cos\phi\big]^2} \quad . \tag{5.97}$$

5.5.2 Encounter with Jupiter (Phase II)

We are interested in the final speed $\boldsymbol{v}_f$ of the probe after its encounter with Jupiter. We will determine the speed v_f and the angle δ which the probe makes with respect to Jupiter after its encounter. According to the geometry of fig. 5.11,

$$\begin{aligned} v_f^2 &= v_J^2 + u^2 - 2uv_J \cos(\chi + \gamma) \\ \cos\delta &= \frac{v_J^2 + v_f^2 - u^2}{2v_f v_J} \end{aligned} \tag{5.98}$$

Note that

$$v_J^2 = \frac{GM_\odot}{a_J} = \frac{a_E}{a_J} \cdot v_E^2 \quad . \tag{5.99}$$

But what are u, χ, and γ?

To determine u, we invoke

$$u^2 = v_J^2 + v_i^2 - 2v_J v_i \cos\beta \quad . \tag{5.100}$$

The initial velocity (in the frame of the Sun) when the probe crosses Jupiter's orbit is given by energy conservation:

$$\tfrac{1}{2}m(\lambda v_E)^2 - \frac{GM_\odot m}{a_E} = \tfrac{1}{2}mv_i^2 - \frac{GM_\odot m}{a_J} \quad , \tag{5.101}$$

which yields

$$v_i^2 = \left(\lambda^2 - 2 + \frac{2a_E}{a_J}\right) v_E^2 \quad . \tag{5.102}$$

As for β, we invoke conservation of angular momentum:

$$\mu(v_i \cos\beta) a_J = \mu(\lambda v_E) a_E \quad \Rightarrow \quad v_i \cos\beta = \lambda \frac{a_E}{a_J} v_E \quad . \tag{5.103}$$

The angle γ is determined from

$$v_J = v_i \cos\beta + u\cos\gamma \quad . \tag{5.104}$$

Putting all this together, we obtain

$$\begin{aligned} v_{\rm i} &= v_{\rm E}\sqrt{\lambda^2-2+2x} \\ u &= v_{\rm E}\sqrt{\lambda^2-2+3x-2\lambda x^{3/2}} \\ \cos\gamma &= \frac{\sqrt{x}-\lambda x}{\sqrt{\lambda^2-2+3x-2\lambda x^{3/2}}} \quad , \end{aligned} \tag{5.105}$$

where $x \equiv a_{\rm E}/a_{\rm J} = 0.1923$.

We next consider the scattering of the probe by the planet Jupiter. In the Jovian frame, we may write

$$r(\phi) = \frac{\kappa R_{\rm J}\,(1+\varepsilon_{\rm J})}{1+\varepsilon_{\rm J}\cos\phi} \quad , \tag{5.106}$$

where perijove occurs at $r(0) = \kappa R_{\rm J}$. Here, κ is a dimensionless quantity, which is simply perijove in units of the Jovian radius. Clearly we require $\kappa > 1$ or else the probe crashes into Jupiter! The probe's energy in this frame is simply $E = \frac{1}{2}mu^2$, which means the probe enters into a hyperbolic orbit about Jupiter. Next, from

$$E = \frac{k}{2}\,\frac{\varepsilon^2-1}{\ell^2/\mu k} \qquad , \qquad \frac{\ell^2}{\mu k} = (1+\varepsilon)\,\kappa R_{\rm J} \tag{5.107}$$

we find

$$\varepsilon_{\rm J} = 1 + \kappa\left(\frac{R_{\rm J}}{a_{\rm E}}\right)\left(\frac{M_\odot}{M_{\rm J}}\right)\left(\frac{u}{v_{\rm E}}\right)^2 \quad . \tag{5.108}$$

The opening angle of the Keplerian hyperbola is then $\phi_{\rm c} = \cos^{-1}\left(\varepsilon_{\rm J}^{-1}\right)$, and the angle χ is related to $\phi_{\rm c}$ through

$$\chi = \pi - 2\phi_{\rm c} = \pi - 2\cos^{-1}\left(\frac{1}{\varepsilon_{\rm J}}\right) \quad . \tag{5.109}$$

Therefore, we may finally write

$$v_{\rm f} = \sqrt{x\,v_{\rm E}^2 + u^2 + 2uv_{\rm E}\sqrt{x}\cos(2\phi_{\rm c}-\gamma)} \tag{5.110}$$

and

$$\cos\delta = \frac{xv_{\rm E}^2 + v_{\rm f}^2 - u^2}{2\,v_{\rm f}\,v_{\rm E}\sqrt{x}} \quad . \tag{5.111}$$

5.5.3 Jupiter to Neptune (Phase III)

Immediately after undergoing gravitational scattering off Jupiter, the energy and angular momentum of the probe are

$$E = \tfrac{1}{2}mv_{\rm f}^2 - \frac{GM_\odot m}{a_{\rm J}} \tag{5.112}$$

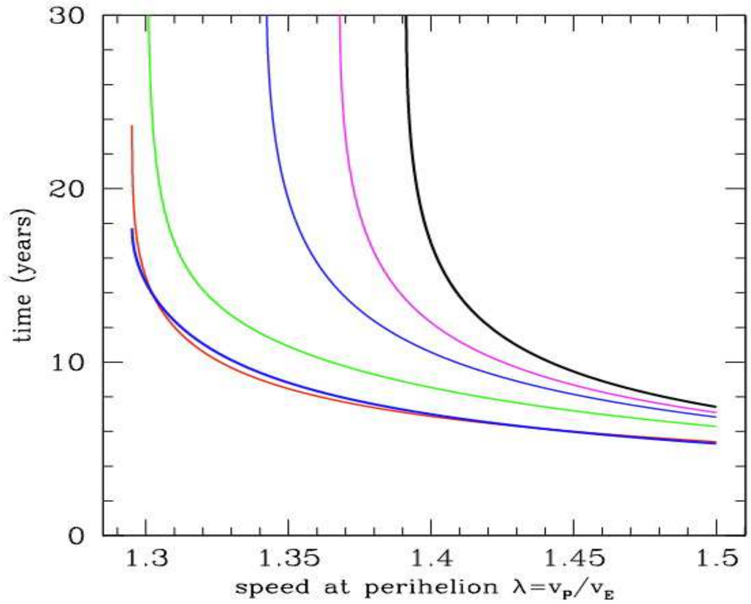


Figure 5.12: Total time for Earth-Neptune mission as a function of dimensionless velocity at perihelion, $\lambda = v_{\rm p}/v_{\rm E}$. Six different values of κ, the value of perijove in units of the Jovian radius, are shown: $\kappa = 1.0$ (thick blue), $\kappa = 5.0$ (red), $\kappa = 20$ (green), $\kappa = 50$ (blue), $\kappa = 100$ (magenta), and $\kappa = \infty$ (thick black).

and $\ell = \mu\, v_{\rm f}\, a_{\rm J} \cos\delta$. We write the geometric equation for the probe's orbit as

$$r(\phi) = \frac{d}{1 + \varepsilon\cos(\phi - \phi_{\rm J} - \alpha)} \quad , \tag{5.113}$$

where

$$d = \frac{\ell^2}{\mu k} = \left(\frac{v_{\rm f}\, a_{\rm J} \cos\delta}{v_{\rm E}\, a_{\rm E}}\right)^2 a_{\rm E} \quad . \tag{5.114}$$

Setting $E = (\varepsilon^2 - 1)(\mu k^2/2\ell^2)$, we obtain the eccentricity

$$\varepsilon = \sqrt{1 + \left(\frac{v_{\rm f}^2}{v_{\rm E}^2} - \frac{2a_{\rm E}}{a_{\rm J}}\right)\frac{d}{a_{\rm E}}} \quad . \tag{5.115}$$

Note that the orbit is hyperbolic – the probe will escape the Sun – if $v_{\rm f} > (2a_{\rm E}/a_{\rm J})^{1/2}\, v_{\rm E}$. The condition that this orbit intersect Jupiter at $\phi = \phi_{\rm J}$ yields

$$\cos\alpha = \frac{1}{\varepsilon}\left(\frac{d}{a_{\rm J}} - 1\right) \quad , \tag{5.116}$$

which determines the angle α. Interception of Neptune occurs at

$$\frac{d}{1 + \varepsilon\cos(\phi_{\rm N} - \phi_{\rm J} - \alpha)} = a_{\rm N} \qquad \Rightarrow \qquad \phi_{\rm N} = \phi_{\rm J} + \alpha + \cos^{-1}\left[\frac{1}{\varepsilon}\left(\frac{d}{a_{\rm N}} - 1\right)\right] \quad . \tag{5.117}$$

We then have

$$\tau_{\rm JN} = \tau_{\rm E} \cdot \left(\frac{d}{a_{\rm E}}\right)^3 \int\limits_{\phi_{\rm J}}^{\phi_{\rm N}} \frac{d\phi}{2\pi} \, \frac{1}{\big[1 + \varepsilon \cos(\phi - \phi_{\rm J} - \alpha)\big]^2} \quad . \tag{5.118}$$

The total time to Neptune is then the sum,

$$\tau_{\rm EN} = \tau_{\rm EJ} + \tau_{\rm JN} \quad . \tag{5.119}$$

Fig. 5.12 shows the mission time $\tau_{\rm EN}$ *versus* the velocity at perihelion, $v_{\rm p} = \lambda v_{\rm E}$, for various values of κ. The value $\kappa = \infty$ corresponds to the case of no Jovian encounter at all.

5.6 Restricted Three-Body Problem

Problem : Consider the 'restricted three body problem' in which a light object of mass m (*e.g.* a satellite) moves in the presence of two celestial bodies of masses m_1 and m_2 (*e.g.* the sun and the earth, or the earth and the moon). Suppose m_1 and m_2 execute stable circular motion about their common center of mass. You may assume $m \ll m_2 \le m_1$.

(a) Show that the angular frequency for the motion of masses 1 and 2 is related to their (constant) relative separation, by $\omega_0^2 = GM/r_0^3$, where $M = m_1 + m_2$ is the total mass.

Solution : For a Kepler potential $U = -k/r$, the circular orbit lies at $r_0 = \ell^2/\mu k$, where $\ell = \mu r^2 \dot\phi$ is the angular momentum and $k = G m_1 m_2$. This gives

$$\omega_0^2 = \frac{\ell^2}{\mu^2\, r_0^4} = \frac{k}{\mu r_0^3} = \frac{GM}{r_0^3} \quad , \tag{5.120}$$

with $\omega_0 = \dot\phi$.

(b) The satellite moves in the combined gravitational field of the two large bodies; the satellite itself is of course much too small to affect their motion. In deriving the motion for the satellilte, it is convenient to choose a reference frame whose origin is the CM and which rotates with angular velocity ω_0. In the rotating frame the masses m_1 and m_2 lie, respectively, at $x_1 = -\alpha r_0$ and $x_2 = \beta r_0$, with

$$\alpha = \frac{m_2}{M} \qquad , \qquad \beta = \frac{m_1}{M} \tag{5.121}$$

and with $y_1 = y_2 = 0$. Note $\alpha + \beta = 1$.

Show that the Lagrangian for the satellite in this rotating frame may be written

$$L = \tfrac{1}{2} m \big(\dot x - \omega_0\, y\big)^2 + \tfrac{1}{2} m \big(\dot y + \omega_0\, x\big)^2 + \frac{G\, m_1\, m}{\sqrt{(x + \alpha r_0)^2 + y^2}} + \frac{G\, m_2\, m}{\sqrt{(x - \beta r_0)^2 + y^2}} \quad . \tag{5.122}$$

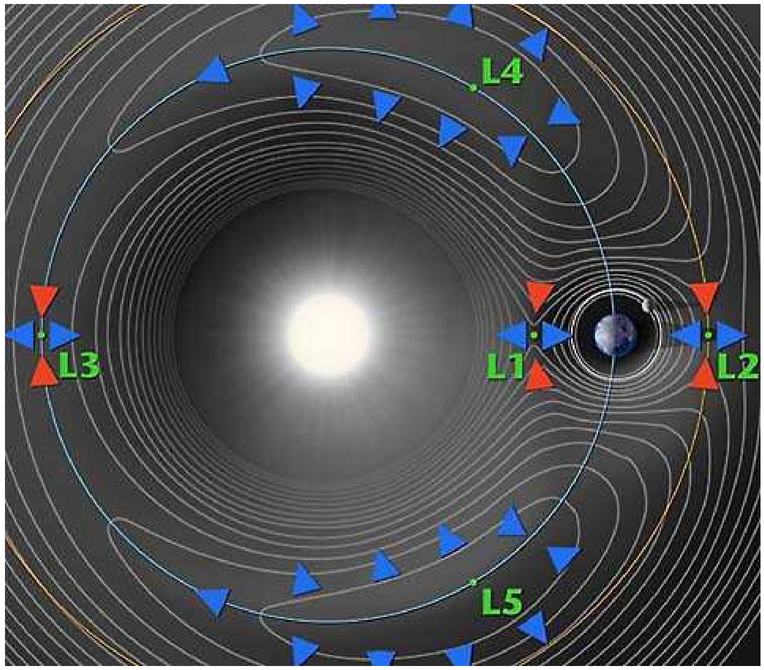


Figure 5.13: The Lagrange points for the earth-sun system. *Credit: WMAP project.*

Solution : Let the original (inertial) coordinates be (x_0, y_0). Then let us define the rotated coordinates (x, y) as

$$\begin{aligned} x &= \cos(\omega_0 t)\, x_0 + \sin(\omega_0 t)\, y_0 \\ y &= -\sin(\omega_0 t)\, x_0 + \cos(\omega_0 t)\, y_0 \quad . \end{aligned} \tag{5.123}$$

Therefore,

$$\begin{aligned} \dot{x} &= \cos(\omega_0 t)\, \dot{x}_0 + \sin(\omega_0 t)\, \dot{y}_0 + \omega_0\, y \\ \dot{y} &= -\sin(\omega_0 t)\, x_0 + \cos(\omega_0 t)\, y_0 - \omega_0\, x \quad . \end{aligned} \tag{5.124}$$

Therefore

$$(\dot{x} - \omega_0\, y)^2 + (\dot{y} + \omega_0\, x)^2 = \dot{x}_0^2 + \dot{y}_0^2 \quad , \tag{5.125}$$

The Lagrangian is then

$$L = \tfrac{1}{2} m \big(\dot{x} - \omega_0\, y\big)^2 + \tfrac{1}{2} m \big(\dot{y} + \omega_0\, x\big)^2 + \frac{G\, m_1\, m}{\sqrt{(x - x_1)^2 + y^2}} + \frac{G\, m_2\, m}{\sqrt{(x - x_2)^2 + y^2}} \quad , \tag{5.126}$$

which, with $x_1 \equiv -\alpha r_0$ and $x_2 \equiv \beta r_0$, agrees with eqn. 5.122

(c) Lagrange discovered that there are five special points where the satellite remains fixed in the rotating frame. These are called the *Lagrange points* {L1, L2, L3, L4, L5}. A sketch of the Lagrange points for the earth-sun system is provided in fig. 5.13. *Observation: In working out the rest of this problem, I found it convenient to measure all distances in units of r_0 and times in units of ω_0^{-1}, and to eliminate G by writing $Gm_1 = \beta\, \omega_0^2\, r_0^3$ and $Gm_2 = \alpha\, \omega_0^2\, r_0^3$.*

Assuming the satellite is stationary in the rotating frame, derive the equations for the positions of the Lagrange points.

Solution : At this stage it is convenient to measure all distances in units of r_0 and times in units of ω_0^{-1} to factor out a term $m\,r_0^2\,\omega_0^2$ from L, writing the dimensionless Lagrangian $\widetilde{L} \equiv L/(m\,r_0^2\,\omega_0^2)$. Using as well the definition of ω_0^2 to eliminate G, we have

$$\widetilde{L} = \tfrac{1}{2}\,(\dot{\xi} - \eta)^2 + \tfrac{1}{2}(\dot{\eta} + \xi)^2 + \frac{\beta}{\sqrt{(\xi+\alpha)^2 + \eta^2}} + \frac{\alpha}{\sqrt{(\xi-\beta)^2 + \eta^2}} \quad , \tag{5.127}$$

with

$$\xi \equiv \frac{x}{r_0} \quad , \quad \eta \equiv \frac{y}{r_0} \quad , \quad \dot{\xi} \equiv \frac{1}{\omega_0\,r_0}\frac{dx}{dt} \quad , \quad \dot{\eta} \equiv \frac{1}{\omega_0\,r_0}\frac{dy}{dt} \quad . \tag{5.128}$$

The equations of motion are then

$$\begin{aligned} \ddot{\xi} - 2\dot{\eta} &= \xi - \frac{\beta(\xi+\alpha)}{d_1^3} - \frac{\alpha(\xi-\beta)}{d_2^3} \\ \ddot{\eta} + 2\dot{\xi} &= \eta - \frac{\beta\eta}{d_1^3} - \frac{\alpha\eta}{d_2^3} \quad , \end{aligned} \tag{5.129}$$

where

$$d_1 = \sqrt{(\xi+\alpha)^2 + \eta^2} \quad , \quad d_2 = \sqrt{(\xi-\beta)^2 + \eta^2} \quad . \tag{5.130}$$

Here, $\xi \equiv x/r_0$, $\xi = y/r_0$, *etc.* Recall that $\alpha + \beta = 1$. Setting the time derivatives to zero yields the static equations for the Lagrange points:

$$\xi = \frac{\beta(\xi+\alpha)}{d_1^3} + \frac{\alpha(\xi-\beta)}{d_2^3} \quad , \quad \eta = \frac{\beta\eta}{d_1^3} + \frac{\alpha\eta}{d_2^3} \quad , \tag{5.131}$$

(d) Show that the Lagrange points with $y = 0$ are determined by a single nonlinear equation. Show graphically that this equation always has three solutions, one with $x < x_1$, a second with $x_1 < x < x_2$, and a third with $x > x_2$. These solutions correspond to the points L3, L1, and L2, respectively.

Solution : If $\eta = 0$ the second equation is automatically satisfied. The first equation then gives

$$\xi = \beta \cdot \frac{\xi+\alpha}{\left|\xi+\alpha\right|^3} + \alpha \cdot \frac{\xi-\beta}{\left|\xi-\beta\right|^3} \quad . \tag{5.132}$$

The RHS of the above equation diverges to $+\infty$ for $\xi = -\alpha + 0^+$ and $\xi = \beta + 0^+$, and diverges to $-\infty$ for $\xi = -\alpha - 0^+$ and $\xi = \beta - 0^+$, where 0^+ is a positive infinitesimal. The situation is depicted in fig. 5.14. Clearly there are three solutions, one with $\xi < -\alpha$, one with $-\alpha < \xi < \beta$, and one with $\xi > \beta$.

(e) Show that the remaining two Lagrange points, L4 and L5, lie along equilateral triangles with the two masses at the other vertices.

Solution : If $\eta \neq 0$, then dividing the second equation by η yields

$$1 = \frac{\beta}{d_1^3} + \frac{\alpha}{d_2^3} \quad . \tag{5.133}$$

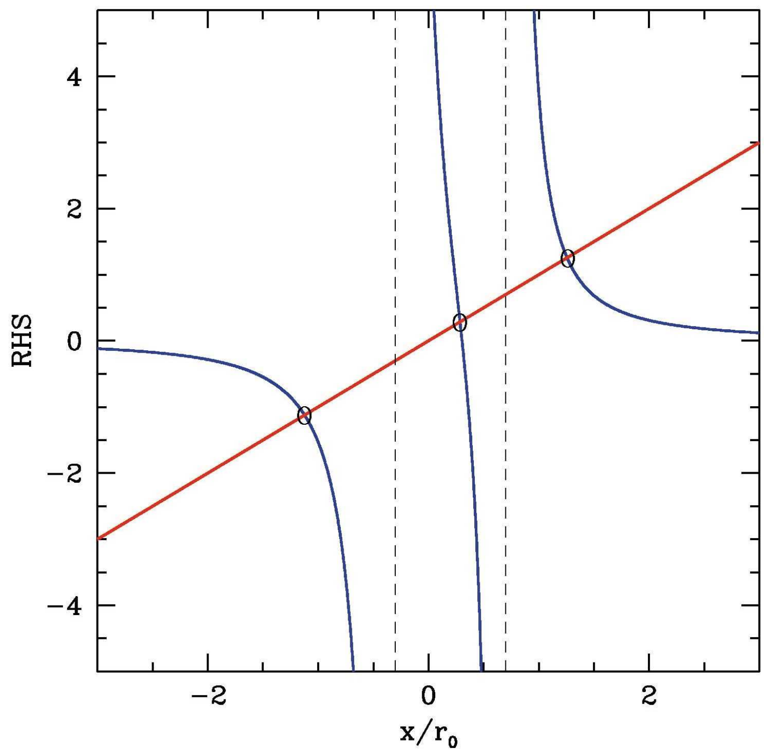


Figure 5.14: Graphical solution for the Lagrange points L1, L2, and L3.

Substituting this into the first equation,

$$\xi = \left(\frac{\beta}{d_1^3} + \frac{\alpha}{d_2^3}\right)\xi + \left(\frac{1}{d_1^3} - \frac{1}{d_2^3}\right)\alpha\beta \quad , \tag{5.134}$$

gives $d_1 = d_2$. Reinserting this into the previous equation then yields $d_1 = d_2 = 1$, which says that each of L4 and L5 lies on an equilateral triangle whose two other vertices are the masses m_1 and m_2. The side length of this equilateral triangle is r_0. Thus, the dimensionless coordinates of L4 and L5 are

$$(\xi_{\rm L4},\, \eta_{\rm L4}) = \left(\tfrac{1}{2} - \alpha,\, \tfrac{\sqrt{3}}{2}\right) \quad , \quad (\xi_{\rm L5},\, \eta_{\rm L5}) = \left(\tfrac{1}{2} - \alpha,\, -\tfrac{\sqrt{3}}{2}\right) \quad . \tag{5.135}$$

It turns out that L1, L2, and L3 are always unstable. Satellites placed in these positions must undergo periodic course corrections in order to remain approximately fixed. The SOlar and Heliopheric Observation satellite, *SOHO*, is located at L1, which affords a continuous unobstructed view of the Sun.

(f) Show that the Lagrange points L4 and L5 are stable (obviously you need only consider one of them) provided that the mass ratio m_1/m_2 is sufficiently large. Determine this critical ratio. Also find the frequency of small oscillations for motion in the vicinity of L4 and L5.

Solution : Now we write

$$\xi = \xi_{\rm L4} + \delta\xi \quad , \quad \eta = \eta_{\rm L4} + \delta\eta \quad , \tag{5.136}$$

and derive the linearized dynamics. Expanding the equations of motion to lowest order in $\delta\xi$ and $\delta\eta$, we have

$$\begin{aligned} \delta\ddot{\xi} - 2\delta\dot{\eta} &= \bigg(1 - \beta + \tfrac{3}{2}\beta\,\frac{\partial d_1}{\partial\xi}\bigg|_{\rm L4} - \alpha - \tfrac{3}{2}\alpha\,\frac{\partial d_2}{\partial\xi}\bigg|_{\rm L4}\bigg)\delta\xi + \bigg(\tfrac{3}{2}\beta\,\frac{\partial d_1}{\partial\eta}\bigg|_{\rm L4} - \tfrac{3}{2}\alpha\,\frac{\partial d_2}{\partial\eta}\bigg|_{\rm L4}\bigg)\delta\eta \\ &= \tfrac{3}{4}\,\delta\xi + \tfrac{3\sqrt{3}}{4}\,\varepsilon\,\delta\eta \end{aligned} \tag{5.137}$$

and

$$\begin{aligned} \delta\ddot{\eta} + 2\delta\dot{\xi} &= \bigg(\tfrac{3\sqrt{3}}{2}\beta\,\frac{\partial d_1}{\partial\xi}\bigg|_{\rm L4} + \tfrac{3\sqrt{3}}{2}\alpha\,\frac{\partial d_2}{\partial\xi}\bigg|_{\rm L4}\bigg)\delta\xi + \bigg(\tfrac{3\sqrt{3}}{2}\beta\,\frac{\partial d_1}{\partial\eta}\bigg|_{\rm L4} + \tfrac{3\sqrt{3}}{2}\alpha\,\frac{\partial d_2}{\partial\eta}\bigg|_{\rm L4}\bigg)\delta\eta \\ &= \tfrac{3\sqrt{3}}{4}\,\varepsilon\,\delta\xi + \tfrac{9}{4}\,\delta\eta \quad , \end{aligned} \tag{5.138}$$

where we have defined

$$\varepsilon \equiv \beta - \alpha = \frac{m_1 - m_2}{m_1 + m_2} \quad . \tag{5.139}$$

As defined, $\varepsilon \in [0,1]$.

Fourier transforming the differential equation, we replace each time derivative by $(-i\nu)$, and thereby obtain

$$\begin{pmatrix} \nu^2 + \frac{3}{4} & -2i\nu + \frac{3}{4}\sqrt{3}\,\varepsilon \\ 2i\nu + \frac{3}{4}\sqrt{3}\,\varepsilon & \nu^2 + \frac{9}{4} \end{pmatrix} \begin{pmatrix} \delta\hat{\xi} \\ \delta\hat{\eta} \end{pmatrix} = 0 \quad . \tag{5.140}$$

Nontrivial solutions exist only when the determinant D vanishes. One easily finds

$$D(\nu^2) = \nu^4 - \nu^2 + \tfrac{27}{16}\left(1 - \varepsilon^2\right) \quad , \tag{5.141}$$

which yields a quadratic equation in ν^2, with roots

$$\nu^2 = \tfrac{1}{2} \pm \tfrac{1}{4}\sqrt{27\,\varepsilon^2 - 23} \quad . \tag{5.142}$$

These frequencies are dimensionless. To convert to dimensionful units, we simply multiply the solutions for ν by ω_0, since we have rescaled time by ω_0^{-1}.

Note that the L4 and L5 points are stable only if $\varepsilon^2 > \frac{23}{27}$. If we define the mass ratio $\gamma \equiv m_1/m_2$, the stability condition is equivalent to

$$\gamma = \frac{m_1}{m_2} > \frac{\sqrt{27} + \sqrt{23}}{\sqrt{27} - \sqrt{23}} = 24.960 \quad , \tag{5.143}$$

which is satisfied for both the Sun-Jupiter system ($\gamma = 1047$) – and hence for the Sun and any planet – and also for the Earth-Moon system ($\gamma = 81.2$).

Objects found at the L4 and L5 points are called *Trojans*, after the three large asteroids Agamemnon, Achilles, and Hector found orbiting in the L4 and L5 points of the Sun-Jupiter system. No large asteroids have been found in the L4 and L5 points of the Sun-Earth system.

Personal aside : David T. Wilkinson

The image in fig. 5.13 comes from the education and outreach program of the Wilkinson Microwave Anisotropy Probe (WMAP) project, a NASA mission, launched in 2001, which has produced some of the most important recent data in cosmology. The project is named in honor of David T. Wilkinson, who was a leading cosmologist at Princeton, and a founder of the Cosmic Background Explorer (COBE) satellite (launched in 1989). WMAP was sent to the L2 Lagrange point, on the night side of the earth, where it can constantly scan the cosmos with an ultra-sensitive microwave detector, shielded by the earth from interfering solar electromagnetic radiation. The L2 point is of course unstable, with a time scale of about 23 days. Satellites located at such points must undergo regular course and attitude corrections to remain situated.

During the summer of 1981, as an undergraduate at Princeton, I was a member of Wilkinson's "gravity group," working under Jeff Kuhn and Ken Libbrecht. It was a pretty big group and Dave – everyone would call him Dave – used to throw wonderful parties at his home, where we'd always play volleyball. I was very fortunate to get to know David Wilkinson a bit – after working in his group that summer I took a class from him the following year. He was a wonderful person, a superb teacher, and a world class physicist.

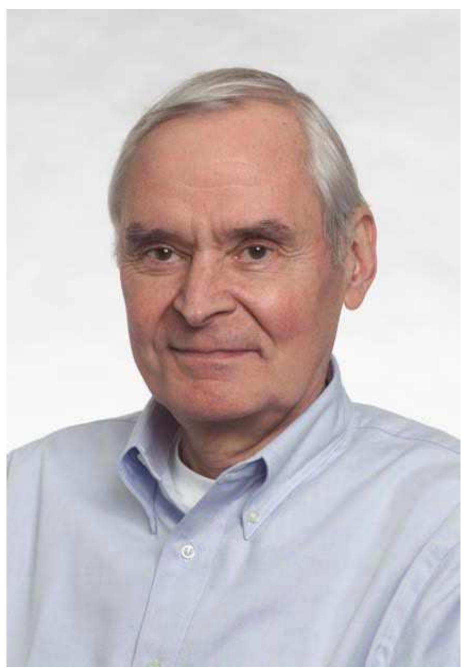


Figure 5.15: David T. Wilkinson (1935 – 2002).

Chapter 6

Linearized Dynamics of Coupled Oscillations

6.1 Basic Objective

Our basic objective in studying small coupled oscillations is to expand the equations of motion to linear order in the n generalized coordinates about a stable equilibrium configuration. This yields a set of n coupled second order differential equations that is both *linear* and *homogeneous*. Such a system may then be solved by elementary linear algebraic means. The general solution may then be written as a sum over n *normal mode oscillations*, each of which oscillates at a particular *eigenfrequency* ω_j, with $j \in \{1, \ldots, n\}$. The set of eigenfrequencies is determined by the form of the linearized equations of motion. The n normal mode amplitudes and n normal mode phase shifts are determined by the $2n$ initial conditions on the generalized coordinates and velocities.

6.2 Euler-Lagrange Equations of Motion

We assume, for a set of n generalized coordinates $\{q_1, \ldots, q_n\}$, that the kinetic energy is a quadratic function of the velocities,

$$T = \tfrac{1}{2}\, T_{\sigma\sigma'}(q_1, \ldots, q_n)\, \dot{q}_\sigma\, \dot{q}_{\sigma'} \quad , \tag{6.1}$$

where the sum on σ and σ' from 1 to n is implied. For example, expressed in terms of polar coordinates (r, θ, ϕ), the matrix $T_{\sigma\sigma'}$ is

$$T_{\sigma\sigma'}(r, \theta, \phi) = m \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\!\theta \end{pmatrix} \quad \Longrightarrow \quad T = \tfrac{1}{2}\, m\big(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\!\theta\, \dot{\phi}^2\big) \quad . \tag{6.2}$$

The potential $U(q_1, \ldots, q_n)$ is assumed to be a function of the generalized coordinates alone: $U = U(q)$. A more general formulation of the problem of small oscillations is given in the appendix, section 6.8.

The generalized momenta are

$$p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma} = T_{\sigma\sigma'}\,\dot{q}_{\sigma'} \quad , \tag{6.3}$$

and the generalized forces are

$$F_\sigma = \frac{\partial L}{\partial q_\sigma} = \frac{1}{2}\,\frac{\partial T_{\sigma'\sigma''}}{\partial q_\sigma}\,\dot{q}_{\sigma'}\,\dot{q}_{\sigma''} - \frac{\partial U}{\partial q_\sigma} \quad . \tag{6.4}$$

The Euler-Lagrange equations are then $\dot{p}_\sigma = F_\sigma$, or

$$T_{\sigma\sigma'}\,\ddot{q}_{\sigma'} + \left(\frac{\partial T_{\sigma\sigma'}}{\partial q_{\sigma''}} - \frac{1}{2}\,\frac{\partial T_{\sigma'\sigma''}}{\partial q_\sigma}\right)\dot{q}_{\sigma'}\,\dot{q}_{\sigma''} = -\frac{\partial U}{\partial q_\sigma} \tag{6.5}$$

which is a set of coupled nonlinear second order ODEs. Here we are using the Einstein 'summation convention', where we automatically sum over any and all repeated indices.

6.3 Expansion about Static Equilibrium

Small oscillation theory begins with the identification of a static equilibrium $\{\bar{q}_1, \ldots, \bar{q}_n\}$, which satisfies the n nonlinear equations

$$\left.\frac{\partial U}{\partial q_\sigma}\right|_{q=\bar{q}} = 0 \quad . \tag{6.6}$$

Once an equilibrium is found (note that there may be more than one static equilibrium), we expand about this equilibrium, writing

$$q_\sigma \equiv \bar{q}_\sigma + \eta_\sigma \quad . \tag{6.7}$$

The coordinates $\{\eta_1, \ldots, \eta_n\}$ represent the *displacements relative to equilibrium.*

We next expand the Lagrangian to quadratic order in the generalized displacements, yielding

$$L = \tfrac{1}{2}\mathsf{T}_{\sigma\sigma'}\,\dot{\eta}_\sigma\,\dot{\eta}_{\sigma'} - \tfrac{1}{2}\mathsf{V}_{\sigma\sigma'}\,\eta_\sigma\eta_{\sigma'} \quad , \tag{6.8}$$

where

$$\mathsf{T}_{\sigma\sigma'} = \left.\frac{\partial^2 T}{\partial\dot{q}_\sigma\,\partial\dot{q}_{\sigma'}}\right|_{q=\bar{q}} \quad , \qquad \mathsf{V}_{\sigma\sigma'} = \left.\frac{\partial^2 U}{\partial q_\sigma\,\partial q_{\sigma'}}\right|_{q=\bar{q}} \quad . \tag{6.9}$$

Writing $\boldsymbol{\eta}^{\mathsf{t}}$ for the row-vector $(\eta_1, \ldots, \eta_n)$, we may suppress indices and write

$$L = \tfrac{1}{2}\,\dot{\boldsymbol{\eta}}^{\mathsf{t}}\,\mathsf{T}\,\dot{\boldsymbol{\eta}} - \tfrac{1}{2}\,\boldsymbol{\eta}^{\mathsf{t}}\,\mathsf{V}\,\boldsymbol{\eta} \quad , \tag{6.10}$$

where T and V are the constant matrices of eqn. 6.9.

6.4 Method of Small Oscillations

The idea behind the method of small oscillations is to effect a coordinate transformation from the generalized displacements $\boldsymbol{\eta}$ to a new set of coordinates $\boldsymbol{\xi}$, which render the Lagrangian particularly simple. All that is required is a linear transformation,

$$\eta_\sigma = \mathsf{A}_{\sigma i}\,\xi_i \quad , \tag{6.11}$$

where both σ and i run from 1 to n. The $n \times n$ matrix $\mathsf{A}_{\sigma i}$ is known as the *modal matrix*. With the substitution $\boldsymbol{\eta} = \mathsf{A}\,\boldsymbol{\xi}$ (hence $\boldsymbol{\eta}^{\mathsf{t}} = \boldsymbol{\xi}^{\mathsf{t}}\,\mathsf{A}^{\mathsf{t}}$, where $\mathsf{A}^{\mathsf{t}}_{i\sigma} = \mathsf{A}_{\sigma i}$ is the matrix transpose), we have

$$L = \tfrac{1}{2}\,\dot{\boldsymbol{\xi}}^{\mathsf{t}}\,\mathsf{A}^{\mathsf{t}}\,\mathsf{T}\,\mathsf{A}\,\dot{\boldsymbol{\xi}} - \tfrac{1}{2}\,\boldsymbol{\xi}^{\mathsf{t}}\,\mathsf{A}^{\mathsf{t}}\,\mathsf{V}\,\mathsf{A}\,\boldsymbol{\xi} \quad . \tag{6.12}$$

We now choose the matrix A such that

$$\begin{aligned} \mathsf{A}^{\mathsf{t}}\,\mathsf{T}\,\mathsf{A} &= \mathbb{I} \\ \mathsf{A}^{\mathsf{t}}\,\mathsf{V}\,\mathsf{A} &= \mathsf{diag}\big(\omega_1^2\,,\,\ldots\,,\,\omega_n^2\big) \quad . \end{aligned} \tag{6.13}$$

With this choice of A, the Lagrangian decouples:

$$L = \tfrac{1}{2}\sum_{i=1}^{n}\Big(\dot{\xi}_i^2 - \omega_i^2\,\xi_i^2\Big) \quad , \tag{6.14}$$

with the solution

$$\xi_i(t) = C_i\cos(\omega_i\,t) + D_i\sin(\omega_i\,t) \quad , \tag{6.15}$$

where $\{C_1,\ldots,C_n\}$ and $\{D_1,\ldots,D_n\}$ are $2n$ constants of integration, determined by the $2n$ initial conditions on $\boldsymbol{\eta}(0)$ and $\dot{\boldsymbol{\eta}}(0)$, and where there is no implied sum on i. Note that

$$\boldsymbol{\xi} = \mathsf{A}^{-1}\boldsymbol{\eta} = \mathsf{A}^{\mathsf{t}}\,\mathsf{T}\boldsymbol{\eta} \quad . \tag{6.16}$$

In terms of the original generalized displacements, the solution is

$$\eta_\sigma(t) = \sum_{i=1}^{n}\mathsf{A}_{\sigma i}\Big\{C_i\cos(\omega_i t) + D_i\sin(\omega_i t)\Big\} \quad , \tag{6.17}$$

and the constants of integration are linearly related to the initial generalized displacements and generalized velocities:

$$\begin{aligned} C_i &= \mathsf{A}^{\mathsf{t}}_{i\sigma}\,\mathsf{T}_{\sigma\sigma'}\,\eta_{\sigma'}(0) \\ D_i &= \omega_i^{-1}\mathsf{A}^{\mathsf{t}}_{i\sigma}\,\mathsf{T}_{\sigma\sigma'}\,\dot{\eta}_{\sigma'}(0) \quad , \end{aligned} \tag{6.18}$$

again with no implied sum on i on the RHS of the second equation, and where we have used $\mathsf{A}^{-1} = \mathsf{A}^{\mathsf{t}}\,\mathsf{T}$, from eqn. 6.13. (The implied sums in eqn. 6.18 are over σ and σ'.)

If all the generalized coordinates have units of length, *i.e.* $[q_\sigma] = L$, then

$$[\mathsf{T}_{\sigma\sigma'}] = M \quad , \quad [\mathsf{V}_{\sigma\sigma'}] = MT^{-2} \quad , \quad [A_{\sigma i}] = M^{-1/2} \quad , \quad [\xi_i] = M^{1/2}L \quad . \tag{6.19}$$

Can you really just choose an A so that both of eqns. 6.13 hold?

Yes.

Er...care to elaborate?

Both T and V are symmetric matrices. Aside from that, there is no special relation between them. In particular, they need not commute, hence they do not necessarily share any eigenvectors. Nevertheless, they may be simultaneously diagonalized as per eqns. 6.13. Here's why:

- Since T is symmetric, it can be diagonalized by an orthogonal transformation. That is, there exists a matrix $O_1 \in O(n)$ such that
$$O_1^t \, T \, O_1 = D \quad , \tag{6.20}$$
where D is diagonal.

- We may safely assume that T is positive definite. Otherwise the kinetic energy can become arbitrarily negative, which is unphysical. Therefore, one may form the matrix $D^{-1/2}$ which is the diagonal matrix whose entries are the inverse square roots of the corresponding entries of D. Consider the linear transformation $O_1 D^{-1/2}$. Its effect on T is
$$D^{-1/2} \, \overbrace{O_1^t \, T \, O_1}^{D} \, D^{-1/2} = \mathbb{I} \quad . \tag{6.21}$$

- Since O_1 and D are wholly derived from T, the only thing we know about
$$\widetilde{V} \equiv D^{-1/2} \, O_1^t \, V \, O_1 \, D^{-1/2} \tag{6.22}$$
is that it is explicitly a symmetric matrix. Therefore, it may be diagonalized by some orthogonal matrix $O_2 \in O(n)$. As T has already been transformed to the identity, the additional orthogonal transformation has no effect there. Thus, we have shown that there exist orthogonal matrices O_1 and O_2 such that
$$\begin{aligned} O_2^t \, D^{-1/2} \, O_1^t \, T \, O_1 \, D^{-1/2} \, O_2 &= \mathbb{I} \\ O_2^t \, D^{-1/2} \, O_1^t \, V \, O_1 \, D^{-1/2} \, O_2 &= \mathrm{diag}(\omega_1^2, \ldots, \omega_n^2) \quad . \end{aligned} \tag{6.23}$$
All that remains is to identify the modal matrix $A = O_1 D^{-1/2} \, O_2$.

Note that it is *not possible* to simultaneously diagonalize *three* symmetric matrices in general.

6.4.1 Finding the modal matrix

While the above proof allows one to construct A by finding the two orthogonal matrices O_1 and O_2, such a procedure is extremely cumbersome. It would be much more convenient if A could be determined in one fell swoop. Fortunately, this is possible.

We start with the equations of motion, $\mathsf{T}\ddot{\boldsymbol{\eta}} + \mathsf{V}\boldsymbol{\eta} = 0$. In component notation, we have

$$\mathsf{T}_{\sigma\sigma'}\,\ddot{\eta}_{\sigma'} + \mathsf{V}_{\sigma\sigma'}\,\eta_{\sigma'} = 0 \quad . \tag{6.24}$$

We now assume that $\boldsymbol{\eta}(t)$ oscillates with a single frequency ω, *i.e.* $\eta_\sigma(t) = \psi_\sigma\, e^{-i\omega t}$. This results in a set of linear algebraic equations for the components ψ_σ:

$$\big(\omega^2\,\mathsf{T}_{\sigma\sigma'} - \mathsf{V}_{\sigma\sigma'}\big)\,\psi_{\sigma'} = 0 \quad . \tag{6.25}$$

These are n equations in n unknowns: one for each value of $\sigma = 1, \ldots, n$. Because the equations are homogeneous and linear, there is always a trivial solution $\boldsymbol{\psi} = 0$. In fact one might think this is the only solution, since

$$\big(\omega^2\,\mathsf{T} - \mathsf{V}\big)\,\boldsymbol{\psi} = 0 \qquad \stackrel{?}{\Longrightarrow} \qquad \boldsymbol{\psi} = \big(\omega^2\,\mathsf{T} - \mathsf{V}\big)^{-1}\,0 = 0 \quad . \tag{6.26}$$

However, this fails when the matrix $\omega^2\,\mathsf{T} - \mathsf{V}$ is defective[1], *i.e.* when

$$\det\big(\omega^2\,\mathsf{T} - \mathsf{V}\big) = 0 \quad . \tag{6.27}$$

Since T and V are of rank n, the above determinant yields an n^{th} order polynomial in ω^2, whose n roots are the desired squared eigenfrequencies $\{\omega_1^2\,, \ldots,\, \omega_n^2\}$.

Once the n eigenfrequencies are obtained, the modal matrix is constructed as follows. Solve the equations

$$\sum_{\sigma'=1}^{n} \big(\omega_i^2\,\mathsf{T}_{\sigma\sigma'} - \mathsf{V}_{\sigma\sigma'}\big)\,\psi^{(i)}_{\sigma'} = 0 \tag{6.28}$$

which are a set of $(n-1)$ linearly independent equations among the n components of the eigenvector $\boldsymbol{\psi}^{(i)}$. That is, there are n equations ($\sigma = 1, \ldots, n$), but one linear dependency since $\det\,(\omega_i^2\,\mathsf{T} - \mathsf{V}) = 0$. The eigenvectors may be chosen to satisfy a generalized orthogonality relationship,

$$\psi^{(i)}_\sigma\,\mathsf{T}_{\sigma\sigma'}\,\psi^{(j)}_{\sigma'} = \delta_{ij} \quad . \tag{6.29}$$

To see this, let us duplicate eqn. 6.28, replacing i with j, and multiply both equations as follows:

$$\begin{aligned} \psi^{(j)}_\sigma \times \big(\omega_i^2\,\mathsf{T}_{\sigma\sigma'} - \mathsf{V}_{\sigma\sigma'}\big)\,\psi^{(i)}_{\sigma'} &= 0 \\ \psi^{(i)}_\sigma \times \big(\omega_j^2\,\mathsf{T}_{\sigma\sigma'} - \mathsf{V}_{\sigma\sigma'}\big)\,\psi^{(j)}_{\sigma'} &= 0 \quad . \end{aligned} \tag{6.30}$$

Using the symmetry of T and V, upon subtracting these equations we obtain

$$(\omega_i^2 - \omega_j^2) \sum_{\sigma,\sigma'=1}^{n} \psi^{(i)}_\sigma\,\mathsf{T}_{\sigma\sigma'}\,\psi^{(j)}_{\sigma'} = 0 \quad , \tag{6.31}$$

[1] The label *defective* has a distastefully negative connotation. In modern parlance, we should instead refer to such a matrix as *determinantally challenged*.

where the sums on i and j have been made explicit. This establishes that eigenvectors $\boldsymbol{\psi}^{(i)}$ and $\boldsymbol{\psi}^{(j)}$ corresponding to distinct eigenvalues $\omega_i^2 \neq \omega_j^2$ are orthogonal: $(\boldsymbol{\psi}^{(i)})^{\mathsf{t}}\, \mathsf{T}\, \boldsymbol{\psi}^{(j)} = 0$. For degenerate eigenvalues, the eigenvectors are not *a priori* orthogonal, but they may be orthogonalized via application of the Gram-Schmidt procedure. The remaining degrees of freedom - one for each eigenvector – are fixed by imposing the condition of normalization:

$$\psi^{(i)}_\sigma \to \psi^{(i)}_\sigma \Big/ \sqrt{\psi^{(i)}_\mu \, \mathsf{T}_{\mu\mu'}\, \psi^{(i)}_{\mu'}} \qquad \Longrightarrow \qquad \psi^{(i)}_\sigma \, \mathsf{T}_{\sigma\sigma'}\, \psi^{(j)}_{\sigma'} = \delta_{ij} \quad . \tag{6.32}$$

The modal matrix is just the matrix of eigenvectors: $\mathsf{A}_{\sigma i} = \psi^{(i)}_\sigma$.

With the eigenvectors $\psi^{(i)}_\sigma$ thusly normalized, we have

$$\begin{aligned} 0 &= \psi^{(i)}_\sigma \left(\omega_j^2\, \mathsf{T}_{\sigma\sigma'} - \mathsf{V}_{\sigma\sigma'}\right) \psi^{(j)}_{\sigma'} \\ &= \omega_j^2\, \delta_{ij} - \psi^{(i)}_\sigma \, \mathsf{V}_{\sigma\sigma'}\, \psi^{(j)}_{\sigma'} \quad , \end{aligned} \tag{6.33}$$

with no sum on j. This establishes the result

$$\mathsf{A}^{\mathsf{t}}\, \mathsf{V}\, \mathsf{A} = \mathsf{diag}\big(\omega_1^2\,,\,\ldots\,,\,\omega_n^2\big) \quad . \tag{6.34}$$

Recall the relation $\eta_\sigma = \mathsf{A}_{\sigma i}\, \xi_i$ between the generalized displacements η_σ and the normal coordinates ξ_i. We can invert this relation to obtain

$$\xi_i = \mathsf{A}^{-1}_{i\sigma}\, \eta_\sigma = \mathsf{A}^{\mathsf{t}}_{i\sigma} \mathsf{T}_{\sigma\sigma'}\, \eta_{\sigma'} \quad . \tag{6.35}$$

Here we have used the result $\mathsf{A}^{\mathsf{t}}\, \mathsf{T}\, \mathsf{A} = 1$ to write

$$\mathsf{A}^{-1} = \mathsf{A}^{\mathsf{t}}\, \mathsf{T} \quad . \tag{6.36}$$

This is a convenient result, because it means that if we ever need to express the normal coordinates in terms of the generalized displacements, we don't have to invert any matrices – we just need to do one matrix multiplication.

6.4.2 Summary of the method

(i) Obtain the T and V matrices,

$$\mathsf{T}_{\sigma\sigma'} = \frac{\partial T}{\partial q_\sigma\, \partial \dot q_{\sigma'}}\bigg|_{\bar q} \qquad , \qquad \mathsf{V}_{\sigma\sigma'} = \frac{\partial U}{\partial q_\sigma\, \partial \dot q_{\sigma'}}\bigg|_{\bar q} \quad , \tag{6.37}$$

where the equilibrium conditions are $\partial U / \partial q_\sigma \big|_{\bar q} = 0$. The quadratic form Lagrangian for small oscillations of the generalized displacements from equilibrium η_σ and their velocities is then

$$L = \tfrac{1}{2}\, \dot\eta_\sigma\, \mathsf{T}_{\sigma\sigma'}\, \dot\eta_{\sigma'} - \tfrac{1}{2}\, \eta_\sigma\, \mathsf{V}_{\sigma\sigma'}\, \eta_{\sigma'} \quad . \tag{6.38}$$

(ii) Solve $\mathsf{det}(\omega^2\mathsf{T} - \mathsf{V}) = 0$, which is an $n^{\rm th}$ order polynomial in ω^2.

(iii) For each root ω_i^2, solve the defective linear system $(\omega_i^2\mathsf{T} - \mathsf{V})\,\boldsymbol{\psi}^{(i)} = 0$.

(iv) Eigenvectors corresponding to different eigenfrequencies will necessarily be orthogonal, *i.e.*

$$\langle\,\boldsymbol{\psi}^{(i)}\,|\,\boldsymbol{\psi}^{(j)}\,\rangle \equiv \psi^{(i)}_\sigma\,\mathsf{T}_{\sigma\sigma'}\,\psi^{(j)}_{\sigma'} = 0 \quad \text{if} \quad \omega_i^2 \neq \omega_j^2 \quad . \tag{6.39}$$

In the case of degenerate eigenvalues, use the Gram-Schmidt method to find an orthogonal basis for the degenerate subspace. Then normalize each eigenvector such that $\langle\,\boldsymbol{\psi}^{(i)}\,|\,\mathsf{T}\,|\,\boldsymbol{\psi}^{(j)}\,\rangle = \delta_{ij}$ for all i and j.

(v) The modal matrix $\mathsf{A}_{\sigma j} = \psi^{(j)}_\sigma$ then satisfies

$$\mathsf{A}^{\mathsf{t}}\,\mathsf{T}\,\mathsf{A} = \mathbb{I} \qquad , \qquad \mathsf{A}^{\mathsf{t}}\,\mathsf{V}\,\mathsf{A} = \mathsf{diag}\big(\omega_1^2\,,\,\ldots\,,\,\omega_n^2\big) \quad . \tag{6.40}$$

Note that $\mathsf{A}^{-1} = \mathsf{A}^{\mathsf{t}}\,\mathsf{T}$. The relation between the generalized displacements η_σ and the normal modes ξ_j is $\eta_\sigma = \mathsf{A}_{\sigma j}\,\xi_j$, which entails $\xi_j = \mathsf{A}^{\mathsf{t}}_{j\sigma}\mathsf{T}_{\sigma\sigma'}\eta_{\sigma'} = \mathsf{A}_{\sigma j}\mathsf{T}_{\sigma\sigma'}\eta_{\sigma'}$. In terms of the normal mode coordinates and their velocities,

$$L = \sum_i \tfrac{1}{2}\big(\dot\xi_i^2 - \omega_i^2\,\xi_i^2\big) \quad , \tag{6.41}$$

and the equations of motion are those of decoupled oscillators: $\ddot\xi_i = -\omega_i^2\,\xi_i$.

(vi) The complete solution for the generalized displacements is then

$$\eta_\sigma(t) = \sum_{i=1}^n \mathsf{A}_{\sigma i}\,\Big\{C_i\cos(\omega_i t) + D_i\sin(\omega_i t)\Big\} \quad , \tag{6.42}$$

with

$$C_i = \mathsf{A}^{\mathsf{t}}_{i\sigma}\,\mathsf{T}_{\sigma\sigma'}\,\eta_{\sigma'}(0) \qquad , \qquad D_i = \omega_i^{-1}\mathsf{A}^{\mathsf{t}}_{i\sigma}\,\mathsf{T}_{\sigma\sigma'}\,\dot\eta_{\sigma'}(0) \quad . \tag{6.43}$$

6.5 Examples

6.5.1 Masses and springs

Two blocks and three springs are configured as in fig. 6.1. All motion is horizontal. When the blocks are at rest, all springs are unstretched.

(a) Choose as generalized coordinates the displacement of each block from its equilibrium position, and write the Lagrangian.

(b) Find the T and V matrices.

(c) Suppose

$$m_1 = 2m \quad , \quad m_2 = m \quad , \quad k_1 = 4k \quad , \quad k_2 = k \quad , \quad k_3 = 2k \quad , \tag{6.44}$$

Find the frequencies of small oscillations.

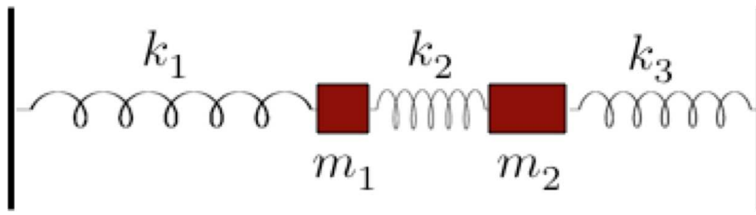


Figure 6.1: A system of masses and springs.

(d) Find the normal modes of oscillation.

(e) At time $t = 0$, mass #1 is displaced by a distance b relative to its equilibrium position. *I.e.* $x_1(0) = b$. The other initial conditions are $x_2(0) = 0$, $\dot{x}_1(0) = 0$, and $\dot{x}_2(0) = 0$. Find t^*, the next time at which x_2 vanishes.

<u>Solution</u> :

(a) The Lagrangian is

$$L = \tfrac{1}{2} m_1\, \dot{x}_1^2 + \tfrac{1}{2} m_2\, \dot{x}_2^2 - \tfrac{1}{2} k_1\, x_1^2 - \tfrac{1}{2} k_2\, (x_2 - x_1)^2 - \tfrac{1}{2} k_3\, x_2^2 \quad , \tag{6.45}$$

which is already a quadratic form. Thus, the full equations of motion are already linear.

(b) The T and V matrices are

$$\mathsf{T}_{ij} = \frac{\partial^2 T}{\partial \dot{x}_i\, \partial \dot{x}_j} = \begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \qquad , \qquad \mathsf{V}_{ij} = \frac{\partial^2 U}{\partial x_i\, \partial x_j} = \begin{pmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{pmatrix} \quad . \tag{6.46}$$

(c) We have $m_1 = 2m$, $m_2 = m$, $k_1 = 4k$, $k_2 = k$, and $k_3 = 2k$. Let us write $\omega^2 \equiv \lambda\, \omega_0^2$, where $\omega_0 \equiv \sqrt{k/m}$. Then

$$\omega^2 \mathsf{T} - \mathsf{V} = k \begin{pmatrix} 2\lambda - 5 & 1 \\ 1 & \lambda - 3 \end{pmatrix} \quad . \tag{6.47}$$

The determinant is

$$\begin{aligned} \det\,(\omega^2 \mathsf{T} - \mathsf{V}) &= (2\lambda^2 - 11\lambda + 14)\, k^2 \\ &= (2\lambda - 7)\,(\lambda - 2)\, k^2 \quad . \end{aligned} \tag{6.48}$$

There are two roots: $\lambda_- = 2$ and $\lambda_+ = \frac{7}{2}$, corresponding to the eigenfrequencies

$$\omega_- = \sqrt{\frac{2k}{m}} \qquad , \qquad \omega_+ = \sqrt{\frac{7k}{2m}} \quad . \tag{6.49}$$

(d) The normal modes are determined from $(\omega_a^2 \mathsf{T} - \mathsf{V})\,\boldsymbol{\psi}^{(a)} = 0$. Plugging in $\lambda = 2$ we have for the normal mode $\boldsymbol{\psi}^{(-)}$

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \psi_1^{(-)} \\ \psi_2^{(-)} \end{pmatrix} = 0 \qquad \Rightarrow \qquad \boldsymbol{\psi}^{(-)} = \mathcal{C}_- \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad . \tag{6.50}$$

Plugging in $\lambda = \frac{7}{2}$ we have for the normal mode $\boldsymbol{\psi}^{(+)}$

$$\begin{pmatrix} 2 & 1 \\ 1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \psi_1^{(+)} \\ \psi_2^{(+)} \end{pmatrix} = 0 \qquad \Rightarrow \qquad \boldsymbol{\psi}^{(+)} = \mathcal{C}_+ \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad . \tag{6.51}$$

The standard normalization $\psi_i^{(a)}\,\mathsf{T}_{ij}\,\psi_j^{(b)} = \delta_{ab}$ gives

$$\mathcal{C}_- = \frac{1}{\sqrt{3m}} \qquad , \qquad \mathcal{C}_+ = \frac{1}{\sqrt{6m}} \quad . \tag{6.52}$$

(e) The general solution is

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos(\omega_- t) + B \begin{pmatrix} 1 \\ -2 \end{pmatrix} \cos(\omega_+ t) + C \begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin(\omega_- t) + D \begin{pmatrix} 1 \\ -2 \end{pmatrix} \sin(\omega_+ t) \quad . \tag{6.53}$$

The initial conditions $x_1(0) = b$, $x_2(0) = \dot{x}_1(0) = \dot{x}_2(0) = 0$ yield

$$A = \tfrac{2}{3} b \quad , \quad B = \tfrac{1}{3} b \quad , \quad C = 0 \quad , \quad D = 0 \quad . \tag{6.54}$$

Thus,

$$\begin{aligned} x_1(t) &= \tfrac{1}{3} b \cdot \Big(2\cos(\omega_- t) + \cos(\omega_+ t) \Big) \\ x_2(t) &= \tfrac{2}{3} b \cdot \Big(\cos(\omega_- t) - \cos(\omega_+ t) \Big) \quad . \end{aligned} \tag{6.55}$$

Setting $x_2(t^*) = 0$, we find

$$\cos(\omega_- t^*) = \cos(\omega_+ t^*) \quad \Rightarrow \quad \pi - \omega_- t = \omega_+ t - \pi \quad \Rightarrow \quad t^* = \frac{2\pi}{\omega_- + \omega_+} \quad . \tag{6.56}$$

6.5.2 Double pendulum

As a second example, consider the double pendulum, with $m_1 = m_2 = m$ and $\ell_1 = \ell_2 = \ell$. The Lagrangian and equations of motion for this problem were discussed in §4.4.5 for the general case of differing masses and lengths. For our simpler version, the kinetic and potential energies are

$$\begin{aligned} T &= m\ell^2 \dot{\theta}_1^2 + m\ell^2 \cos(\theta_1 - \theta_1)\, \dot{\theta}_1 \dot{\theta}_2 + \tfrac{1}{2} m\ell^2 \dot{\theta}_2^2 \\ U &= -2mg\ell \cos\theta_1 - mg\ell \cos\theta_2 \quad . \end{aligned} \tag{6.57}$$

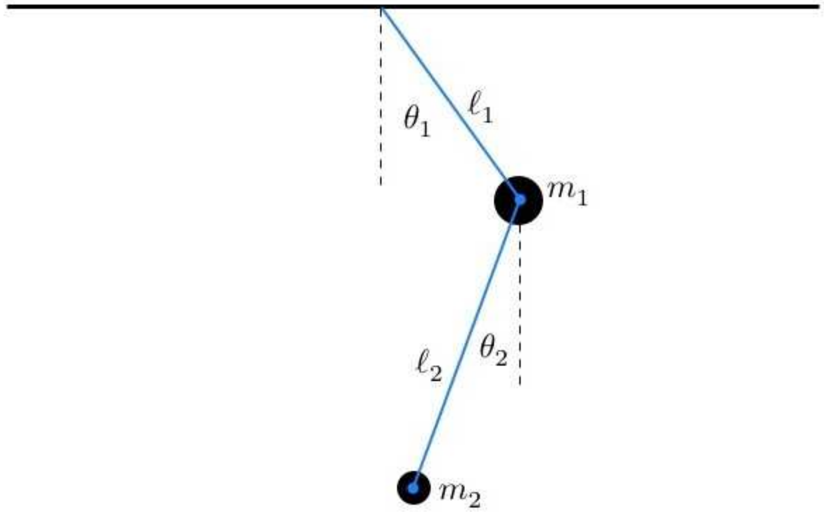


Figure 6.2: The double pendulum (again).

Equilibrium is at $\theta_1 = \theta_2 = 0$, and the T and V matrices are given by

$$\mathsf{T} = \frac{\partial^2 T}{\partial\dot{\theta}_i\,\partial\dot{\theta}_j}\bigg|_{\bar{\theta}} = \begin{pmatrix} 2m\ell^2 & m\ell^2 \\ m\ell^2 & m\ell^2 \end{pmatrix} \quad , \quad \mathsf{V} = \frac{\partial^2 U}{\partial\theta_i\,\partial\theta_j}\bigg|_{\bar{\theta}} = \begin{pmatrix} 2mg\ell & 0 \\ 0 & mg\ell \end{pmatrix} \quad . \tag{6.58}$$

Then

$$\omega^2\mathsf{T} - \mathsf{V} = m\ell^2 \begin{pmatrix} 2\omega^2 - 2\omega_0^2 & \omega^2 \\ \omega^2 & \omega^2 - \omega_0^2 \end{pmatrix} \quad , \tag{6.59}$$

with $\omega_0 = \sqrt{g/\ell}$. Setting the determinant to zero gives

$$2(\omega^2 - \omega_0^2)^2 - \omega^4 = 0 \quad \Rightarrow \quad \omega^2 = (2 \pm \sqrt{2})\,\omega_0^2 \quad . \tag{6.60}$$

We find the unnormalized eigenvectors by setting $(\omega_i^2\,\mathsf{T} - V)\,\psi^{(i)} = 0$. This gives

$$\psi^+ = C_+ \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} \quad , \quad \psi^- = C_- \begin{pmatrix} 1 \\ +\sqrt{2} \end{pmatrix} \quad , \tag{6.61}$$

where $C_\pm$ are constants. One can check $\mathsf{T}_{\sigma\sigma'}\,\psi_\sigma^{(i)}\,\psi_{\sigma'}^{(j)}$ vanishes for $i \neq j$. We then normalize by demanding $\mathsf{T}_{\sigma\sigma'}\,\psi_\sigma^{(i)}\,\psi_{\sigma'}^{(i)} = 1$ (no sum on i), which determines the coefficients $C_\pm = \frac{1}{2}\sqrt{(2 \pm \sqrt{2})/m\ell^2}$. Thus, the modal matrix is

$$\mathsf{A} = \begin{pmatrix} \psi_1^+ & \psi_1^- \\ \psi_2^+ & \psi_2^- \end{pmatrix} = \frac{1}{2\sqrt{m\ell^2}} \begin{pmatrix} \sqrt{2+\sqrt{2}} & \sqrt{2-\sqrt{2}} \\ -\sqrt{4+2\sqrt{2}} & +\sqrt{4-2\sqrt{2}} \end{pmatrix} \quad . \tag{6.62}$$

6.6 Zero Modes

6.6.1 Noether's theorem and zero modes

Recall Noether's theorem, which says that for every continuous one-parameter family of coordinate transformations,

$$q_\sigma \longrightarrow \tilde{q}_\sigma(q,\zeta) \qquad , \qquad \tilde{q}_\sigma(q,\zeta=0) = q_\sigma \quad , \tag{6.63}$$

which leaves the Lagrangian invariant, *i.e.* $dL/d\zeta = 0$, there is an associated conserved quantity,

$$\Lambda = \sum_\sigma \frac{\partial L}{\partial \dot{q}_\sigma}\, \frac{\partial \tilde{q}_\sigma}{\partial \zeta}\bigg|_{\zeta=0} \qquad \text{satisfies} \qquad \frac{d\Lambda}{dt} = 0 \quad . \tag{6.64}$$

For small oscillations, we write $q_\sigma = \bar{q}_\sigma + \eta_\sigma$, hence

$$\Lambda_k = \sum_\sigma C_{k\sigma}\, \dot{\eta}_\sigma \quad , \tag{6.65}$$

where k labels the one-parameter families (in the event there is more than one continuous symmetry), and where

$$C_{k\sigma} = \sum_{\sigma'} \mathsf{T}_{\sigma\sigma'}\, \frac{\partial \tilde{q}_{\sigma'}}{\partial \zeta_k}\bigg|_{\zeta=0} \quad . \tag{6.66}$$

Therefore, we can define the (unnormalized) normal mode

$$\xi_k = \sum_\sigma C_{k\sigma}\, \eta_\sigma \quad , \tag{6.67}$$

which satisfies $\ddot{\xi}_k = 0$. Thus, in systems with continuous symmetries, to each such continuous symmetry there is an associated zero mode of the small oscillations problem, *i.e.* a mode with $\omega_k^2 = 0$.

6.6.2 Examples of zero modes

The simplest example of a zero mode would be a pair of masses m_1 and m_2 moving frictionlessly along a line and connected by a spring of force constant k and unstretched length a. We know from our study of central forces that the Lagrangian may be written

$$\begin{aligned} L &= \tfrac{1}{2} m_1\, \dot{x}_1^2 + \tfrac{1}{2} m_2\, \dot{x}_2^2 - \tfrac{1}{2} k\, (x_1 - x_2 - a)^2 \\ &= \tfrac{1}{2} M \dot{X}^2 + \tfrac{1}{2}\mu \dot{x}^2 - \tfrac{1}{2} k (x-a)^2 \quad , \end{aligned} \tag{6.68}$$

where $X = (m_1 x_1 + m_2 x_2)/(m_1 + m_2)$ is the center of mass position, $x = x_1 - x_2$ is the relative coordinate, $M = m_1 + m_2$ is the total mass, and $\mu = m_1 m_2/(m_1 + m_2)$ is the reduced mass. The relative coordinate obeys $\ddot{x} = -\omega_0^2\, x$, where the oscillation frequency is $\omega_0 = \sqrt{k/\mu}$. The center of mass coordinate obeys $\ddot{X} = 0$, *i.e.* its oscillation frequency is zero. The center of mass motion is a zero mode.

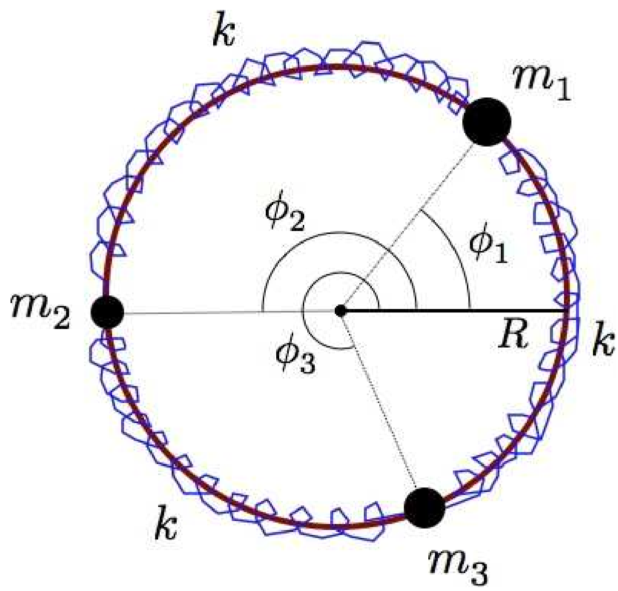


Figure 6.3: Coupled oscillations of three masses on a frictionless hoop of radius R. All three springs have the same force constant k, but the masses are all distinct.

Another example is furnished by the system depicted in fig. 6.3, where three distinct masses m_1, m_2, and m_3 move around a frictionless hoop of radius R. The masses are connected to their neighbors by identical springs of force constant k. We choose as generalized coordinates the angles ϕ_σ ($\sigma = 1, 2, 3$), with the convention that

$$\phi_3 - 2\pi < \phi_1 \le \phi_2 \le \phi_3 \le 2\pi + \phi_1 \quad . \tag{6.69}$$

The kinetic energy is

$$T = \tfrac{1}{2} R^2 \big(m_1\, \dot\phi_1^2 + m_2\, \dot\phi_2^2 + m_3\, \dot\phi_3^3 \big) \quad . \tag{6.70}$$

Let $R\chi$ be the equilibrium length for each of the springs. Then the potential energy is

$$\begin{aligned} U &= \tfrac{1}{2} k R^2 \Big\{ (\phi_2 - \phi_1 - \chi)^2 + (\phi_3 - \phi_2 - \chi)^2 + (2\pi + \phi_1 - \phi_3 - \chi)^2 \Big\} \\ &= \tfrac{1}{2} k R^2 \Big\{ (\phi_2 - \phi_1)^2 + (\phi_3 - \phi_2)^2 + (2\pi + \phi_1 - \phi_3)^2 + 3\chi^2 - 4\pi\chi \Big\} \quad . \end{aligned} \tag{6.71}$$

Note that the equilibrium angle χ enters only in an additive constant to the potential energy. Thus, for the calculation of the equations of motion, it is irrelevant. It doesn't matter whether or not the equilibrium configuration is unstretched ($\chi = 2\pi/3$) or not ($\chi \ne 2\pi/3$).

The equilibrium configuration is

$$\bar\phi_1 = \zeta \quad , \quad \bar\phi_1 = \zeta + \tfrac{2\pi}{3} \quad , \quad \bar\phi_1 = \zeta + \tfrac{4\pi}{3} \quad , \tag{6.72}$$

where ζ is an arbitrary real number, corresponding to continuous translational invariance of the entire system around the ring. The T and V matrices are then

$$\mathsf{T} = \begin{pmatrix} m_1 R^2 & 0 & 0 \\ 0 & m_2 R^2 & 0 \\ 0 & 0 & m_3 R^2 \end{pmatrix} \quad , \quad \mathsf{V} = \begin{pmatrix} 2kR^2 & -kR^2 & -kR^2 \\ -kR^2 & 2kR^2 & -kR^2 \\ -kR^2 & -kR^2 & 2kR^2 \end{pmatrix} \quad . \tag{6.73}$$

We then have

$$\omega^2\,\mathsf{T}-\mathsf{V}=kR^2\begin{pmatrix}\frac{\omega^2}{\Omega_1^2}-2 & 1 & 1\\ 1 & \frac{\omega^2}{\Omega_2^2}-2 & 1\\ 1 & 1 & \frac{\omega^2}{\Omega_3^2}-2\end{pmatrix}\quad, \tag{6.74}$$

where $\Omega_j^2\equiv k/m_j$. We compute the determinant to find the characteristic polynomial:

$$\begin{aligned}&P(\omega)=\mathsf{det}(\omega^2\,\mathsf{T}-\mathsf{V})\equiv(kR^2)^3\,\widetilde{P}(\omega)\\ &\widetilde{P}(\omega)=\frac{\omega^6}{\Omega_1^2\,\Omega_2^2\,\Omega_3^2}-2\left(\frac{1}{\Omega_1^2\,\Omega_2^2}+\frac{1}{\Omega_2^2\,\Omega_3^2}+\frac{1}{\Omega_1^2\,\Omega_3^2}\right)\omega^4+3\left(\frac{1}{\Omega_1^2}+\frac{1}{\Omega_2^2}+\frac{1}{\Omega_3^2}\right)\omega^2\quad.\end{aligned} \tag{6.75}$$

The equation $\widetilde{P}(\omega)=0$ yields a cubic equation in ω^2, but clearly ω^2 is a factor, and when we divide this out we obtain a quadratic equation. One root obviously is $\omega_1^2=0$. The other two roots are solutions to the quadratic equation:

$$\omega_{2,3}^2=\Omega_1^2+\Omega_2^2+\Omega_3^2\pm\sqrt{\tfrac{1}{2}\big(\Omega_1^2-\Omega_2^2\big)^2+\tfrac{1}{2}\big(\Omega_2^2-\Omega_3^2\big)^2+\tfrac{1}{2}\big(\Omega_1^2-\Omega_3^2\big)^2}\quad. \tag{6.76}$$

To find the eigenvectors and the modal matrix, we set

$$\begin{pmatrix}\frac{\omega_j^2}{\Omega_1^2}-2 & 1 & 1\\ 1 & \frac{\omega_j^2}{\Omega_2^2}-2 & 1\\ 1 & 1 & \frac{\omega_j^2}{\Omega_3^2}-2\end{pmatrix}\begin{pmatrix}\psi_1^{(j)}\\ \psi_2^{(j)}\\ \psi_3^{(j)}\end{pmatrix}=0\quad, \tag{6.77}$$

Writing down the three coupled equations for the components of $\boldsymbol{\psi}^{(j)}$, we find

$$\left(\frac{\omega_j^2}{\Omega_1^2}-3\right)\psi_1^{(j)}=\left(\frac{\omega_j^2}{\Omega_2^2}-3\right)\psi_2^{(j)}=\left(\frac{\omega_j^2}{\Omega_3^2}-3\right)\psi_3^{(j)}\quad. \tag{6.78}$$

We therefore conclude

$$\boldsymbol{\psi}^{(j)}=\mathcal{C}_j\begin{pmatrix}\left(\frac{\omega_j^2}{\Omega_1^2}-3\right)^{-1}\\ \left(\frac{\omega_j^2}{\Omega_2^2}-3\right)^{-1}\\ \left(\frac{\omega_j^2}{\Omega_3^2}-3\right)^{-1}\end{pmatrix}\quad. \tag{6.79}$$

The normalization condition $\psi_\sigma^{(i)}\,\mathsf{T}_{\sigma\sigma'}\,\psi_{\sigma'}^{(j)}=\delta_{ij}$ then fixes the constants $\mathcal{C}_j$:

$$\left[m_1\left(\frac{\omega_j^2}{\Omega_1^2}-3\right)^{-2}+\,m_2\left(\frac{\omega_j^2}{\Omega_2^2}-3\right)^{-2}+\,m_3\left(\frac{\omega_j^2}{\Omega_3^2}-3\right)^{-2}\right]\big|\mathcal{C}_j\big|^2=1\quad. \tag{6.80}$$

The Lagrangian is invariant under the one-parameter family of transformations

$$\phi_\sigma\longrightarrow\phi_\sigma+\zeta \tag{6.81}$$

for all $\sigma = 1, 2, 3$. The associated conserved quantity is

$$
\begin{aligned}
\Lambda &= \sum_\sigma \frac{\partial L}{\partial \dot{\phi}_\sigma} \frac{\partial \tilde{\phi}_\sigma}{\partial \zeta} \\
&= R^2 \left(m_1 \dot{\phi}_1 + m_2 \dot{\phi}_2 + m_3 \dot{\phi}_3 \right) \quad ,
\end{aligned}
\tag{6.82}
$$

which is, of course, the total angular momentum relative to the center of the ring. We stress that Λ is a constant in general, and not only in the limit of small deviations from static equilibrium. From

$$
\xi_1 = \mathcal{C} \left(m_1 \eta_1 + m_2 \eta_2 + m_3 \eta_3 \right) \quad ,
\tag{6.83}
$$

where $\mathcal{C}$ is a constant. Recall the relation $\eta_\sigma = \mathsf{A}_{\sigma i} \, \xi_i$ between the generalized displacements η_σ and the normal coordinates ξ_i, which may be inverted to yield $\xi_i = \mathsf{A}^{-1}_{i\sigma} \, \eta_\sigma = \mathsf{A}^{\mathsf{t}}_{i\sigma}$. In our case here, the T matrix is diagonal, so the multiplication is trivial. From eqns. 6.83 and 6.35, we conclude that the matrix $\mathsf{A}^{\mathsf{t}}\,\mathsf{T}$ must have a first *row* which is proportional to (m_1, m_2, m_3). Since these are the very diagonal entries of T, we conclude that A^{t} itself must have a first row which is proportional to $(1, 1, 1)$, which means that the first *column* of A is proportional to $(1, 1, 1)$. But this is confirmed by eqn. 6.78 when we take $j = 1$, since $\omega^2_{j=1} = 0$: $\psi^{(1)}_1 = \psi^{(1)}_2 = \psi^{(1)}_3$.

6.7 Chain of Mass Points

6.7.1 Lagrangian and equations of motion

Next consider an infinite chain of identical masses, connected by identical springs of spring constant k and equilibrium length a. The Lagrangian is

$$
\begin{aligned}
L &= \tfrac{1}{2} m \sum_n \dot{x}_n^2 - \tfrac{1}{2} k \sum_n (x_{n+1} - x_n - a)^2 \\
&= \tfrac{1}{2} m \sum_n \dot{u}_n^2 - \tfrac{1}{2} k \sum_n (u_{n+1} - u_n)^2 \quad ,
\end{aligned}
\tag{6.84}
$$

where $u_n \equiv x_n - na + \zeta$ is the displacement from equilibrium of the n^{th} mass. The constant ζ is arbitrary and is cyclic in L, reflecting overall translational invariance with a consequent zero mode according to Noether's theorem. The Euler-Lagrange equations are

$$
\begin{aligned}
m \ddot{u}_n &= \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{u}_n} \right) = \frac{\partial L}{\partial u_n} \\
&= k \left(u_{n+1} - u_n \right) - k \left(u_n - u_{n-1} \right) = k \left(u_{n+1} + u_{n-1} - 2 u_n \right) \quad .
\end{aligned}
\tag{6.85}
$$

Now let us assume that the system is placed on a large ring of circumference Na, where $N \gg 1$. Then $u_{n+N} = u_n$ and we may shift to Fourier coefficients,

$$
u_n = \frac{1}{\sqrt{N}} \sum_q e^{iqan} \, \hat{u}_q \qquad , \qquad \hat{u}_q = \frac{1}{\sqrt{N}} \sum_n e^{-iqan} \, u_n \quad ,
\tag{6.86}
$$

where $q_j = 2\pi j/Na$, and both sums are over the set $j, n \in \{1, \ldots, N\}$. Expressed in terms of the $\{\hat{u}_q\}$, the equations of motion become

$$\begin{aligned}\ddot{\hat{u}}_q &= \frac{1}{\sqrt{N}} \sum_n e^{-iqna}\, \ddot{u}_n = \frac{k}{m} \frac{1}{\sqrt{N}} \sum_n e^{-iqan} \left(u_{n+1} + u_{n-1} - 2u_n\right) \\ &= \frac{k}{m} \frac{1}{\sqrt{N}} \sum_n e^{-iqan} \left(e^{-iqa} + e^{+iqa} - 2\right) u_n = -\frac{4k}{m} \sin^2\!\left(\tfrac{1}{2}qa\right) \hat{u}_q\end{aligned} \tag{6.87}$$

Thus, the $\{\hat{u}_q\}$ *are* the normal modes of the system (up to a normalization constant), and the eigenfrequencies are

$$\omega_q = 2\sqrt{\frac{k}{m}}\, \left|\sin\left(\tfrac{1}{2}qa\right)\right| \quad . \tag{6.88}$$

This means that the modal matrix is

$$\mathsf{A}_{nq} = \frac{1}{\sqrt{Nm}}\, e^{iqan} \quad , \tag{6.89}$$

where we've included the $\frac{1}{\sqrt{m}}$ factor for a proper normalization. The normal modes themselves are then $\xi_q = \mathsf{A}^\dagger_{qn} \mathsf{T}_{nn'} u_{n'} = \sqrt{m}\, \hat{u}_q$. For complex A, we have $\mathsf{A}^\dagger \mathsf{T} \mathsf{A} = \mathbb{I}$ and $\mathsf{A}^\dagger \mathsf{V} \mathsf{A} = \mathrm{diag}(\omega_1^2, \ldots, \omega_N^2)$.

Note that

$$\begin{aligned}\mathsf{T}_{nn'} &= m\, \delta_{n,n'} \\ \mathsf{V}_{nn'} &= 2k\, \delta_{n,n'} - k\, \delta_{n,n'+1} - k\, \delta_{n,n'-1}\end{aligned} \tag{6.90}$$

and that

$$\begin{aligned}(\mathsf{A}^\dagger \mathsf{T} \mathsf{A})_{qq'} &= \sum_{n=1}^N \sum_{n'=1}^N \mathsf{A}^*_{nq} \mathsf{T}_{nn'} \mathsf{A}_{n'q'} \\ &= \frac{1}{Nm} \sum_{n=1}^N \sum_{n'=1}^N e^{-iqan}\, m\, \delta_{nn'}\, e^{iq'an'} = \frac{1}{N} \sum_{n=1}^N e^{i(q'-q)an} = \delta_{qq'} \quad ,\end{aligned} \tag{6.91}$$

and

$$\begin{aligned}(\mathsf{A}^\dagger \mathsf{V} \mathsf{A})_{qq'} &= \sum_{n=1}^N \sum_{n'=1}^N \mathsf{A}^*_{nq} \mathsf{T}_{nn'} \mathsf{A}_{n'q'} \\ &= \frac{1}{Nm} \sum_{n=1}^N \sum_{n'=1}^N e^{-iqan} \Big(2k\, \delta_{n,n'} - k\, \delta_{n,n'+1} - k\, \delta_{n,n'-1}\Big)\, e^{iq'an'} \\ &= \frac{k}{m} \frac{1}{N} \sum_{n=1}^N e^{i(q'-q)an} \left(2 - e^{-iq'a} - e^{iq'a}\right) = \frac{4k}{m} \sin^2\!\left(\tfrac{1}{2}qa\right) \delta_{qq'} = \omega_q^2\, \delta_{qq'}\end{aligned} \tag{6.92}$$

Since $\hat{x}_{q+G} = \hat{x}_q$, where $G = 2\pi/a$, we may choose any set of q values such that no two are separated by an integer multiple of G. The set of points $\{jG\}$ with $j \in \mathbb{Z}$ is called the *reciprocal lattice*. For a linear chain, the reciprocal lattice is itself a linear chain[2]. One natural set to choose is $q \in \big[-\frac{\pi}{a}, \frac{\pi}{a}\big]$. This is known as the *first Brillouin zone* of the reciprocal lattice.

Finally, we can write the Lagrangian itself in terms of the $\{u_q\}$. One easily finds

$$L = \tfrac{1}{2}\, m \sum_q \dot{\hat{u}}_q^*\, \dot{\hat{u}}_q - \, k \sum_q (1 - \cos qa)\, \hat{u}_q^*\, \hat{u}_q \quad , \tag{6.93}$$

where the sum is over q in the first Brillouin zone. Note that

$$\hat{u}_{-q} = \hat{u}_{-q+G} = \hat{u}_q^* \quad . \tag{6.94}$$

This means that we can restrict the sum to half the Brillouin zone:

$$L = \sum_{q \in [0, \frac{\pi}{a}]} \Big\{ m\, \dot{\hat{u}}_q^*\, \dot{\hat{u}}_q - 4k\, \sin^2\!\big(\tfrac{1}{2} qa\big)\, \hat{u}_q^*\, \hat{u}_q \Big\} \quad . \tag{6.95}$$

Now $\hat{u}_q$ and $\hat{u}_q^*$ may be regarded as linearly independent, as one regards complex variables z and z^*. The Euler-Lagrange equation for $\hat{u}_q^*$ gives

$$\frac{d}{dt}\bigg(\frac{\partial L}{\partial \dot{\hat{u}}_q^*}\bigg) = \frac{\partial L}{\partial \hat{u}_q^*} \quad \Rightarrow \quad \ddot{\hat{u}}_q = -\omega_q^2\, \hat{u}_q \quad . \tag{6.96}$$

Extremizing with respect to $\hat{u}_q$ gives the complex conjugate equation.

6.7.2 Continuum limit

Let us take $N \to \infty$, $a \to 0$, with $L_0 = Na$ fixed. We'll write $u_n(t) \to u(x = na, t)$, in which case

$$\begin{aligned} T &= \tfrac{1}{2} m \sum_n \dot{u}_n^2 \quad \longrightarrow \quad \tfrac{1}{2} m \int \frac{dx}{a} \bigg(\frac{\partial u}{\partial t}\bigg)^{\!2} \\ V &= \tfrac{1}{2} k \sum_n (u_{n+1} - u_n)^2 \quad \longrightarrow \quad \tfrac{1}{2} k \int \frac{dx}{a} \bigg(\frac{u(x+a) - u(x)}{a}\bigg)^{\!2} a^2 \end{aligned} \tag{6.97}$$

Recognizing the spatial derivative above, we finally obtain

$$\begin{aligned} L &= \int\! dx\, \mathcal{L}(u, \partial_t u, \partial_x u) \\ \mathcal{L} &= \tfrac{1}{2}\, \mu \bigg(\frac{\partial u}{\partial t}\bigg)^{\!2} - \tfrac{1}{2}\, \tau \bigg(\frac{\partial u}{\partial x}\bigg)^{\!2} \quad , \end{aligned} \tag{6.98}$$

where $\mu = m/a$ is the linear mass density and $\tau = ka$ is the tension[3]. The quantity $\mathcal{L}$ is the *Lagrangian density*; it depends on the field $u(x,t)$ as well as its partial derivatives $\partial_t u$ and $\partial_x u$[4]. The action is

$$S\big[u(x,t)\big] = \int\limits_{t_a}^{t_b}\! dt \int\limits_{x_a}^{x_b}\! dx\, \mathcal{L}(u, \partial_t u, \partial_x u) \quad , \tag{6.99}$$

[2]For higher dimensional Bravais lattices, the reciprocal lattice is often different than the real space ("direct") lattice. For example, the reciprocal lattice of a face-centered cubic structure is a body-centered cubic lattice.

[3]For a proper limit, we demand μ and τ be neither infinite nor infinitesimal.

[4]$\mathcal{L}$ may also depend explicitly on x and t.

where $\{x_a, x_b\}$ are the limits on the x coordinate. Setting $\delta S = 0$ gives the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial u} - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_t u\right)}\right) - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_x u\right)}\right) = 0 \quad . \tag{6.100}$$

For our system, this yields the Helmholtz equation,

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \quad , \tag{6.101}$$

where $c = \sqrt{\tau/\mu}$ is the velocity of wave propagation. This is a linear equation, solutions of which are of the form

$$u(x,t) = C\, e^{iqx}\, e^{-i\omega t} \quad , \tag{6.102}$$

where $\omega = \pm cq$. Note that in the continuum limit $a \to 0$, the dispersion relation derived for the chain becomes

$$\omega_q^2 = \frac{4k}{m}\sin^2\!\left(\tfrac{1}{2}qa\right) \longrightarrow \frac{ka^2}{m}\, q^2 = c^2 q^2 \quad , \tag{6.103}$$

and so the results agree.

6.8 General Formulation of Small Oscillations

In the development in section 6.2, we assumed that the kinetic energy T is a homogeneous function of degree 2, and the potential energy U a homogeneous function of degree 0, in the generalized velocities $\dot q_\sigma$. However, we've encountered situations where this is not so: problems with time-dependent holonomic constraints, such as the mass point on a rotating hoop, and problems involving charged particles moving in magnetic fields. The general Lagrangian is of the form

$$L = \tfrac{1}{2}\, T_2^{\sigma\sigma'}(q)\,\dot q_\sigma\,\dot q_{\sigma'} + T_1^\sigma(q)\,\dot q_\sigma + T_0(q) - U_1^\sigma(q)\,\dot q_\sigma - U_0(q) \quad , \tag{6.104}$$

where the subscript 0, 1, or 2 labels the degree of homogeneity of each term in the generalized velocities. The generalized momenta are then

$$p_\sigma = \frac{\partial L}{\partial \dot q_\sigma} = T_2^{\sigma\sigma'}\,\dot q_{\sigma'} + T_1^\sigma(q) - U_1^\sigma(q) \tag{6.105}$$

and the generalized forces are

$$F_\sigma = \frac{\partial L}{\partial q_\sigma} = \frac{\partial (T_0 - U_0)}{\partial q_\sigma} + \frac{\partial (T_1^{\sigma'} - U_1^{\sigma'})}{\partial q_\sigma}\,\dot q_{\sigma'} + \frac{1}{2}\,\frac{\partial T_2^{\sigma\sigma'}}{\partial q_\sigma}\,\dot q_{\sigma'}\,\dot q_{\sigma''} \quad , \tag{6.106}$$

and the equations of motion are again $\dot p_\sigma = F_\sigma$

In equilibrium, we seek a time-independent solution of the form $q_\sigma(t) = \bar q_\sigma$. This entails

$$\left.\frac{\partial\left\{U_0(q) - T_0(q)\right\}}{\partial q_\sigma}\right|_{q=\bar q} = 0 \quad , \tag{6.107}$$

which give us n equations in the n unknowns $(\bar{q}_1, \ldots, \bar{q}_n)$. We then write $q_\sigma = \bar{q}_\sigma + \eta_\sigma$ and expand in the notionally small quantities η_σ. It is important to understand that we assume η and all of its time derivatives as well are small. Thus, we can expand L to quadratic order in $(\eta, \dot{\eta})$ to obtain

$$L = \tfrac{1}{2}\,\mathsf{T}_{\sigma\sigma'}\,\dot{\eta}_\sigma\,\dot{\eta}_{\sigma'} - \tfrac{1}{2}\,\mathsf{B}_{\sigma\sigma'}\,\eta_\sigma\,\dot{\eta}_{\sigma'} - \tfrac{1}{2}\,\mathsf{V}_{\sigma\sigma'}\,\eta_\sigma\,\eta_{\sigma'} \quad , \tag{6.108}$$

where

$$\mathsf{T}_{\sigma\sigma'} = T_2^{\sigma\sigma'}(\bar{q}) \quad , \quad \mathsf{B}_{\sigma\sigma'} = 2\,\frac{\partial\big(U_1^{\sigma'} - T_1^{\sigma'}\big)}{\partial q_\sigma}\bigg|_{q=\bar{q}} \quad , \quad \mathsf{V}_{\sigma\sigma'} = \frac{\partial^2\big(U_0 - T_0\big)}{\partial q_\sigma\,\partial q_{\sigma'}}\bigg|_{q=\bar{q}} \quad . \tag{6.109}$$

Note that the T and V matrices are symmetric. The $\mathsf{B}_{\sigma\sigma'}$ term is new.

Now we can always write $\mathsf{B} = \frac{1}{2}(\mathsf{B}^{\mathsf{s}} + \mathsf{B}^{\mathsf{a}})$ as a sum over symmetric and antisymmetric parts, with $\mathsf{B}^{\mathsf{s}} = \frac{1}{2}(\mathsf{B} + \mathsf{B}^{\mathsf{t}})$ and $\mathsf{B}^{\mathsf{a}} = \frac{1}{2}(\mathsf{B} - \mathsf{B}^{\mathsf{t}})$. Since,

$$\mathsf{B}_{\sigma\sigma'}\,\eta_\sigma\,\dot{\eta}_{\sigma'} = \frac{d}{dt}\Big(\tfrac{1}{2}\,\mathsf{B}^{\mathsf{s}}_{\sigma\sigma'}\,\eta_\sigma\,\eta_{\sigma'}\Big) \quad , \tag{6.110}$$

any symmetric part to B contributes a total time derivative to L, and thus has no effect on the equations of motion. Therefore, we can project V onto its antisymmetric part, writing

$$\mathsf{B}_{\sigma\sigma'} \equiv \mathsf{B}^{\mathsf{A}}_{\sigma\sigma'} = \left(\frac{\partial\big(U_1^{\sigma'} - T_1^{\sigma'}\big)}{\partial q_\sigma} - \frac{\partial\big(U_1^{\sigma} - T_1^{\sigma}\big)}{\partial q_{\sigma'}}\right)_{q=\bar{q}} \quad . \tag{6.111}$$

We now have

$$p_\sigma = \frac{\partial L}{\partial \dot{\eta}_\sigma} = \mathsf{T}_{\sigma\sigma'}\,\dot{\eta}_{\sigma'} + \tfrac{1}{2}\,\mathsf{B}_{\sigma\sigma'}\,\eta_{\sigma'} \quad , \tag{6.112}$$

and

$$F_\sigma = \frac{\partial L}{\partial \eta_\sigma} = -\tfrac{1}{2}\,\mathsf{B}_{\sigma\sigma'}\,\dot{\eta}_{\sigma'} - \mathsf{V}_{\sigma\sigma'}\,\eta_{\sigma'} \quad . \tag{6.113}$$

The equations of motion, $\dot{p}_\sigma = F_\sigma$, then yield

$$\mathsf{T}_{\sigma\sigma'}\,\ddot{\eta}_{\sigma'} + \mathsf{B}_{\sigma\sigma'}\,\dot{\eta}_{\sigma'} + \mathsf{V}_{\sigma\sigma'}\,\eta_{\sigma'} = 0 \quad . \tag{6.114}$$

Let us write $\boldsymbol{\eta}(t) = \boldsymbol{\eta}\,e^{-i\omega t}$. We then have

$$\big(\omega^2\,\mathsf{T} + i\omega\,\mathsf{B} - \mathsf{V}\big)\boldsymbol{\eta} = 0 \quad . \tag{6.115}$$

To solve eqn. 6.115, we set $P(\omega) = 0$, where $P(\omega) = \det\big[\mathrm{Q}(\omega)\big]$, with

$$\mathrm{Q}(\omega) \equiv \omega^2\,\mathsf{T} + i\omega\,\mathsf{B} - \mathsf{V} \quad . \tag{6.116}$$

Since T, B, and V are real-valued matrices, and since $\det(M) = \det(M^{\mathsf{t}})$ for any matrix M, we can use $\mathsf{B}^{\mathsf{t}} = -\mathsf{B}$ to obtain $P(-\omega) = P(\omega)$ and $P(\omega^*) = \big[P(\omega)\big]^*$. This establishes that if $P(\omega) = 0$, *i.e.* if ω is an eigenfrequency, then $P(-\omega) = 0$ and $P(\omega^*) = 0$, *i.e.* $-\omega$ and ω^* and $-\omega^*$ are also eigenfrequencies. Furthermore, $P(\omega)$ must again be a polynomial of order n in ω^2.

Example

As an example, consider the following Lagrangian, which is a function of four generalized coordinates $\{x_1, y_1, x_2, y_2\}$ and their corresponding velocities:

$$L = \tfrac{1}{2}m\big(\dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2\big) - \tfrac{1}{2}\kappa\big(x_1 - x_2\big)^2 - \tfrac{1}{2}b\big(y_1^2 + y_2^2\big) + \tfrac{1}{2}m\omega_{\rm c}\big(x_1\,\dot{y}_1 - y_1\,\dot{x}_1 + x_2\,\dot{y}_2 - y_2\,\dot{x}_2\big) \quad . \tag{6.117}$$

The last term, which is linear in the generalized velocities, arises if the masses are also equally charged and in the presence of a magnetic field $\boldsymbol{B} = B\hat{\boldsymbol{z}}$. The quantity $\omega_{\rm c} = qB/mc$, where q is the charge, is called the *cyclotron frequency*. We then have

$$\begin{aligned}
p_{x,1} &= m\dot{x}_1 - \tfrac{1}{2}m\omega_{\rm c}\,y_1 \quad , & F_{x,1} &= -\kappa\big(x_1 - x_2\big) + \tfrac{1}{2}m\omega_{\rm c}\,\dot{y}_1 \\
p_{y,1} &= m\dot{y}_1 + \tfrac{1}{2}m\omega_{\rm c}\,x_1 \quad , & F_{y,1} &= -by_1 - \tfrac{1}{2}m\omega_{\rm c}\,\dot{x}_1 \\
p_{x,2} &= m\dot{x}_2 - \tfrac{1}{2}m\omega_{\rm c}\,y_2 \quad , & F_{x,2} &= -\kappa\big(x_2 - x_1\big) + \tfrac{1}{2}m\omega_{\rm c}\,\dot{y}_2 \\
p_{y,2} &= m\dot{y}_2 + \tfrac{1}{2}m\omega_{\rm c}\,x_2 \quad , & F_{y,2} &= -by_2 - \tfrac{1}{2}m\omega_{\rm c}\,\dot{x}_2 \quad .
\end{aligned} \tag{6.118}$$

Defining $\nu^2 \equiv \kappa/m$ and $\varOmega^2 \equiv b/m$, we have the equations of motion

$$\begin{aligned}
\ddot{x}_1 - \omega_{\rm c}\,\dot{y}_1 &= -\nu^2\big(x_1 - x_2\big) \\
\ddot{y}_1 + \omega_{\rm c}\,\dot{x}_1 &= -\varOmega^2\,y_1 \\
\ddot{x}_2 - \omega_{\rm c}\,\dot{y}_2 &= -\nu^2\big(x_2 - x_1\big) \\
\ddot{y}_2 + \omega_{\rm c}\,\dot{x}_2 &= -\varOmega^2\,y_2 \quad .
\end{aligned} \tag{6.119}$$

From these equations, we read off the matrices

$$\mathsf{T} = \begin{pmatrix} m & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & m & 0 \\ 0 & 0 & 0 & m \end{pmatrix} \quad , \quad \mathsf{B} = \begin{pmatrix} 0 & -m\omega_{\rm c} & 0 & 0 \\ m\omega_{\rm c} & 0 & 0 & 0 \\ 0 & 0 & 0 & m\omega_{\rm c} \\ 0 & 0 & -m\omega_{\rm c} & 0 \end{pmatrix} \tag{6.120}$$

and

$$\mathsf{V} = \begin{pmatrix} m\nu^2 & 0 & -m\nu^2 & 0 \\ 0 & m\varOmega^2 & 0 & 0 \\ -m\nu^2 & 0 & m\nu^2 & 0 \\ 0 & 0 & 0 & m\varOmega^2 \end{pmatrix} \quad , \tag{6.121}$$

where the rows and columns correspond to the coordinates $\{x_1, y_1, x_2, y_2\}$, respectively. If we define the CM and relative coordinates

$$X \equiv \tfrac{1}{2}(x_1 + x_2) \quad , \quad Y \equiv \tfrac{1}{2}(y_1 + y_2) \quad , \quad x \equiv x_1 - x_2 \quad , \quad y \equiv y_1 - y_2 \quad , \tag{6.122}$$

the equations of motion decouple into two 2×2 systems, *viz.*

$$\ddot{X} - \omega_{\rm c}\,\dot{Y} = 0 \quad , \quad \ddot{Y} + \omega_{\rm c}\,\dot{X} = -\varOmega^2\,Y \quad , \quad \ddot{x} - \omega_{\rm c}\,\dot{y} = -2\nu^2 x \quad , \quad \ddot{y} + \omega_{\rm c}\,\dot{x} = -\varOmega^2\,y \quad . \tag{6.123}$$

Thus, for the (X, Y) system we have

$$\det\begin{pmatrix}\omega^2 & -i\omega\omega_c \\ i\omega\omega_c & \omega^2-\Omega^2\end{pmatrix}=0 \qquad \Rightarrow \qquad \omega_1^2=0 \quad , \quad \omega_2^2=\Omega^2+\omega_c^2 \quad , \tag{6.124}$$

while for the (x, y) system we have

$$\det\begin{pmatrix}\omega^2-2\nu^2 & -i\omega\omega_c \\ i\omega\omega_c & \omega^2-\Omega^2\end{pmatrix}=0 \qquad \Rightarrow \qquad \omega_{3,4}^2=\tfrac{1}{2}\big(2\nu^2+\Omega^2+\omega_c^2\big)\pm\tfrac{1}{2}\sqrt{\big(2\nu^2+\Omega^2+\omega_c^2\big)^2-8\nu^2\Omega^2} \quad . \tag{6.125}$$

When $\omega_c=0$, we have the zero mode X with frequency $\omega_1=0$, the relative coordinate x with frequency $\omega_4=\sqrt{2}\nu$, and two independent y and Y oscillations with degenerate frequencies $\omega_2=\omega_3=\Omega$. Nonzero ω_c couples x to y and X to Y, and shifts the eigenfrequencies $\omega_{2,3,4}$ according to the above results.

Note that zero mode frequency is unaffected by a finite ω_c. If we write the Lagrangian in terms of the CM and relative coordinates, we obtain $L=L_{CM}+L_{rel}$, with

$$\begin{aligned} L_{CM}&=m\big(\dot{X}^2+\dot{Y}^2\big)+m\omega_c\big(X\dot{Y}-Y\dot{X}\big)-bY^2 \\ L_{rel}&=\tfrac{1}{4}m\big(\dot{x}^2+\dot{y}^2\big)+\tfrac{1}{4}m\omega_c\big(x\dot{y}-y\dot{x}\big)-\tfrac{1}{2}\kappa x^2-\tfrac{1}{4}by^2 \quad . \end{aligned} \tag{6.126}$$

At first, it seems that the zero mode should be lifted by finite ω_c since the coordinate X is no longer cyclic in L. However, X may be made cyclic by a different *choice of gauge* for the electromagnetic vector potential. Our choice had been $\boldsymbol{A}(\boldsymbol{r})=\frac{1}{2}B\hat{\boldsymbol{z}}\times\boldsymbol{r}=\frac{1}{2}B(x\hat{\boldsymbol{y}}-y\hat{\boldsymbol{x}})$, but had we instead chosen $\boldsymbol{A}=-By\hat{\boldsymbol{x}}$, we would have had $\frac{q}{c}\boldsymbol{A}\cdot\dot{\boldsymbol{r}}=-\frac{qB}{c}\,y\,\dot{x}$ and only the velocities $\dot{x}_{1,2}$ would have entered here for each particle, so X would have been cyclic. Equivalently, in L_{CM} we could write

$$m\omega_c\big(X\dot{Y}-Y\dot{X}\big)=\frac{d}{dt}\big(m\omega_c\,XY\big)-2m\omega_c Y\dot{X} \quad , \tag{6.127}$$

and the total time derivative term may be dropped from L_{CM}. The resulting CM Lagrangian is then cyclic in X, so the zero mode survives!

6.9 Additional Examples

6.9.1 Right triatomic molecule

A molecule consists of three identical atoms located at the vertices of a $45°$ right triangle. Each pair of atoms interacts by an effective spring potential, with all spring constants equal to k. Consider only planar motion of this molecule.

(a) Find three 'zero modes' for this system (*i.e.* normal modes whose associated eigenfrequencies vanish).

(b) Find the remaining three normal modes.

Solution

It is useful to choose the following coordinates:

$$\begin{aligned}(X_1, Y_1) &= (x_1\ ,\ y_1)\\ (X_2, Y_2) &= (a + x_2\ ,\ y_2)\\ (X_3, Y_3) &= (x_3\ ,\ a + y_3) \quad .\end{aligned} \tag{6.128}$$

The three separations are then

$$\begin{aligned}d_{12} &= \sqrt{(a + x_2 - x_1)^2 + (y_2 - y_1)^2}\\ &= a + x_2 - x_1 + \ldots\end{aligned} \tag{6.129}$$

and

$$\begin{aligned}d_{23} &= \sqrt{(-a + x_3 - x_2)^2 + (a + y_3 - y_2)^2}\\ &= \sqrt{2}\,a - \tfrac{1}{\sqrt{2}}\big(x_3 - x_2\big) + \tfrac{1}{\sqrt{2}}\big(y_3 - y_2\big) + \ldots\end{aligned} \tag{6.130}$$

and

$$\begin{aligned}d_{13} &= \sqrt{(x_3 - x_1)^2 + (a + y_3 - y_1)^2}\\ &= a + y_3 - y_1 + \ldots \quad .\end{aligned} \tag{6.131}$$

The potential is then

$$\begin{aligned}U &= \tfrac{1}{2}k\,\big(d_{12} - a\big)^2 + \tfrac{1}{2}k\,\big(d_{23} - \sqrt{2}\,a\big)^2 + \tfrac{1}{2}k\,\big(d_{13} - a\big)^2\\ &= \tfrac{1}{2}k\big(x_2 - x_1\big)^2 + \tfrac{1}{4}k\big(x_3 - x_2\big)^2 + \tfrac{1}{4}k\big(y_3 - y_2\big)^2\\ &\qquad - \tfrac{1}{2}k\big(x_3 - x_2\big)\big(y_3 - y_2\big) + \tfrac{1}{2}k\big(y_3 - y_1\big)^2\end{aligned} \tag{6.132}$$

Defining the row vector

$$\boldsymbol{\eta}^{\mathsf{t}} \equiv \big(x_1\ ,\ y_1\ ,\ x_2\ ,\ y_2\ ,\ x_3\ ,\ y_3\big) \quad , \tag{6.133}$$

we have that U is a quadratic form:

$$U = \tfrac{1}{2}\eta_\sigma \mathsf{V}_{\sigma\sigma'}\eta_{\sigma'} = \tfrac{1}{2}\boldsymbol{\eta}^{\mathsf{t}}\,\mathsf{V}\,\boldsymbol{\eta}, \tag{6.134}$$

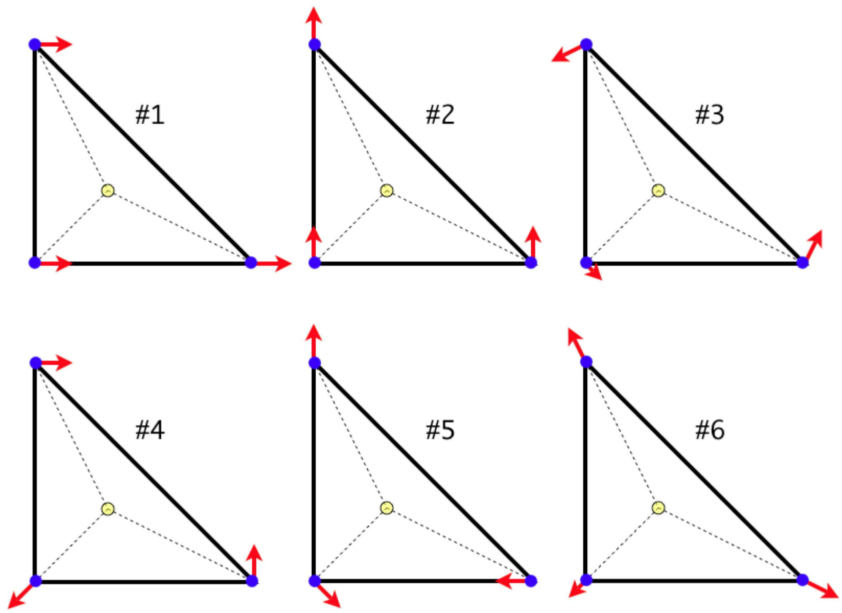


Figure 6.4: Normal modes of the $45°$ right triangle. The yellow circle is the location of the CM of the triangle. The labels for the vertices are 1 (lower left), 2 (lower right), and 3 (upper left).

with

$$\mathsf{V}_{\sigma\sigma'} = \frac{\partial^2 U}{\partial q_\sigma \, \partial q_{\sigma'}}\bigg|_{\bar{q}} = k \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ -1 & 0 & \frac{3}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & -1 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{3}{2} \end{pmatrix} \tag{6.135}$$

The kinetic energy is simply

$$T = \tfrac{1}{2} m \big(\dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2 + \dot{x}_3^2 + \dot{y}_3^2 \big) \quad , \tag{6.136}$$

hence

$$\mathsf{T}_{\sigma\sigma'} = m \, \delta_{\sigma\sigma'} \quad . \tag{6.137}$$

(b) The three zero modes correspond to x-translation, y-translation, and rotation. Their eigenvectors, respectively, are

$$\psi_1 = \frac{1}{\sqrt{3m}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad , \quad \psi_2 = \frac{1}{\sqrt{3m}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \quad , \quad \psi_3 = \frac{1}{2\sqrt{3m}} \begin{pmatrix} 1 \\ -1 \\ 1 \\ 2 \\ -2 \\ -1 \end{pmatrix} \quad . \tag{6.138}$$

Thus $\omega_1 = \omega_2 = \omega_3 = 0$. To find the unnormalized rotation vector, we find the CM of the triangle, located at $\big(\frac{a}{3}, \frac{a}{3}\big)$, and sketch orthogonal displacements $\hat{\boldsymbol{z}} \times (\boldsymbol{R}_a - \boldsymbol{R}_{\text{CM}})$ at the position of mass point a.

(c) The remaining modes may be determined by symmetry, and are given by

$$\psi_4 = \frac{1}{2\sqrt{m}} \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \quad , \quad \psi_5 = \frac{1}{2\sqrt{m}} \begin{pmatrix} 1 \\ -1 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad , \quad \psi_6 = \frac{1}{2\sqrt{3m}} \begin{pmatrix} -1 \\ -1 \\ 2 \\ -1 \\ -1 \\ 2 \end{pmatrix} \quad , \tag{6.139}$$

with

$$\omega_4 = \sqrt{\frac{k}{m}} \quad , \quad \omega_5 = \sqrt{\frac{2k}{m}} \quad , \quad \omega_6 = \sqrt{\frac{3k}{m}} \quad . \tag{6.140}$$

Since $\mathsf{T} = m \cdot 1$ is a multiple of the unit matrix, the orthogonormality relation $\psi^{(i)}_\sigma \, \mathsf{T}_{\sigma\sigma'} \, \psi^{(j)}_{\sigma'} = \delta^{ij}$ entails $\boldsymbol{\psi}^{(i)} \cdot \boldsymbol{\psi}^{(j)} = m^{-1} \, \delta_{ij}$, *i.e.*the eigenvectors are mutually orthogonal in the conventional dot product sense. One can check that the eigenvectors listed here satisfy this condition.

The simplest of the set $\{\boldsymbol{\psi}_4, \boldsymbol{\psi}_5, \boldsymbol{\psi}_6\}$ to find is the uniform dilation $\boldsymbol{\psi}_6$, sometimes called the *breathing mode*. This must keep the triangle in the same shape, which means that the deviations at each mass point are proportional to the distance to the CM. Next simplest to find is $\boldsymbol{\psi}_4$, in which the long and short sides of the triangle oscillate out of phase. Finally, the mode $\boldsymbol{\psi}_5$ must be orthogonal to all the remaining modes. No heavy lifting (*e.g.* `Mathematica`) is required!

6.9.2 Triple pendulum

Consider a triple pendulum consisting of three identical masses m and three identical rigid massless rods of length ℓ, as depicted in fig. 6.5.

(a) Find the T and V matrices.

(b) Find the equation for the eigenfrequencies.

(c) Numerically solve the eigenvalue equation for ratios ω_j^2/ω_0^2, where $\omega_0 = \sqrt{g/\ell}$. Find the three normal modes.

Solution

The Cartesian coordinates for the three masses are

$$x_1 = \ell\,\sin\theta_1 \qquad y_1 = -\ell\,\cos\theta_1 \tag{6.141}$$

$$x_2 = \ell\,\sin\theta_1 + \ell\,\sin\theta_2 \qquad y_2 = -\ell\,\cos\theta_1 - \ell\,\cos\theta_2 \tag{6.142}$$

$$x_3 = \ell\,\sin\theta_1 + \ell\,\sin\theta_2 + \ell\,\sin\theta_3 \qquad y_3 = -\ell\,\cos\theta_1 - \ell\,\cos\theta_2 - \ell\,\cos\theta_3 \quad . \tag{6.143}$$

By inspection, we can write down the kinetic energy:

$$\begin{aligned} T &= \tfrac{1}{2}m\big(\dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2 + \dot{x}_3^2 + \dot{y}_3^2\big) \\ &= \tfrac{1}{2}m\,\ell^2\,\Big\{3\dot{\theta}_1^2 + 2\dot{\theta}_2^2 + \dot{\theta}_3^2 + 4\cos(\theta_1-\theta_2)\,\dot{\theta}_1\,\dot{\theta}_2 + 2\cos(\theta_1-\theta_3)\,\dot{\theta}_1\,\dot{\theta}_3 + 2\cos(\theta_2-\theta_3)\,\dot{\theta}_2\,\dot{\theta}_3\Big\} \end{aligned} \tag{6.144}$$

The potential energy is

$$U = -mg\ell\,\Big\{3\cos\theta_1 + 2\cos\theta_2 + \cos\theta_3\Big\} \quad , \tag{6.145}$$

and the Lagrangian is $L = T - U$:

$$\begin{aligned} L = \tfrac{1}{2}m\,\ell^2\,\Big\{&3\,\dot{\theta}_1^2 + 2\,\dot{\theta}_2^2 + \dot{\theta}_3^2 + 4\cos(\theta_1-\theta_2)\,\dot{\theta}_1\,\dot{\theta}_2 + 2\cos(\theta_1-\theta_3)\,\dot{\theta}_1\,\dot{\theta}_3 \\ &+ 2\cos(\theta_2-\theta_3)\,\dot{\theta}_2\,\dot{\theta}_3\Big\} + mg\ell\,\Big\{3\cos\theta_1 + 2\cos\theta_2 + \cos\theta_3\Big\} \quad . \end{aligned} \tag{6.146}$$

The canonical momenta are given by

$$\begin{aligned} \pi_1 &= \frac{\partial L}{\partial\dot{\theta}_1} = m\,\ell^2\,\Big\{3\dot{\theta}_1 + 2\cos(\theta_1-\theta_2)\,\dot{\theta}_2 + \cos(\theta_1-\theta_3)\,\dot{\theta}_3\Big\} \\ \pi_2 &= \frac{\partial L}{\partial\dot{\theta}_2} = m\,\ell^2\,\Big\{2\dot{\theta}_2 + 2\cos(\theta_1-\theta_2)\,\dot{\theta}_1 + \cos(\theta_2-\theta_3)\,\dot{\theta}_3\Big\} \\ \pi_3 &= \frac{\partial L}{\partial\dot{\theta}_2} = m\,\ell^2\,\Big\{\dot{\theta}_3 + \cos(\theta_1-\theta_3)\,\dot{\theta}_1 + \cos(\theta_2-\theta_3)\,\dot{\theta}_2\Big\} \quad . \end{aligned} \tag{6.147}$$

The only conserved quantity is the total energy, $E = T + U$.

(a) As for the T and V matrices, we have

$$\mathsf{T}_{\sigma\sigma'} = \frac{\partial^2 T}{\partial\theta_\sigma\,\partial\theta_{\sigma'}}\bigg|_{\boldsymbol{\theta}=0} = m\ell^2 \begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix} \tag{6.148}$$

and

$$\mathsf{V}_{\sigma\sigma'} = \frac{\partial^2 U}{\partial\theta_\sigma\,\partial\theta_{\sigma'}}\bigg|_{\boldsymbol{\theta}=0} = mg\ell \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad . \tag{6.149}$$

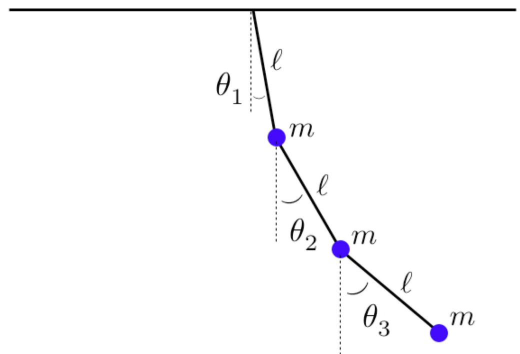


Figure 6.5: The triple pendulum.

(b) The eigenfrequencies are roots of the equation $\det(\omega^2\,\mathsf{T}-\mathsf{V})=0$. Defining $\omega_0\equiv\sqrt{g/\ell}$, we have

$$\omega^2\,\mathsf{T}-\mathsf{V}=m\ell^2\begin{pmatrix}3(\omega^2-\omega_0^2) & 2\,\omega^2 & \omega^2\\ 2\,\omega^2 & 2(\omega^2-\omega_0^2) & \omega^2\\ \omega^2 & \omega^2 & (\omega^2-\omega_0^2)\end{pmatrix} \tag{6.150}$$

and hence

$$\begin{aligned}\widetilde{P}(\omega)\equiv\det(\omega^2\mathsf{T}-\mathsf{V})/(m\ell^2)^3&=3(\omega^2-\omega_0^2)\cdot\Big[2(\omega^2-\omega_0^2)^2-\omega^4\Big]\\ &\qquad-2\,\omega^2\cdot\Big[2\,\omega^2(\omega^2-\omega_0^2)-\omega^4\Big]+\omega^2\cdot\Big[2\,\omega^4-2\,\omega^2(\omega^2-\omega_0^2)\Big]\\ &=6\,(\omega^2-\omega_0^2)^3-9\,\omega^4\,(\omega^2-\omega_0^2)+4\omega^6\\ &=\omega^6-9\,\omega_0^2\,\omega^4+18\,\omega_0^4\,\omega^2-6\,\omega_0^6\quad.\end{aligned} \tag{6.151}$$

(c) The equation for the eigenfrequencies is

$$\lambda^3-9\,\lambda^2+18\,\lambda-6=0\quad, \tag{6.152}$$

where $\omega^2=\lambda\,\omega_0^2$. This is a cubic equation in λ. Numerically solving for the roots, one finds

$$\omega_1^2=0.415774\,\omega_0^2\quad,\quad\omega_2^2=2.29428\,\omega_0^2\quad,\quad\omega_3^2=6.28995\,\omega_0^2\quad. \tag{6.153}$$

I find the (unnormalized) eigenvectors to be

$$\boldsymbol{\psi}_1=\begin{pmatrix}1\\ 1.2921\\ 1.6312\end{pmatrix}\quad,\quad\boldsymbol{\psi}_2=\begin{pmatrix}1\\ 0.35286\\ -2.3981\end{pmatrix}\quad,\quad\boldsymbol{\psi}_3=\begin{pmatrix}1\\ -1.6450\\ 0.76690\end{pmatrix}\quad. \tag{6.154}$$

6.9.3 Equilateral linear triatomic molecule

Consider the vibrations of an equilateral triangle of mass points, depicted in figure 6.6 . The system is confined to the (x,y) plane, and in equilibrium all the strings are unstretched and of length a.

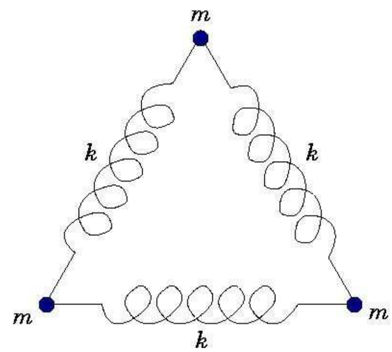


Figure 6.6: An equilateral triangle of identical mass points and springs. We label the sites as 1 (lower left), 2 (lower right), and 3 (upper).

(a) Choose as generalized coordinates the Cartesian displacements (x_i, y_i) with respect to equilibrium. Write down the exact potential energy.

(b) Find the T and V matrices.

(c) There are three normal modes of oscillation for which the corresponding eigenfrequencies all vanish: $\omega_a = 0$. Write down these modes explicitly, and provide a physical interpretation for why $\omega_a = 0$. Since this triplet is degenerate, there is no unique answer – any linear combination will also serve as a valid 'zero mode'. However, if you think physically, a natural set should emerge.

(d) The three remaining modes all have finite oscillation frequencies. They correspond to distortions of the triangular shape. One such mode is the "breathing mode" in which the triangle uniformly expands and contracts. Write down the eigenvector associated with this normal mode and compute its associated oscillation frequency.

(e) The fifth and sixth modes are degenerate. They must be orthogonal (with respect to the inner product defined by T) to all the other modes. See if you can figure out what these modes are, and compute their oscillation frequencies. As in (a), any linear combination of these modes will also be an eigenmode.

(f) Write down the modal matrix $\mathsf{A}_{\sigma i}$, and check that it is correct by using `Mathematica`.

Solution

Choosing as generalized coordinates the Cartesian displacements relative to equilibrium, we have the following:

$$
\begin{aligned}
&\#1 : (x_1, y_1) \\
&\#2 : (a + x_2, y_2) \\
&\#3 : \big(\tfrac{1}{2}a + x_3, \tfrac{\sqrt{3}}{2}a + y_3\big) \quad .
\end{aligned}
$$

Let d_{ij} be the separation of particles i and j. The potential energy of the spring connecting them is then $\frac{1}{2}k(d_{ij}-a)^2$.

$$\begin{aligned} d_{12}^2 &= \big(a + x_2 - x_1\big)^2 + \big(y_2 - y_1\big)^2 \\ d_{23}^2 &= \big(-\tfrac{1}{2}a + x_3 - x_2\big)^2 + \big(\tfrac{\sqrt{3}}{2}a + y_3 - y_2\big)^2 \\ d_{13}^2 &= \big(\tfrac{1}{2}a + x_3 - x_1\big)^2 + \big(\tfrac{\sqrt{3}}{2}a + y_3 - y_1\big)^2 \quad . \end{aligned} \tag{6.155}$$

The full potential energy is

$$U = \tfrac{1}{2}\,k\,\big(d_{12}-a\big)^2 + \tfrac{1}{2}\,k\,\big(d_{23}-a\big)^2 + \tfrac{1}{2}\,k\,\big(d_{13}-a\big)^2 \quad . \tag{6.156}$$

This is a cumbersome expression, involving square roots.

To find T and V, we need to write T and V as quadratic forms, neglecting higher order terms. Therefore, we must expand $d_{ij}-a$ to linear order in the generalized coordinates. This results in the following:

$$\begin{aligned} d_{12} &= a + \big(x_2 - x_1\big) + \ldots \\ d_{23} &= a - \tfrac{1}{2}\,\big(x_3 - x_2\big) + \tfrac{\sqrt{3}}{2}\,\big(y_3 - y_2\big) + \ldots \\ d_{13} &= a + \tfrac{1}{2}\,\big(x_3 - x_1\big) + \tfrac{\sqrt{3}}{2}\,\big(y_3 - y_1\big) + \ldots \quad . \end{aligned} \tag{6.157}$$

Thus,

$$\begin{aligned} U = \tfrac{1}{2}k\big(x_2 - x_1\big)^2 &+ \tfrac{1}{8}k\big(x_2 - x_3 - \sqrt{3}\,y_2 + \sqrt{3}\,y_3\big)^2 \\ &+ \tfrac{1}{8}k\big(x_3 - x_1 + \sqrt{3}\,y_3 - \sqrt{3}\,y_1\big)^2 + \text{higher order terms} \quad . \end{aligned} \tag{6.158}$$

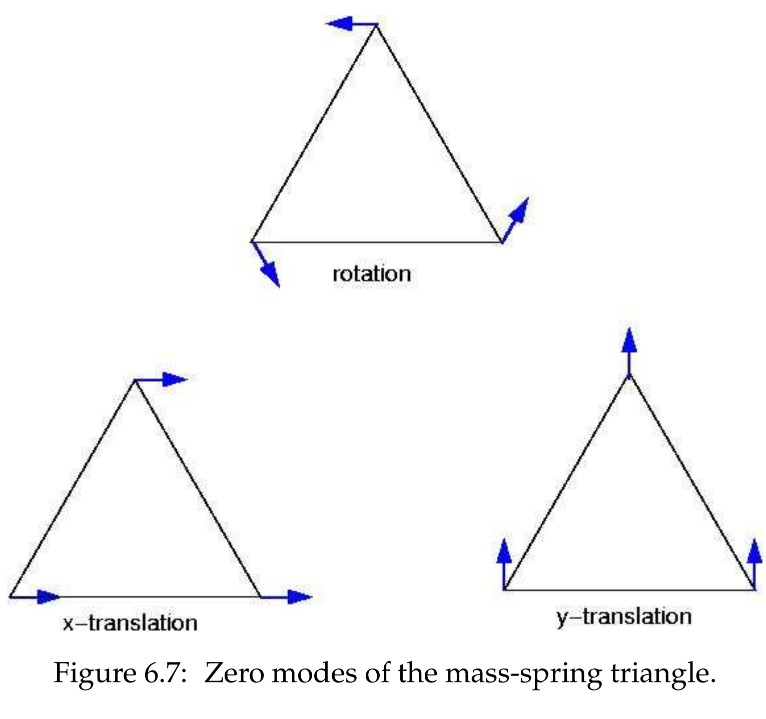


Figure 6.7: Zero modes of the mass-spring triangle.

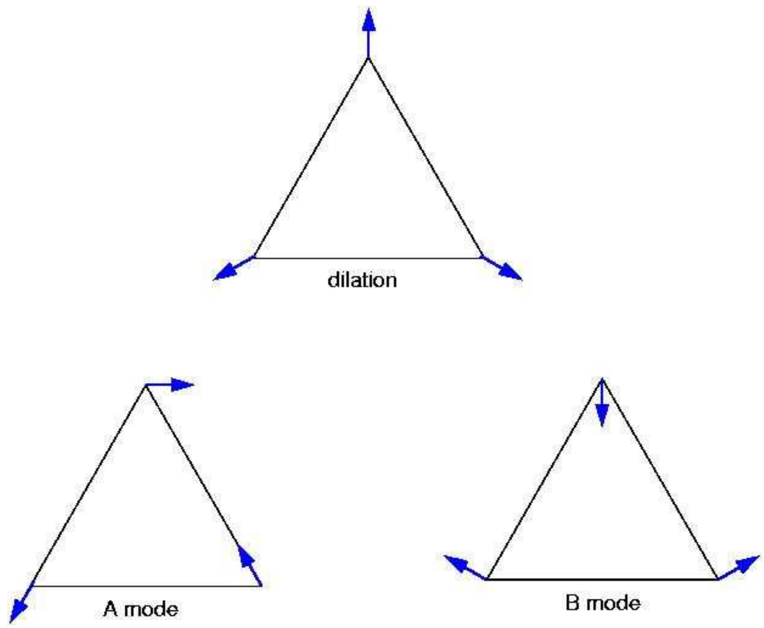


Figure 6.8: Finite oscillation frequency modes of the mass-spring triangle.

Defining

$$\big(q_1, q_2, q_3, q_4, q_5, q_6\big) = \big(x_1, y_1, x_2, y_2, x_3, y_3\big) \quad , \tag{6.159}$$

we may now read off

$$\mathsf{V}_{\sigma\sigma'} = \frac{\partial^2 U}{\partial q_\sigma \, \partial q_{\sigma'}}\bigg|_{\bar{q}} = k \begin{pmatrix} 5/4 & \sqrt{3}/4 & -1 & 0 & -1/4 & -\sqrt{3}/4 \\ \sqrt{3}/4 & 3/4 & 0 & 0 & -\sqrt{3}/4 & -3/4 \\ -1 & 0 & 5/4 & -\sqrt{3}/4 & -1/4 & \sqrt{3}/4 \\ 0 & 0 & -\sqrt{3}/4 & 3/4 & \sqrt{3}/4 & -3/4 \\ -1/4 & -\sqrt{3}/4 & -1/4 & \sqrt{3}/4 & 1/2 & 0 \\ -\sqrt{3}/4 & -3/4 & \sqrt{3}/4 & -3/4 & 0 & 3/2 \end{pmatrix} \tag{6.160}$$

The T matrix is trivial. From

$$T = \tfrac{1}{2} m \big(\dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2 + \dot{x}_3^2 + \dot{y}_3^2\big) \quad . \tag{6.161}$$

we obtain

$$\mathsf{T}_{\sigma\sigma'} = \frac{\partial^2 T}{\partial \dot{q}_\sigma \, \partial \dot{q}_{\sigma'}} = m \, \delta_{\sigma\sigma'} \quad , \tag{6.162}$$

and $\mathsf{T} = m \cdot \mathbb{I}$ is a multiple of the unit matrix.

Figure 6.9: *John Henry*, statue by Charles O. Cooper (1972). "Now the man that invented the steam drill, he thought he was mighty fine. But John Henry drove fifteen feet, and the steam drill only made nine." - from *The Ballad of John Henry*.

The zero modes are depicted graphically in figure 6.7. Explicitly, we have

$$\boldsymbol{\xi}^x = \frac{1}{\sqrt{3m}}\begin{pmatrix}1\\0\\1\\0\\1\\0\end{pmatrix} \quad , \quad \boldsymbol{\xi}^y = \frac{1}{\sqrt{3m}}\begin{pmatrix}0\\1\\0\\1\\0\\1\end{pmatrix} \quad , \quad \boldsymbol{\xi}^{\rm rot} = \frac{1}{\sqrt{3m}}\begin{pmatrix}1/2\\-\sqrt{3}/2\\1/2\\\sqrt{3}/2\\-1\\0\end{pmatrix} \quad . \tag{6.163}$$

That these are indeed zero modes may be verified by direct multiplication:

$$\mathsf{V}\,\boldsymbol{\xi}^{x.y} = \mathsf{V}\,\boldsymbol{\xi}^{\rm rot} = 0 \quad . \tag{6.164}$$

The three modes with finite oscillation frequency are depicted graphically in figure 6.8. Explicitly, we have

$$\boldsymbol{\xi}^{\rm A} = \frac{1}{\sqrt{3m}}\begin{pmatrix}-1/2\\-\sqrt{3}/2\\-1/2\\\sqrt{3}/2\\1\\0\end{pmatrix} \quad , \quad \boldsymbol{\xi}^{\rm B} = \frac{1}{\sqrt{3m}}\begin{pmatrix}-\sqrt{3}/2\\1/2\\\sqrt{3}/2\\1/2\\0\\-1\end{pmatrix} \quad , \quad \boldsymbol{\xi}^{\rm dil} = \frac{1}{\sqrt{3m}}\begin{pmatrix}-\sqrt{3}/2\\-1/2\\\sqrt{3}/2\\-1/2\\0\\1\end{pmatrix} \quad . \tag{6.165}$$

The oscillation frequencies of these modes are easily checked by multiplying the eigenvectors by the matrix V. Since $\mathsf{T} = m \cdot \mathbb{I}$ is diagonal, we have $\mathsf{V}\,\boldsymbol{\xi}^{(j)} = m\omega_j^2\,\boldsymbol{\xi}^{(j)}$. One finds

$$\omega_{\rm A} = \omega_{\rm B} = \sqrt{\frac{3k}{2m}} \quad , \quad \omega_{\rm dil} = \sqrt{\frac{3k}{m}} \quad . \tag{6.166}$$

`Mathematica`? I don't need no stinking `Mathematica`.

6.10 Aside: Christoffel Symbols

The coupled equations in eqn. 6.5 may be written in the form

$$\ddot{q}_\sigma + \Gamma^\sigma_{\mu\nu}\,\dot{q}_\mu\,\dot{q}_\nu = W_\sigma \quad , \tag{6.167}$$

with

$$\Gamma^\sigma_{\mu\nu} = \tfrac{1}{2}\,T^{-1}_{\sigma\alpha}\bigg(\frac{\partial T_{\alpha\mu}}{\partial q_\nu} + \frac{\partial T_{\alpha\nu}}{\partial q_\mu} - \frac{\partial T_{\mu\nu}}{\partial q_\alpha}\bigg) \tag{6.168}$$

and

$$W_\sigma = -T^{-1}_{\sigma\alpha}\,\frac{\partial U}{\partial q_\alpha} \quad . \tag{6.169}$$

The components of the rank-three tensor $\Gamma^\sigma_{\alpha\beta}$ are known as *Christoffel symbols*, in the case where $T_{\mu\nu}(q)$ defines a *metric* on the space of generalized coordinates.

Chapter 7

Elastic Collisions

7.1 Center of Mass Frame

A collision or 'scattering event' is said to be *elastic* if it results in no change in the internal state of any of the particles involved. Thus, no internal energy is liberated or captured in an elastic process.

Consider the elastic scattering of two particles. Recall the relation between laboratory coordinates $\{\boldsymbol{r}_1, \boldsymbol{r}_2\}$ and the CM and relative coordinates $\{\boldsymbol{R}, \boldsymbol{r}\}$:

$$\boldsymbol{R} = \frac{m_1 \boldsymbol{r}_1 + m_2 \boldsymbol{r}_2}{m_1 + m_2} \qquad\qquad \boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2 \tag{7.1}$$

$$\boldsymbol{r}_1 = \boldsymbol{R} + \frac{m_2}{m_1 + m_2}\,\boldsymbol{r} \qquad\qquad \boldsymbol{r}_2 = \boldsymbol{R} - \frac{m_1}{m_1 + m_2}\,\boldsymbol{r} \tag{7.2}$$

If external forces are negligible, the CM momentum $\boldsymbol{P} = M\dot{\boldsymbol{R}}$ is constant, and therefore the frame of reference whose origin is tied to the CM position is an inertial frame of reference. In this frame,

$$\boldsymbol{v}_1^{\rm CM} = \frac{m_2\,\boldsymbol{v}}{m_1 + m_2} \quad , \quad \boldsymbol{v}_2^{\rm CM} = -\frac{m_1\,\boldsymbol{v}}{m_1 + m_2} \quad , \tag{7.3}$$

where $\boldsymbol{v} = \boldsymbol{v}_1 - \boldsymbol{v}_2 = \boldsymbol{v}_1^{\rm CM} - \boldsymbol{v}_2^{\rm CM}$ is the relative velocity, which is the same in both L and CM frames. Note that the CM momenta satisfy

$$\begin{aligned} \boldsymbol{p}_1^{\rm CM} &= m_1 \boldsymbol{v}_1^{\rm CM} = \mu \boldsymbol{v} \\ \boldsymbol{p}_2^{\rm CM} &= m_2 \boldsymbol{v}_2^{\rm CM} = -\mu \boldsymbol{v} \quad , \end{aligned} \tag{7.4}$$

where $\mu = m_1 m_2/(m_1 + m_2)$ is the reduced mass. Thus, $\boldsymbol{p}_1^{\rm CM} + \boldsymbol{p}_2^{\rm CM} = 0$ and the total momentum in the CM frame is zero. We may then write

$$\boldsymbol{p}_1^{\rm CM} \equiv p_0 \hat{\boldsymbol{n}} \quad , \quad \boldsymbol{p}_2^{\rm CM} \equiv -p_0 \hat{\boldsymbol{n}} \quad \Rightarrow \quad E^{\rm CM} = \frac{p_0^2}{2m_1} + \frac{p_0^2}{2m_2} = \frac{p_0^2}{2\mu} \quad . \tag{7.5}$$

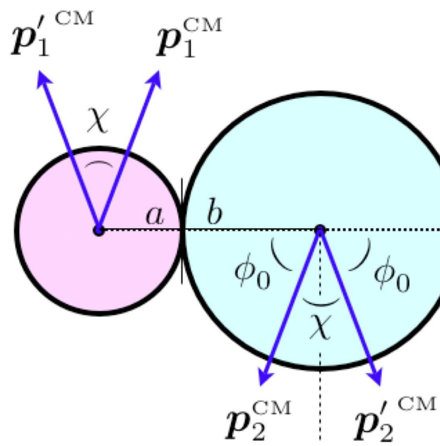


Figure 7.1: The scattering of two hard spheres of radii a and b. The scattering angle is χ.

The energy is evaluated when the particles are asymptotically far from each other, in which case the potential energy is assumed to be negligible. After the collision, energy and momentum conservation require

$$\boldsymbol{p}_1'^{\,\rm CM} \equiv p_0\hat{\boldsymbol{n}}' \quad , \quad \boldsymbol{p}_2'^{\,\rm CM} \equiv -p_0\hat{\boldsymbol{n}}' \quad \Rightarrow \quad E'^{\rm CM} = E^{\rm CM} = \frac{p_0^2}{2\mu} \quad . \tag{7.6}$$

The angle between $\boldsymbol{n}$ and $\boldsymbol{n}'$ is the *scattering angle* χ:

$$\boldsymbol{n}\cdot\boldsymbol{n}' \equiv \cos\chi \quad . \tag{7.7}$$

The value of χ depends on the details of the scattering process, *i.e.* on the interaction potential $U(r)$. As an example, consider the scattering of two hard spheres, depicted in fig. 7.1. The potential is

$$U(r) = \begin{cases} \infty & \text{if } r \le a+b \\ 0 & \text{if } r > a+b \quad . \end{cases} \tag{7.8}$$

Clearly the scattering angle is $\chi = \pi - 2\phi_0$, where ϕ_0 is the angle between the initial momentum of either sphere and a line containing their two centers at the moment of contact.

There is a simple geometric interpretation of these results, depicted in fig. 7.2. We have

$$\boldsymbol{p}_1 = m_1\boldsymbol{V} + p_0\hat{\boldsymbol{n}} \qquad\qquad \boldsymbol{p}_1' = m_1\boldsymbol{V} + p_0\hat{\boldsymbol{n}}' \tag{7.9}$$

$$\boldsymbol{p}_2 = m_2\boldsymbol{V} - p_0\hat{\boldsymbol{n}} \qquad\qquad \boldsymbol{p}_2' = m_2\boldsymbol{V} - p_0\hat{\boldsymbol{n}}' \quad . \tag{7.10}$$

So draw a circle of radius p_0 whose center is the origin. The vectors $p_0\hat{\boldsymbol{n}}$ and $p_0\hat{\boldsymbol{n}}'$ must both lie along this circle. We define the angle ψ between $\boldsymbol{V}$ and $\boldsymbol{n}$:

$$\hat{\boldsymbol{V}}\cdot\boldsymbol{n} = \cos\psi \quad . \tag{7.11}$$

It is now an exercise in geometry, using the law of cosines, to determine everything of interest in terms

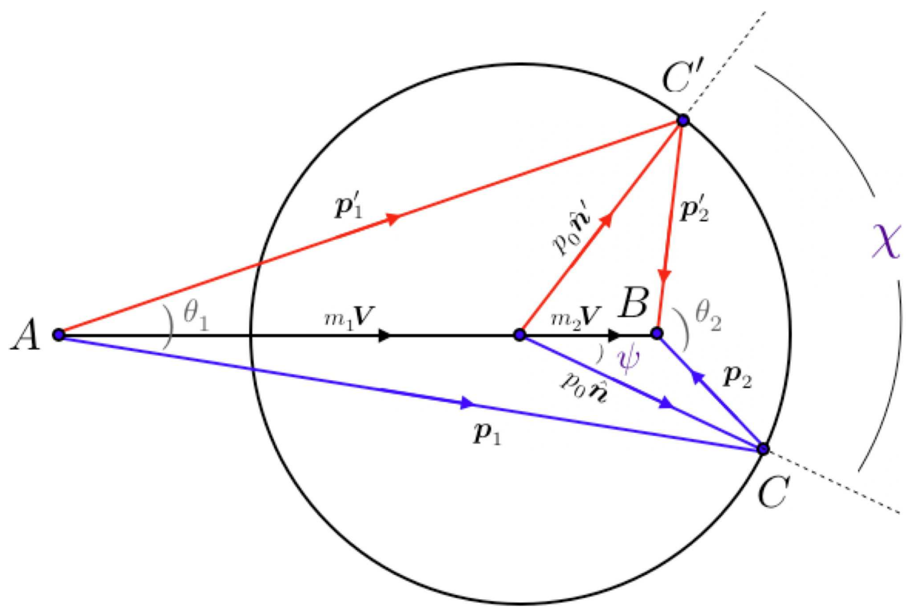


Figure 7.2: Scattering of two particles of masses m_1 and m_2. The scattering angle χ is the angle between $\hat{\boldsymbol{n}}$ and $\hat{\boldsymbol{n}}'$.

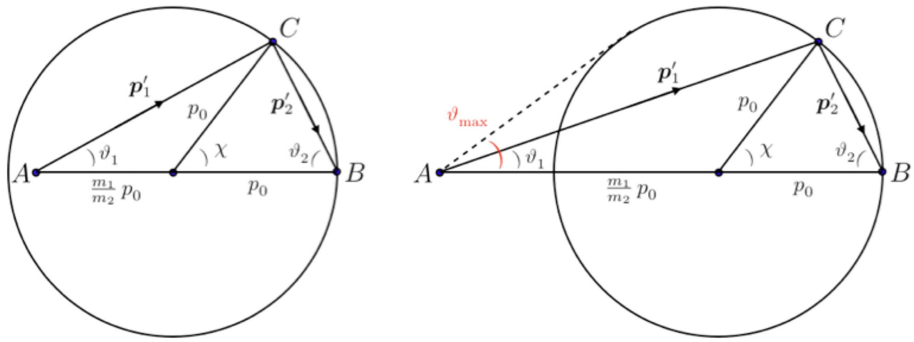
of the quantities V, v, ψ, and χ. For example, the momenta are

$$
\begin{aligned}
p_1 &= \sqrt{m_1^2 V^2 + \mu^2 v^2 + 2m_1 \mu V v \cos\psi} \\
p_1' &= \sqrt{m_1^2 V^2 + \mu^2 v^2 + 2m_1 \mu V v \cos(\chi - \psi)} \\
p_2 &= \sqrt{m_2^2 V^2 + \mu^2 v^2 - 2m_2 \mu V v \cos\psi} \\
p_2' &= \sqrt{m_2^2 V^2 + \mu^2 v^2 - 2m_2 \mu V v \cos(\chi - \psi)} \quad ,
\end{aligned}
\tag{7.12}
$$

and the scattering angles are

$$
\begin{aligned}
\theta_1 &= \tan^{-1}\!\left(\frac{\mu v \sin\psi}{\mu v \cos\psi + m_1 V}\right) + \tan^{-1}\!\left(\frac{\mu v \sin(\chi - \psi)}{\mu v \cos(\chi - \psi) + m_1 V}\right) \\
\theta_2 &= \tan^{-1}\!\left(\frac{\mu v \sin\psi}{\mu v \cos\psi - m_2 V}\right) + \tan^{-1}\!\left(\frac{\mu v \sin(\chi - \psi)}{\mu v \cos(\chi - \psi) - m_2 V}\right) \quad .
\end{aligned}
\tag{7.13}
$$

If particle 2, say, is initially at rest, the situation is somewhat simpler. In this case, $\boldsymbol{V} = m_1 \boldsymbol{V}/(m_1 + m_2)$ and $m_2 \boldsymbol{V} = \mu \boldsymbol{v}$, which means the point B lies on the circle in fig. 7.3 ($m_1 \neq m_2$) and fig. 7.4 ($m_1 = m_2$).



(a) $m_1 < m_2$ (b) $m_1 > m_2$

Figure 7.3: Scattering when particle 2 is initially at rest.

Let $\vartheta_{1,2}$ be the angles between the directions of motion after the collision and the direction $\boldsymbol{V}$ of impact. The scattering angle χ is the angle through which particle 1 turns in the CM frame. Clearly

$$\tan\vartheta_1 = \frac{\sin\chi}{\frac{m_1}{m_2} + \cos\chi} \qquad , \qquad \vartheta_2 = \tfrac{1}{2}(\pi - \chi) \quad . \tag{7.14}$$

We can also find the speeds v_1' and v_2' in terms of v and χ, from

$$p_1'^{\,2} = p_0^2 + \big(\tfrac{m_1}{m_2}\,p_0\big)^2 - 2\,\tfrac{m_1}{m_2}\,p_0^2\,\cos(\pi - \chi) \tag{7.15}$$

and

$$p_2^2 = 2\,p_0^2\,(1 - \cos\chi) \quad . \tag{7.16}$$

These equations yield

$$v_1' = \frac{\sqrt{m_1^2 + m_2^2 + 2m_1 m_2\cos\chi}}{m_1 + m_2}\,v \qquad , \qquad v_2' = \frac{2m_1 v}{m_1 + m_2}\,\sin(\tfrac{1}{2}\chi) \quad . \tag{7.17}$$

The angle $\vartheta_{\rm max}$ from fig. 7.3(b) is given by $\sin\vartheta_{\rm max} = \frac{m_2}{m_1}$. Note that when $m_1 = m_2$ we have $\vartheta_1 + \vartheta_2 = \pi$. A sketch of the orbits in the cases of both repulsive and attractive scattering, in both the laboratory and CM frames, in shown in fig. 7.5.

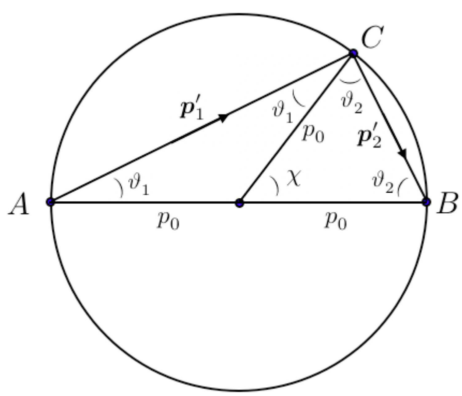


Figure 7.4: Scattering of identical mass particles when particle 2 is initially at rest.

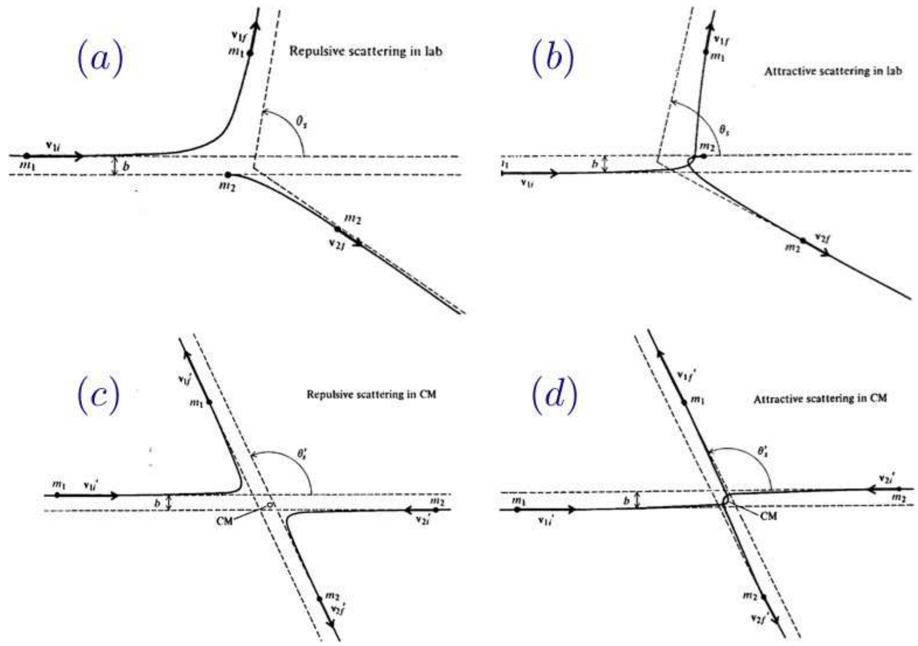


Figure 7.5: Repulsive (A,C) and attractive (B,D) scattering in the lab (A,B) and CM (C,D) frames, assuming particle 2 starts from rest in the lab frame. (From Barger and Olsson.)

7.2 Central Force Scattering

Consider a single particle of mass μ movng in a central potential $U(r)$, or a two body central force problem in which μ is the reduced mass. Recall that

$$\frac{dr}{dt} = \frac{d\phi}{dt} \cdot \frac{dr}{d\phi} = \frac{\ell}{\mu r^2} \cdot \frac{dr}{d\phi} \quad , \tag{7.18}$$

and therefore

$$\begin{aligned} E &= \tfrac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + U(r) \\ &= \frac{\ell^2}{2\mu r^4}\left(\frac{dr}{d\phi}\right)^2 + \frac{\ell^2}{2\mu r^2} + U(r) \quad . \end{aligned} \tag{7.19}$$

Solving for $\frac{dr}{d\phi}$, we obtain

$$\frac{dr}{d\phi} = \pm\sqrt{\frac{2\mu r^4}{\ell^2}\big(E - U(r)\big) - r^2} \quad , \tag{7.20}$$

Consulting fig. 7.6, we have that

$$\phi_0 = \frac{\ell}{\sqrt{2\mu}} \int\limits_{r_{\rm p}}^{\infty} \frac{dr}{r^2\sqrt{E - U_{\rm eff}(r)}} \quad , \tag{7.21}$$

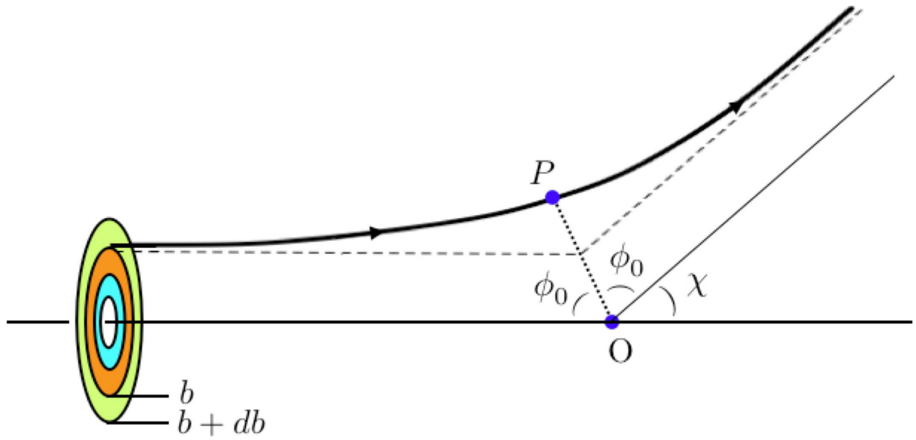


Figure 7.6: Scattering in the CM frame. O is the force center and P is the point of periapsis. The impact parameter is b, and χ is the scattering angle. ϕ_0 is the angle through which the relative coordinate moves between periapsis and infinity.

where r_p is the radial distance at periapsis, and where

$$U_{\text{eff}}(r) = \frac{\ell^2}{2\mu r^2} + U(r) \tag{7.22}$$

is the effective potential, as before. From fig. 7.6, we conclude that the scattering angle is

$$\chi = \left|\pi - 2\phi_0\right| \quad . \tag{7.23}$$

It is convenient to define the *impact parameter* b as the distance of the asymptotic trajectory from a parallel line containing the force center. The geometry is shown again in fig. 7.6. Note that the energy and angular momentum, which are conserved, can be evaluated at infinity using the impact parameter:

$$E = \tfrac{1}{2}\mu v_\infty^2 \qquad , \qquad \ell = \mu v_\infty b \quad . \tag{7.24}$$

Substituting for $\ell(b)$, we have

$$\phi_0(E,b) = \int\limits_{r_p}^{\infty} \frac{dr}{r^2} \, \frac{b}{\sqrt{1 - \frac{b^2}{r^2} - \frac{U(r)}{E}}} \quad , \tag{7.25}$$

In physical applications, we are often interested in the deflection of a beam of incident particles by a scattering center. We define the *differential scattering cross section* $d\sigma$ by

$$d\sigma = \frac{\text{\# of particles scattered into solid angle } d\Omega \text{ per unit time}}{\text{incident flux}} \quad . \tag{7.26}$$

Now for particles of a given energy E there is a unique relationship between the scattering angle χ and the impact parameter b, as we have just derived in eqn. 7.25. The differential solid angle is given by $d\Omega = 2\pi \sin\chi \, d\chi$, hence

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\chi} \left|\frac{db}{d\chi}\right| = \left|\frac{d\left(\frac{1}{2}b^2\right)}{d\cos\chi}\right| \quad . \tag{7.27}$$

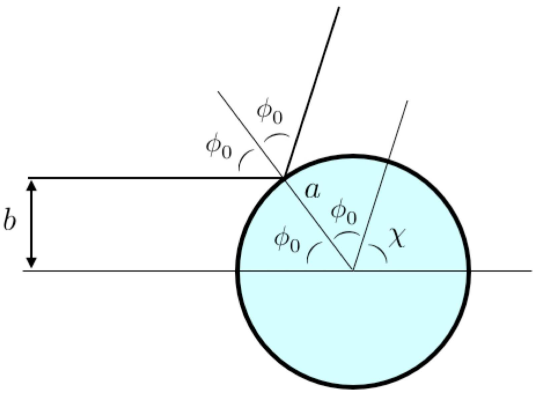


Figure 7.7: Geometry of hard sphere scattering.

Note that $\frac{d\sigma}{d\Omega}$ has dimensions of area. The integral of $\frac{d\sigma}{d\Omega}$ over all solid angle is the *total scattering cross section*,

$$\sigma_{\rm T} = 2\pi \int_0^\pi d\chi \, \sin\chi \, \frac{d\sigma}{d\Omega} \quad . \tag{7.28}$$

7.2.1 Hard sphere scattering

Consider a point particle scattering off a hard sphere of radius a, or two hard spheres of radii a_1 and a_2 scattering off each other, with $a \equiv a_1 + a_2$. From the geometry of fig. 7.7, we have $b = a \sin\phi_0$ and $\phi_0 = \frac{1}{2}(\pi - \chi)$, so

$$b^2 = a^2 \sin^2\big(\tfrac{1}{2}\pi - \tfrac{1}{2}\chi\big) = \tfrac{1}{2}a^2\,(1 + \cos\chi) \quad . \tag{7.29}$$

We therefore have

$$\frac{d\sigma}{d\Omega} = \frac{d\,(\frac{1}{2}b^2)}{d\cos\chi} = \tfrac{1}{4}\,a^2 \tag{7.30}$$

and $\sigma_{\rm T} = \pi a^2$. The total scattering cross section is simply the area of a sphere of radius a projected onto a plane perpendicular to the incident flux.

7.2.2 Rutherford scattering

Consider scattering by the Kepler potential $U(r) = -\frac{k}{r}$. We assume that the orbits are unbound, *i.e.* they are Keplerian hyperbolae with $E > 0$, described by the equation

$$r(\phi) = \frac{a\,(\varepsilon^2 - 1)}{\pm 1 + \varepsilon\,\cos\phi} \quad \Rightarrow \quad \cos\phi_0 = \pm\,\frac{1}{\varepsilon} \quad . \tag{7.31}$$

Recall that the eccentricity is given by

$$\varepsilon^2 = 1 + \frac{2E\ell^2}{\mu k^2} = 1 + \left(\frac{\mu b v_\infty}{k}\right)^2 \quad . \tag{7.32}$$

We then have

$$\begin{aligned}\left(\frac{\mu b v_\infty}{k}\right)^2 &= \varepsilon^2 - 1\\ &= \sec^2\!\phi_0 - 1 = \tan^2\!\phi_0 = \text{ctn}^2\big(\tfrac{1}{2}\chi\big) \quad .\end{aligned} \tag{7.33}$$

Therefore

$$b(\chi) = \frac{k}{\mu v_\infty^2}\,\text{ctn}\big(\tfrac{1}{2}\chi\big) \tag{7.34}$$

We finally obtain

$$\begin{aligned}\frac{d\sigma}{d\Omega} = \frac{d\,(\frac{1}{2}b^2)}{d\cos\chi} &= \frac{1}{2}\left(\frac{k}{\mu v_\infty^2}\right)^2 \frac{d\,\text{ctn}^2\big(\frac{1}{2}\chi\big)}{d\cos\chi}\\ &= \frac{1}{2}\left(\frac{k}{\mu v_\infty^2}\right)^2 \frac{d}{d\cos\chi}\left(\frac{1+\cos\chi}{1-\cos\chi}\right)\\ &= \left(\frac{k}{2\mu v_\infty^2}\right)^2 \csc^4\big(\tfrac{1}{2}\chi\big) \quad ,\end{aligned} \tag{7.35}$$

which is the same as

$$\frac{d\sigma}{d\Omega} = \left(\frac{k}{4E}\right)^2 \csc^4\big(\tfrac{1}{2}\chi\big) \quad . \tag{7.36}$$

Since $\frac{d\sigma}{d\Omega} \propto \chi^{-4}$ as $\chi \to 0$, the total cross section $\sigma_{\rm T}$ diverges! This is a consequence of the long-ranged nature of the Kepler/Coulomb potential. In electron-atom scattering, the Coulomb potential of the nucleus is *screened* by the electrons of the atom, and the $1/r$ behavior is cut off at large distances.

7.2.3 Transformation to laboratory coordinates

We previously derived the relation

$$\tan\vartheta = \frac{\sin\chi}{\gamma + \cos\chi} \quad , \tag{7.37}$$

where $\vartheta \equiv \vartheta_1$ is the scattering angle for particle 1 in the laboratory frame, and $\gamma = \frac{m_1}{m_2}$ is the ratio of the masses. We now derive the differential scattering cross section in the laboratory frame. To do so, we note that particle conservation requires

$$\left(\frac{d\sigma}{d\Omega}\right)_{\rm L} \cdot\, 2\pi\sin\vartheta\,d\vartheta = \left(\frac{d\sigma}{d\Omega}\right)_{\rm CM} \cdot\, 2\pi\sin\chi\,d\chi \quad , \tag{7.38}$$

which says

$$\left(\frac{d\sigma}{d\Omega}\right)_{\rm L} = \left(\frac{d\sigma}{d\Omega}\right)_{\rm CM} \cdot\, \frac{d\cos\chi}{d\cos\vartheta} \quad . \tag{7.39}$$

From

$$\cos\vartheta = \frac{1}{\sqrt{1+\tan^2\!\vartheta}} = \frac{\gamma + \cos\chi}{\sqrt{1+\gamma^2+2\gamma\cos\chi}} \quad , \tag{7.40}$$

we derive

$$\frac{d\cos\vartheta}{d\cos\chi} = \frac{1+\gamma\cos\chi}{\left(1+\gamma^2+2\gamma\cos\chi\right)^{3/2}} \tag{7.41}$$

and, accordingly,

$$\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{L}} = \frac{\left(1+\gamma^2+2\gamma\cos\chi\right)^{3/2}}{1+\gamma\cos\chi}\cdot\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{CM}} \quad . \tag{7.42}$$

Chapter 8

Noninertial Reference Frames

8.1 Accelerated Coordinate Systems

A reference frame which is fixed with respect to a rotating rigid body is not inertial. The parade example of this is an observer fixed on the surface of the earth. Due to the rotation of the earth, such an observer is in a noninertial frame, and there are corresponding corrections to Newton's laws of motion which must be accounted for in order to correctly describe mechanical motion in the observer's frame. As is well known, these corrections involve fictitious centrifugal and Coriolis forces.

Consider an inertial frame with a fixed set of coordinate axes $\hat{\mathbf{e}}_\mu$, where μ runs from 1 to d, the dimension of space, and a noninertial frame with axes $\hat{\mathbf{e}}'_\mu$. Any vector $\boldsymbol{A}$ may be written in either basis:

$$\boldsymbol{A} = \sum_\mu A_\mu \, \hat{\mathbf{e}}_\mu = \sum_\mu A'_\mu \, \hat{\mathbf{e}}'_\mu \quad , \tag{8.1}$$

where $A_\mu = \boldsymbol{A} \cdot \hat{\mathbf{e}}_\mu$ and $A'_\mu = \boldsymbol{A} \cdot \hat{\mathbf{e}}'_\mu$ are projections onto the different coordinate axes. We may now write

$$\begin{aligned} \left(\frac{d\boldsymbol{A}}{dt}\right)_{\text{inertial}} &= \sum_\mu \frac{dA_\mu}{dt} \, \hat{\mathbf{e}}_\mu \\ &= \sum_i \frac{dA'_\mu}{dt} \, \hat{\mathbf{e}}'_\mu + \sum_\mu A'_\mu \, \frac{d\hat{\mathbf{e}}'_\mu}{dt} \quad . \end{aligned} \tag{8.2}$$

The first term on the RHS is $(d\boldsymbol{A}/dt)_{\text{body}}$, the time derivative of $\boldsymbol{A}$ along body-fixed axes, *i.e.* as seen by an observer rotating with the body. But what is $d\hat{\mathbf{e}}'_i/dt$? Well, we can always expand it in the $\{\hat{\mathbf{e}}'_i\}$ basis:

$$d\hat{\mathbf{e}}'_\mu = \sum_\nu d\Omega_{\mu\nu} \, \hat{\mathbf{e}}'_\nu \quad \Longleftrightarrow \quad d\Omega_{\mu\nu} \equiv d\hat{\mathbf{e}}'_\mu \cdot \hat{\mathbf{e}}'_\nu \quad . \tag{8.3}$$

Note that $d\Omega_{\mu\nu} = -d\Omega_{\nu\mu}$ is antisymmetric, because

$$0 = d\big(\hat{\mathbf{e}}'_\mu \cdot \hat{\mathbf{e}}'_\nu\big) = d\Omega_{\nu\mu} + d\Omega_{\mu\nu} \quad , \tag{8.4}$$

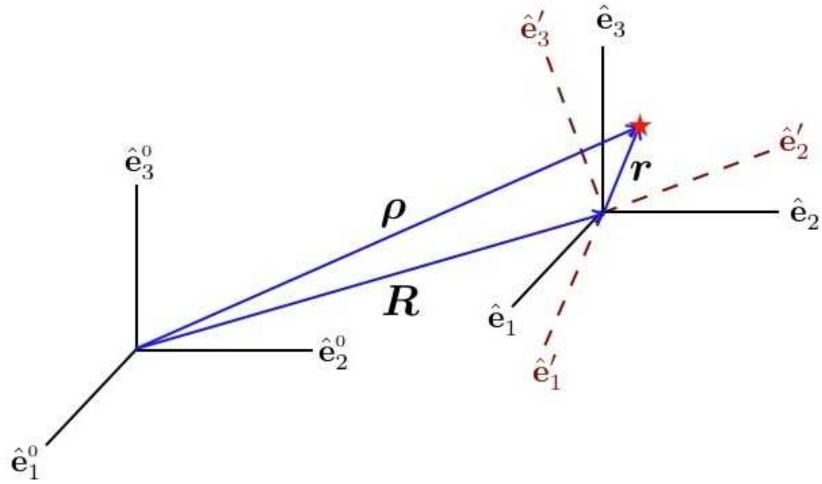


Figure 8.1: Reference frames related by both translation and rotation. Note $\hat{\mathbf{e}}_\mu = \hat{\mathbf{e}}_\mu^0$.

because $\hat{\mathbf{e}}_\mu' \cdot \hat{\mathbf{e}}_\nu' = \delta_{\mu\nu}$ is a constant. Now we may define $d\Omega_{12} \equiv d\Omega_3$, *et cyc.*, so that

$$d\Omega_{\mu\nu} = \sum_\sigma \epsilon_{\mu\nu\sigma}\, d\Omega_\sigma \quad , \quad \omega_\sigma \equiv \frac{d\Omega_\sigma}{dt} \quad , \tag{8.5}$$

which yields

$$\frac{d\hat{\mathbf{e}}_\mu'}{dt} = \boldsymbol{\omega} \times \hat{\mathbf{e}}_\mu' \quad . \tag{8.6}$$

Finally, we obtain the important result

$$\left(\frac{d\boldsymbol{A}}{dt}\right)_{\text{inertial}} = \left(\frac{d\boldsymbol{A}}{dt}\right)_{\text{body}} + \boldsymbol{\omega} \times \boldsymbol{A} \quad , \tag{8.7}$$

which is valid for any vector $\boldsymbol{A}$.

Applying this result to the position vector $\boldsymbol{r}$, we have

$$\left(\frac{d\boldsymbol{r}}{dt}\right)_{\text{inertial}} = \left(\frac{d\boldsymbol{r}}{dt}\right)_{\text{body}} + \boldsymbol{\omega} \times \boldsymbol{r} \quad . \tag{8.8}$$

Applying it twice,

$$\begin{aligned}\left(\frac{d^2\boldsymbol{r}}{dt^2}\right)_{\text{inertial}} &= \left(\frac{d}{dt}\bigg|_{\text{body}} + \boldsymbol{\omega} \times\right)\left(\frac{d}{dt}\bigg|_{\text{body}} + \boldsymbol{\omega} \times\right)\boldsymbol{r} \\ &= \left(\frac{d^2\boldsymbol{r}}{dt^2}\right)_{\text{body}} + \frac{d\boldsymbol{\omega}}{dt} \times \boldsymbol{r} + 2\,\boldsymbol{\omega} \times \left(\frac{d\boldsymbol{r}}{dt}\right)_{\text{body}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) \quad .\end{aligned} \tag{8.9}$$

Note that $d\boldsymbol{\omega}/dt$ appears with no "inertial" or "body" label. This is because, upon invoking eq. 8.7,

$$\left(\frac{d\boldsymbol{\omega}}{dt}\right)_{\text{inertial}} = \left(\frac{d\boldsymbol{\omega}}{dt}\right)_{\text{body}} + \boldsymbol{\omega} \times \boldsymbol{\omega} \quad , \tag{8.10}$$

and since $\boldsymbol{\omega} \times \boldsymbol{\omega} = 0$, inertial and body-fixed observers will agree on the value of $\dot{\boldsymbol{\omega}}_{\text{inertial}} = \dot{\boldsymbol{\omega}}_{\text{body}} \equiv \dot{\boldsymbol{\omega}}$.

8.1.1 Translations

Suppose that frame K moves with respect to an inertial frame K^0, such that the origin of K lies at $\boldsymbol{R}(t)$. Suppose further that frame K' rotates with respect to K, but shares the same origin (see fig. 8.1). Consider the motion of an object lying at position $\boldsymbol{\rho}$ relative to the origin of K^0, and $\boldsymbol{r}$ relative to the origin of K/K'. Thus,

$$\boldsymbol{\rho} = \boldsymbol{R} + \boldsymbol{r} \quad , \tag{8.11}$$

and

$$\begin{aligned}\left(\frac{d\boldsymbol{\rho}}{dt}\right)_{\text{inertial}} &= \left(\frac{d\boldsymbol{R}}{dt}\right)_{\text{inertial}} + \left(\frac{d\boldsymbol{r}}{dt}\right)_{\text{body}} + \boldsymbol{\omega} \times \boldsymbol{r} \\ \left(\frac{d^2\boldsymbol{\rho}}{dt^2}\right)_{\text{inertial}} &= \left(\frac{d^2\boldsymbol{R}}{dt^2}\right)_{\text{inertial}} + \left(\frac{d^2\boldsymbol{r}}{dt^2}\right)_{\text{body}} + \frac{d\boldsymbol{\omega}}{dt} \times \boldsymbol{r} + 2\,\boldsymbol{\omega} \times \left(\frac{d\boldsymbol{r}}{dt}\right)_{\text{body}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) \quad .\end{aligned} \tag{8.12}$$

Here, $\boldsymbol{\omega}$ is the angular velocity in the frame K or K'.

8.1.2 Motion on the surface of the earth

The earth both rotates about its axis and orbits the Sun. If we add the infinitesimal effects of the two rotations,

$$\begin{aligned} d\boldsymbol{r}_1 &= \boldsymbol{\omega}_1 \times \boldsymbol{r}\, dt \\ d\boldsymbol{r}_2 &= \boldsymbol{\omega}_2 \times (\boldsymbol{r} + d\boldsymbol{r}_1)\, dt \\ d\boldsymbol{r} &= d\boldsymbol{r}_1 + d\boldsymbol{r}_2 = (\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2)\, dt \times \boldsymbol{r} + \mathcal{O}\big((dt)^2\big) \quad . \end{aligned} \tag{8.13}$$

Thus, *infinitesimal rotations add.* Dividing by dt, this means that

$$\boldsymbol{\omega} = \sum_i \boldsymbol{\omega}_i \quad , \tag{8.14}$$

where the sum is over all the rotations. For the earth, $\boldsymbol{\omega} = \boldsymbol{\omega}_{\text{rot}} + \boldsymbol{\omega}_{\text{orb}}$.

- The rotation about earth's axis, $\boldsymbol{\omega}_{\text{rot}}$ has magnitude $\omega_{\text{rot}} = 2\pi/(1\,\text{day}) = 7.29 \times 10^{-5}\,\text{s}^{-1}$. The radius of the earth is $R_{\text{e}} = 6.37 \times 10^3$ km.
- The orbital rotation about the Sun, $\boldsymbol{\omega}_{\text{orb}}$ has magnitude $\omega_{\text{orb}} = 2\pi/(1\,\text{yr}) = 1.99 \times 10^{-7}\,\text{s}^{-1}$. The radius of the earth's orbit is $a_{\text{e}} = 1.50 \times 10^8$ km.

Thus, $\omega_{\text{rot}}/\omega_{\text{orb}} = T_{\text{orb}}/T_{\text{rot}} = 365.25$, which is of course the number of days (*i.e.* rotational periods) in a year (*i.e.* orbital period). There is also a very slow precession of the earth's axis of rotation, the period of which is about 25,000 years, which we will ignore. Note $\dot{\boldsymbol{\omega}} = 0$ for the earth. Thus, applying Newton's second law and then invoking eq. 8.12, we arrive at

$$m\left(\frac{d^2\boldsymbol{r}}{dt^2}\right)_{\text{earth}} = \boldsymbol{F}^{(\text{tot})} - m\left(\frac{d^2\boldsymbol{R}}{dt^2}\right)_{\text{Sun}} - 2m\,\boldsymbol{\omega} \times \left(\frac{d\boldsymbol{r}}{dt}\right)_{\text{earth}} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) \quad , \tag{8.15}$$

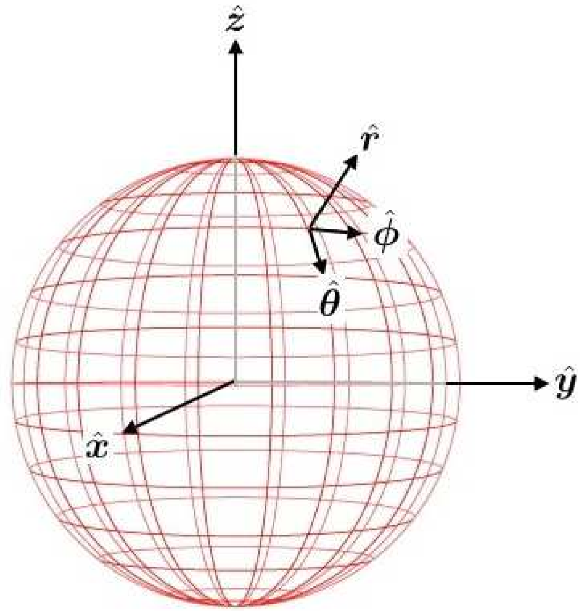


Figure 8.2: The locally orthonormal triad $\{\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}\}$.

where $\boldsymbol{\omega} = \boldsymbol{\omega}_{\rm rot} + \boldsymbol{\omega}_{\rm orb}$, and where $\ddot{\boldsymbol{R}}_{\rm Sun}$ is the acceleration of the center of the earth around the Sun, assuming the Sun-fixed frame to be inertial. The force $\boldsymbol{F}^{\rm (tot)}$ is the total force on the object, and arises from three parts: (i) gravitational pull of the Sun, (ii) gravitational pull of the earth, and (iii) other earthly forces, such as springs, rods, surfaces, electric fields, *etc.*

On the earth's surface, the ratio of the Sun's gravity to the earth's is

$$\frac{F_\odot}{F_{\rm e}} = \frac{GM_\odot m}{a_{\rm e}^2} \bigg/ \frac{GM_{\rm e} m}{R_{\rm e}^2} = \frac{M_\odot}{M_{\rm e}} \left(\frac{R_{\rm e}}{a_{\rm e}}\right)^2 \approx 6.02 \times 10^{-4} \quad . \tag{8.16}$$

In fact, it is clear that the Sun's field precisely cancels with the term $m\,\ddot{\boldsymbol{R}}_{\rm Sun}$ at the earth's center, leaving only gradient contributions of even lower order, *i.e.* multiplied by another factor of $R_{\rm e}/a_{\rm e} \approx 4.25 \times 10^{-5}$. Thus, to an excellent approximation, we may neglect the Sun entirely and write

$$\frac{d^2\boldsymbol{r}}{dt^2} = \frac{\boldsymbol{F}'}{m} + \boldsymbol{g} - 2\,\boldsymbol{\omega} \times \frac{d\boldsymbol{r}}{dt} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) \quad . \tag{8.17}$$

Note that we've dropped the 'earth' label here and henceforth. We define $\boldsymbol{g} = -GM_{\rm e}\,\hat{\boldsymbol{r}}/r^2$, the acceleration due to gravity; $\boldsymbol{F}'$ is the sum of all earthly forces other than the earth's gravity. The last two terms on the RHS are corrections to $m\ddot{\boldsymbol{r}} = \boldsymbol{F}$ due to the noninertial frame of the earth, and are recognized as the Coriolis and centrifugal acceleration terms, respectively.

8.2 Spherical Polar Coordinates

The locally orthonormal triad $\{\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}\}$ varies with position. In terms of the body-fixed triad $\{\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}\}$, we have

$$\begin{aligned}\hat{\boldsymbol{r}} &= \sin\theta\cos\phi\,\hat{\boldsymbol{x}} + \sin\theta\sin\phi\,\hat{\boldsymbol{y}} + \cos\theta\,\hat{\boldsymbol{z}} \\ \hat{\boldsymbol{\theta}} &= \cos\theta\cos\phi\,\hat{\boldsymbol{x}} + \cos\theta\sin\phi\,\hat{\boldsymbol{y}} - \sin\theta\,\hat{\boldsymbol{z}} \\ \hat{\boldsymbol{\phi}} &= -\sin\phi\,\hat{\boldsymbol{x}} + \cos\phi\,\hat{\boldsymbol{y}}\end{aligned} \tag{8.18}$$

where $\theta = \frac{\pi}{2} - \lambda$ is the *colatitude* (*i.e.* $\lambda \in \left[-\frac{\pi}{2}, +\frac{\pi}{2}\right]$ is the latitude). Inverting the relation between the triads $\{\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}\}$ and $\{\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}\}$, we obtain

$$\begin{aligned}\hat{\boldsymbol{x}} &= \sin\theta\,\cos\phi\,\hat{\boldsymbol{r}} + \cos\theta\,\cos\phi\,\hat{\boldsymbol{\theta}} - \sin\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\boldsymbol{y}} &= \sin\theta\,\sin\phi\,\hat{\boldsymbol{r}} + \cos\theta\,\sin\phi\,\hat{\boldsymbol{\theta}} + \cos\phi\,\hat{\boldsymbol{\phi}} \\ \hat{\boldsymbol{z}} &= \cos\theta\,\hat{\boldsymbol{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \quad .\end{aligned} \tag{8.19}$$

The differentials of these unit vectors are

$$\begin{aligned}d\hat{\boldsymbol{r}} &= \hat{\boldsymbol{\theta}}\,d\theta + \sin\theta\,\hat{\boldsymbol{\phi}}\,d\phi \\ d\hat{\boldsymbol{\theta}} &= -\hat{\boldsymbol{r}}\,d\theta + \cos\theta\,\hat{\boldsymbol{\phi}}\,d\phi \\ d\hat{\boldsymbol{\phi}} &= -\sin\theta\,\hat{\boldsymbol{r}}\,d\phi - \cos\theta\,\hat{\boldsymbol{\theta}}\,d\phi \quad .\end{aligned} \tag{8.20}$$

Thus,

$$\begin{aligned}\dot{\boldsymbol{r}} &= \frac{d}{dt}\big(r\,\hat{\boldsymbol{r}}\big) = \dot{r}\,\hat{\boldsymbol{r}} + r\,\dot{\hat{\boldsymbol{r}}} \\ &= \dot{r}\,\hat{\boldsymbol{r}} + r\dot{\theta}\,\hat{\boldsymbol{\theta}} + r\sin\theta\,\dot{\phi}\,\hat{\boldsymbol{\phi}} \quad .\end{aligned} \tag{8.21}$$

If we differentiate a second time, we find, after some tedious accounting,

$$\begin{aligned}\ddot{\boldsymbol{r}} = \big(\ddot{r} - r\,\dot{\theta}^2 - r\,\sin^2\!\theta\,\dot{\phi}^2\big)\,\hat{\boldsymbol{r}} &+ \big(2\,\dot{r}\,\dot{\theta} + r\,\ddot{\theta} - r\,\sin\theta\cos\theta\,\dot{\phi}^2\big)\,\hat{\boldsymbol{\theta}} \\ &+ \big(2\,\dot{r}\,\dot{\phi}\,\sin\theta + 2\,r\,\dot{\theta}\,\dot{\phi}\,\cos\theta + r\,\sin\theta\,\ddot{\phi}\,\big)\,\hat{\boldsymbol{\phi}} \quad .\end{aligned} \tag{8.22}$$

8.3 Centrifugal Force

One major distinction between the Coriolis and centrifugal forces is that the Coriolis force acts only on moving particles, whereas the centrifugal force is present even when $\dot{\boldsymbol{r}} = 0$. Thus, the equation for stationary equilibrium on the earth's surface is

$$m\boldsymbol{g} + \boldsymbol{F}' - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) = 0 \quad , \tag{8.23}$$

involves the centrifugal term. We can write this as $\boldsymbol{F}' + m\widetilde{\boldsymbol{g}} = 0$, where

$$\begin{aligned}\widetilde{\boldsymbol{g}} &= -\frac{GM_{\rm e}\,\hat{\boldsymbol{r}}}{r^2} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) \\ &= -\big(g_0 - \omega^2 R_{\rm e}\,\sin^2\theta\big)\,\hat{\boldsymbol{r}} + \omega^2\,R_{\rm e}\,\sin\theta\cos\theta\,\hat{\boldsymbol{\theta}} \quad ,\end{aligned} \tag{8.24}$$

where $g_0 = GM_e/R_e^2 = 980\,\text{cm/s}^2$. Thus, on the equator, where $\theta = \frac{\pi}{2}$, we have $\tilde{\boldsymbol{g}} = -\big(g_0 - \omega^2 R_e\big)\,\hat{\boldsymbol{r}}$, with $\omega^2 R_e \approx 3.39\,\text{cm/s}^2$, a small but significant correction. You therefore weigh less on the equator. Note also that $\tilde{\boldsymbol{g}}$ has a component along $\hat{\boldsymbol{\theta}}$. This means that a plumb bob suspended from a general point above the earth's surface won't point exactly toward the earth's center. Moreover, if the earth were replaced by an equivalent mass of fluid, the fluid would rearrange itself so as to make its surface locally perpendicular to $\tilde{\boldsymbol{g}}$. Indeed, the earth (and Sun) do exhibit quadrupolar distortions in their mass distributions – both are oblate spheroids. In fact, the observed difference $\tilde{g}(\theta = 0) - \tilde{g}(\theta = \frac{\pi}{2}) \approx 5.2\,\text{cm/s}^2$, which is 53% greater than the naïvely expected value of $3.39\,\text{cm/s}^2$. The earth's oblateness enhances the effect.

8.3.1 Rotating cylinder of fluid

Consider a cylinder filled with a liquid, rotating with angular frequency ω about its symmetry axis $\hat{\boldsymbol{z}}$. In steady state, the fluid is stationary in the rotating frame, and we may write, for any given element of fluid

$$0 = \boldsymbol{f}' + \boldsymbol{g} - \omega^2\,\hat{\boldsymbol{z}} \times (\hat{\boldsymbol{z}} \times \boldsymbol{r}) \quad , \tag{8.25}$$

where $\boldsymbol{f}'$ is the force per unit mass on the fluid element. Now consider a fluid element on the surface. Since there is no static friction to the fluid, any component of $\boldsymbol{f}'$ parallel to the fluid's surface will cause the fluid to flow in that direction. This contradicts the steady state assumption. Therefore, we must have $\boldsymbol{f}' = f'\,\hat{\boldsymbol{n}}$, where $\hat{\boldsymbol{n}}$ is the local unit normal to the fluid surface. We write the equation for the fluid's surface as $z = z(\rho)$. Thus, with $\boldsymbol{r} = \rho\,\hat{\boldsymbol{\rho}} + z(\rho)\,\hat{\boldsymbol{z}}$, Newton's second law yields

$$f'\,\hat{\boldsymbol{n}} = g\,\hat{\boldsymbol{z}} - \omega^2\,\rho\,\hat{\boldsymbol{\rho}} \quad , \tag{8.26}$$

where $\boldsymbol{g} = -g\,\hat{\boldsymbol{z}}$ is assumed. From this, we conclude that the unit normal to the fluid surface and the force per unit mass are given by

$$\hat{\boldsymbol{n}}(\rho) = \frac{g\,\hat{\boldsymbol{z}} - \omega^2\,\rho\,\hat{\boldsymbol{\rho}}}{\sqrt{g^2 + \omega^4\,\rho^2}} \qquad , \qquad f'(\rho) = \sqrt{g^2 + \omega^4 \rho^2} \quad . \tag{8.27}$$

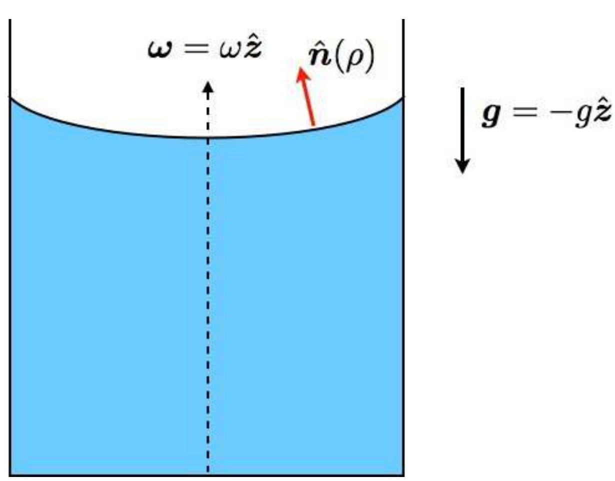


Figure 8.3: A rotating cylinder of fluid.

Now suppose $\boldsymbol{r}(\rho,\phi) = \rho\,\hat{\boldsymbol{\rho}} + z(\rho)\,\hat{\boldsymbol{z}}$ is a point on the surface of the fluid. We have that

$$d\boldsymbol{r} = \hat{\boldsymbol{\rho}}\,d\rho + z'(\rho)\,\hat{\boldsymbol{z}}\,d\rho + \rho\,\hat{\boldsymbol{\phi}}\,d\phi \quad , \tag{8.28}$$

where $z' = dz/d\rho$, and where we have used $d\hat{\boldsymbol{\rho}} = \hat{\boldsymbol{\phi}}\,d\phi$, which follows from the first of eqn. 8.20 after setting $\theta = \frac{\pi}{2}$. Now $d\boldsymbol{r}$ must lie along the surface, therefore $\hat{\boldsymbol{n}} \cdot d\boldsymbol{r} = 0$, which says

$$g\,\frac{dz}{d\rho} = \omega^2\,\rho \quad . \tag{8.29}$$

Integrating this equation, we obtain the shape of the surface:

$$z(\rho) = z_0 + \frac{\omega^2\rho^2}{2g} \quad . \tag{8.30}$$

8.4 The Coriolis Force

8.4.1 Projectile motion

The Coriolis force is given by $\boldsymbol{F}_{\rm Cor} = -2m\,\boldsymbol{\omega} \times \dot{\boldsymbol{r}}$. According to (8.17), the acceleration of a free particle ($\boldsymbol{F}' = 0$) isn't along $\widetilde{\boldsymbol{g}}$ – an orthogonal component is generated by the Coriolis force. To actually solve the coupled equations of motion is difficult because the unit vectors $\{\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}\}$ change with position, and hence with time. The following standard problem highlights some of the effects of the Coriolis and centrifugal forces.

PROBLEM: A cannonball is dropped from the top of a tower of height h located at a northerly latitude of λ. Assuming the cannonball is initially at rest with respect to the tower, and neglecting air resistance, calculate its deflection (magnitude and direction) due to (a) centrifugal and (b) Coriolis forces by the time it hits the ground. Evaluate for the case $h = 100$ m, $\lambda = 45°$. The radius of the earth is $R_{\rm e} = 6.4 \times 10^6$ m.

SOLUTION: The equation of motion for a particle near the earth's surface is

$$\ddot{\boldsymbol{r}} = -2\,\boldsymbol{\omega} \times \dot{\boldsymbol{r}} - g_0\,\hat{\boldsymbol{r}} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) \quad , \tag{8.31}$$

where $\boldsymbol{\omega} = \omega\,\hat{\boldsymbol{z}}$, with $\omega = 2\pi/(24\,{\rm hrs}) = 7.3 \times 10^{-5}\,{\rm rad/s}$. Here, $g_0 = GM_{\rm e}/R_{\rm e}^2 = 980\,{\rm cm/s^2}$. We use a locally orthonormal coordinate system $\{\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}\}$ and write

$$\boldsymbol{r} = x\,\hat{\boldsymbol{\theta}} + y\,\hat{\boldsymbol{\phi}} + (R_{\rm e} + z)\,\hat{\boldsymbol{r}} \quad , \tag{8.32}$$

where $R_{\rm e} = 6.4 \times 10^6$ m is the radius of the earth. Expressing $\hat{\boldsymbol{z}}$ in terms of our chosen orthonormal triad,

$$\hat{\boldsymbol{z}} = \cos\theta\,\hat{\boldsymbol{r}} - \sin\theta\,\hat{\boldsymbol{\theta}} \quad , \tag{8.33}$$

where $\theta = \frac{\pi}{2} - \lambda$ is the polar angle, or 'colatitude'. Since the height of the tower and the deflections are all very small on the scale of $R_{\rm e}$, we may regard the orthonormal triad as fixed and time-independent, although, in general, these unit vectors change as a function of $\boldsymbol{r}$. Thus, we have $\dot{\boldsymbol{r}} \simeq \dot{x}\,\hat{\boldsymbol{\theta}} + \dot{y}\,\hat{\boldsymbol{\phi}} + \dot{z}\,\hat{\boldsymbol{r}}$, and we find

$$\begin{aligned} \hat{\boldsymbol{z}} \times \dot{\boldsymbol{r}} &= (\cos\theta\,\hat{\boldsymbol{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}) \times (\dot{x}\,\hat{\boldsymbol{\theta}} + \dot{y}\,\hat{\boldsymbol{\phi}} + \dot{z}\,\hat{\boldsymbol{r}}) \\ &= -\dot{y}\,\cos\theta\,\hat{\boldsymbol{\theta}} + (\dot{x}\,\cos\theta + \dot{z}\,\sin\theta)\,\hat{\boldsymbol{\phi}} - \dot{y}\sin\theta\,\hat{\boldsymbol{r}} \end{aligned} \tag{8.34}$$

and

$$\begin{aligned} \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}) &= \omega^2 (\cos\theta\,\hat{\boldsymbol{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}) \times \Big((\cos\theta\,\hat{\boldsymbol{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}) \times (R_{\rm e}\,\hat{\boldsymbol{r}} + \overbrace{x\,\hat{\boldsymbol{\theta}} + y\,\hat{\boldsymbol{\phi}} + z\,\hat{\boldsymbol{r}}}^{\text{negligible}}) \Big) \\ &\approx \omega^2 (\cos\theta\,\hat{\boldsymbol{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}) \times R_{\rm e}\sin\theta\,\hat{\boldsymbol{\phi}} \\ &= -\omega^2 R_{\rm e} \sin\theta\cos\theta\,\hat{\boldsymbol{\theta}} - \omega^2 R_{\rm e}\sin^2\!\theta\,\hat{\boldsymbol{r}} \quad . \end{aligned} \tag{8.35}$$

Note that the distances x, y, and z are all extremely small in magnitude compared with $R_{\rm e}$.

The equations of motion, written in components, are then

$$\begin{aligned} \dot{v}_x &= g_1 \sin\theta\cos\theta + 2\omega\cos\theta\,v_y \\ \dot{v}_y &= -2\omega\cos\theta\,v_x - 2\omega\sin\theta\,v_z \\ \dot{v}_z &= -g_0 + g_1\sin^2\!\theta + 2\omega\sin\theta\,v_y \quad , \end{aligned} \tag{8.36}$$

with $g_1 \equiv \omega^2 R_{\rm e}$. While these (inhomogeneous) equations are linear, they also are coupled, so an exact analytical solution is not trivial to obtain (but see below). Fortunately, the deflections are small, so we can solve this perturbatively. To do so, let us write $\boldsymbol{v}(t)$ as a power series in t. For each component, we write

$$v_\alpha(t) = \sum_{n=0}^{\infty} v_{\alpha,n}\,t^n \quad , \tag{8.37}$$

with $v_{\alpha,0} = v_\alpha(t=0) \equiv v_\alpha^0$. Eqns. 8.36 then may be written as the coupled hierarchy

$$\begin{aligned} n v_{x,n} &= g_1 \sin\theta\cos\theta\,\delta_{n,1} + 2\omega\cos\theta\,v_{y,n-1} \\ n v_{y,n} &= -2\omega\cos\theta\,v_{x,n-1} - 2\omega\sin\theta\,v_{z,n-1} \\ n v_{z,n} &= -(g_0 - g_1\sin^2\!\theta)\,\delta_{n,1} + 2\omega\sin\theta\,v_{y,n-1} \quad . \end{aligned} \tag{8.38}$$

Integrating $\boldsymbol{v}(t)$, we obtain the displacements,

$$x_\alpha(t) = x_\alpha^0 + \sum_{n=0}^{\infty} \frac{v_{\alpha,n}}{n+1}\,t^{n+1} \quad . \tag{8.39}$$

Now let's roll up our sleeves and solve for the coefficients $v_{\alpha,n}$ for $n = 0, 1, 2$. This will give us the displacements up to terms of order t^3. For $n = 0$ we already have $v_{\alpha,0} = v_\alpha^0$. For $n = 1$, we use Eqns. 8.38 with $n = 1$ to obtain

$$\begin{aligned} v_{x,1} &= 2\omega\cos\theta\,v_y^0 + g_1\sin\theta\cos\theta \\ v_{y,1} &= -2\omega\cos\theta\,v_x^0 - 2\omega\sin\theta\,v_z^0 \\ v_{z,1} &= 2\omega\sin\theta\,v_y^0 - g_0 + g_1\sin^2\!\theta \quad . \end{aligned} \tag{8.40}$$

Finally, at level $n = 2$, we have

$$\begin{aligned} v_{x,2} &= \omega \cos\theta \, v_{y,1} = -2\omega^2 \cos\theta \, (\cos\theta \, v_x^0 + \sin\theta \, v_z^0) \\ v_{y,2} &= -2\omega \cos\theta \, v_{x,1} - 2\omega \sin\theta \, v_{z,1} = -2\omega^2 v_y^0 + \omega \sin\theta \, (g_0 - g_1) \\ v_{z,2} &= \omega \sin\theta \, v_{y,1} = -2\omega^2 \sin\theta \, (\cos\theta \, v_x^0 + \sin\theta \, v_z^0) \quad . \end{aligned} \tag{8.41}$$

Thus, the displacements are given by

$$\begin{aligned} x(t) &= x(0) + v_x^0 \, t + \tfrac{1}{2}\big(2\omega \cos\theta \, v_y^0 + g_1 \sin\theta \cos\theta\big) \, t^2 - \tfrac{2}{3} \, \omega^2 \cos\theta \, (\cos\theta \, v_x^0 + \sin\theta \, v_z^0) \, t^3 + \mathcal{O}(t^4) \\ y(t) &= y(0) + v_y^0 \, t - \omega \, (\cos\theta \, v_x^0 + \sin\theta \, v_z^0) \, t^2 - \tfrac{2}{3} \, \omega^2 v_y^0 \, t^3 + \tfrac{1}{3} \, \omega \sin\theta \, (g_0 - g_1) \, t^3 + \mathcal{O}(t^4) \\ z(t) &= z(0) + v_z^0 \, t + \tfrac{1}{2}\big(2\omega \sin\theta \, v_y^0 - g_0 + g_1 \sin^2\!\theta\big) \, t^2 - \tfrac{2}{3} \, \omega^2 \sin\theta \, (\cos\theta \, v_x^0 + \sin\theta \, v_z^0) \, t^3 + \mathcal{O}(t^4) \quad . \end{aligned} \tag{8.42}$$

When dropped from rest, with $x(0) = y(0) = 0$ and $z(0) = h_0$, we have

$$\begin{aligned} x(t) &= \tfrac{1}{2} g_1 \sin\theta \cos\theta \, t^2 + \mathcal{O}(t^4) \\ y(t) &= \tfrac{1}{3} \, \omega \sin\theta \, (g_0 - g_1) \, t^3 + \mathcal{O}(t^4) \\ z(t) &= h_0 - \tfrac{1}{2} \, (g_0 - g_1 \sin^2\!\theta) \, t^2 + \mathcal{O}(t^4) \quad . \end{aligned} \tag{8.43}$$

Recall $g_1 = \omega^2 R_{\rm e}$, so if we neglect the rotation of the earth and set $\omega = 0$, we have $\omega = g_1 = 0$, and $z(t) = h_0 - \frac{1}{2} g_0 t^2$ with $x(t) = y(t) = 0$. This is the familiar high school physics result. As we see, in the noninertial reference frame of the rotating earth, there are deflections along $\hat{\boldsymbol{\theta}}$ given by $x(t)$, along $\hat{\boldsymbol{\phi}}$ given by $y(t)$, and also a correction $\Delta z(t) = \frac{1}{2} g_1 \sin^2\theta \, t^2 + \mathcal{O}(t^4)$ to the motion along $\hat{\boldsymbol{r}}$. To find the deflection of an object dropped from a height h_0, solve $z(t^*) = 0$ to obtain $t^* = \sqrt{2h/(g_0 - g_1 \sin^2\!\theta)}$ for the drop time, and substitute. For $h_0 = 100\,{\rm m}$ and $\lambda = \frac{\pi}{2}$, find $\delta x(t^*) = 17\,{\rm cm}$ south (centrifugal) and $\delta y(t^*) = 1.6\,{\rm cm}$ east (Coriolis). Note that the centrifugal term dominates the deflection in this example. Why is the Coriolis deflection always to the east? The earth rotates eastward, and an object starting from rest in the earth's frame has initial angular velocity equal to that of the earth. To conserve angular momentum, the object must speed up as it falls.

Exact solution for velocities

In fact, an exact solution to (8.36) is readily obtained, via the following analysis. The equations of motion may be written $\dot{\boldsymbol{v}} = 2i\omega \mathcal{J} \boldsymbol{v} + \boldsymbol{b}$, or

$$\begin{pmatrix} \dot v_x \\ \dot v_y \\ \dot v_x \end{pmatrix} = 2i\,\omega \overbrace{\begin{pmatrix} 0 & -i\cos\theta & 0 \\ i\cos\theta & 0 & i\sin\theta \\ 0 & -i\sin\theta & 0 \end{pmatrix}}^{\mathcal{J}} \begin{pmatrix} v_x \\ v_y \\ v_x \end{pmatrix} + \overbrace{\begin{pmatrix} g_1 \sin\theta\cos\theta \\ 0 \\ -g_0 + g_1 \sin^2\!\theta \end{pmatrix}}^{\boldsymbol{b}} \quad . \tag{8.44}$$

Note that $\mathcal{J}^\dagger = \mathcal{J}$, *i.e.* $\mathcal{J}$ is a Hermitian matrix. The formal solution is

$$\boldsymbol{v}(t) = e^{2i\omega\mathcal{J}t} \, \boldsymbol{v}(0) + \left(\frac{e^{2i\omega\mathcal{J}t} - 1}{2i\omega}\right) \mathcal{J}^{-1} \, \boldsymbol{b} \quad . \tag{8.45}$$

When working with matrices, it is convenient to work in an eigenbasis. The characteristic polynomial for $\mathcal{J}$ is $P(\lambda) = \det(\lambda \cdot 1 - \mathcal{J}) = \lambda(\lambda^2 - 1)$, hence the eigenvalues are $\lambda_1 = 0$, $\lambda_2 = +1$, and $\lambda_3 = -1$. The corresponding eigenvectors are easily found to be

$$\boldsymbol{\psi}_1 = \begin{pmatrix} \sin\theta \\ 0 \\ -\cos\theta \end{pmatrix} \quad , \quad \boldsymbol{\psi}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos\theta \\ i \\ \sin\theta \end{pmatrix} \quad , \quad \boldsymbol{\psi}_3 = \frac{1}{\sqrt{2}} \begin{pmatrix} \cos\theta \\ -i \\ \sin\theta \end{pmatrix} \quad . \tag{8.46}$$

Note that $\boldsymbol{\psi}_a^\dagger \cdot \boldsymbol{\psi}_{a'} = \delta_{aa'}$.

Expanding $\boldsymbol{v}$ and $\boldsymbol{b}$ in this eigenbasis, we have $\dot{u}_a = 2i\omega\lambda_a u_a + b_a$, where $u_a = \psi^*_{ia}\, v_i$ and $b_a = \psi^*_{ia}\, b_i$. The solution is

$$u_a(t) = u_a(0)\, e^{2i\lambda_a \omega t} + \left(\frac{e^{2i\lambda_a\omega t} - 1}{2i\lambda_a \omega} \right) b_a \quad . \tag{8.47}$$

Since the eigenvectors of $\mathcal{J}$ are orthonormal, $u_a = \psi^*_{ia}\, v_i$ entails $v_i = \psi_{ia} u_a$, hence

$$v_i(t) = \sum_j \left(\sum_a \psi_{ia}\, e^{2i\lambda_a\omega t}\, \psi^*_{ja} \right) v_j(0) + \sum_j \left(\sum_a \psi_{ia} \left(\frac{e^{2i\lambda_a\omega t} - 1}{2i\lambda_a\omega} \right) \psi^*_{ja} \right) b_j \quad . \tag{8.48}$$

Doing the requisite matrix multiplications, and assuming $\boldsymbol{v}(0) = 0$, we obtain

$$\begin{pmatrix} v_x(t) \\ v_y(t) \\ v_z(t) \end{pmatrix} = \begin{pmatrix} t\sin^2\theta + \frac{\sin 2\omega t}{2\omega}\cos^2\theta & \frac{\sin^2\omega t}{\omega}\cos\theta & -\frac{1}{2}t\sin 2\theta + \frac{\sin 2\omega t}{4\omega}\sin 2\theta \\ -\frac{\sin^2\omega t}{\omega}\cos\theta & \frac{\sin 2\omega t}{2\omega} & -\frac{\sin^2\omega t}{\omega}\sin\theta \\ -\frac{1}{2}t\sin 2\theta + \frac{\sin 2\omega t}{4\omega}\sin 2\theta & \frac{\sin^2\omega t}{\omega}\sin\theta & t\cos^2\theta + \frac{\sin 2\omega t}{2\omega}\sin^2\theta \end{pmatrix} \begin{pmatrix} g_1 \sin\theta\cos\theta \\ 0 \\ -g_0 + g_1\sin^2\theta \end{pmatrix} \quad , \tag{8.49}$$

which says

$$\begin{aligned} v_x(t) &= \left(\tfrac{\sin 2\omega t}{2\omega t} - 1 \right) g_0 t \sin\theta\cos\theta + \tfrac{\sin 2\omega t}{2\omega t}\, g_1 t \sin\theta\cos\theta \\ v_y(t) &= \tfrac{\sin^2\omega t}{\omega t}\,(g_0 - g_1)\, t\sin\theta \\ v_z(t) &= -\left(\cos^2\theta + \tfrac{\sin 2\omega t}{2\omega t}\sin^2\theta \right) g_0 t + \tfrac{\sin^2\omega t}{2\omega t}\, g_1 t \sin^2\theta \quad . \end{aligned} \tag{8.50}$$

One can check that by expanding in a power series in t we recover the results of the previous section.

8.4.2 Foucault's pendulum

A pendulum swinging over one of the poles moves in a fixed inertial plane while the earth rotates underneath. Relative to the earth, the plane of motion of the pendulum makes one revolution every day. What happens at a general latitude? Assume the pendulum is located at colatitude θ and longitude ϕ. Assuming the length scale of the pendulum is small compared to R_e, we can regard the local triad $\{\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}, \hat{\boldsymbol{r}}\}$ as fixed. The situation is depicted in fig. 8.4. We write

$$\boldsymbol{r} = x\,\hat{\boldsymbol{\theta}} + y\,\hat{\boldsymbol{\phi}} + z\,\hat{\boldsymbol{r}} \quad , \tag{8.51}$$

with

$$x = \ell\,\sin\psi\,\cos\alpha \quad , \quad y = \ell\,\sin\psi\,\sin\alpha \quad , \quad z = \ell\,(1 - \cos\psi) \quad . \tag{8.52}$$

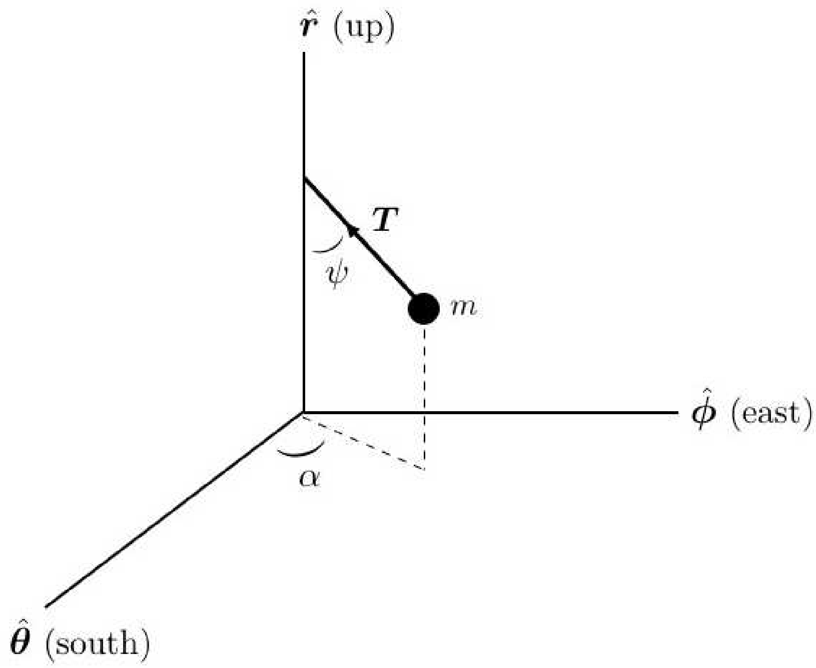


Figure 8.4: Foucault's pendulum.

In our analysis we will ignore centrifugal effects, which are of higher order in ω, and we take $\boldsymbol{g} = -g\,\hat{\boldsymbol{r}}$. We also idealize the pendulum, and consider the suspension rod to be of negligible mass.

The total force on the mass m is due to gravity and tension:

$$\begin{aligned} \boldsymbol{F} &= m\boldsymbol{g} + \boldsymbol{T} \\ &= \big(-T\sin\psi\,\cos\alpha,\ -T\sin\psi\,\sin\alpha,\ T\cos\psi - mg\big) \\ &= \big(-Tx/\ell,\ -Ty/\ell,\ T - Mg - Tz/\ell\big) \quad . \end{aligned} \tag{8.53}$$

The Coriolis term is

$$\begin{aligned} \boldsymbol{F}_{\rm Cor} &= -2m\,\boldsymbol{\omega}\times\dot{\boldsymbol{r}} \\ &= -2m\,\omega\big(\cos\theta\,\hat{\boldsymbol{r}} - \sin\theta\,\hat{\boldsymbol{\theta}}\big)\times\big(\dot{x}\,\hat{\boldsymbol{\theta}} + \dot{y}\,\hat{\boldsymbol{\phi}} + \dot{z}\,\hat{\boldsymbol{r}}\big) \\ &= 2m\omega\,\big(\dot{y}\cos\theta,\ -\dot{x}\cos\theta - \dot{z}\sin\theta,\ \dot{y}\sin\theta\big) \quad . \end{aligned} \tag{8.54}$$

The equations of motion are $m\ddot{\boldsymbol{r}} = \boldsymbol{F} + \boldsymbol{F}_{\rm Cor}$:

$$\begin{aligned} m\ddot{x} &= -Tx/\ell + 2m\omega\cos\theta\,\dot{y} \\ m\ddot{y} &= -Ty/\ell - 2m\omega\cos\theta\,\dot{x} - 2m\omega\sin\theta\,\dot{z} \\ m\ddot{z} &= T - mg - Tz/\ell + 2m\omega\sin\theta\,\dot{y} \quad . \end{aligned} \tag{8.55}$$

These three equations are to be solved for the three unknowns x, y, and T. Note that

$$x^2 + y^2 + (\ell - z)^2 = \ell^2 \quad , \tag{8.56}$$

so $z = z(x, y)$ is not an independent degree of freedom. This equation may be recast in the form $z = (x^2+y^2+z^2)/2\ell$ which shows that if x and y are both small, then z is at least of second order in smallness. Therefore, we will approximate $z \simeq 0$, in which case $\dot{z}$ may be neglected from the second equation of motion. The third equation is used to solve for T:

$$T \simeq mg - 2m\omega \sin\theta\, \dot{y} \quad . \tag{8.57}$$

Adding the first plus i times the second then gives the complexified equation

$$\begin{aligned} \ddot{\xi} &= -\frac{T}{m\ell}\,\xi - 2i\omega\cos\theta\,\dot{\xi} \\ &\approx -\omega_0^2\,\xi - 2i\omega\cos\theta\,\dot{\xi} \end{aligned} \tag{8.58}$$

where $\xi \equiv x + iy$, and where $\omega_0 = \sqrt{g/\ell}$. Note that we have approximated $T \approx mg$ in deriving the second line.

It is now a trivial matter to solve the homogeneous linear ODE of eq. 8.58. Writing

$$\xi = \xi_0\, e^{-i\Omega t} \tag{8.59}$$

and plugging in to find Ω, we obtain

$$\Omega^2 - 2\omega_\perp \Omega - \omega_0^2 = 0 \quad , \tag{8.60}$$

with $\omega_\perp \equiv \omega\cos\theta$. The roots are

$$\Omega_\pm = \omega_\perp \pm \sqrt{\omega_0^2 + \omega_\perp^2} \quad , \tag{8.61}$$

hence the most general solution is

$$\xi(t) = A_+\, e^{-i\Omega_+ t} + A_-\, e^{-i\Omega_- t} \quad . \tag{8.62}$$

Finally, if we take as initial conditions $x(0) = a$, $y(0) = 0$, $\dot{x}(0) = 0$, and $\dot{y}(0) = 0$, we obtain

$$\begin{aligned} x(t) &= \left(\frac{a}{\nu}\right) \cdot \Big\{ \omega_\perp\, \sin(\omega_\perp t)\, \sin(\nu t) + \nu\, \cos(\omega_\perp t)\, \cos(\nu t) \Big\} \\ y(t) &= \left(\frac{a}{\nu}\right) \cdot \Big\{ \omega_\perp\, \cos(\omega_\perp t)\, \sin(\nu t) - \nu\, \sin(\omega_\perp t)\, \cos(\nu t) \Big\} \quad , \end{aligned} \tag{8.63}$$

with $\nu = \sqrt{\omega_0^2 + \omega_\perp^2}$. Typically $\omega_0 \gg \omega_\perp$, since $\omega = 7.3 \times 10^{-5}\,\mathrm{s}^{-1}$. In the limit $\omega_\perp \ll \omega_0$, then, we have $\nu \approx \omega_0$ and

$$x(t) \simeq a\, \cos(\omega_\perp t)\, \cos(\omega_0 t) \quad , \quad y(t) \simeq -a\, \sin(\omega_\perp t)\, \cos(\omega_0 t) \quad , \tag{8.64}$$

and the plane of motion rotates with angular frequency $-\omega_\perp$, *i.e.* the period is $|\sec\theta\,|$ days. Viewed from above, the rotation is clockwise in the northern hemisphere, where $\cos\theta > 0$ and counterclockwise in the southern hemisphere, where $\cos\theta < 0$.

Chapter 9

Rigid Body Motion and Rotational Dynamics

9.1 Rigid Bodies

A rigid body consists of a group of particles whose separations are all fixed in magnitude. Six independent coordinates are required to completely specify the position and orientation of a rigid body. For example, the location of the first particle is specified by three coordinates. A second particle requires only two coordinates since the distance to the first is fixed. Finally, a third particle requires only one coordinate, since its distance to the first two particles is fixed (think about the intersection of two spheres). The positions of all the remaining particles are then determined by their distances from the first three. Usually, one takes these six coordinates to be the center-of-mass position $\boldsymbol{R} = (X, Y, Z)$ and three angles specifying the orientation of the body (*e.g.* the Euler angles).

As derived previously, the equations of motion are

$$\begin{aligned} \boldsymbol{P} &= \sum_i m_i\,\dot{\boldsymbol{r}}_i \quad , \quad \dot{\boldsymbol{P}} = \boldsymbol{F}^{(\text{ext})} \\ \boldsymbol{L} &= \sum_i m_i\,\boldsymbol{r}_i \times \dot{\boldsymbol{r}}_i \quad , \quad \dot{\boldsymbol{L}} = \boldsymbol{N}^{(\text{ext})} \quad . \end{aligned} \tag{9.1}$$

These equations determine the motion of a rigid body.

9.1.1 Examples of rigid bodies

Our first example of a rigid body is of a wheel rolling with constant angular velocity $\dot{\phi} = \omega$, and without slipping, This is shown in fig. 9.1. The no-slip condition is $dx = R\,d\phi$, so $\dot{x} = V_{\text{CM}} = R\omega$. The velocity of a point within the wheel is

$$\boldsymbol{v} = \boldsymbol{V}_{\text{CM}} + \boldsymbol{\omega} \times \boldsymbol{r} \quad , \tag{9.2}$$

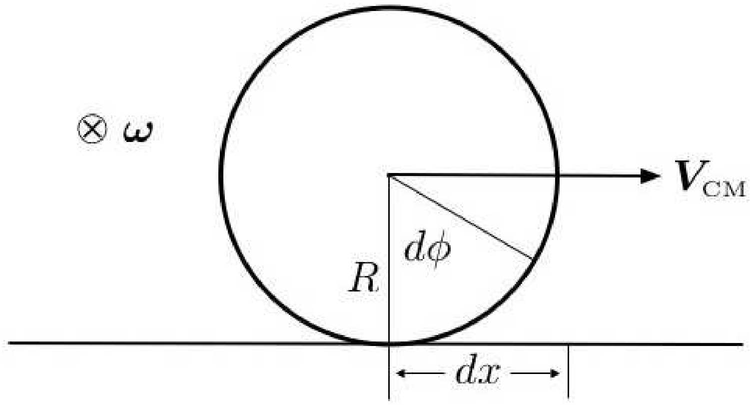


Figure 9.1: A wheel rolling to the right without slipping.

where $\boldsymbol{r}$ is measured from the center of the disk. The velocity of a point on the surface is then given by $\boldsymbol{v} = \omega R\big(\hat{\boldsymbol{x}} + \hat{\boldsymbol{\omega}} \times \hat{\boldsymbol{r}}\big)$.

As a second example, consider a bicycle wheel of mass M and radius R affixed to a light, firm rod of length d, as shown in fig. 9.2. Assuming $\boldsymbol{L}$ lies in the (x, y) plane, one computes the gravitational torque $\boldsymbol{N} = \boldsymbol{r} \times (M\boldsymbol{g}) = Mgd\,\hat{\boldsymbol{\phi}}$. The angular momentum vector then rotates with angular frequency $\dot{\phi}$. Thus,

$$d\phi = \frac{dL}{L} \quad \Longrightarrow \quad \dot{\phi} = \frac{Mgd}{L} \quad . \tag{9.3}$$

But $L = MR^2\omega$, so the precession frequency is

$$\omega_{\rm p} = \dot{\phi} = \frac{gd}{\omega R^2} \quad . \tag{9.4}$$

For $R = d = 30\,{\rm cm}$ and $\omega/2\pi = 200\,{\rm rpm}$, find $\omega_{\rm p}/2\pi \approx 15\,{\rm rpm}$. Note that we have here ignored the contribution to $\boldsymbol{L}$ from the precession itself, which lies along $\hat{\boldsymbol{z}}$, resulting in the *nutation* of the wheel. This is justified if $L_{\rm p}/L = (d^2/R^2) \cdot (\omega_{\rm p}/\omega) \ll 1$.

9.2 The Inertia Tensor

Suppose first that a point within the body itself is fixed. This eliminates the translational degrees of freedom from consideration. We now have

$$\left(\frac{d\boldsymbol{r}}{dt}\right)_{\rm inertial} = \boldsymbol{\omega} \times \boldsymbol{r} \quad , \tag{9.5}$$

since $\dot{\boldsymbol{r}}_{\rm body} = 0$. The kinetic energy is then

$$\begin{aligned} T &= \tfrac{1}{2} \sum_i m_i \left(\frac{d\boldsymbol{r}_i}{dt}\right)^2_{\rm inertial} = \tfrac{1}{2} \sum_i m_i \, (\boldsymbol{\omega} \times \boldsymbol{r}_i) \cdot (\boldsymbol{\omega} \times \boldsymbol{r}_i) \\ &= \tfrac{1}{2} \sum_i m_i \Big[\omega^2\, \boldsymbol{r}_i^2 - (\boldsymbol{\omega} \cdot \boldsymbol{r}_i)^2\Big] \equiv \tfrac{1}{2} I_{\alpha\beta}\, \omega_\alpha\, \omega_\beta \quad , \end{aligned} \tag{9.6}$$

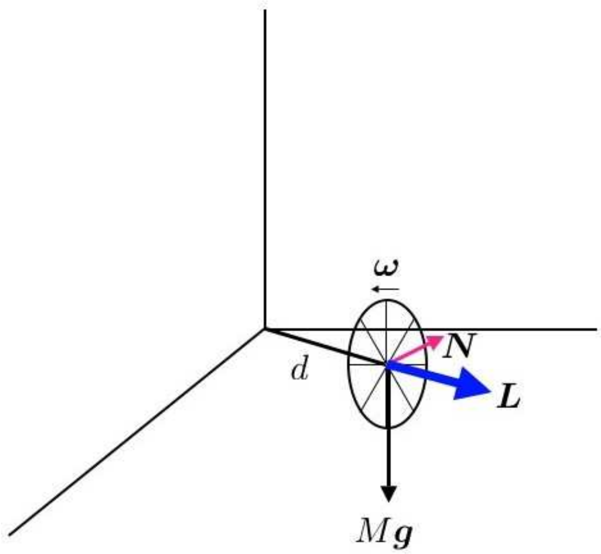


Figure 9.2: Precession of a spinning bicycle wheel.

where ω_α is the component of $\boldsymbol{\omega}$ along the body-fixed axis $\boldsymbol{e}_\alpha$. The quantity $I_{\alpha\beta}$ is the *inertia tensor,*

$$
\begin{aligned}
I_{\alpha\beta} &= \sum_i m_i \Big(\boldsymbol{r}_i^2 \, \delta_{\alpha\beta} - r_{i,\alpha} \, r_{i,\beta} \Big) \\
&= \int \! d^d r \, \varrho(\boldsymbol{r}) \, \Big(\boldsymbol{r}^2 \, \delta_{\alpha\beta} - r_\alpha \, r_\beta \Big) \quad \text{(continuous media)} \quad .
\end{aligned}
\tag{9.7}
$$

The angular momentum is

$$
\begin{aligned}
\boldsymbol{L} &= \sum_i m_i \, \boldsymbol{r}_i \times \left(\frac{d\boldsymbol{r}_i}{dt} \right)_{\text{inertial}} \\
&= \sum_i m_i \, \boldsymbol{r}_i \times (\boldsymbol{\omega} \times \boldsymbol{r}_i) = I_{\alpha\beta} \, \omega_\beta \quad .
\end{aligned}
\tag{9.8}
$$

The diagonal elements of $I_{\alpha\beta}$ are called the *moments of inertia*, while the off-diagonal elements are called the *products of inertia.*

9.2.1 Coordinate transformations

Consider the basis transformation

$$
\hat{\mathbf{e}}'_\alpha = \mathcal{R}_{\alpha\beta} \, \hat{\mathbf{e}}_\beta \quad .
\tag{9.9}
$$

We demand $\hat{\mathbf{e}}'_\alpha \cdot \hat{\mathbf{e}}'_\beta = \delta_{\alpha\beta}$, which means $\mathcal{R} \in \mathrm{O}(d)$ is an orthogonal matrix, *i.e.* $\mathcal{R}^{\mathrm{t}} = \mathcal{R}^{-1}$. Thus the inverse transformation is $\boldsymbol{e}_\alpha = \mathcal{R}^{\mathrm{t}}_{\alpha\beta} \boldsymbol{e}'_\beta$. Consider next a general vector $\boldsymbol{A} = A_\beta \, \hat{\mathbf{e}}_\beta$. Expressed in terms of the new basis $\{\hat{\mathbf{e}}'_\alpha\}$, we have

$$
\boldsymbol{A} = A_\beta \, \overbrace{\mathcal{R}^{\mathrm{t}}_{\beta\alpha} \, \hat{\mathbf{e}}'_\alpha}^{\hat{\mathbf{e}}_\beta} = \overbrace{\mathcal{R}_{\alpha\beta} A_\beta}^{A'_\alpha} \, \hat{\mathbf{e}}'_\alpha
\tag{9.10}
$$

Thus, the components of $\boldsymbol{A}$ transform as $A'_\alpha = \mathcal{R}_{\alpha\beta}\, A_\beta$. This is true for any vector.

Under a rotation, the density $\rho(\boldsymbol{r})$ must satisfy $\rho'(\boldsymbol{r}') = \rho(\boldsymbol{r})$. This is the transformation rule for scalars. The inertia tensor therefore obeys

$$\begin{aligned} I'_{\alpha\beta} &= \int\! d^3r'\, \rho'(\boldsymbol{r}') \Big[\boldsymbol{r}'^2\, \delta_{\alpha\beta} - r'_\alpha\, r'_\beta\Big] \\ &= \int\! d^3r\, \rho(\boldsymbol{r}) \Big[\boldsymbol{r}^2\, \delta_{\alpha\beta} - \big(\mathcal{R}_{\alpha\mu} r_\mu\big)\big(\mathcal{R}_{\beta\nu} r_\nu\big)\Big] \\ &= \mathcal{R}_{\alpha\mu}\, I_{\mu\nu}\, \mathcal{R}^{\rm t}_{\nu\beta} \quad . \end{aligned} \tag{9.11}$$

I.e. $I' = \mathcal{R} I \mathcal{R}^{\rm t}$, the transformation rule for tensors. The angular frequency $\boldsymbol{\omega}$ is a vector, so $\omega'_\alpha = \mathcal{R}_{\alpha\mu}\, \omega_\mu$. The angular momentum $\boldsymbol{L}$ also transforms as a vector. The kinetic energy is $T = \frac{1}{2}\, \boldsymbol{\omega}^{\rm t} \cdot I \cdot \boldsymbol{\omega}$, which transforms as a scalar.

9.2.2 The case of no fixed point

If there is no fixed point, we can let $\boldsymbol{r}'$ denote the distance from the center-of-mass (CM), which will serve as the instantaneous origin in the body-fixed frame. We then adopt the notation where $\boldsymbol{R}$ is the CM position of the rotating body, as observed in an inertial frame, and is computed from the expression

$$\boldsymbol{R} = \frac{1}{M} \sum_i m_i\, \boldsymbol{\rho}_i = \frac{1}{M} \int\! d^3r\, \rho(\boldsymbol{r})\, \boldsymbol{r} \quad , \tag{9.12}$$

where the total mass is of course

$$M = \sum_i m_i = \int\! d^3r\, \rho(\boldsymbol{r}) \quad . \tag{9.13}$$

The kinetic energy and angular momentum are then

$$\begin{aligned} T &= \tfrac{1}{2} M \dot{\boldsymbol{R}}^2 + \tfrac{1}{2} I_{\alpha\beta}\, \omega_\alpha\, \omega_\beta \\ L_\alpha &= \epsilon_{\alpha\beta\gamma} M R_\beta \dot{R}_\gamma + I_{\alpha\beta}\, \omega_\beta \quad , \end{aligned} \tag{9.14}$$

where $I_{\alpha\beta}$ is given in eqs. 9.7, where the origin is the CM.

9.3 Parallel Axis Theorem

Suppose $I_{\alpha\beta}$ is given in a body-fixed frame. If we displace the origin in the body-fixed frame by $\boldsymbol{d}$, then let $I_{\alpha\beta}(\boldsymbol{d})$ be the inertial tensor with respect to the new origin. If, relative to the origin at $\boldsymbol{0}$ a mass element lies at position $\boldsymbol{r}$, then relative to an origin at $\boldsymbol{d}$ it will lie at $\boldsymbol{r} - \boldsymbol{d}$. We then have

$$I_{\alpha\beta}(\boldsymbol{d}) = \sum_i m_i \Big\{ (\boldsymbol{r}_i^2 - 2\boldsymbol{d} \cdot \boldsymbol{r}_i + \boldsymbol{d}^2)\, \delta_{\alpha\beta} - (r_{i,\alpha} - d_\alpha)(r_{i,\beta} - d_\beta) \Big\} \quad . \tag{9.15}$$

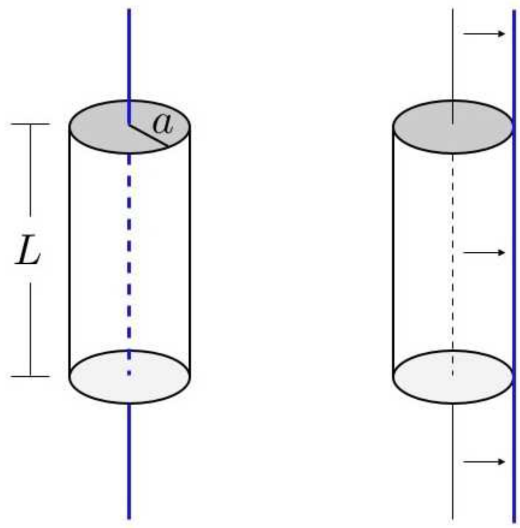


Figure 9.3: Application of the parallel axis theorem to a cylindrically symmetric mass distribution.

If $\boldsymbol{r}_i$ is measured with respect to the CM, then

$$\sum_i m_i \, \boldsymbol{r}_i = 0 \tag{9.16}$$

and

$$I_{\alpha\beta}(\boldsymbol{d}) = I_{\alpha\beta}(0) + M\big(\boldsymbol{d}^2 \delta_{\alpha\beta} - d_\alpha d_\beta\big) \quad , \tag{9.17}$$

a result known as the *parallel axis theorem*.

As an example of the theorem, consider the situation depicted in fig. 9.3, where a cylindrically symmetric mass distribution is rotated about is symmetry axis, and about an axis tangent to its side. The component I_{zz} of the inertia tensor is easily computed when the origin lies along the symmetry axis:

$$\begin{aligned} I_{zz} &= \int d^3\!r \, \rho(\boldsymbol{r}) \, (\boldsymbol{r}^2 - z^2) = \rho L \cdot 2\pi \int\limits_0^a dr_\perp \, r_\perp^3 \\ &= \tfrac{\pi}{2} \rho L a^4 = \tfrac{1}{2} M a^2 \quad , \end{aligned} \tag{9.18}$$

where $M = \pi a^2 L \rho$ is the total mass. If we compute I_{zz} about a vertical axis which is tangent to the cylinder, the parallel axis theorem tells us that

$$I'_{zz} = I_{zz} + M a^2 = \tfrac{3}{2} M a^2 \quad . \tag{9.19}$$

Doing this calculation by explicit integration of $\int dm \, r_\perp^2$ would be tedious!

9.3.1 Example

`Problem:` Compute the CM and the inertia tensor for the planar right triangle of fig. 9.4, assuming it to be of uniform two-dimensional mass density ρ.

Solution: The total mass is $M=\frac{1}{2}\rho\,ab$. The x-coordinate of the CM is then

$$\begin{aligned}X&=\frac{1}{M}\int\limits_0^a dx\int\limits_0^{b(1-\frac{x}{a})}dy\,\rho\,x=\frac{\rho}{M}\int\limits_0^a dx\,b\big(1-\tfrac{x}{a}\big)\,x\\&=\frac{\rho\,a^2 b}{M}\int\limits_0^1 du\,u(1-u)=\frac{\rho\,a^2 b}{6M}=\tfrac{1}{3}\,a\quad.\end{aligned}\tag{9.20}$$

Clearly we must then have $Y=\frac{1}{3}\,b$, which may be verified by explicit integration.

We now compute the inertia tensor, with the origin at $(0,0,0)$. Since the figure is planar, $z=0$ everywhere, hence $I_{xz}=I_{zx}=0$, $I_{yz}=I_{zy}=0$, and also $I_{zz}=I_{xx}+I_{yy}$. We now compute the remaining independent elements:

$$\begin{aligned}I_{xx}&=\rho\int\limits_0^a dx\int\limits_0^{b(1-\frac{x}{a})}dy\,y^2=\rho\int\limits_0^a dx\,\tfrac{1}{3}\,b^3\,\big(1-\tfrac{x}{a}\big)^3\\&=\tfrac{1}{3}\rho\,ab^3\int\limits_0^1 du\,(1-u)^3=\tfrac{1}{12}\rho\,ab^3=\tfrac{1}{6}Mb^2\end{aligned}\tag{9.21}$$

and

$$\begin{aligned}I_{xy}&=-\rho\int\limits_0^a dx\int\limits_0^{b(1-\frac{x}{a})}dy\,x\,y=-\tfrac{1}{2}\rho\,b^2\int\limits_0^a dx\,x\,\big(1-\tfrac{x}{a}\big)^2\\&=-\tfrac{1}{2}\rho\,a^2b^2\int\limits_0^1 du\,u\,(1-u)^2=-\tfrac{1}{24}\rho\,a^2b^2=-\tfrac{1}{12}Mab\quad.\end{aligned}\tag{9.22}$$

Thus,

$$I=\frac{M}{6}\begin{pmatrix}b^2 & -\frac{1}{2}ab & 0\\ -\frac{1}{2}ab & a^2 & 0\\ 0 & 0 & a^2+b^2\end{pmatrix}\quad.\tag{9.23}$$

Suppose we wanted the inertia tensor relative in a coordinate system where the CM lies at the origin. What we computed in eqn. 9.23 is $I(\boldsymbol{d})$, with $\boldsymbol{d}=-\frac{a}{3}\hat{\boldsymbol{x}}-\frac{b}{3}\,\hat{\boldsymbol{y}}$. Thus,

$$\boldsymbol{d}^2\delta_{\alpha\beta}-d_\alpha\,d_\beta=\frac{1}{9}\begin{pmatrix}b^2 & -ab & 0\\ -ab & a^2 & 0\\ 0 & 0 & a^2+b^2\end{pmatrix}\quad.\tag{9.24}$$

Since

$$I(\boldsymbol{d})=I^{\rm CM}+M\Big(\boldsymbol{d}^2\delta_{\alpha\beta}-d_\alpha\,d_\beta\Big)\quad,\tag{9.25}$$

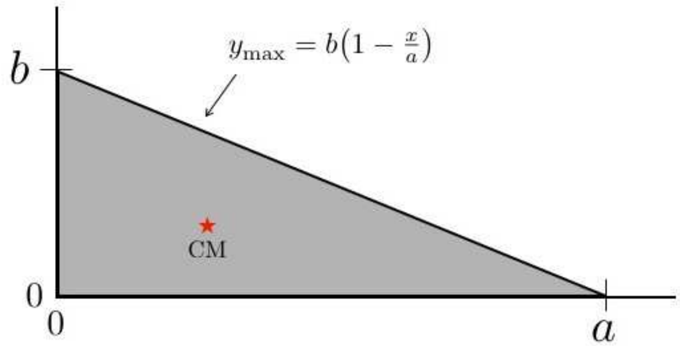


Figure 9.4: A planar mass distribution in the shape of a triangle.

we have that

$$\begin{aligned} I^{\rm CM} &= I(\boldsymbol{d}) - M\Big(\boldsymbol{d}^2\delta_{\alpha\beta} - d_\alpha\, d_\beta\Big) \\ &= \frac{M}{18}\begin{pmatrix} b^2 & \frac{1}{2}ab & 0 \\ \frac{1}{2}ab & a^2 & 0 \\ 0 & 0 & a^2+b^2 \end{pmatrix} \quad . \end{aligned} \tag{9.26}$$

9.3.2 General planar mass distribution

For a general planar mass distribution,

$$\rho(x,y,z) = \sigma(x,y)\,\delta(z) \quad , \tag{9.27}$$

which is confined to the plane $z=0$, we have $I_{xz} = I_{yz} = 0$, and

$$\begin{aligned} I_{xx} &= \int\! dx \!\int\! dy\; \sigma(x,y)\, y^2 \\ I_{yy} &= \int\! dx \!\int\! dy\; \sigma(x,y)\, x^2 \\ I_{xy} &= -\int\! dx \!\int\! dy\; \sigma(x,y)\, xy \quad . \end{aligned} \tag{9.28}$$

Furthermore, $I_{zz} = I_{xx} + I_{yy}$, regardless of the two-dimensional mass distribution $\sigma(x,y)$.

9.4 Principal Axes of Inertia

We found that an orthogonal transformation to a new set of axes $\hat{\mathbf{e}}'_\alpha = \mathcal{R}_{\alpha\beta}\hat{\mathbf{e}}_\beta$ entails $I' = \mathcal{R} I \mathcal{R}^{\rm t}$ for the inertia tensor. Since $I = I^{\rm t}$ is manifestly a symmetric matrix, it can be brought to diagonal form by such an orthogonal transformation. To find $\mathcal{R}$, follow this recipe:

1. Find the diagonal elements of I' by setting $P(\lambda) = 0$, where

$$P(\lambda) = \det\big(\lambda \cdot 1 - I\big) \quad , \tag{9.29}$$

is the characteristic polynomial for I, and 1 is the unit matrix.

2. For each eigenvalue λ_a, solve the d equations

$$\sum_\nu I_{\mu\nu}\, \psi^a_\nu = \lambda_a\, \psi^a_\mu \quad . \tag{9.30}$$

Here, ψ^a_μ is the $\mu^{\rm th}$ component of the $a^{\rm th}$ eigenvector. Since $(\lambda \cdot 1 - I)$ is degenerate, these equations are linearly dependent, which means that the first $d-1$ components may be determined in terms of the $d^{\rm th}$ component.

3. Because $I = I^{\rm t}$, eigenvectors corresponding to different eigenvalues are orthogonal. In cases of degeneracy, the eigenvectors may be chosen to be orthogonal, *e.g.* via the Gram-Schmidt procedure.

4. Due to the underdetermined aspect to step 2, we may choose an arbitrary normalization for each eigenvector. It is conventional to choose the eigenvectors to be orthonormal: $\sum_\mu \psi^a_\mu\, \psi^b_\mu = \delta^{ab}$.

5. The matrix $\mathcal{R}$ is explicitly given by $\mathcal{R}_{a\mu} = \psi^a_\mu$, the matrix whose row vectors are the eigenvectors $\boldsymbol{\psi}^a$. Of course $\mathcal{R}^{\rm t}$ is then the corresponding matrix of column vectors.

6. The eigenvectors form a complete basis. The resolution of unity may be expressed as

$$\sum_a \psi^a_\mu\, \psi^a_\nu = \delta_{\mu\nu} \quad . \tag{9.31}$$

As an example, consider the inertia tensor for a general planar mass distribution, which is of the form

$$I = \begin{pmatrix} I_{xx} & I_{xy} & 0 \\ I_{yx} & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix} \quad , \tag{9.32}$$

where $I_{yx} = I_{xy}$ and $I_{zz} = I_{xx} + I_{yy}$. Define

$$\begin{aligned} A &= \tfrac{1}{2}\big(I_{xx} + I_{yy}\big) \\ B &= \sqrt{\tfrac{1}{4}\big(I_{xx} - I_{yy}\big)^2 + I_{xy}^2} \\ \vartheta &= \tan^{-1}\left(\frac{2I_{xy}}{I_{xx} - I_{yy}}\right) \quad , \end{aligned} \tag{9.33}$$

so that

$$I = \begin{pmatrix} A + B\cos\vartheta & B\sin\vartheta & 0 \\ B\sin\vartheta & A - B\cos\vartheta & 0 \\ 0 & 0 & 2A \end{pmatrix} \quad , \tag{9.34}$$

The characteristic polynomial is found to be

$$P(\lambda) = (\lambda - 2A)\left[(\lambda - A)^2 - B^2\right] \quad , \tag{9.35}$$

which gives $\lambda_1 = A + B$, $\lambda_2 = A - B$, and $\lambda_3 = 2A$. The corresponding normalized eigenvectors are

$$\boldsymbol{\psi}^1 = \begin{pmatrix} \cos\frac{1}{2}\vartheta \\ \sin\frac{1}{2}\vartheta \\ 0 \end{pmatrix} \quad , \quad \boldsymbol{\psi}^2 = \begin{pmatrix} -\sin\frac{1}{2}\vartheta \\ \cos\frac{1}{2}\vartheta \\ 0 \end{pmatrix} \quad , \quad \boldsymbol{\psi}^3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{9.36}$$

and therefore

$$\mathcal{R} = \begin{pmatrix} \cos\frac{1}{2}\vartheta & \sin\frac{1}{2}\vartheta & 0 \\ -\sin\frac{1}{2}\vartheta & \cos\frac{1}{2}\vartheta & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad . \tag{9.37}$$

We then have

$$I' = \mathcal{R} I \mathcal{R}^{\mathrm{t}} = \begin{pmatrix} A+B & 0 & 0 \\ 0 & A-B & 0 \\ 0 & 0 & 2A \end{pmatrix} \quad . \tag{9.38}$$

9.5 Euler's Equations

9.5.1 Derivation of Euler's equations

The equations of motion are

$$\begin{aligned} \boldsymbol{N}^{\mathrm{ext}} &= \left(\frac{d\boldsymbol{L}}{dt}\right)_{\mathrm{inertial}} \\ &= \left(\frac{d\boldsymbol{L}}{dt}\right)_{\mathrm{body}} + \boldsymbol{\omega} \times \boldsymbol{L} = I\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (I\boldsymbol{\omega}) \quad . \end{aligned} \tag{9.39}$$

Let us now choose our coordinate axes to be the principal axes of inertia, with the CM at the origin. We may then write

$$\boldsymbol{\omega} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \quad , \quad I = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} \quad \Longrightarrow \quad \boldsymbol{L} = \begin{pmatrix} I_1\,\omega_1 \\ I_2\,\omega_2 \\ I_3\,\omega_3 \end{pmatrix} \quad . \tag{9.40}$$

From $I\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (I\boldsymbol{\omega}) = \boldsymbol{N}^{\mathrm{ext}}$, we arrive at *Euler's equations:*

$$\begin{aligned} I_1\,\dot{\omega}_1 &= (I_2 - I_3)\,\omega_2\,\omega_3 + N_1^{\mathrm{ext}} \\ I_2\,\dot{\omega}_2 &= (I_3 - I_1)\,\omega_3\,\omega_1 + N_2^{\mathrm{ext}} \\ I_3\,\dot{\omega}_3 &= (I_1 - I_2)\,\omega_1\,\omega_2 + N_3^{\mathrm{ext}} \quad , \end{aligned} \tag{9.41}$$

where $N_{1,2,3}^{\mathrm{ext}}$ are the components of $\boldsymbol{N}^{\mathrm{ext}}$ along the body-fixed principal axes. These equations are coupled and nonlinear. We can however make progress in the case where $\boldsymbol{N}^{\mathrm{ext}} = 0$, *i.e.* when there are no

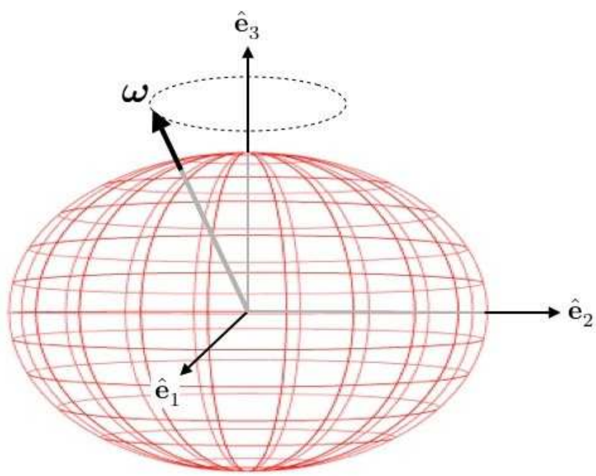


Figure 9.5: Wobbling of a torque-free symmetric top.

external torques. This is true for a body in free space, or in a uniform gravitational field. In the latter case,

$$\boldsymbol{N}^{\rm ext} = \sum_i \boldsymbol{r}_i \times (m_i\, \boldsymbol{g}) = \Big(\sum_i m_i \boldsymbol{r}_i\Big) \times \boldsymbol{g} \quad , \tag{9.42}$$

where $\boldsymbol{g}$ is the uniform gravitational acceleration. In a body-fixed frame whose origin is the CM, we have $\sum_i m_i \boldsymbol{r}_i = 0$, and the external torque vanishes!

9.5.2 Precession of torque-free symmetric tops

Consider a body which has a symmetry axis $\hat{\mathrm{e}}_3$. This guarantees $I_1 = I_2$, but in general we still have $I_1 \neq I_3$. In the absence of external torques, the last of Euler's equations says $\dot{\omega}_3 = 0$, so ω_3 is a constant. The remaining two equations are then

$$\dot{\omega}_1 = \left(\frac{I_1 - I_3}{I_1}\right)\omega_3\, \omega_2 \quad , \quad \dot{\omega}_2 = \left(\frac{I_3 - I_1}{I_1}\right)\omega_3\, \omega_1 \quad . \tag{9.43}$$

I.e. $\dot{\omega}_1 = -\varOmega\, \omega_2$ and $\dot{\omega}_2 = +\varOmega\, \omega_1$, with

$$\varOmega = \left(\frac{I_3 - I_1}{I_1}\right)\omega_3 \quad , \tag{9.44}$$

which are the equations of a harmonic oscillator. The solution is easily obtained:

$$\omega_1(t) = \omega_\perp \cos\big(\varOmega t + \delta\big) \quad , \quad \omega_2(t) = \omega_\perp \sin\big(\varOmega t + \delta\big) \quad , \quad \omega_3(t) = \omega_3 \quad , \tag{9.45}$$

where $\omega_\perp$ and δ are constants of integration, and where $|\boldsymbol{\omega}| = (\omega_\perp^2 + \omega_3^2)^{1/2}$. This motion is sketched in fig. 9.5. Note that the perpendicular components of $\boldsymbol{\omega}$ oscillate harmonically, and that the angle $\boldsymbol{\omega}$ makes with respect to $\hat{\mathrm{e}}_3$ is $\lambda = \tan^{-1}(\omega_\perp/\omega_3)$.

For the earth, $(I_3 - I_1)/I_1 \approx \frac{1}{305}$ and $\omega_\perp \ll \omega_3$, so $\Omega \approx \omega/305$, *i.e.* a precession period of 305 days, or roughly 10 months. Astronomical observations reveal such a precession, known as the *Chandler wobble.* The precession angle is $\lambda_{\rm Chandler} \simeq 6 \times 10^{-7}$ rad, which means that the North Pole moves by about 4 meters during the wobble. The Chandler wobble has a period of about 14 months, so the naïve prediction of 305 days is off by a substantial amount. This discrepancy is attributed to the mechanical properties of the earth: elasticity and fluidity. The earth is not solid![1]

9.5.3 Asymmetric tops

Next, consider the torque-free motion of an asymmetric top, where $I_1 \neq I_2 \neq I_3 \neq I_1$. Unlike the symmetric case, there is no conserved component of $\boldsymbol{\omega}$. True, we can invoke conservation of energy and angular momentum,

$$\begin{aligned} E &= \tfrac{1}{2} I_1\,\omega_1^2 + \tfrac{1}{2} I_2\,\omega_2^2 + \tfrac{1}{2} I_3\,\omega_3^2 \\ \boldsymbol{L}^2 &= I_1^2\,\omega_1^2 + I_2^2\,\omega_2^2 + I_3^2\,\omega_3^2 \quad , \end{aligned} \tag{9.46}$$

and, in principle, solve for ω_1 and ω_2 in terms of ω_3, and then invoke Euler's equations (which must honor these conservation laws). This results in a nonlinear first order ODE of the form $\dot\omega_3 = f(\omega_3)$ which is fairly awkward.

We can, however, find a *particular* solution quite easily – one in which the rotation is about a single axis. Thus, $\omega_1 = \omega_2 = 0$ and $\omega_3 = \omega_0$ is indeed a solution for all time, according to Euler's equations. Let us now perturb about this solution, to explore its stability. We write

$$\boldsymbol{\omega} = \omega_0\,\hat{\mathbf{e}}_3 + \delta\boldsymbol{\omega} \quad , \tag{9.47}$$

and we invoke Euler's equations, linearizing by dropping terms quadratic in $\delta\boldsymbol{\omega}$. This yield

$$\begin{aligned} I_1\,\delta\dot\omega_1 &= (I_2 - I_3)\,\omega_0\,\delta\omega_2 + \mathcal{O}(\delta\omega_2\,\delta\omega_3) \\ I_2\,\delta\dot\omega_2 &= (I_3 - I_1)\,\omega_0\,\delta\omega_1 + \mathcal{O}(\delta\omega_3\,\delta\omega_1) \\ I_3\,\delta\dot\omega_3 &= 0 + \mathcal{O}(\delta\omega_1\,\delta\omega_2) \quad . \end{aligned} \tag{9.48}$$

Taking the time derivative of the first equation and invoking the second, and *vice versa,* yields

$$\delta\ddot\omega_1 = -\Omega^2\,\delta\omega_1 \quad , \quad \delta\ddot\omega_2 = -\Omega^2\,\delta\omega_2 \quad , \tag{9.49}$$

with

$$\Omega^2 = \frac{(I_3 - I_2)(I_3 - I_1)}{I_1\,I_2} \cdot \omega_0^2 \quad . \tag{9.50}$$

The solution is then $\delta\omega_1(t) = C\,\cos(\Omega t + \delta)$.

If $\Omega^2 > 0$, then Ω is real, and the deviation results in a harmonic precession. This occurs if I_3 is either the largest or the smallest of the moments of inertia. If, however, I_3 is the middle moment, then $\Omega^2 < 0$, and Ω is purely imaginary. The perturbation will in general increase exponentially with time, which means that the initial solution to Euler's equations is *unstable* with respect to small perturbations. This result can be vividly realized using a tennis racket, and sometimes goes by the name of the "tennis racket theorem."

[1]The earth is a layered like a *Mozartkugel,* with a solid outer shell, an inner fluid shell, and a solid (iron) core.

9.5.4 Example: The giant asteroid

PROBLEM: A unsuspecting solid spherical planet of mass M_0 rotates with angular velocity ω_0. Suddenly, a giant asteroid of mass αM_0 smashes into and sticks to the planet at a location which is at polar angle θ relative to the initial rotational axis. The new mass distribution is no longer spherically symmetric, and the rotational axis will precess. Recall Euler's equation,

$$\frac{d\boldsymbol{L}}{dt} + \boldsymbol{\omega} \times \boldsymbol{L} = \boldsymbol{N}^{\rm ext} \tag{9.51}$$

for rotations in a body-fixed frame.

(a) What is the new inertia tensor $I_{\alpha\beta}$ along principal center-of-mass frame axes? Don't forget that the CM is no longer at the center of the sphere! Recall $I = \frac{2}{5}MR^2$ for a solid sphere.

(b) What is the period of precession of the rotational axis in terms of the original length of the day $2\pi/\omega_0$?

SOLUTION: Let's choose body-fixed axes with $\hat{\boldsymbol{z}}$ pointing from the center of the planet to the smoldering asteroid. The CM lies a distance

$$d = \frac{\alpha M_0 \cdot R + M_0 \cdot 0}{(1+\alpha)M_0} = \frac{\alpha}{1+\alpha}\, R \tag{9.52}$$

from the center of the sphere. Thus, relative to the center of the sphere, we have

$$I = \tfrac{2}{5} M_0 R^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \alpha M_0 R^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad . \tag{9.53}$$

Now we shift to a frame with the CM at the origin, using the parallel axis theorem,

$$I_{\alpha\beta}(\boldsymbol{d}) = I^{\rm CM}_{\alpha\beta} + M\big(\boldsymbol{d}^2\, \delta_{\alpha\beta} - d_\alpha d_\beta\big) \quad . \tag{9.54}$$

Thus, with $\boldsymbol{d} = d\hat{\boldsymbol{z}}$,

$$\begin{aligned} I^{\rm CM}_{\alpha\beta} &= \tfrac{2}{5} M_0 R^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \alpha M_0 R^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} - (1+\alpha) M_0 d^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= M_0 R^2 \begin{pmatrix} \frac{2}{5} + \frac{\alpha}{1+\alpha} & 0 & 0 \\ 0 & \frac{2}{5} + \frac{\alpha}{1+\alpha} & 0 \\ 0 & 0 & \frac{2}{5} \end{pmatrix} \quad . \end{aligned} \tag{9.55}$$

In the absence of external torques, Euler's equations along principal axes read

$$\begin{aligned} I_1 \frac{d\omega_1}{dt} &= (I_2 - I_3)\, \omega_2\, \omega_3 \\ I_2 \frac{d\omega_2}{dt} &= (I_3 - I_1)\, \omega_3\, \omega_1 \\ I_3 \frac{d\omega_3}{dt} &= (I_1 - I_2)\, \omega_1\, \omega_2 \end{aligned} \tag{9.56}$$

Since $I_1 = I_2$, $\omega_3(t) = \omega_3(0) = \omega_0 \cos\theta$ is a constant. We then obtain $\dot\omega_1 = \Omega\omega_2$, and $\dot\omega_2 = -\Omega\omega_1$, with

$$\Omega = \frac{I_2 - I_3}{I_1}\,\omega_3 = \frac{5\alpha}{7\alpha+2}\,\omega_3 \quad . \tag{9.57}$$

The period of precession τ in units of the pre-cataclysmic day is

$$\frac{\tau}{T} = \frac{\omega}{\Omega} = \frac{7\alpha+2}{5\alpha\cos\theta} \quad . \tag{9.58}$$

9.6 Euler's Angles

9.6.1 Definition of the Euler angles

In d dimensions, an orthogonal matrix $\mathcal{R} \in \mathrm{O}(d)$ has $\frac{1}{2}d(d-1)$ independent parameters. To see this, consider the constraint $\mathcal{R}^{\mathrm{t}}\mathcal{R} = 1$. The matrix $\mathcal{R}^{\mathrm{t}}\mathcal{R}$ is manifestly symmetric, so it has $\frac{1}{2}d(d+1)$ independent entries (*e.g.* on the diagonal and above the diagonal). This amounts to $\frac{1}{2}d(d+1)$ constraints on the d^2 components of $\mathcal{R}$, resulting in $\frac{1}{2}d(d-1)$ freedoms. Thus, in $d = 3$ rotations are specified by three parameters. The *Euler angles* $\{\phi, \theta, \psi\}$ provide one such convenient parameterization.

A general rotation $\mathcal{R}(\phi,\theta,\psi)$ is built up in three steps. We start with an orthonormal triad $\hat{\mathbf{e}}^0_\mu$ of body-fixed axes. The first step is a rotation by an angle ϕ about $\hat{\mathbf{e}}^0_3$:

$$\hat{\mathbf{e}}'_\mu = \mathcal{R}_{\mu\nu}\big(\hat{\mathbf{e}}^0_3, \phi\big)\,\hat{\mathbf{e}}^0_\nu \quad , \quad \mathcal{R}\big(\hat{\mathbf{e}}^0_3, \phi\big) = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad . \tag{9.59}$$

This step is shown in panel (a) of fig. 9.6. The second step is a rotation by θ about the new axis $\hat{\mathbf{e}}'_1$:

$$\hat{\mathbf{e}}''_\mu = \mathcal{R}_{\mu\nu}\big(\hat{\mathbf{e}}'_1, \theta\big)\,\hat{\mathbf{e}}'_\nu \quad , \quad \mathcal{R}\big(\hat{\mathbf{e}}'_1, \theta\big) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \quad . \tag{9.60}$$

This step is shown in panel (b). The third and final step is a rotation by ψ about the new axis $\hat{\mathbf{e}}''_3$:

$$\hat{\mathbf{e}}'''_\mu = \mathcal{R}_{\mu\nu}\big(\hat{\mathbf{e}}''_3, \psi\big)\,\hat{\mathbf{e}}''_\nu \quad , \quad \mathcal{R}\big(\hat{\mathbf{e}}''_3, \psi\big) = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad . \tag{9.61}$$

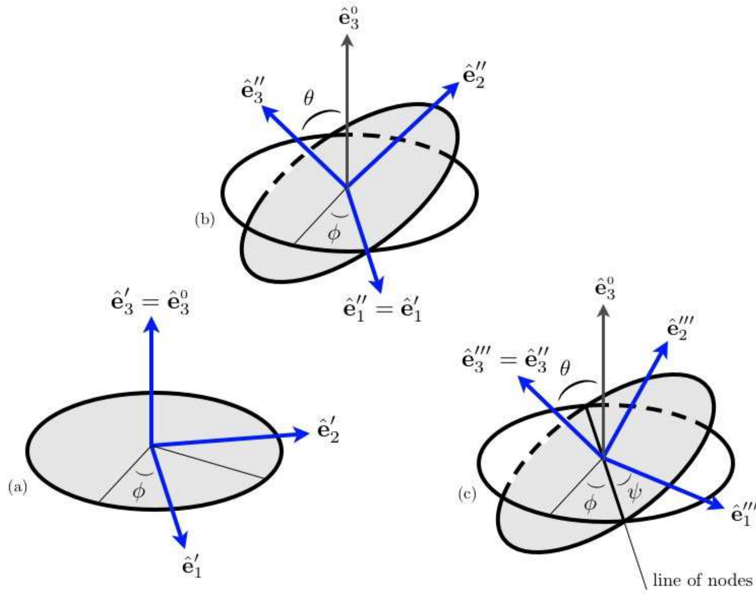


Figure 9.6: A general rotation, defined in terms of the Euler angles $\{\phi, \theta, \psi\}$. Three successive steps of the transformation are shown.

This step is shown in panel (c). Putting this all together,

$$\begin{aligned}\mathcal{R}(\phi,\theta,\psi) &= \mathcal{R}\big(\hat{\mathbf{e}}_3'',\psi\big)\,\mathcal{R}\big(\hat{\mathbf{e}}_1',\theta\big)\,\mathcal{R}\big(\hat{\mathbf{e}}_3^0,\phi\big) \\ &= \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos\psi\cos\phi - \sin\psi\cos\theta\sin\phi & \cos\psi\sin\phi + \sin\psi\cos\theta\cos\phi & \sin\psi\sin\theta \\ -\sin\psi\cos\phi - \cos\psi\cos\theta\sin\phi & -\sin\psi\sin\phi + \cos\psi\cos\theta\cos\phi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{pmatrix} \quad .\end{aligned} \tag{9.62}$$

Note that the order of our rotations was ZXZ. We could have chosen ZYZ instead, or any of XZX, XYX, YXY, and YZY. Any such rotation protocol is referred to as based on *proper Euler angles*. An equivalent system is to adopt one of the following protocols: XYZ, XZY, YZX, YXZ, ZXY, or ZYX, corresponding to the so-called *Tait-Bryan angles*. The latter are used, inter alia, in aeronautics, where they are known respectively as *roll*, *pitch*, and *yaw* (see Fig. 9.7).

Gimbal locking

A *gimbal* is a ring which rotates about a fixed axis. In a gyroscope, the inner rotor typically rotates at a very high angular velocity such that its spin axis is fixed in an inertial frame. The orientation of the

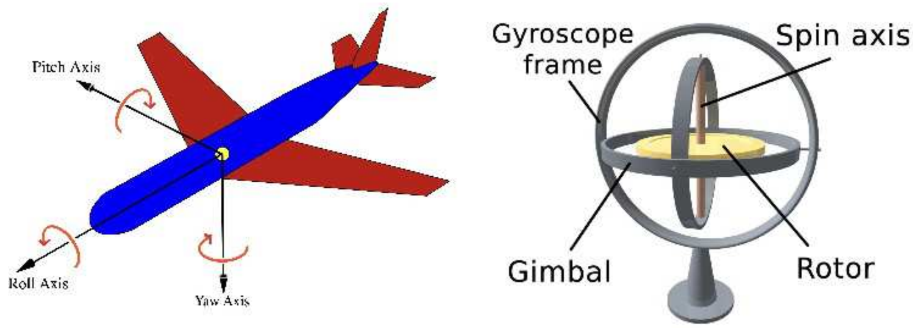


Figure 9.7: Left: Roll, pitch, and yaw. Right: A gyroscope with two gimbals. If the gyroscope frame is free to rotate about its axis, this serves as a third gimbal. *(Image credits: Wikipedia)*

gimbal axes in an *inertial measurement unit* (IMU) may then be used to determine attitude and angular velocity. Replacing the rotor with a camera, the gimbals can be rotated to achieve a desired orientation, say for tracking an object as it moves in three-dimensional space.

A problem arises, though, when the orientation of an object is described using the Euler angles. This is illustrated in Eqn. 9.62 when the angle θ is set to zero, in which case

$$\mathcal{R}(\phi,\theta=0,\psi)=\begin{pmatrix}\cos(\phi+\psi) & \sin(\phi+\psi) & 0\\ -\sin(\phi+\psi) & \cos(\phi+\psi) & 0\\ 0 & 0 & 1\end{pmatrix} \quad . \tag{9.63}$$

We would expect that fixing one of the Euler angles would still allow us to independently rotate about two remaining axes, but we see above that when $\theta = 0$, the space of rotation matrices becomes one-dimensional! A similar problem occurs when $\theta = \pi$, where

$$\mathcal{R}(\phi,\theta=\pi,\psi)=\begin{pmatrix}\cos(\phi-\psi) & \sin(\phi-\psi) & 0\\ \sin(\phi-\psi) & -\cos(\phi-\psi) & 0\\ 0 & 0 & -1\end{pmatrix} \quad . \tag{9.64}$$

Thus, the two-dimensional space of points (ϕ,ψ) maps to a *one*-dimensional subset of $\mathsf{SO}(3)$. You might wonder whether the problem goes away if we choose Tait-Bryan angles instead. It doesn't.

What we are encountering here is a *coordinate singularity* associated with the way we are coordinatizing the $\mathsf{SO}(3)$ manifold[2]. Something analogous happens when we coordinatize the two-sphere S^2 using polar (θ) and azimuthal (ϕ) angles. Precisely at the poles $\theta = 0$ and $\theta = \pi$, the all azimuthal angles ϕ map to the same point, and we have a zero-dimensional rather than a one-dimensional space. In the context of Euler angles, this coordinate singularity is referred to as *gimbal lock*. Navigational difficulties associated with gimbal lock can be avoided by adding a redundant fourth gimbal to an IMU[3], which can be cumbersome, or by using different coordinates on $\mathsf{SO}(3)$, such as *unit quaternions*.

[2] $\mathsf{SO}(3)$ is a *Lie group*, meaning that it is a manifold with a group structure such that the group operations of multiplication and inverse are smooth.

[3] About two hours after the Apollo 11 moon landing on July 20, 1969, NASA Mission Control in Houston contacted Command Module pilot Michael Collins to inform him that he was "maneuvering very close to gimbal lock" and suggesting he back away, whereupon Collins replied saying that he was trying to avoid this situation, adding wryly, "How about sending me a fourth gimbal for Christmas?"

9.6.2 Precession, nutation, and axial rotation

Next, we'd like to relate the components $\omega_\mu = \boldsymbol{\omega} \cdot \hat{\mathbf{e}}_\mu$ (with $\hat{\mathbf{e}}_\mu \equiv \hat{\mathbf{e}}'''_\mu$) of the rotation in the body-fixed frame to the derivatives $\dot{\phi}$, $\dot{\theta}$, and $\dot{\psi}$. To do this, we write

$$\boldsymbol{\omega} = \dot{\phi}\,\hat{\mathbf{e}}_\phi + \dot{\theta}\,\hat{\mathbf{e}}_\theta + \dot{\psi}\,\hat{\mathbf{e}}_\psi \quad , \tag{9.65}$$

where

$$\begin{aligned} \hat{\mathbf{e}}^0_3 = \hat{\mathbf{e}}_\phi &= \sin\theta\sin\psi\,\hat{\mathbf{e}}_1 + \sin\theta\cos\psi\,\hat{\mathbf{e}}_2 + \cos\theta\,\hat{\mathbf{e}}_3 \\ \hat{\mathbf{e}}_\theta &= \cos\psi\,\hat{\mathbf{e}}_1 - \sin\psi\,\hat{\mathbf{e}}_2 \qquad (\text{"line of nodes"}) \\ \hat{\mathbf{e}}_\psi &= \hat{\mathbf{e}}_3 \quad . \end{aligned} \tag{9.66}$$

The first of these follows from the relation $\hat{\mathbf{e}}_\mu = \mathcal{R}_{\mu\nu}(\phi,\theta,\psi)\,\hat{\mathbf{e}}^0_\nu$, whose inverse is $\hat{\mathbf{e}}^0_\mu = \mathcal{R}^{\rm t}_{\mu\nu}(\phi,\theta,\psi)\,\hat{\mathbf{e}}_\nu$, since $\mathcal{R}^{-1} = \mathcal{R}^{\rm t}$. Thus the coefficients of $\hat{\mathbf{e}}_{1,2,3}$ in $\hat{\mathbf{e}}^0_3$ are the elements of the rightmost ($\nu = 3$) *column* of $\mathcal{R}(\phi,\theta,\psi)$. We may now read off

$$\begin{aligned} \omega_1 &= \boldsymbol{\omega}\cdot\hat{\mathbf{e}}_1 = \dot{\phi}\,\sin\theta\sin\psi + \dot{\theta}\cos\psi \\ \omega_2 &= \boldsymbol{\omega}\cdot\hat{\mathbf{e}}_2 = \dot{\phi}\,\sin\theta\cos\psi - \dot{\theta}\,\sin\psi \\ \omega_3 &= \boldsymbol{\omega}\cdot\hat{\mathbf{e}}_3 = \dot{\phi}\cos\theta + \dot{\psi} \quad . \end{aligned} \tag{9.67}$$

Note that

$$\dot{\phi} \leftrightarrow \text{precession} \quad , \quad \dot{\theta} \leftrightarrow \text{nutation} \quad , \quad \dot{\psi} \leftrightarrow \text{axial rotation} \quad . \tag{9.68}$$

The general form of the kinetic energy is then

$$T = \tfrac{1}{2} I_1 \big(\dot{\phi}\,\sin\theta\sin\psi + \dot{\theta}\cos\psi\big)^2 + \tfrac{1}{2} I_2\,\big(\dot{\phi}\,\sin\theta\cos\psi - \dot{\theta}\,\sin\psi\big)^2 + \tfrac{1}{2} I_3\,\big(\dot{\phi}\cos\theta + \dot{\psi}\big)^2 \quad .$$

Note that

$$\boldsymbol{L} = p_\phi\,\hat{\mathbf{e}}_\phi + p_\theta\,\hat{\mathbf{e}}_\theta + p_\psi\,\hat{\mathbf{e}}_\psi \quad , \tag{9.69}$$

which may be verified by explicit computation.

9.6.3 Torque-free symmetric top

A body falling in a gravitational field experiences no net torque about its CM:

$$\boldsymbol{N}^{\rm ext} = \sum_i \boldsymbol{r}_i \times (-m_i\,\boldsymbol{g}) = \boldsymbol{g} \times \sum_i m_i\,\boldsymbol{r}_i = 0 \quad . \tag{9.70}$$

For a symmetric top with $I_1 = I_2$, we have

$$T = \tfrac{1}{2} I_1 \big(\dot{\theta}^2 + \dot{\phi}^2\,\sin^2\!\theta\big) + \tfrac{1}{2} I_3 \big(\dot{\phi}\cos\theta + \dot{\psi}\big)^2 \quad . \tag{9.71}$$

The potential is cyclic in the Euler angles, hence the equations of motion are

$$\frac{d}{dt}\,\frac{\partial T}{\partial(\dot{\phi},\dot{\theta},\dot{\psi})} = \frac{\partial T}{\partial(\phi,\theta,\psi)} \quad . \tag{9.72}$$

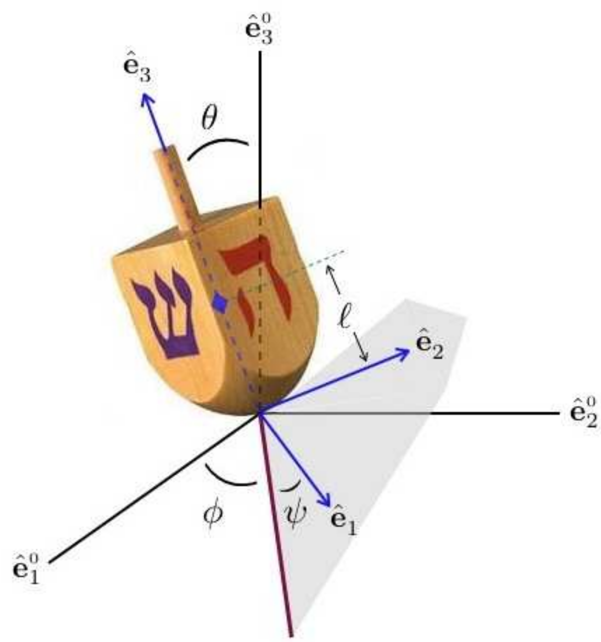


Figure 9.8: A *dreidl* is a symmetric top. The four-fold symmetry axis guarantees $I_1 = I_2$. The blue diamond represents the center-of-mass and lies within the object.

Since ϕ and ψ are cyclic in T, their conjugate momenta are conserved:

$$\begin{aligned} p_\phi &= \frac{\partial L}{\partial \dot\phi} = I_1\,\dot\phi\,\sin^2\!\theta + I_3\,(\dot\phi\cos\theta + \dot\psi)\,\cos\theta \\ p_\psi &= \frac{\partial L}{\partial \dot\psi} = I_3\,(\dot\phi\cos\theta + \dot\psi) \quad . \end{aligned} \tag{9.73}$$

Note that $p_\psi = I_3\,\omega_3$, hence ω_3 is constant, as we have already seen.

To solve for the motion, we first note that $\boldsymbol{L}$ is conserved in the inertial frame. We are therefore permitted to define $\hat{\boldsymbol{L}} = \hat{\mathbf{e}}_3^0 = \hat{\mathbf{e}}_\phi$. Thus, $p_\phi = L$. Since $\hat{\mathbf{e}}_\phi \cdot \hat{\mathbf{e}}_\psi = \cos\theta$, we have that $p_\psi = \boldsymbol{L}\cdot\hat{\mathbf{e}}_\psi = L\cos\theta$. Finally, $\hat{\mathbf{e}}_\phi \cdot \hat{\mathbf{e}}_\theta = 0$, which means $p_\theta = \boldsymbol{L}\cdot\hat{\mathbf{e}}_\theta = 0$. From the equations of motion,

$$\dot p_\theta = I_1\,\ddot\theta = \big(I_1\,\dot\phi\cos\theta - p_\psi\big)\,\dot\phi\sin\theta \quad , \tag{9.74}$$

hence we must have

$$\dot\theta = 0 \quad , \quad \dot\phi = \frac{p_\psi}{I_1\cos\theta} \quad . \tag{9.75}$$

Note that $\dot\theta = 0$ follows from conservation of $p_\psi = L\cos\theta$. From the equation for p_ψ, we may now conclude

$$\dot\psi = \frac{p_\psi}{I_3} - \frac{p_\psi}{I_1} = \left(\frac{I_3 - I_1}{I_3}\right)\omega_3 \quad , \tag{9.76}$$

which recapitulates (9.44), with $\dot\psi = \Omega$.

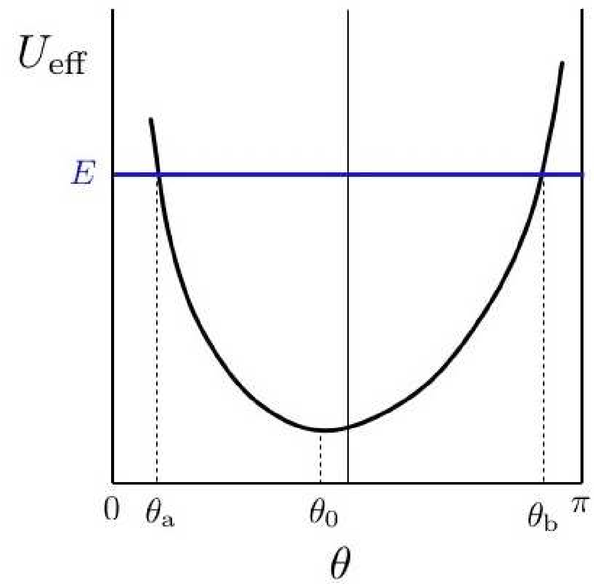


Figure 9.9: The effective potential of eq. 9.85.

9.6.4 Symmetric top with one point fixed

Consider the case of a symmetric top with one point fixed, as depicted in fig. 9.8. The Lagrangian is

$$L = \tfrac{1}{2}I_1\big(\dot\theta^2 + \dot\phi^2\,\sin^2\!\theta\big) + \tfrac{1}{2}I_3\big(\dot\phi\cos\theta + \dot\psi\big)^2 - Mg\ell\cos\theta \quad . \tag{9.77}$$

Here, ℓ is the distance from the fixed point to the CM, and the inertia tensor is defined along principal axes whose origin lies at the fixed point (not the CM!). Gravity now supplies a torque, but as in the torque-free case, the Lagrangian is still cyclic in ϕ and ψ, so

$$\begin{aligned} p_\phi &= (I_1\sin^2\!\theta + I_3\cos^2\!\theta)\,\dot\phi + I_3\cos\theta\,\dot\psi \\ p_\psi &= I_3\cos\theta\,\dot\phi + I_3\,\dot\psi \end{aligned} \tag{9.78}$$

are each conserved. We can invert these relations to obtain $\dot\phi$ and $\dot\psi$ in terms of $\{p_\phi, p_\psi, \theta\}$:

$$\dot\phi = \frac{p_\phi - p_\psi\,\cos\theta}{I_1\,\sin^2\!\theta} \quad , \quad \dot\psi = \frac{p_\psi}{I_3} - \frac{(p_\phi - p_\psi\,\cos\theta)\,\cos\theta}{I_1\,\sin^2\!\theta} \quad . \tag{9.79}$$

In addition, since $\partial L/\partial t = 0$, the total energy is conserved:

$$E = T + U = \tfrac{1}{2}I_1\,\dot\theta^2 + \overbrace{\frac{(p_\phi - p_\psi\,\cos\theta)^2}{2I_1\,\sin^2\!\theta} + \frac{p_\psi^2}{2I_3} + Mg\ell\cos\theta}^{U_{\rm eff}(\theta)} \quad , \tag{9.80}$$

where the term under the brace is the effective potential $U_{\rm eff}(\theta)$.

The problem thus reduces to the one-dimensional dynamics of $\theta(t)$, *i.e.*

$$I_1\,\ddot\theta = -\frac{\partial U_{\rm eff}}{\partial\theta} \quad , \tag{9.81}$$

with

$$U_{\text{eff}}(\theta) = \frac{(p_\phi - p_\psi\,\cos\theta)^2}{2I_1\,\sin^2\!\theta} + \frac{p_\psi^2}{2I_3} + Mg\ell\cos\theta \quad . \tag{9.82}$$

Using energy conservation, we may write

$$dt = \pm\sqrt{\frac{I_1}{2}}\,\frac{d\theta}{\sqrt{E - U_{\text{eff}}(\theta)}} \quad . \tag{9.83}$$

and thus the problem is reduced to quadratures:

$$t(\theta) = t(\theta_0) \pm \sqrt{\frac{I_1}{2}}\int\limits_{\theta_0}^{\theta} d\vartheta\,\frac{1}{\sqrt{E - U_{\text{eff}}(\vartheta)}} \quad . \tag{9.84}$$

We can gain physical insight into the motion by examining the shape of the effective potential,

$$U_{\text{eff}}(\theta) = \frac{(p_\phi - p_\psi\,\cos\theta)^2}{2I_1\,\sin^2\!\theta} + Mg\ell\cos\theta + \frac{p_\psi^2}{2I_3} \quad , \tag{9.85}$$

over the interval $\theta \in [0, \pi]$. Clearly $U_{\text{eff}}(0) = U_{\text{eff}}(\pi) = \infty$, so the motion must be bounded. What is not yet clear, but what is nonetheless revealed by some additional analysis, is that $U_{\text{eff}}(\theta)$ has a single minimum on this interval, at $\theta = \theta_0$. The turning points for the θ motion are at $\theta = \theta_{\text{a}}$ and $\theta = \theta_{\text{b}}$, where $U_{\text{eff}}(\theta_{\text{a}}) = U_{\text{eff}}(\theta_{\text{b}}) = E$. Clearly if we expand about θ_0 and write $\theta = \theta_0 + \eta$, the η motion will be harmonic, with

$$\eta(t) = \eta_0\,\cos(\Omega t + \delta) \quad , \quad \Omega = \sqrt{\frac{U''_{\text{eff}}(\theta_0)}{I_1}} \quad . \tag{9.86}$$

To prove that $U_{\text{eff}}(\theta)$ has these features, let us define $u \equiv \cos\theta$. Then $\dot{u} = -\dot{\theta}\,\sin\theta$, and from $E = \frac{1}{2}I_1\,\dot{\theta}^2 + U_{\text{eff}}(\theta)$ we derive

$$\dot{u}^2 = \left(\frac{2E}{I_1} - \frac{p_\psi^2}{I_1 I_3}\right)(1-u^2) - \frac{2Mg\ell}{I_1}\,(1-u^2)\,u - \left(\frac{p_\phi - p_\psi\,u}{I_1}\right)^2 \equiv f(u) \quad . \tag{9.87}$$

The turning points occur at $f(u) = 0$. The function $f(u)$ is cubic, and the coefficient of the cubic term is $2Mg\ell/I_1$, which is positive. Clearly $f(u = \pm 1) = -(p_\phi \mp p_\psi)^2/I_1^2$ is negative, so there must be at least one solution to $f(u) = 0$ on the interval $u \in (1, \infty)$. Clearly there can be at most three real roots for $f(u)$, since the function is cubic in u, hence there are at most two turning points on the interval $u \in [-1, 1]$. Thus, $U_{\text{eff}}(\theta)$ has the form depicted in fig. 9.9.

To apprehend the full motion of the top in an inertial frame, let us follow the symmetry axis $\hat{\mathbf{e}}_3$:

$$\hat{\mathbf{e}}_3 = \sin\theta\sin\phi\,\hat{\mathbf{e}}_1^0 - \sin\theta\cos\phi\,\hat{\mathbf{e}}_2^0 + \cos\theta\,\hat{\mathbf{e}}_3^0 \quad . \tag{9.88}$$

Once we know $\theta(t)$ and $\phi(t)$ we're done. The motion $\theta(t)$ is described above: θ oscillates between turning points at θ_{a} and θ_{b}. As for $\phi(t)$, we have already derived the result

$$\dot{\phi} = \frac{p_\phi - p_\psi\,\cos\theta}{I_1\,\sin^2\!\theta} \quad . \tag{9.89}$$

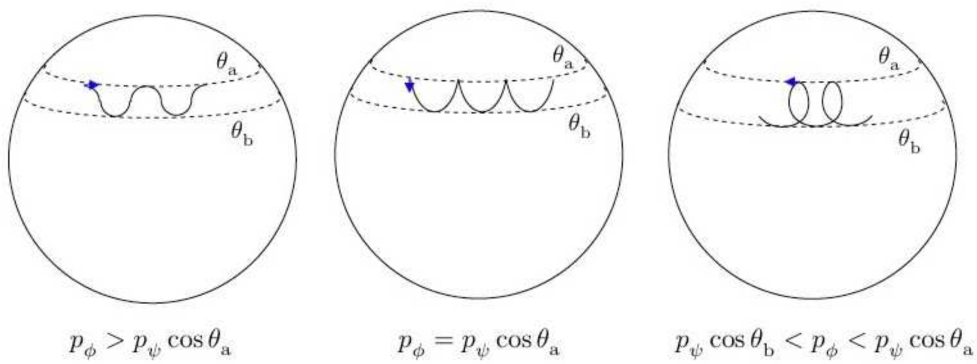


Figure 9.10: Precession and nutation of the symmetry axis of a symmetric top.

Thus, if $p_\phi > p_\psi \cos\theta_{\rm a}$, then $\dot\phi$ will remain positive throughout the motion. If, on the other hand, we have

$$p_\psi \cos\theta_{\rm b} < p_\phi < p_\psi \cos\theta_{\rm a} \quad , \tag{9.90}$$

then $\dot\phi$ changes sign at an angle $\theta^* = \cos^{-1}\big(p_\phi/p_\psi\big)$. The motion is depicted in fig. 9.10. An extensive discussion of this problem is given in H. Goldstein, *Classical Mechanics*.

9.7 Rolling and Skidding Motion of Real Tops

The material in this section is based on the corresponding sections from V. Barger and M. Olsson, *Classical Mechanics: A Modern Perspective*. This is an excellent book which contains many interesting applications and examples.

9.7.1 Rolling tops

In most tops, the point of contact rolls or skids along the surface. Consider the peg end top of fig. 9.11, executing a circular rolling motion, as sketched in fig. 9.12. There are three components to the force acting on the top: gravity, the normal force from the surface, and friction. The frictional force is perpendicular to the CM velocity, and results in centripetal acceleration of the top:

$$f = M\varOmega^2 \rho \le \mu M g \quad , \tag{9.91}$$

where $\varOmega$ is the frequency of the CM motion and μ is the coefficient of friction. If the above inequality is violated, the top starts to slip.

The frictional and normal forces combine to produce a torque $N = Mg\ell\sin\theta - f\ell\cos\theta$ about the CM[4]. This torque is tangent to the circular path of the CM, and causes $\boldsymbol{L}$ to precess. We assume that the top is spinning rapidly, so that $\boldsymbol{L}$ very nearly points along the symmetry axis of the top itself. (As we'll see, this

[4]Gravity of course produces no net torque about the CM.

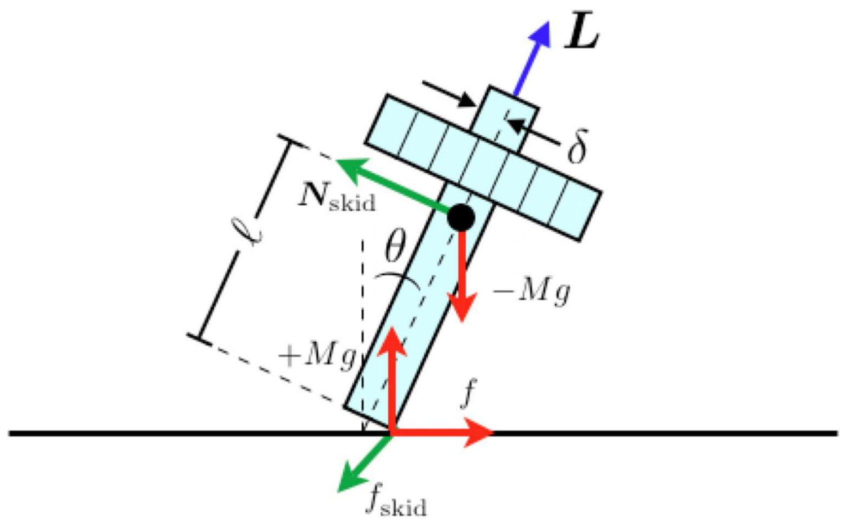


Figure 9.11: A top with a peg end. The frictional forces $\boldsymbol{f}$ and $\boldsymbol{f}_{\rm skid}$ are shown. When the top rolls without skidding, $\boldsymbol{f}_{\rm skid} = 0$.

is true for slow precession but not for fast precession, where the precession frequency is proportional to ω_3.) The precession is then governed by the equation

$$\begin{aligned} N &= Mg\ell \sin\theta - f\ell\cos\theta \\ &= \big|\dot{\boldsymbol{L}}\big| = \big|\boldsymbol{\Omega}\times\boldsymbol{L}\big| \approx \Omega\, I_3\,\omega_3 \sin\theta \quad , \end{aligned} \tag{9.92}$$

where $\hat{e}_3$ is the instantaneous symmetry axis of the top. Substituting $f = M\Omega^2\rho$,

$$\frac{Mg\ell}{I_3\,\omega_3}\left(1 - \frac{\Omega^2\rho}{g}\,\text{ctn}\,\theta\right) = \Omega \quad , \tag{9.93}$$

which is a quadratic equation for Ω. We supplement this with the 'no slip' condition,

$$\omega_3\,\delta = \Omega\,\big(\rho + \ell\sin\theta\big) \quad , \tag{9.94}$$

resulting in two equations for the two unknowns Ω and ρ.

Substituting for $\rho(\Omega)$ and solving for Ω, we obtain

$$\Omega = \frac{I_3\,\omega_3}{2M\ell^2\cos\theta}\left\{1 + \frac{Mg\ell\delta}{I_3}\,\text{ctn}\,\theta \pm \sqrt{\left(1 + \frac{Mg\ell\delta}{I_3}\,\text{ctn}\,\theta\right)^2 - \frac{4M\ell^2}{I_3}\cdot\frac{Mg\ell}{I_3\,\omega_3^2}}\right\} \quad . \tag{9.95}$$

This in order to have a real solution we must have

$$\omega_3 \ge \frac{2M\ell^2\sin\theta}{I_3\sin\theta + Mg\ell\delta\cos\theta}\sqrt{\frac{g}{\ell}} \quad . \tag{9.96}$$

If the inequality is satisfied, there are two possible solutions for Ω, corresponding to fast and slow precession. Usually one observes slow precession. Note that it is possible that $\rho < 0$, in which case the CM and the peg end lie on opposite sides of a circle from each other.

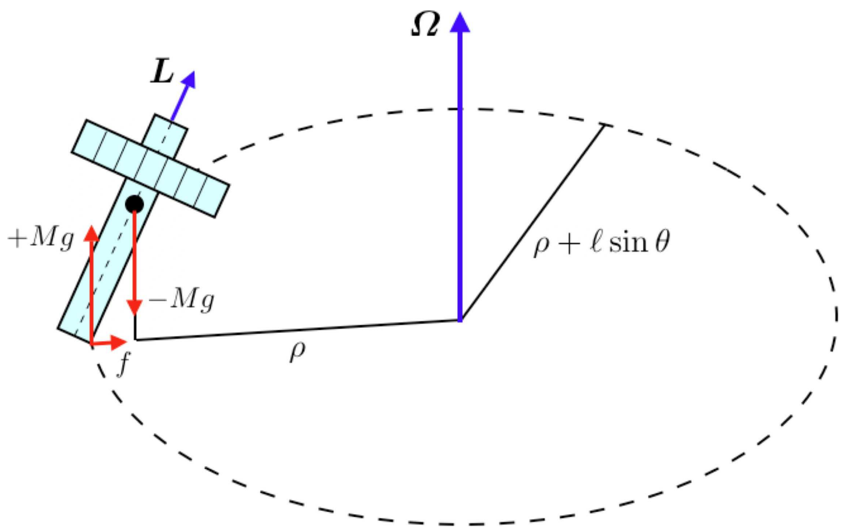


Figure 9.12: Circular rolling motion of the peg top.

9.7.2 Skidding tops

A skidding top experiences a frictional force which opposes the skidding velocity, until $v_{\text{skid}} = 0$ and a pure rolling motion sets in. This force provides a torque which makes the top *rise:*

$$\dot{\theta} = -\frac{N_{\text{skid}}}{L} = -\frac{\mu M g \ell}{I_3\,\omega_3} \quad . \tag{9.97}$$

Suppose $\delta \approx 0$, in which case $\rho + \ell \sin\theta = 0$, from eqn. 9.94, and the point of contact remains fixed. Now recall the effective potential for a symmetric top with one point fixed:

$$U_{\text{eff}}(\theta) = \frac{(p_\phi - p_\psi\,\cos\theta)^2}{2I_1\,\sin^2\!\theta} + \frac{p_\psi^2}{2I_3} + Mg\ell\cos\theta \quad . \tag{9.98}$$

We demand $U'_{\text{eff}}(\theta_0) = 0$, which yields

$$\cos\theta_0 \cdot \beta^2 - p_\psi \sin^2\!\theta_0 \cdot \beta + Mg\ell I_1 \sin^4\!\theta_0 = 0 \quad , \tag{9.99}$$

where

$$\beta \equiv p_\phi - p_\psi \cos\theta_0 = I_1 \sin^2\!\theta_0\,\dot{\phi} \quad . \tag{9.100}$$

Solving the quadratic equation for β, we find

$$\dot{\phi} = \frac{I_3\,\omega_3}{2I_1\cos\theta_0}\left(1 \pm \sqrt{1 - \frac{4Mg\ell I_1\cos\theta_0}{I_3^2\,\omega_3^2}}\right) \quad . \tag{9.101}$$

This is simply a recapitulation of eqn. 9.95, with $\delta = 0$ and with $M\ell^2$ replaced by I_1. Note $I_1 = M\ell^2$ by the parallel axis theorem if $I_1^{\text{CM}} = 0$. But to the extent that $I_1^{\text{CM}} \neq 0$, our treatment of the peg top was incorrect. It turns out to be OK, however, if the precession is slow, *i.e.* if $\Omega/\omega_3 \ll 1$.

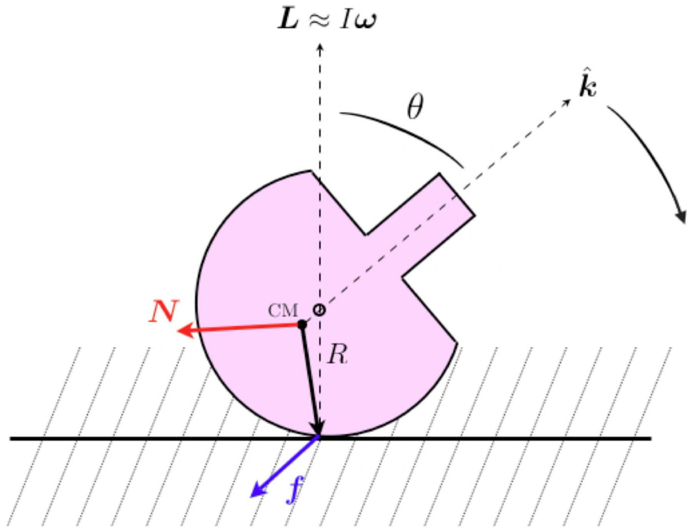


Figure 9.13: The tippie-top behaves in a counterintuitive way. Once started spinning with the peg end up, the peg axis rotates downward. Eventually the peg scrapes the surface and the top rises to the vertical in an inverted orientation.

On a level surface, $\cos\theta_0 > 0$, and therefore we must have

$$\omega_3 \geq \frac{2}{I_3}\sqrt{Mg\ell I_1 \cos\theta_0} \quad . \tag{9.102}$$

Thus, if the top spins too slowly, it cannot maintain precession. Eqn. 9.101 says that there are two possible precession frequencies. When ω_3 is large, we have

$$\dot{\phi}_{\rm slow} = \frac{Mg\ell}{I_3\,\omega_3} + \mathcal{O}(\omega_3^{-1}) \qquad , \qquad \dot{\phi}_{\rm fast} = \frac{I_3\,\omega_3}{I_1\cos\theta_0} + \mathcal{O}(\omega_3^{-3}) \quad . \tag{9.103}$$

Again, one usually observes slow precession.

A top with $\omega_3 > \frac{2}{I_3}\sqrt{Mg\ell I_1}$ may 'sleep' in the vertical position with $\theta_0 = 0$. Due to the constant action of frictional forces, ω_3 will eventually drop below this value, at which time the vertical position is no longer stable. The top continues to slow down and eventually falls.

9.7.3 Tippie-top

A particularly nice example from the Barger and Olsson book is that of the tippie-top, a truncated sphere with a peg end, sketched in fig. 9.13 The CM is close to the center of curvature, which means that there is almost no gravitational torque acting on the top. The frictional force $\boldsymbol{f}$ opposes slipping, but as the top spins $\boldsymbol{f}$ rotates with it, and hence the time-averaged frictional force $\langle \boldsymbol{f} \rangle \approx 0$ has almost no effect on the motion of the CM. A similar argument shows that the frictional torque, which is nearly horizontal, also time averages to zero:

$$\left\langle \frac{d\boldsymbol{L}}{dt} \right\rangle_{\rm inertial} \approx \ 0 \quad . \tag{9.104}$$

In the *body*-fixed frame, however, $\boldsymbol{N}$ is roughly constant, with magnitude $N \approx \mu M g R$, where R is the radius of curvature and μ the coefficient of sliding friction. Now we invoke

$$\boldsymbol{N} = \frac{d\boldsymbol{L}}{dt}\bigg|_{\rm body} + \ \boldsymbol{\omega} \times \boldsymbol{L} \quad . \tag{9.105}$$

The second term on the RHS is very small, because the tippie-top is almost spherical, hence inertia tensor is very nearly diagonal, and this means

$$\boldsymbol{\omega} \times \boldsymbol{L} \approx \boldsymbol{\omega} \times I\boldsymbol{\omega} = 0 \quad . \tag{9.106}$$

Thus, $\dot{\boldsymbol{L}}_{\rm body} \approx \boldsymbol{N}$, and taking the dot product of this equation with the unit vector $\hat{\boldsymbol{k}}$, we obtain

$$-N \sin\theta = \hat{\boldsymbol{k}} \cdot \boldsymbol{N} = \frac{d}{dt}\Big(\hat{\boldsymbol{k}} \cdot \boldsymbol{L}_{\rm body}\Big) = -L \sin\theta\,\dot{\theta} \quad . \tag{9.107}$$

Thus,

$$\dot{\theta} = \frac{N}{L} \approx \frac{\mu M g R}{I\omega} \quad . \tag{9.108}$$

Once the stem scrapes the table, the tippie-top rises to the vertical just like any other rising top.

Chapter 10

Continuum Mechanics

10.1 Continuum Mechanics of the String

10.1.1 Lagrangian formulation

Consider a string of linear mass density $\mu(x)$ under tension $\tau(x)$.[1] Let the string move in a plane, such that its shape is described by a smooth function $y(x)$, the vertical displacement of the string at horizontal position x, as depicted in fig. 10.1. The action is a functional of the height $y(x,t)$, where the coordinate along the string, x, and time, t, are the two independent variables. Consider a differential element of the string extending from x to $x + dx$. The change in length relative to the unstretched $(y = 0)$ configuration is

$$d\ell = \sqrt{dx^2 + dy^2} - dx = \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2 dx + \mathcal{O}\big(dx^2\big) \quad . \tag{10.1}$$

The differential potential energy is then

$$dU = \tau(x)\, d\ell = \tfrac{1}{2}\, \tau(x) \left(\frac{\partial y}{\partial x}\right)^2 dx \quad . \tag{10.2}$$

The differential kinetic energy is simply

$$dT = \tfrac{1}{2}\, \mu(x) \left(\frac{\partial y}{\partial t}\right)^2 dx \quad . \tag{10.3}$$

We can then write

$$L = \int dx\, \mathcal{L} \quad , \tag{10.4}$$

where the *Lagrangian density* $\mathcal{L}$ is

$$\mathcal{L}(y, \dot{y}, y'; x, t) = \tfrac{1}{2}\, \mu(x)\left(\frac{\partial y}{\partial t}\right)^2 - \tfrac{1}{2}\, \tau(x)\left(\frac{\partial y}{\partial x}\right)^2 \quad . \tag{10.5}$$

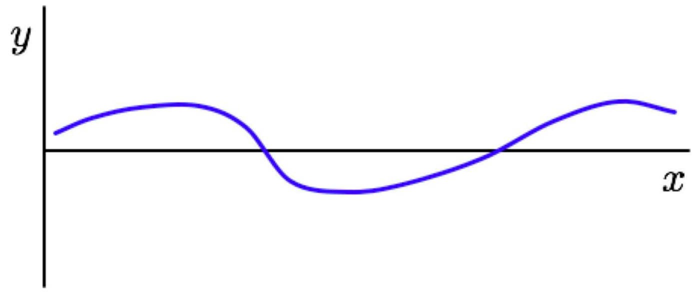


Figure 10.1: A string is described by the vertical displacement field $y(x,t)$.

The action for the string is now a double integral,

$$S = \int_{t_a}^{t_b}\!dt \int_{x_a}^{x_b}\!dx\; \mathcal{L}(y, \dot{y}, y'; x, t) \quad , \tag{10.6}$$

where $y(x,t)$ is the vertical displacement field. Typically, we have $\mathcal{L} = \frac{1}{2}\mu\dot{y}^2 - \frac{1}{2}\tau y'^2$. The first variation of S is

$$\delta S = \int_{t_a}^{t_b}\!dt \int_{x_a}^{x_b}\!dx \left[\frac{\partial \mathcal{L}}{\partial y} - \frac{\partial}{\partial x}\bigg(\frac{\partial \mathcal{L}}{\partial y'}\bigg) - \frac{\partial}{\partial t}\bigg(\frac{\partial \mathcal{L}}{\partial \dot{y}}\bigg)\right]\delta y + \int_{x_a}^{x_b}\!dx \left[\frac{\partial \mathcal{L}}{\partial \dot{y}}\,\delta y\right]_{t=t_a}^{t=t_b} + \int_{t_a}^{t_b}\!dt \left[\frac{\partial \mathcal{L}}{\partial y'}\,\delta y\right]_{x=x_a}^{x=x_b} \quad , \tag{10.7}$$

which simply recapitulates the general result from eqn. 10.181. There are two boundary terms, one of which is an integral over time and the other an integral over space. The first boundary term vanishes provided $\delta y(x,t_a) = \delta y(x,t_b) = 0$. The second boundary term vanishes provided $\tau(x)\, y'(x)\, \delta y(x) = 0$ at $x = x_a$ and $x = x_b$, for all t. Assuming $\tau(x)$ does not vanish, this can happen in one of two ways: at each endpoint either $y(x)$ is fixed or $y'(x)$ vanishes.

Assuming that either $y(x)$ is fixed or $y'(x) = 0$ at the endpoints $x = x_a$ and $x = x_b$, the Euler-Lagrange equations for the string are obtained by setting $\delta S = 0$:

$$\begin{aligned} 0 = \frac{\delta S}{\delta y(x,t)} &= \frac{\partial \mathcal{L}}{\partial y} - \frac{\partial}{\partial t}\bigg(\frac{\partial \mathcal{L}}{\partial \dot{y}}\bigg) - \frac{\partial}{\partial x}\bigg(\frac{\partial \mathcal{L}}{\partial y'}\bigg) \\ &= \frac{\partial}{\partial x}\bigg[\tau(x)\,\frac{\partial y}{\partial x}\bigg] - \mu(x)\,\frac{\partial^2 y}{\partial t^2} \quad , \end{aligned} \tag{10.8}$$

where $y' = \frac{\partial y}{\partial x}$ and $\dot{y} = \frac{\partial y}{\partial t}$. When $\tau(x) = \tau$ and $\mu(x) = \mu$ are both constants, we obtain the Helmholtz equation,

$$\frac{1}{c^2}\,\frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} = 0 \quad , \tag{10.9}$$

which is the wave equation for the string, where $c = \sqrt{\tau/\mu}$ has dimensions of velocity. We will now see that c is the speed of wave propagation on the string.

[1] As an example of a string with a position-dependent tension, consider a string of length ℓ freely suspended from one end at $z = 0$ in a gravitational field. The tension is then $\tau(z) = \mu g\,(\ell - z)$.

10.1.2 d'Alembert's solution to the wave equation

Let us define two new variables,

$$u \equiv x - ct \qquad , \qquad v \equiv x + ct \quad . \tag{10.10}$$

We then have

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial u}{\partial x}\,\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\,\frac{\partial}{\partial v} = \frac{\partial}{\partial u} + \frac{\partial}{\partial v} \\ \frac{1}{c}\,\frac{\partial}{\partial t} &= \frac{1}{c}\,\frac{\partial u}{\partial t}\,\frac{\partial}{\partial u} + \frac{1}{c}\,\frac{\partial v}{\partial t}\,\frac{\partial}{\partial v} = -\frac{\partial}{\partial u} + \frac{\partial}{\partial v} \quad . \end{aligned} \tag{10.11}$$

Thus,

$$\frac{1}{c^2}\,\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} = -4\,\frac{\partial^2}{\partial u\,\partial v} \quad . \tag{10.12}$$

Thus, the wave equation may be solved:

$$\frac{\partial^2 y}{\partial u\,\partial v} = 0 \qquad \Longrightarrow \qquad y(u,v) = f(u) + g(v) \quad , \tag{10.13}$$

where $f(u)$ and $g(v)$ are arbitrary functions. For the moment, we work with an infinite string, so we have no spatial boundary conditions to satisfy. Note that $f(u)$ describes a right-moving disturbance, and $g(v)$ describes a left-moving disturbance:

$$y(x,t) = f(x-ct) + g(x+ct) \quad . \tag{10.14}$$

We do, however, have boundary conditions in time. At $t=0$, the configuration of the string is given by $y(x,0)$, and its instantaneous vertical velocity is $\dot{y}(x,0)$. We then have

$$y(x,0) = f(x) + g(x) \qquad , \qquad \dot{y}(x,0) = -c\,f'(x) + c\,g'(x) \quad , \tag{10.15}$$

hence

$$f'(x) = \tfrac{1}{2}\,y'(x,0) - \tfrac{1}{2c}\,\dot{y}(x,0) \qquad , \qquad g'(x) = \tfrac{1}{2}\,y'(x,0) + \tfrac{1}{2c}\,\dot{y}(x,0) \quad , \tag{10.16}$$

and integrating we obtain the right and left moving components

$$\begin{aligned} f(\xi) &= \tfrac{1}{2}\,y(\xi,0) - \frac{1}{2c}\int\limits_0^\xi \! d\xi'\;\dot{y}(\xi',0) - \mathcal{C} \\ g(\xi) &= \tfrac{1}{2}\,y(\xi,0) + \frac{1}{2c}\int\limits_0^\xi \! d\xi'\;\dot{y}(\xi',0) + \mathcal{C} \quad , \end{aligned} \tag{10.17}$$

where $\mathcal{C}$ is an arbitrary constant. Adding these together, we obtain the full solution

$$y(x,t) = \frac{1}{2}\Big[y(x-ct,0) + y(x+ct,0)\Big] + \frac{1}{2c}\int\limits_{x-ct}^{x+ct}\! d\xi\;\dot{y}(\xi,0) \quad , \tag{10.18}$$

valid for all times.

10.1.3 Energy density and energy current

The Hamiltonian density for a string is

$$\mathcal{H} = \wp\, \dot{y} - \mathcal{L} \quad , \tag{10.19}$$

where $\wp = \partial \mathcal{L}/\partial \dot{y} = \mu\, \dot{y}$ is the momentum density *transverse* to the string. Thus,

$$\mathcal{H} = \frac{\wp^2}{2\mu} + \tfrac{1}{2}\tau y'^2 \quad . \tag{10.20}$$

Expressed in terms of $\dot{y}$ rather than $\wp$, this is the energy density $\mathcal{E}$,

$$\mathcal{E} = \tfrac{1}{2}\mu\, \dot{y}^2 + \tfrac{1}{2}\tau y'^2 \quad . \tag{10.21}$$

We now evaluate $\dot{\mathcal{E}}$ for a solution to the equations of motion:

$$\begin{aligned}\frac{\partial \mathcal{E}}{\partial t} = \mu\, \frac{\partial y}{\partial t}\, \frac{\partial^2 y}{\partial t^2} + \tau\, \frac{\partial y}{\partial x}\, \frac{\partial^2 y}{\partial t\, \partial x} &= \frac{\partial y}{\partial t}\, \frac{\partial}{\partial x}\bigg(\tau\, \frac{\partial y}{\partial x}\bigg) + \tau\, \frac{\partial y}{\partial x}\, \frac{\partial^2 y}{\partial t\, \partial x} \\ &= \frac{\partial}{\partial x}\bigg[\tau\, \frac{\partial y}{\partial x}\, \frac{\partial y}{\partial t}\bigg] \equiv -\frac{\partial j_{\mathcal{E}}}{\partial x} \quad ,\end{aligned} \tag{10.22}$$

where the *energy current density* (or energy flux) *along the string* is

$$j_{\mathcal{E}} = -\tau\, \frac{\partial y}{\partial x}\, \frac{\partial y}{\partial t} \quad . \tag{10.23}$$

We therefore have that solutions of the equation of motion also obey the energy *continuity equation*

$$\frac{\partial \mathcal{E}}{\partial t} + \frac{\partial j_{\mathcal{E}}}{\partial x} = 0 \quad . \tag{10.24}$$

Let us integrate the above equation between points x_1 and x_2. We obtain

$$\frac{\partial}{\partial t} \int\limits_{x_1}^{x_2} dx\, \mathcal{E}(x,t) = -\int\limits_{x_1}^{x_2} dx\, \frac{\partial j_{\mathcal{E}}(x,t)}{\partial x} = j_{\mathcal{E}}(x_1,t) - j_{\mathcal{E}}(x_2,t) \quad , \tag{10.25}$$

which says that the time rate of change of the energy contained in the interval $\big[x_1, x_2\big]$ is equal to the difference between the entering and exiting energy flux.

When $\tau(x) = \tau$ and $\mu(x) = \mu$, we have

$$y(x,t) = f(x - ct) + g(x + ct) \tag{10.26}$$

and we find

$$\begin{aligned}\mathcal{E}(x,t) &= \tau\big[f'(x - ct)\big]^2 + \tau\big[g'(x + ct)\big]^2 \\ j_{\mathcal{E}}(x,t) &= c\tau\big[f'(x - ct)\big]^2 - c\tau\big[g'(x + ct)\big]^2 \quad ,\end{aligned} \tag{10.27}$$

which are each sums over right-moving and left-moving contributions.

Another example is the Klein-Gordon system, for which the Lagrangian density is

$$\mathcal{L} = \tfrac{1}{2}\mu\dot{y}^2 - \tfrac{1}{2}\tau y'^2 - \tfrac{1}{2}\beta y^2 \quad . \tag{10.28}$$

One obtains the equation of motion $\mu\ddot{y} = \tau y'' - \beta y$ and the energy density

$$\mathcal{E} = \tfrac{1}{2}\mu\dot{y}^2 + \tfrac{1}{2}\tau y'^2 + \tfrac{1}{2}\beta y^2 \quad . \tag{10.29}$$

It is left as an exercise to the student to check that the energy current, $j_{\mathcal{E}}$, is the same as in the Helmholtz case: $j_{\mathcal{E}} = -\tau\,\dot{y}\,y'$. Energy continuity is again given by $\partial_t\,\mathcal{E} + \partial_x\,j_{\mathcal{E}} = 0$. Note that solutions to the Klein-Gordon equation of motion are not of the D'Alembert form.

Momentum flux density and stress energy tensor

Let's now examine the spatial derivative $\mathcal{E}'$. For the Helmholtz equation, $\mathcal{E} = \frac{1}{2}\mu\dot{y}^2 + \frac{1}{2}\tau y'^2$. We assume $\mu(x) = \mu$ and $\tau(x) = \tau$ are constant. Then

$$\frac{\partial\mathcal{E}}{\partial x} = \mu\,\dot{y}\,\dot{y}' + \tau\,y'y'' = \frac{\partial}{\partial t}\big(\mu\,\dot{y}\,y'\big) \quad , \tag{10.30}$$

where we have invoked the equation of motion $\tau\,y'' = \mu\,\ddot{y}$. Thus, we may write

$$\frac{\partial\mathit{\Pi}}{\partial t} + \frac{\partial j_{\mathit{\Pi}}}{\partial x} = 0 \quad , \tag{10.31}$$

where

$$\mathit{\Pi} = -\mu\,\frac{\partial y}{\partial t}\,\frac{\partial y}{\partial x} = \frac{j_{\mathcal{E}}}{c^2} \qquad , \qquad j_{\mathit{\Pi}} = \mathcal{E} \quad . \tag{10.32}$$

$\mathit{\Pi}$ is the *momentum flux density along the string*. Eqn. 10.31 is thus a continuity equation for momentum, with the *energy density* playing the role of the *momentum current*. Note that $\mathit{\Pi}$ and $\wp = \mu\dot{y}$ have the same dimensions, but the former is the momentum density *along the string* while the latter is the momentum density *transverse to the string*. We may now write

$$\begin{pmatrix} \frac{\partial}{\partial t} & \frac{\partial}{\partial x} \end{pmatrix} \overbrace{\begin{pmatrix} \mathcal{E} & -\mathit{\Pi} \\ j_{\mathcal{E}} & -j_{\mathit{\Pi}} \end{pmatrix}}^{T^{\mu}{}_{\nu}} = 0 \quad , \tag{10.33}$$

where $\mathit{\Pi} = j_{\mathcal{E}}/c^2$ and $j_{\mathit{\Pi}} = \mathcal{E}$ for the Helmholtz model. In component notation this is neatly expressed as $\partial_\mu\,T^{\mu}{}_{\nu} = 0$, where $T^{\mu}{}_{\nu}$ is the *stress-energy tensor* and $\partial_\mu = \big(\partial_t\,,\partial_x\big)$.

Below in eqn. 10.184, we will see how the general result for the stress-energy tensor is

$$T^{\mu}{}_{\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\,y)}\,\partial_\nu\,y - \delta^{\mu}_{\nu}\,\mathcal{L} \quad , \tag{10.34}$$

where $\mu,\nu \in \{0,1\}$. For $\mathcal{L} = \frac{1}{2}\mu\dot{y}^2 - \frac{1}{2}\tau y'^2$, we recover the stress-energy tensor for the Helmholtz model in eqn. 10.33. For the Klein-Gordon model, $\mathcal{L} = \frac{1}{2}\mu\dot{y}^2 - \frac{1}{2}\tau y'^2 - \frac{1}{2}\beta y^2$, we find once again $T^{0}{}_{1} = -\mathit{\Pi}$ but $T^{1}{}_{1} = -\frac{1}{2}\mu\dot{y}^2 - \frac{1}{2}\tau y'^2 + \frac{1}{2}\beta y^2$ so $T^{1}{}_{1} \neq -\mathcal{E}$.

Energy and momentum continuity in electrodynamics

A similar energy continuity equation pertains in electrodynamics. Recall $\mathcal{E} = \frac{1}{8\pi}(\boldsymbol{E}^2 + \boldsymbol{B}^2)$ is the energy density. We then have

$$\begin{aligned}\frac{\partial\mathcal{E}}{\partial t} &= \frac{1}{4\pi}\bigg(\boldsymbol{E}\cdot\frac{\partial\boldsymbol{E}}{\partial t} + \boldsymbol{B}\cdot\frac{\partial\boldsymbol{B}}{\partial t}\bigg)\\ &= \frac{1}{4\pi}\,\boldsymbol{E}\cdot(c\,\boldsymbol{\nabla}\times\boldsymbol{B} - 4\pi\boldsymbol{J}) + \frac{1}{4\pi}\boldsymbol{B}\cdot(-c\,\boldsymbol{\nabla}\times\boldsymbol{E}) \qquad (10.35)\\ &= -\boldsymbol{E}\cdot\boldsymbol{J} - \boldsymbol{\nabla}\cdot\overbrace{\Big(\frac{c}{4\pi}\,\boldsymbol{E}\times\boldsymbol{B}\Big)}^{\text{Poynting vector }\boldsymbol{S}} \quad .\end{aligned}$$

Thus,

$$\frac{\partial\mathcal{E}}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{S} = -\boldsymbol{J}\cdot\boldsymbol{E} \quad , \tag{10.36}$$

which resembles a continuity equation, but with a 'sink' term on the RHS to account for the local power dissipated. If $\boldsymbol{J} = \sigma\boldsymbol{E}$, where σ is the conductivity, then $\boldsymbol{J}{\cdot}\boldsymbol{E} = \sigma\boldsymbol{E}^2$, which accounts for *Ohmic dissipation*.

The stress-energy tensor for Maxwell theory is given by

$$T^\mu_{\ \nu} = \begin{pmatrix} \mathcal{E} & -S_x/c & -S_y/c & -S_z/c \\ S_x/c & \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ S_y/c & \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ S_z/c & \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \tag{10.37}$$

where $\mathcal{E} = \frac{1}{8\pi}(\boldsymbol{E}^2 + \boldsymbol{B}^2)$ is the energy density, $\boldsymbol{S} = \frac{c}{4\pi}\boldsymbol{E}\times\boldsymbol{B}$ is the Poynting vector, and

$$\sigma_{ij} = \frac{1}{4\pi}\Big\{-E_i\,E_j - B_i\,B_j + \tfrac{1}{2}\delta_{ij}\big(\boldsymbol{E}^2 + \boldsymbol{B}^2\big)\Big\} \tag{10.38}$$

is the *Maxwell stress tensor*. One again has $\partial_\mu\,T^\mu_{\ \nu} = 0$, this time with $\partial_\mu = \big(\frac{1}{c}\partial_t\,,\partial_x\,,\partial_y\,,\partial_z\big)$.

10.1.4 Reflection at an interface

Consider a semi-infinite string on the interval $\big[0,\infty\big]$, with $y(0,t) = 0$. We can still invoke d'Alembert's solution, $y(x,t) = f(x-ct) + g(x+ct)$, but we must demand

$$y(0,t) = f(-ct) + g(ct) = 0 \quad\Rightarrow\quad f(\xi) = -g(-\xi) \quad . \tag{10.39}$$

Thus,

$$y(x,t) = g(ct+x) - g(ct-x) \quad . \tag{10.40}$$

Now suppose $g(\xi)$ describes a pulse, and is nonzero only within a neighborhood of $\xi = 0$. For large negative values of t, the right-moving part, $-g(ct-x)$, is negligible everywhere, since $x > 0$ means that the argument $ct - x$ is always large and negative. On the other hand, the left moving part $g(ct+x)$ is nonzero for $x \approx -ct > 0$. Thus, for $t < 0$ we have a left-moving pulse incident from the right. For $t > 0$,

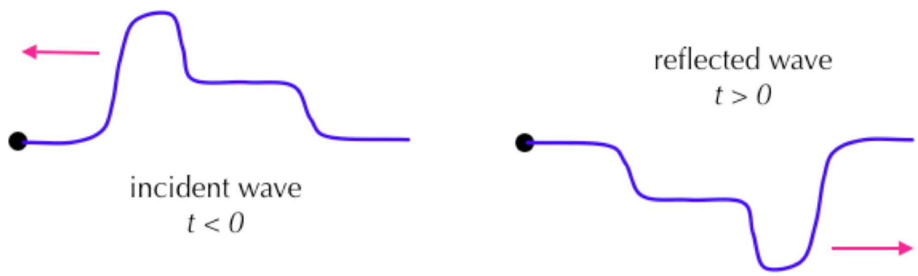


Figure 10.2: Reflection of a pulse at an interface at $x = 0$, with $y(0, t) = 0$.

the situation is reversed, and the left-moving component is negligible, and we have a right moving reflected wave. However, the minus sign in eqn. 10.39 means that the reflected wave is *inverted.*

If instead of fixing the endpoint at $x = 0$ we attach this end of the string to a massless ring which frictionlessly slides up and down a vertical post, then we must have $y'(0, t) = 0$, else there is a finite vertical force on the massless ring, resulting in infinite acceleration. We again write $y(x, t) = f(x - ct) + g(x + ct)$, and we invoke

$$y'(0, t) = f'(-ct) + g'(ct) \quad \Rightarrow \quad f'(\xi) = -g'(-\xi) \quad , \tag{10.41}$$

which, upon integration, yields $f(\xi) = g(-\xi)$, and therefore

$$y(x, t) = g(ct + x) + g(ct - x) \quad . \tag{10.42}$$

The reflected pulse is now 'right-side up', in contrast to the situation with a fixed endpoint.

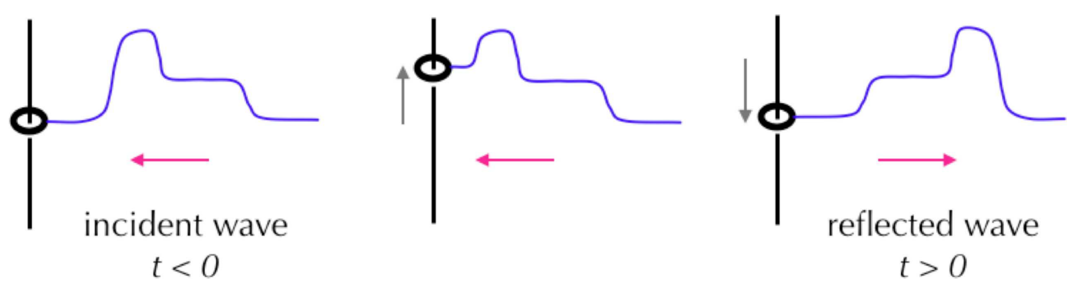


Figure 10.3: Reflection of a pulse at an interface at $x = 0$, with $y'(0, t) = 0$.

10.1.5 Mass point on a string

Next, consider the case depicted in fig. 10.4, where a point mass m is affixed to an infinite string at $x = 0$. Let us suppose that at large negative values of t, a right moving wave $f(ct - x)$ is incident from the left. The full solution may then be written as a sum of incident, reflected, and transmitted waves:

$$\begin{aligned} x < 0 \quad &: \quad y(x, t) = f(ct - x) + g(ct + x) \\ x > 0 \quad &: \quad y(x, t) = h(ct - x) \quad . \end{aligned} \tag{10.43}$$

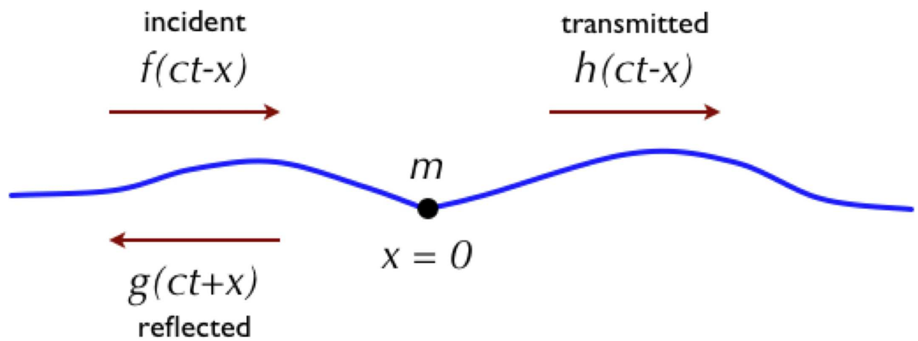


Figure 10.4: Reflection and transmission at an impurity. A point mass m is affixed to an infinite string at $x = 0$.

At $x = 0$, we invoke Newton's second Law, $F = ma$:

$$m\,\ddot{y}(0,t) = \tau\,y'(0^+,t) - \tau\,y'(0^-,t) \quad . \tag{10.44}$$

Any discontinuity in the derivative $y'(x,t)$ at $x = 0$ results in an acceleration of the point mass. Note that

$$y'(0^-,t) = -f'(ct) + g'(ct) \qquad , \qquad y'(0^+,t) = -h'(ct) \quad . \tag{10.45}$$

Further invoking continuity at $x = 0$, *i.e.* $y(0^-,t) = y(0^+,t)$, we have

$$h(\xi) = f(\xi) + g(\xi) \quad , \tag{10.46}$$

and eqn. 10.44 becomes

$$g''(\xi) + \frac{2\tau}{mc^2}\,g'(\xi) = -f''(\xi) \quad . \tag{10.47}$$

We solve this equation by Fourier analysis:

$$f(\xi) = \int\limits_{-\infty}^{\infty} \frac{dk}{2\pi}\,\hat{f}(k)\,e^{ik\xi} \qquad , \qquad \hat{f}(k) = \int\limits_{-\infty}^{\infty} d\xi\,f(\xi)\,e^{-ik\xi} \quad . \tag{10.48}$$

Defining $Q \equiv 2\tau/mc^2 = 2\mu/m$, we have

$$\big[-k^2 + iQk\big]\,\hat{g}(k) = k^2\,\hat{f}(k) \quad . \tag{10.49}$$

We may now write $\hat{g}(k) = \hat{r}(k)\,\hat{f}(k)$ and $\hat{h}(k) = \hat{t}(k)\,\hat{f}(k)$, where

$$\hat{r}(k) = -\frac{k}{k - iQ} \qquad , \qquad \hat{t}(k) = -\frac{iQ}{k - iQ} \tag{10.50}$$

are the *reflection and transmission amplitudes*, respectively.

Energy conservation

Note that $\hat{t}(k) = 1 + \hat{r}(k)$. This relation follows from continuity at $x = 0$, which entails $h(\xi) = f(\xi) + g(\xi)$, hence $\hat{h}(k) = \hat{f}(k) + \hat{g}(k)$. What is also true – *if there is no dissipation* – is

$$\big|\hat{r}(k)\big|^2 + \big|\hat{t}(k)\big|^2 = 1 \quad , \tag{10.51}$$

which is a statement of energy conservation. Integrating the energy density of the string itself, one finds

$$\begin{aligned} E_{\text{string}}(t) &= \int\limits_{-\infty}^{\infty} \! dx \, \Big(\tfrac{1}{2}\mu \dot{y}^2 + \tfrac{1}{2}\tau y'^2\Big) \\ &= \tau \int\limits_{ct}^{\infty} \! d\xi \, \big[f'(\xi)\big]^2 + \tau \int\limits_{-\infty}^{ct} \! d\xi \, \Big(\big[g'(\xi)\big]^2 + \big[h'(\xi)\big]^2\Big) \quad . \end{aligned} \tag{10.52}$$

What is missing from this expression is the kinetic energy of the mass point. However, as $t \to \pm\infty$, the kinetic energy of the mass point vanishes; it starts from rest, and as $t \to \infty$ it shakes off all its energy into waves on the string. Therefore

$$\begin{aligned} E_{\text{string}}(-\infty) &= \tau \int\limits_{-\infty}^{\infty} \! d\xi \, \big[f'(\xi)\big]^2 = \tau \int\limits_{-\infty}^{\infty} \! \frac{dk}{2\pi} \, k^2 \, \big|\hat{f}(k)\big|^2 \\ E_{\text{string}}(+\infty) &= \tau \int\limits_{-\infty}^{\infty} \! d\xi \, \Big(\big[g'(\xi)\big]^2 + \big[h'(\xi)\big]^2\Big) = \tau \int\limits_{-\infty}^{\infty} \! \frac{dk}{2\pi} \, k^2 \, \Big(\big|\hat{r}(k)\big|^2 + \big|\hat{t}(k)\big|^2\Big) \big|\hat{f}(k)\big|^2 \quad , \end{aligned} \tag{10.53}$$

and since the profile $\hat{f}(k)$ is arbitrary we conclude that eqn. 10.51 must hold *for every possible value of the wavevector* k. It must be stressed energy conservation holds *only if there is no dissipation*. Dissipation could be modeled by adding a friction term $-\gamma \, \dot{y}(0,t)$ to the RHS of eqn. 10.44. In this case, $dE_{\text{string}}(t)/dt$ would be negative, corresponding to the energy loss due to friction.

Real space form of the solution

Getting back to our solution, in real space we have

$$\begin{aligned} h(\xi) &= \int\limits_{-\infty}^{\infty} \! \frac{dk}{2\pi} \, \hat{t}(k) \, \hat{f}(k) \, e^{ik\xi} \\ &= \int\limits_{-\infty}^{\infty} \! d\xi' \left[\int\limits_{-\infty}^{\infty} \! \frac{dk}{2\pi} \, \hat{t}(k) \, e^{ik(\xi-\xi')} \right] f(\xi') \equiv \int\limits_{-\infty}^{\infty} \! d\xi' \, t(\xi - \xi') \, f(\xi') \quad , \end{aligned} \tag{10.54}$$

where

$$t(\xi - \xi') = \int\limits_{-\infty}^{\infty} \! \frac{dk}{2\pi} \, \hat{t}(k) \, e^{ik(\xi-\xi')} \quad , \tag{10.55}$$

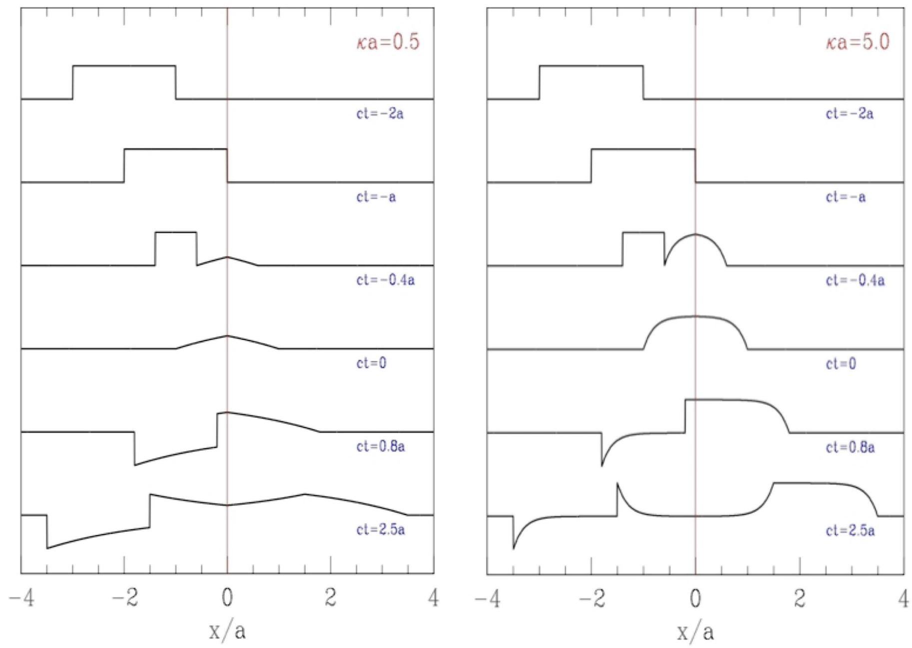


Figure 10.5: Reflection and transmission of a square wave pulse by a point mass at $x = 0$. The configuration of the string is shown for six different times, for $Qa = 0.5$ (left panel; note $\kappa \equiv Q$) and $Qa = 5.0$ (right panel). Note that the $Qa = 0.5$ case, which corresponds to a large mass $m = 2\mu/Q$, results in strong reflection with inversion, and weak transmission. For large Q, corresponding to small mass m, the reflection is weak and the transmission is strong.

is the transmission kernel in real space. For our example with $\hat{t}(k) = -iQ/(k - iQ)$, the integral is done easily using the method of contour integration:

$$t(\xi - \xi') = \int\limits_{-\infty}^{\infty} \frac{dk}{2\pi}\, \frac{-iQ}{k - iQ}\, e^{ik(\xi - \xi')} = Q\, e^{-Q(\xi - \xi')}\, \Theta(\xi - \xi') \quad . \tag{10.56}$$

Therefore,

$$h(\xi) = Q \int\limits_{-\infty}^{\xi} d\xi'\, e^{-Q(\xi - \xi')}\, f(\xi') \quad , \tag{10.57}$$

and of course $g(\xi) = h(\xi) - f(\xi)$. Note that $m = \infty$ means $Q = 0$, in which case $\hat{r}(k) = -1$ and $\hat{t}(k) = 0$. Thus we recover the inversion of the pulse shape under reflection found earlier.

For example, let the incident pulse shape be $f(\xi) = b\,\Theta\big(a - |\xi|\big)$. Then

$$\begin{aligned} h(\xi) &= Q \int\limits_{-\infty}^{\xi} d\xi'\, e^{-Q(\xi - \xi')}\, b\,\Theta(a - \xi')\,\Theta(a + \xi') \\ &= b\, e^{-Q\xi} \Big[e^{Q \min(a,\xi)} - e^{-Qa} \Big]\, \Theta(\xi + a) \quad . \end{aligned} \tag{10.58}$$

Taking cases,

$$h(\xi)=\begin{cases}0 & \text{if } \xi<-a\\ b\left(1-e^{-Q(a+\xi)}\right) & \text{if } -a<\xi<a\\ 2b\,e^{-Q\xi}\,\sinh(Qa) & \text{if } \xi>a\quad.\end{cases} \tag{10.59}$$

In fig. 10.5 we show the reflection and transmission of this square pulse for two different values of Qa.

10.1.6 Interface between strings of different mass density

Consider the situation in fig. 10.6, where the string for $x<0$ is of density $\mu_{\rm L}$ and for $x>0$ is of density $\mu_{\rm R}$. The d'Alembert solution in the two regions, with an incoming wave from the left, is

$$\begin{aligned} x<0:&\quad y(x,t)=f(c_{\rm L}t-x)+g(c_{\rm L}t+x)\\ x>0:&\quad y(x,t)=h(c_{\rm R}t-x)\quad. \end{aligned} \tag{10.60}$$

At $x=0$ we have

$$\begin{aligned} f(c_{\rm L}t)+g(c_{\rm L}t)&=h(c_{\rm R}t)\\ -f'(c_{\rm L}t)+g'(c_{\rm L}t)&=-h'(c_{\rm R}t)\quad, \end{aligned} \tag{10.61}$$

where the second equation follows from $\tau\,y'(0^+,t)=\tau\,y'(0^-,t)$, so there is no finite vertical force on the infinitesimal interval bounding $x=0$, which contains infinitesimal mass. Defining $\alpha\equiv c_{\rm R}/c_{\rm L}$, we integrate the second of these equations and have

$$f(\xi)+g(\xi)=h(\alpha\,\xi)\qquad,\qquad f(\xi)-g(\xi)=\alpha^{-1}\,h(\alpha\,\xi)\quad. \tag{10.62}$$

Note that $y(\pm\infty,0)=0$ fixes the constant of integration. The solution is then

$$g(\xi)=\frac{\alpha-1}{\alpha+1}\,f(\xi)\qquad,\qquad h(\xi)=\frac{2\alpha}{\alpha+1}\,f(\xi/\alpha)\quad. \tag{10.63}$$

Thus,

$$\begin{aligned} x<0:&\quad y(x,t)=f\big(c_{\rm L}t-x\big)+\left(\frac{\alpha-1}{\alpha+1}\right)f\big(c_{\rm L}t+x\big)\\ x>0:&\quad y(x,t)=\frac{2\alpha}{\alpha+1}\,f\big((c_{\rm R}t-x)/\alpha\big)\quad. \end{aligned} \tag{10.64}$$

It is instrutive to compute the total energy in the string. For large negative values of the time t, the entire disturbance is confined to the region $x<0$. The energy is

$$E(-\infty)=\tau\int\limits_{-\infty}^{\infty}\!d\xi\,\big[f'(\xi)\big]^2\quad. \tag{10.65}$$

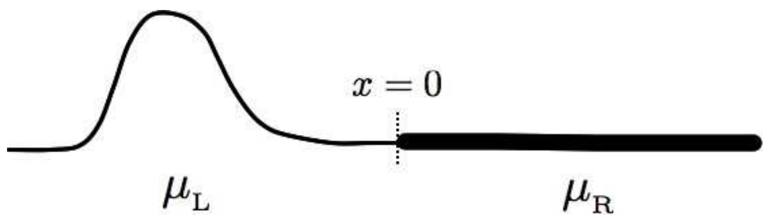


Figure 10.6: A string formed from two semi-infinite regions of different densities.

For large positive times, the wave consists of the left-moving reflected $g(\xi)$ component in the region $x < 0$ and the right-moving transmitted component $h(\xi)$ in the region $x > 0$. The energy in the reflected wave is

$$E_{\rm L}(+\infty) = \tau \left(\frac{\alpha - 1}{\alpha + 1}\right)^2 \int\limits_{-\infty}^{\infty} d\xi \, \big[f'(\xi)\big]^2 \quad . \tag{10.66}$$

For the transmitted portion, we use

$$y'(x > 0, t) = \frac{2}{\alpha + 1} \, f'\big((c_{\rm R} t - x)/\alpha\big) \tag{10.67}$$

to obtain

$$E_{\rm R}(\infty) = \frac{4\tau}{(\alpha+1)^2} \int\limits_{-\infty}^{\infty} d\xi \, \big[f'(\xi/\alpha)\big]^2 = \frac{4\alpha\tau}{(\alpha+1)^2} \int\limits_{-\infty}^{\infty} d\xi \, \big[f'(\xi)\big]^2 \quad . \tag{10.68}$$

Thus, $E_{\rm L}(\infty) + E_{\rm R}(\infty) = E(-\infty)$, and energy is conserved.

10.1.7 Finite Strings: Bernoulli's solution

Suppose $x_a = 0$ and $x_b = L$ are the boundaries of the string, where $y(0,t) = y(L,t) = 0$. Again we write

$$y(x,t) = f(x - ct) + g(x + ct) \quad . \tag{10.69}$$

Applying the boundary condition at $x_a = 0$ gives, as earlier,

$$y(x,t) = g(ct + x) - g(ct - x) \quad . \tag{10.70}$$

Next, we apply the boundary condition at $x_b = L$, which results in

$$g(ct + L) - g(ct - L) = 0 \quad \Longrightarrow \quad g(\xi) = g(\xi + 2L) \quad . \tag{10.71}$$

Thus, $g(\xi)$ is periodic, with period $2L$. Any such function may be written as a Fourier sum,

$$g(\xi) = \sum_{n=1}^{\infty} \left\{ \mathcal{A}_n \cos\left(\frac{n\pi\xi}{L}\right) + \mathcal{B}_n \sin\left(\frac{n\pi\xi}{L}\right) \right\} \quad . \tag{10.72}$$

The full solution for $y(x,t)$ is then

$$\begin{aligned} y(x,t) &= g(ct+x) - g(ct-x) \\ &= \left(\frac{2}{\mu L}\right)^{1/2} \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \left\{ A_n \cos\left(\frac{n\pi ct}{L}\right) + B_n \sin\left(\frac{n\pi ct}{L}\right) \right\} \quad , \end{aligned} \tag{10.73}$$

where $A_n = \sqrt{2\mu L}\,\mathcal{B}_n$ and $B_n = -\sqrt{2\mu L}\,\mathcal{A}_n$. This is known as Bernoulli's solution.

We define the functions

$$\psi_n(x) \equiv \left(\frac{2}{\mu L}\right)^{1/2} \sin\left(\frac{n\pi x}{L}\right) \quad . \tag{10.74}$$

We also write

$$k_n \equiv \frac{n\pi}{L} \quad , \quad \omega_n \equiv \frac{n\pi c}{L} \quad , \quad n = 1, 2, 3, \ldots, \infty \quad . \tag{10.75}$$

Thus, $\psi_n(x) = \sqrt{2/\mu L}\ \sin(k_n x)$ has $(n+1)$ nodes at $x = jL/n$, for $j \in \{0, \ldots, n\}$. Note that

$$\langle \psi_m \,|\, \psi_n \rangle \equiv \int_0^L dx\, \mu\, \psi_m(x)\, \psi_n(x) = \delta_{mn} \quad . \tag{10.76}$$

Furthermore, this basis is complete:

$$\mu \sum_{n=1}^{\infty} \psi_n(x)\, \psi_n(x') = \delta(x - x') \quad . \tag{10.77}$$

Our general solution is thus equivalent to

$$y(x,0) = \sum_{n=1}^{\infty} A_n\, \psi_n(x) \qquad , \qquad \dot{y}(x,0) = \sum_{n=1}^{\infty} \omega_n\, B_n\, \psi_n(x) \quad . \tag{10.78}$$

The Fourier coefficients $\{A_n, B_n\}$ may be extracted from the initial data using the orthonormality of the basis functions and their associated resolution of unity:

$$A_n = \int_0^L dx\, \mu\, \psi_n(x)\, y(x,0) \qquad , \qquad B_n = \frac{1}{\omega_n} \int_0^L dx\, \mu\, \psi_n(x)\, \dot{y}(x,0) \quad . \tag{10.79}$$

As an example, suppose our initial configuration is a triangle, with

$$y(x,0) = \begin{cases} 2bx/L & \text{if } 0 \le x \le \frac{1}{2}L \\ \\ 2b(L-x)/L & \text{if } \frac{1}{2}L \le x \le L \quad , \end{cases} \tag{10.80}$$

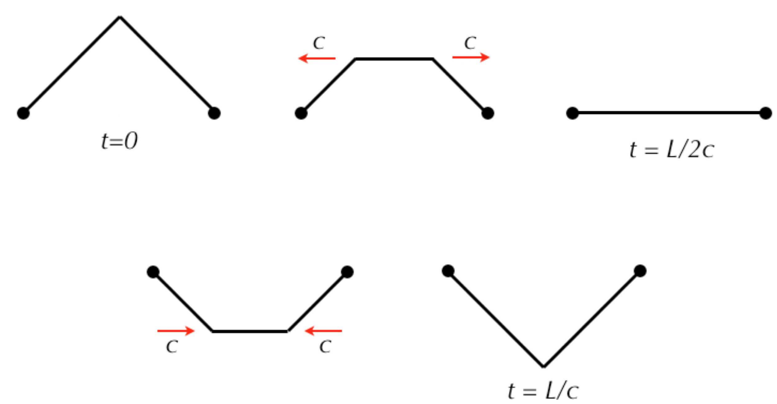


Figure 10.7: Evolution of a string with fixed ends starting from an isosceles triangle shape.

and $\dot{y}(x,0) = 0$. Then $B_n = 0$ for all n, while

$$\begin{aligned} A_n &= \left(\frac{2\mu}{L}\right)^{1/2} \cdot \frac{2b}{L} \left\{ \int\limits_0^{L/2} \! dx \, x \, \sin\left(\frac{n\pi x}{L}\right) + \int\limits_{L/2}^{L} \! dx \, (L-x) \, \sin\left(\frac{n\pi x}{L}\right) \right\} \\ &= (2\mu L)^{1/2} \cdot \frac{4b}{n^2\pi^2} \, \sin\left(\tfrac{1}{2}n\pi\right) \quad , \end{aligned} \tag{10.81}$$

after changing variables to $x = L\theta/n\pi$ and using $\theta \, \sin\theta \, d\theta = d\big(\sin\theta - \theta\, \cos\theta\big)$. Another way to write this is to separately give the results for even and odd coefficients:

$$A_{2k} = 0 \qquad , \qquad A_{2k+1} = \frac{4b}{\pi^2} \, (2\mu L)^{1/2} \cdot \frac{(-1)^k}{(2k+1)^2} \quad . \tag{10.82}$$

Note that each $\psi_{2k}(x) = -\psi_{2k}(L-x)$ is antisymmetric about the midpoint $x = \frac{1}{2}L$, for all k. Since our initial conditions are that $y(x,0)$ is symmetric about $x = \frac{1}{2}L$, none of the even order eigenfunctions can enter into the expansion, precisely as we have found. The d'Alembert solution to this problem is particularly simple and is shown in fig. 10.7. Note that $g(x) = \frac{1}{2}y(x,0)$ must be extended to the entire real line. We know that $g(x) = g(x+2L)$ is periodic with spatial period $2L$, but how do we extend $g(x)$ from the interval $\big[0,L\big]$ to the interval $\big[-L,0\big]$? To do this, we use $y(x,0) = g(x) - g(-x)$, which says that $g(x)$ must be *antisymmetric*, *i.e.* $g(x) = -g(-x)$. Equivalently, $\dot{y}(x,0) = cg'(x) - cg'(-x) = 0$, which integrates to $g(x) = -g(-x)$.

10.2 Sturm-Liouville Theory

10.2.1 Mathematical formalism

Consider the Lagrangian density

$$\mathcal{L} = \tfrac{1}{2}\,\mu(x)\,\dot{y}^2 - \tfrac{1}{2}\,\tau(x)\,{y'}^2 - \tfrac{1}{2}\,v(x)\,y^2 \quad . \tag{10.83}$$

The last term is new and has the physical interpretation of a harmonic potential which attracts the string to the line $y = 0$. The Euler-Lagrange equations are then

$$-\frac{\partial}{\partial x}\left[\tau(x)\,\frac{\partial y}{\partial x}\right] + v(x)\,y = -\mu(x)\,\frac{\partial^2 y}{\partial t^2} \quad . \tag{10.84}$$

This equation is invariant under time translation. Thus, if $y(x,t)$ is a solution, then so is $y(x,t+t_0)$, for any t_0. This means that the solutions can be chosen to be eigenstates of the operator ∂_t, which is to say $y(x,t) = \psi(x)\,e^{-i\omega t}$. Because the coefficients are real, both y and y^* are solutions, and taking linear combinations we have

$$y(x,t) = \psi(x)\,\cos(\omega t + \phi) \quad . \tag{10.85}$$

Plugging this into eqn. 10.84, we obtain

$$-\frac{d}{dx}\Big[\tau(x)\,\psi'(x)\Big] + v(x)\,\psi(x) = \omega^2\,\mu(x)\,\psi(x) \quad . \tag{10.86}$$

This is the Sturm-Liouville equation. There are four types of boundary conditions that we shall consider:

1. Fixed endpoint: $\psi(x) = 0$, where $x = x_{a,b}$.
2. Natural: $\tau(x)\,\psi'(x) = 0$, where $x = x_{a,b}$.
3. Periodic: $\psi(x) = \psi(x+L)$, where $L = x_b - x_a$, with $\tau(x) = \tau(x+L)$ as well.
4. Mixed homogeneous: $\alpha\,\psi(x) + \beta\,\psi'(x) = 0$, where $x = x_{a,b}$.

The Sturm-Liouville equation is an eigenvalue equation. The eigenfunctions $\{\psi_n(x)\}$ satisfy

$$-\frac{d}{dx}\Big[\tau(x)\,\psi_n'(x)\Big] + v(x)\,\psi_n(x) = \omega_n^2\,\mu(x)\,\psi_n(x) \quad . \tag{10.87}$$

Now suppose we have a second solution $\psi_m(x)$, satisfying

$$-\frac{d}{dx}\Big[\tau(x)\,\psi_m'(x)\Big] + v(x)\,\psi_m(x) = \omega_m^2\,\mu(x)\,\psi_m(x) \quad . \tag{10.88}$$

Now multiply (10.87)* by $\psi_m(x)$ and (10.88) by $\psi_n^*(x)$ and subtract, yielding

$$\begin{aligned}\psi_n^*\,\frac{d}{dx}\Big[\tau\,\psi_m'\Big] - \psi_m\,\frac{d}{dx}\Big[\tau\,\psi'^*_n\Big] &= \big(\omega_n^{*2} - \omega_m^2\big)\,\mu\,\psi_m\,\psi_n^* \\ &= \frac{d}{dx}\Big[\tau\,\psi_n^*\,\psi_m' - \tau\,\psi_m\,\psi'^*_n\Big] \quad .\end{aligned} \tag{10.89}$$

We integrate this equation over the length of the string, to get

$$\big(\omega_n^{*2} - \omega_m^2\big)\int\limits_{x_a}^{x_b}\!dx\,\mu(x)\,\psi_n^*(x)\,\psi_m(x) = \Big[\tau(x)\,\psi_n^*(x)\,\psi_m'(x) - \tau(x)\,\psi_m(x)\,\psi'^*_n(x)\Big]_{x_a}^{x_b} = 0 \quad . \tag{10.90}$$

The term in square brackets vanishes for any of the four types of boundary conditions articulated above. Thus, we have

$$\big(\omega_n^{*\,2} - \omega_m^2\big)\, \langle\, \psi_n \,|\, \psi_m \,\rangle = 0 \quad , \tag{10.91}$$

where the inner product is defined as

$$\langle\, \psi \,|\, \phi \,\rangle \equiv \int\limits_{x_a}^{x_b}\! dx\, \mu(x)\, \psi^*(x)\, \phi(x) \quad . \tag{10.92}$$

The distribution $\mu(x)$ is non-negative definite. Setting $m = n$, we have $\langle\, \psi_n \,|\, \psi_n \,\rangle \geq 0$, and hence $\omega_n^{*\,2} = \omega_n^2$, which says that $\omega_n^2 \in \mathbb{R}$. When $\omega_m^2 \neq \omega_n^2$, the eigenfunctions are orthogonal with respect to the above inner product. In the case of degeneracies, we may invoke the Gram-Schmidt procedure, which orthogonalizes the eigenfunctions within a given degenerate subspace. Since the Sturm-Liouville equation is linear, we may normalize the eigenfunctions, taking

$$\langle\, \psi_m \,|\, \psi_n \,\rangle = \delta_{mn}. \tag{10.93}$$

Finally, since the coefficients in the Sturm-Liouville equation are all real, we can and henceforth do choose the eigenfunctions themselves to be real.

Another important result, which we will not prove here, is the *completeness* of the eigenfunction basis. Completeness means

$$\mu(x) \sum_n \psi_n^*(x)\, \psi_n(x') = \delta(x - x') \quad . \tag{10.94}$$

Thus, any function can be expanded in the eigenbasis, *viz.*

$$\phi(x) = \sum_n C_n\, \psi_n(x) \qquad , \qquad C_n = \langle\, \psi_n \,|\, \phi \,\rangle \quad . \tag{10.95}$$

10.2.2 Variational method

Consider the functional

$$\omega^2\big[\psi(x)\big] = \frac{\frac{1}{2}\int\limits_{x_a}^{x_b} dx\, \Big\{ \tau(x)\, \psi'^2(x) + v(x)\, \psi^2(x) \Big\}}{\frac{1}{2}\int\limits_{x_a}^{x_b} dx\, \mu(x)\, \psi^2(x)} \equiv \frac{\mathcal{N}}{\mathcal{D}} \quad . \tag{10.96}$$

The variation is

$$\delta\omega^2 = \frac{\delta\mathcal{N}}{\mathcal{D}} - \frac{\mathcal{N}\,\delta\mathcal{D}}{\mathcal{D}^2} = \frac{\delta\mathcal{N} - \omega^2\,\delta\mathcal{D}}{\mathcal{D}} \quad . \tag{10.97}$$

Thus, $\delta\omega^2 = 0$ requires $\delta\mathcal{N} = \omega^2\,\delta\mathcal{D}$, which says

$$-\frac{d}{dx}\bigg[\tau(x)\,\frac{d\psi(x)}{dx}\bigg] + v(x)\,\psi(x) = \omega^2\,\mu(x)\,\psi(x) \quad , \tag{10.98}$$

which is the Sturm-Lioiuville equation. In obtaining this equation, we have dropped a boundary term from $\delta\mathcal{N}$, given by

$$\delta\mathcal{N}\Big|_{\rm bdry} = \Big[\,\tau(x)\,\psi'(x)\,\delta\psi(x)\,\Big]_{x=x_a}^{x=x_b} \quad . \tag{10.99}$$

One can check that this expression vanishes for any of the first three classes of boundary conditions: (fixed endpoint, natural, and periodic). For the fourth class of boundary conditions, $\alpha\psi + \beta\psi' = 0$ (mixed homogeneous), the Sturm-Liouville equation may still be derived, provided one uses a slightly different functional,

$$\omega^2\big[\psi(x)\big] = \frac{\widetilde{\mathcal{N}}}{\mathcal{D}} \qquad \text{with} \qquad \widetilde{\mathcal{N}} = \mathcal{N} + \frac{\alpha}{2\beta}\Big[\tau\big(x_b\big)\,\psi^2\big(x_b\big) - \tau\big(x_a\big)\,\psi^2\big(x_a\big)\Big] \quad , \tag{10.100}$$

since then

$$\begin{aligned}\delta\widetilde{\mathcal{N}} - \widetilde{\mathcal{N}}\,\delta D = \int\limits_{x_a}^{x_b}\! dx \left\{ -\frac{d}{dx}\bigg[\tau(x)\,\frac{d\psi(x)}{dx}\bigg] + v(x)\,\psi(x) - \omega^2\mu(x)\,\psi(x)\right\}\delta\psi(x) \\ + \left[\tau(x)\bigg(\psi'(x) + \frac{\alpha}{\beta}\,\psi(x)\bigg)\,\delta\psi(x)\right]_{x=x_a}^{x=x_b} \quad ,\end{aligned} \tag{10.101}$$

and the last term vanishes as a result of the boundary conditions.

For all four classes of boundary conditions we may write

$$\omega^2\big[\psi(x)\big] = \frac{\int\limits_{x_a}^{x_b}\! dx\,\psi(x)\,\overbrace{\Big[-\frac{d}{dx}\,\tau(x)\,\frac{d}{dx} + v(x)\Big]}^{K}\,\psi(x)}{\int\limits_{x_a}^{x_b}\! dx\,\mu(x)\,\psi^2(x)} \tag{10.102}$$

If we expand $\psi(x)$ in the basis of eigenfunctions of the Sturm-Liouville operator K,

$$\psi(x) = \sum_{n=1}^{\infty} \mathcal{C}_n\,\psi_n(x) \quad , \tag{10.103}$$

we obtain

$$\omega^2\big[\psi(x)\big] = \omega^2(\mathcal{C}_1,\ldots,\mathcal{C}_\infty) = \frac{\sum_{j=1}^{\infty} |\mathcal{C}_j|^2\,\omega_j^2}{\sum_{k=1}^{\infty} |\mathcal{C}_k|^2} \quad . \tag{10.104}$$

If $\omega_1^2 \le \omega_2^2 \le \ldots$, then we see that $\omega^2 \ge \omega_1^2$, so an arbitrary function $\psi(x)$ will always yield an upper bound to the lowest eigenvalue.

As an example, consider a violin string ($v = 0$) with a mass m affixed in the center. We write $\mu(x) = \mu + m\,\delta(x - \frac{1}{2}L)$, hence

$$\omega^2\big[\psi(x)\big] = \frac{\tau\!\int\limits_0^L\! dx\,\psi'^2(x)}{m\,\psi^2(\frac{1}{2}L) + \mu\!\int\limits_0^L\! dx\,\psi^2(x)} \tag{10.105}$$

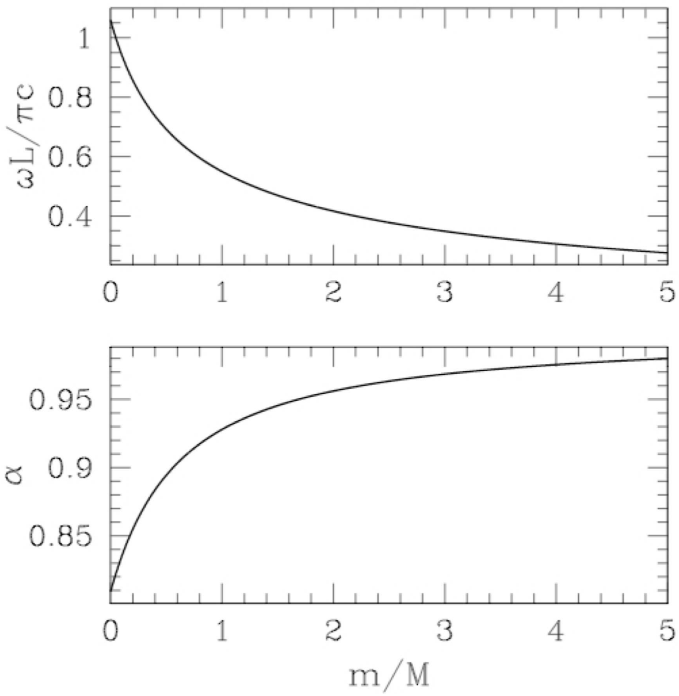


Figure 10.8: One-parameter variational solution for a string with a point mass m at $x = \frac{1}{2}L$.

Now consider a trial function

$$\psi(x) = \begin{cases} A\,x^\alpha & \text{if} \quad 0 \le x \le \frac{1}{2}L \\ \\ A\,(L-x)^\alpha & \text{if} \quad \frac{1}{2}L \le x \le L \quad . \end{cases} \tag{10.106}$$

The functional $\omega^2\big[\psi(x)\big]$ now becomes an ordinary function of the trial parameter α, with

$$\omega^2(\alpha) = \frac{2\tau\int_0^{L/2} dx\,\alpha^2\,x^{2\alpha-2}}{m\,\big(\frac{1}{2}L\big)^{2\alpha} + 2\mu\int\limits_0^{L/2} dx\,x^{2\alpha}} = \left(\frac{2c}{L}\right)^2 \cdot \frac{\alpha^2(2\alpha+1)}{(2\alpha-1)\big[1+(2\alpha+1)\frac{m}{M}\big]} \quad , \tag{10.107}$$

where $M = \mu L$ is the mass of the string alone. We minimize $\omega^2(\alpha)$ to obtain the optimal solution of this form:

$$\frac{d}{d\alpha}\,\omega^2(\alpha) = 0 \qquad \Longrightarrow \qquad 4\alpha^2 - 2\alpha - 1 + (2\alpha+1)^2\,(\alpha-1)\,\frac{m}{M} = 0 \quad . \tag{10.108}$$

For $m/M \to 0$, we obtain $\alpha = \frac{1}{4}\big(1+\sqrt{5}\big) \approx 0.809$. The variational estimate for the eigenvalue is then 6.00% larger than the exact answer $\omega_1^0 = \pi c/L$. In the opposite limit, $m/M \to \infty$, the inertia of the string may be neglected. The normal mode is then piecewise linear, in the shape of an isosceles triangle with base L and height y. The equation of motion is then $m\ddot{y} = -2\tau \cdot (y/\frac{1}{2}L)$, assuming $|y/L| \ll 1$. Thus, $\omega_1 = (2c/L)\sqrt{M/m}$. This is reproduced exactly by the variational solution, for which $\alpha \to 1$ as $m/M \to \infty$.

10.3 Continua in Higher Dimensions

10.3.1 General formalism

In higher dimensions, we generalize the operator K as follows:

$$K = -\frac{\partial}{\partial x^\alpha}\,\tau_{\alpha\beta}(\boldsymbol{x})\,\frac{\partial}{\partial x^\beta} + v(\boldsymbol{x}) \quad . \tag{10.109}$$

The eigenvalue equation is again

$$K\psi(\boldsymbol{x}) = \omega^2\,\mu(\boldsymbol{x})\,\psi(\boldsymbol{x}) \quad , \tag{10.110}$$

and the *Green's function* (see §10.7) satisfies

$$\Big[K - \omega^2\,\mu(\boldsymbol{x})\Big]\,G_\omega(\boldsymbol{x},\boldsymbol{x}') = \delta(\boldsymbol{x}-\boldsymbol{x}') \quad , \tag{10.111}$$

and has the eigenfunction expansion,

$$G_\omega(\boldsymbol{x},\boldsymbol{x}') = \sum_{n=1}^\infty \frac{\psi_n(\boldsymbol{x})\,\psi_n(\boldsymbol{x}')}{\omega_n^2-\omega^2} \quad . \tag{10.112}$$

The eigenfunctions form a complete and orthonormal basis:

$$\begin{gathered}\mu(\boldsymbol{x})\sum_{n=1}^\infty \psi_n(\boldsymbol{x})\,\psi_n(\boldsymbol{x}') = \delta(\boldsymbol{x}-\boldsymbol{x}')\\ \int\limits_\Omega \! d\boldsymbol{x}\;\mu(\boldsymbol{x})\,\psi_m(\boldsymbol{x})\,\psi_n(\boldsymbol{x}) = \delta_{mn} \quad ,\end{gathered} \tag{10.113}$$

where Ω is the region of space in which the continuous medium exists. For purposes of simplicity, we consider here fixed boundary conditions $u(\boldsymbol{x},t)\big|_{\partial\Omega} = 0$, where $\partial\Omega$ is the boundary of Ω. The general solution to the wave equation

$$\bigg[\mu(\boldsymbol{x})\,\frac{\partial^2}{\partial t^2} - \frac{\partial}{\partial x^\alpha}\,\tau_{\alpha\beta}(\boldsymbol{x})\,\frac{\partial}{\partial x^\beta} + v(\boldsymbol{x})\bigg]\,u(\boldsymbol{x},t) = 0 \tag{10.114}$$

is

$$u(\boldsymbol{x},t) = \sum_{n=1}^\infty \mathcal{C}_n\,\psi_n(\boldsymbol{x})\,\cos(\omega_n\,t+\delta_n) \quad . \tag{10.115}$$

The variational approach generalizes as well. We define

$$\mathcal{N}\big[\psi(\boldsymbol{x})\big] = \int\limits_\Omega \! d\boldsymbol{x}\,\bigg[\tau_{\alpha\beta}\,\frac{\partial\psi}{\partial x^\alpha}\,\frac{\partial\psi}{\partial x^\beta} + v\,\psi^2\bigg] \tag{10.116}$$

$$\mathcal{D}\big[\psi(\boldsymbol{x})\big] = \int\limits_\Omega \! d\boldsymbol{x}\;\mu\,\psi^2 \quad , \tag{10.117}$$

and

$$\omega^2\big[\psi(\boldsymbol{x})\big] = \frac{\mathcal{N}\big[\psi(\boldsymbol{x})\big]}{\mathcal{D}\big[\psi(\boldsymbol{x})\big]} \quad . \tag{10.118}$$

Setting the variation $\delta\omega^2 = 0$ recovers the eigenvalue equation $K\psi = \omega^2\mu\,\psi$.

10.3.2 Membranes

Consider a surface where the height z is a function of the lateral coordinates x and y:

$$z = u(x,y) \quad . \tag{10.119}$$

The equation of the surface is then

$$F(x,y,z) = z - u(x,y) = 0 \quad . \tag{10.120}$$

Let the differential element of surface area be dS. The projection of this element onto the (x,y) plane is

$$dA = dx\,dy = \hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{z}}\, dS \quad . \tag{10.121}$$

The unit normal $\hat{\boldsymbol{n}}$ is given by

$$\hat{\boldsymbol{n}} = \frac{\boldsymbol{\nabla} F}{\big|\boldsymbol{\nabla} F\big|} = \frac{\hat{\boldsymbol{z}} - \boldsymbol{\nabla} u}{\sqrt{1 + (\boldsymbol{\nabla} u)^2}} \quad . \tag{10.122}$$

Thus,

$$dS = \frac{dx\,dy}{\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{z}}} = \sqrt{1 + (\boldsymbol{\nabla} u)^2}\, dx\,dy \quad . \tag{10.123}$$

The potential energy for a deformed surface can take many forms. In the case we shall consider here, we consider only the effect of surface tension σ, and we write the potential energy functional as

$$U\big[u(x,y,t)\big] = \sigma \!\int\! dS = U_0 + \tfrac{1}{2}\sigma \!\int\! dA\, (\boldsymbol{\nabla} u)^2 + \ldots \quad . \tag{10.124}$$

The kinetic energy functional is

$$T\big[u(x,y,t)\big] = \tfrac{1}{2} \!\int\! dA\, \mu(\boldsymbol{x})\, (\partial_t u)^2 \quad . \tag{10.125}$$

Thus, the action is

$$S\big[u(\boldsymbol{x},t)\big] = \int\! d^2\!x\, \mathcal{L}(u, \boldsymbol{\nabla} u, \partial_t u, \boldsymbol{x}) \quad , \tag{10.126}$$

where the Lagrangian density is

$$\mathcal{L} = \tfrac{1}{2}\mu(\boldsymbol{x})\, (\partial_t u)^2 - \tfrac{1}{2}\sigma(\boldsymbol{x})\, (\boldsymbol{\nabla} u)^2 \quad , \tag{10.127}$$

where here we have allowed both $\mu(\boldsymbol{x})$ and $\sigma(\boldsymbol{x})$ to depend on the spatial coordinates. The equations of motion are

$$\begin{aligned} 0 &= \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial\, (\partial_t u)} + \boldsymbol{\nabla} \cdot \frac{\partial \mathcal{L}}{\partial\, \boldsymbol{\nabla} u} - \frac{\partial \mathcal{L}}{\partial u} \\ &= \mu(\boldsymbol{x})\, \frac{\partial^2 u}{\partial t^2} - \boldsymbol{\nabla} \cdot \Big\{ \sigma(\boldsymbol{x})\, \boldsymbol{\nabla} u \Big\} \quad . \end{aligned} \tag{10.128}$$

10.3.3 Helmholtz equation

When μ and σ are each constant, we obtain the Helmholtz equation:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) u(\boldsymbol{x},t) = 0 \quad , \tag{10.129}$$

with $c = \sqrt{\sigma/\mu}$. The d'Alembert solution still works – waves of arbitrary shape can propagate *in a fixed direction* $\hat{\boldsymbol{k}}$:

$$u(\boldsymbol{x},t) = f(\hat{\boldsymbol{k}}\cdot\boldsymbol{x} - ct) \quad . \tag{10.130}$$

This is called a *plane wave* because the three dimensional generalization of this wave has wavefronts which are planes. In our case, it might better be called a *line wave*, but people will look at you funny if you say that, so we'll stick with *plane wave*. Note that the locus of points of constant f satisfies

$$\phi(\boldsymbol{x},t) = \hat{\boldsymbol{k}}\cdot\boldsymbol{x} - ct = \text{constant} \quad , \tag{10.131}$$

and setting $d\phi = 0$ gives

$$\hat{\boldsymbol{k}}\cdot\frac{d\boldsymbol{x}}{dt} = c \quad , \tag{10.132}$$

which means that the velocity along $\hat{\boldsymbol{k}}$ is c. The component of $\boldsymbol{x}$ perpendicular to $\hat{\boldsymbol{k}}$ is arbitrary, hence the regions of constant ϕ correspond to lines which are orthogonal to $\hat{\boldsymbol{k}}$.

Owing to the linearity of the wave equation, we can construct arbitrary superpositions of plane waves. The most general solution is written

$$u(\boldsymbol{x},t) = \int\!\frac{d^2\!k}{(2\pi)^2}\left[A(\boldsymbol{k})\,e^{i(\boldsymbol{k}\cdot\boldsymbol{x}-ckt)} + B(\boldsymbol{k})\,e^{i(\boldsymbol{k}\cdot\boldsymbol{x}+ckt)}\right] \quad . \tag{10.133}$$

The first term in the bracket on the RHS corresponds to a plane wave moving in the $+\hat{\boldsymbol{k}}$ direction, and the second term to a plane wave moving in the $-\hat{\boldsymbol{k}}$ direction.

10.3.4 Rectangles

Consider a rectangular membrane where $x \in [0,a]$ and $y \in [0,b]$, and subject to the boundary conditions $u(0,y) = u(a,y) = u(x,0) = u(x,b) = 0$. We try a solution of the form

$$u(x,y,t) = X(x)\,Y(y)\,T(t) \quad . \tag{10.134}$$

This technique is known as *separation of variables*. Dividing the Helmholtz equation by u then gives

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = \frac{1}{c^2}\frac{1}{T}\frac{\partial^2 T}{\partial t^2} \quad . \tag{10.135}$$

The first term on the LHS depends only on x. The second term on the LHS depends only on y. The RHS depends only on t. Therefore, each of these terms must individually be constant. We write

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -k_x^2 \quad , \quad \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -k_y^2 \quad , \quad \frac{1}{T}\frac{\partial^2 T}{\partial t^2} = -\omega^2 \quad , \tag{10.136}$$

with

$$k_x^2 + k_y^2 = \frac{\omega^2}{c^2} \quad . \tag{10.137}$$

Thus, $\omega = \pm c|\boldsymbol{k}|$. The most general solution is then

$$\begin{aligned} X(x) &= A\,\cos(k_x x) + B\sin(k_x x) \\ Y(y) &= C\,\cos(k_y y) + D\sin(k_y y) \\ T(t) &= E\,\cos(\omega t) + B\sin(\omega t) \quad . \end{aligned} \tag{10.138}$$

The boundary conditions now demand

$$A = 0 \quad , \quad C = 0 \quad , \quad \sin(k_x a) = 0 \quad , \quad \sin(k_y b) = 0 \quad . \tag{10.139}$$

Thus, the most general solution subject to the boundary conditions is

$$u(x, y, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \mathcal{A}_{mn} \sin\!\left(\frac{m\pi x}{a}\right) \sin\!\left(\frac{n\pi y}{b}\right) \cos\!\big(\omega_{mn} t + \delta_{mn}\big) \quad , \tag{10.140}$$

where

$$\omega_{mn} = \sqrt{\left(\frac{m\pi c}{a}\right)^2 + \left(\frac{n\pi c}{b}\right)^2} \quad . \tag{10.141}$$

10.3.5 Circles

For a circular membrane, such as a drumhead, it is convenient to work in two-dimensional polar coordinates (r, φ). The Laplacian is then

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\, r\, \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} \quad . \tag{10.142}$$

We seek a solution to the Helmholtz equation which satisfies the boundary conditions $u(r = a, \varphi, t) = 0$. Once again, we invoke the separation of variables method, writing

$$u(r, \varphi, t) = R(r)\,\Phi(\varphi)\,T(t) \quad , \tag{10.143}$$

resulting in

$$\frac{1}{R}\frac{1}{r}\frac{\partial}{\partial r}\left(r\,\frac{\partial R}{\partial r}\right) + \frac{1}{r^2}\frac{1}{\Phi}\frac{\partial^2 \Phi}{\partial \varphi^2} = \frac{1}{c^2}\frac{1}{T}\frac{\partial^2 T}{\partial t^2} \quad . \tag{10.144}$$

The azimuthal and temporal functions are

$$\Phi(\varphi) = e^{im\varphi} \quad , \quad T(t) = \cos(\omega t + \delta) \quad , \tag{10.145}$$

where m is an integer in order that the function $u(r, \varphi, t)$ be single-valued. The radial equation is then

$$\frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\frac{\partial R}{\partial r} + \left(\frac{\omega^2}{c^2} - \frac{m^2}{r^2}\right) R = 0 \quad . \tag{10.146}$$

This is Bessel's equation, with solution

$$R(r) = A\,J_m\Big(\frac{\omega r}{c}\Big) + B\,N_m\Big(\frac{\omega r}{c}\Big) \quad , \tag{10.147}$$

where $J_m(z)$ and $N_m(z)$ are the Bessel and Neumann functions of order m, respectively. Since the Neumann functions diverge at $r = 0$, we must exclude them, setting $B = 0$ for each m.

We now invoke the boundary condition $u(r = a, \varphi, t) = 0$. This requires

$$J_m\Big(\frac{\omega a}{c}\Big) = 0 \qquad \Longrightarrow \qquad \omega = \omega_{m\ell} = x_{m\ell}\,\frac{c}{a} \quad , \tag{10.148}$$

where $J_m(x_{m\ell}) = 0$, *i.e.* $x_{m\ell}$ is the $\ell^{\rm th}$ zero of $J_m(x)$. The mose general solution is therefore

$$u(r, \varphi, t) = \sum_{m=0}^\infty \sum_{\ell=1}^\infty \mathcal{A}_{m\ell}\, J_m(x_{m\ell}\, r/a)\, \cos\big(m\varphi + \beta_{m\ell}\big)\, \cos(\omega_{m\ell}\, t + \delta_{m\ell}) \quad . \tag{10.149}$$

10.3.6 Sound in fluids

Let $\varrho(\boldsymbol{x}, t)$ and $\boldsymbol{v}(\boldsymbol{x}, t)$ be the density and velocity fields in a fluid. Mass conservation requires

$$\frac{\partial \varrho}{\partial t} + \boldsymbol{\nabla} \cdot (\varrho\, \boldsymbol{v}) = 0 \quad . \tag{10.150}$$

This is the continuity equation for mass.

Focus now on a small packet of fluid of infinitesimal volume dV. The total force on this fluid element is $d\boldsymbol{F} = \big(-\boldsymbol{\nabla} p + \varrho\,\boldsymbol{g}\big)\, dV$. By Newton's Second Law,

$$d\boldsymbol{F} = \big(\varrho\, dV\big)\,\frac{d\boldsymbol{v}}{dt} \tag{10.151}$$

Note that the chain rule gives

$$\frac{d\boldsymbol{v}}{dt} = \frac{\partial \boldsymbol{v}}{\partial t} + \big(\boldsymbol{v} \cdot \boldsymbol{\nabla}\big)\boldsymbol{v} \quad . \tag{10.152}$$

Thus, dividing eqn, 10.151 by dV, we obtain

$$\varrho\bigg(\frac{\partial \boldsymbol{v}}{\partial t} + \big(\boldsymbol{v} \cdot \boldsymbol{\nabla}\big)\boldsymbol{v}\bigg) = -\boldsymbol{\nabla} p + \varrho\,\boldsymbol{g} \quad . \tag{10.153}$$

This is the inviscid (*i.e.* zero viscosity) form of the Navier-Stokes equation.

Locally the fluid can also be described in terms of thermodynamic variables $p(\boldsymbol{x}, t)$ (pressure) and $T(\boldsymbol{x}, t)$ (temperature). For a one-component fluid there is necessarily an equation of state of the form $p = p(\varrho, T)$. Thus, we may write

$$dp = \frac{\partial p}{\partial \varrho}\bigg|_T d\varrho + \frac{\partial p}{\partial T}\bigg|_\varrho dT \quad . \tag{10.154}$$

We now make the following approximations. First, we assume that the fluid is close to equilibrium at $\boldsymbol{v} = 0$, meaning we write $p = \bar{p} + \delta p$ and $\varrho = \bar{\varrho} + \delta\varrho$, and assume that δp, $\delta\varrho$, and $\boldsymbol{v}$ are small. The smallness of $\boldsymbol{v}$ means we can neglect the nonlinear term $(\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{v}$ in eqn. 10.153. Second, we neglect gravity (more on this later). The continuity equation then takes the form

$$\frac{\partial\,\delta\varrho}{\partial t} + \bar{\varrho}\,\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0 \quad , \tag{10.155}$$

and the Navier-Stokes equation becomes

$$\bar{\varrho}\,\frac{\partial \boldsymbol{v}}{\partial t} = -\boldsymbol{\nabla}\delta p \quad . \tag{10.156}$$

Taking the time derivative of the former, and then invoking the latter of these equations yields

$$\frac{\partial^2\,\delta\varrho}{\partial t^2} = \nabla^2 p = \left(\frac{\partial p}{\partial\varrho}\right)\nabla^2\,\delta\varrho \equiv c^2\,\nabla^2\delta\varrho \quad . \tag{10.157}$$

The speed of wave propagation, *i.e.* the speed of sound, is given by

$$c = \sqrt{\frac{\partial p}{\partial\varrho}} \quad . \tag{10.158}$$

Finally, we must make an assumption regarding the conditions under which the derivative $\partial p/\partial\varrho$ is computed. If the fluid is an excellent conductor of heat, then the temperature will equilibrate quickly and it is a good approximation to take the derivative at fixed temperature. The resulting value of c is called the *isothermal* sound speed c_T. If, on the other hand, the fluid is a poor conductor of heat, as is the case for air, then it is more appropriate to take the derivative at constant entropy, yielding the *adiabatic* sound speed. Thus,

$$c_T = \sqrt{\left(\frac{\partial p}{\partial\varrho}\right)_T} \quad , \quad c_S = \sqrt{\left(\frac{\partial p}{\partial\varrho}\right)_S} \quad . \tag{10.159}$$

In an ideal gas, $c_S/c_T = \sqrt{\gamma}$, where $\gamma = c_p/c_V$ is the ratio of the specific heat at constant pressure to that at constant volume. For a (mostly) diatomic gas like air (comprised of N_2 and O_2 and just a little Ar), $\gamma = \frac{7}{5}$. Note that one can write $c^2 = 1/\varrho\kappa$, where

$$\kappa = \frac{1}{\varrho}\left(\frac{\partial\varrho}{\partial p}\right) \tag{10.160}$$

is the *compressibility*, which is the inverse of the *bulk modulus*. Again, one must specify whether one is talking about κ_T or κ_S. For reference in air at $T = 293\,\mathrm{K}$, using $M = 28.8\,\mathrm{g/mol}$, one obtains $c_T = 290.8\,\mathrm{m/s}$ and $c_S = 344.0\,\mathrm{m/s}$. In H_2O at 293 K, $c = 1482\,\mathrm{m/s}$. In Al at 273 K, $c = 6420\,\mathrm{m/s}$.

If we retain gravity, the wave equation becomes

$$\frac{\partial^2\delta\varrho}{\partial t^2} = c^2\,\nabla^2\delta\varrho - \boldsymbol{g}\cdot\boldsymbol{\nabla}\delta\varrho \quad . \tag{10.161}$$

The dispersion relation is then

$$\omega(\boldsymbol{k}) = \sqrt{c^2k^2 + i\boldsymbol{g}\cdot\boldsymbol{k}} \quad . \tag{10.162}$$

We are permitted to ignore the effects of gravity so long as $c^2k^2 \gg gk$. In terms of the wavelength $\lambda = 2\pi/k$, this requires

$$\lambda \ll \frac{2\pi c^2}{g} = 75.9\,\mathrm{km}\ (\mathrm{at}\ T = 293\,\mathrm{K}) \quad . \tag{10.163}$$

10.4 Dispersion

10.4.1 Helmholtz *versus* Klein-Gordon equations

The one-dimensional Helmholtz equation $\ddot{y} = c^2\, y''$ is solved by a plane wave

$$y(x,t) = A\, e^{ikx}\, e^{-i\omega t} \quad , \tag{10.164}$$

provided $\omega = \pm ck$. We say that there are *two branches* to the *dispersion relation* $\omega(k)$ for this equation. In general, we may add solutions, due to the linearity of the Helmholtz equation. The most general solution is then

$$\begin{aligned} y(x,t) &= \int\limits_{-\infty}^{\infty} \frac{dk}{2\pi} \Big[\hat{f}(k)\, e^{ik(x-ct)} + \hat{g}(k)\, e^{ik(x+ct)} \Big] \\ &= f(x-ct) + g(x+ct) \quad , \end{aligned} \tag{10.165}$$

which is consistent with d'Alembert's solution.

The Klein-Gordon equation, $\ddot{\phi} = c^2\, \phi'' - \gamma^2\, \phi$, also has a plane wave solution as in eqn. 10.164, but with dispersion branches $\omega = \pm W(k)$ with $W(k) = \pm(\gamma^2 + c^2k^2)^{1/2}$. The general solution is then

$$\phi(x,t) = \int\limits_{-\infty}^{\infty} \frac{dk}{2\pi} \Big[\hat{A}(k)\, e^{ikx} e^{-iW(k)t} + \hat{B}(k)\, e^{ikx} e^{iW(k)t} \Big] \quad , \tag{10.166}$$

which is not of the D'Alembert form.

10.4.2 Schrödinger's equation

Consider now the free particle Schrödinger equation in one space dimension,

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\, \frac{\partial^2 \psi}{\partial x^2} \quad . \tag{10.167}$$

The function $\psi(x,t)$ is the quantum mechanical wavefunction for a particle of mass m moving freely along a one-dimensional line. The *probability density* for finding the particle at position x at time t is

$$\rho(x,t) = \big|\psi(x,t)\big|^2 \quad . \tag{10.168}$$

Conservation of probability therefore requires

$$\int\limits_{-\infty}^{\infty} dx \, \big|\psi(x,t)\big|^2 = 1 \quad . \tag{10.169}$$

This condition must hold at all times t.

As is the case with the Helmholtz and Klein-Gordon equations, the Schrödinger equation is solved by a plane wave of the form

$$\psi(x,t) = A \, e^{ikx} \, e^{-i\omega t} \quad , \tag{10.170}$$

where the dispersion relation now only has one branch, and is given by

$$\omega(k) = \frac{\hbar k^2}{2m} \quad . \tag{10.171}$$

The most general solution is then

$$\psi(x,t) = \int\limits_{-\infty}^{\infty} \frac{dk}{2\pi} \, \hat{\psi}(k) \, e^{ikx} \, e^{-i\hbar k^2 t/2m} \quad . \tag{10.172}$$

Let's suppose we start at time $t = 0$ with a Gaussian wavepacket,

$$\psi(x,0) = \big(\pi\ell_0^2\big)^{-1/4} \, e^{-x^2/2\ell_0^2} \, e^{ik_0 x} \quad . \tag{10.173}$$

To find the amplitude $\hat{\psi}(k)$, we perform the Fourier transform:

$$\hat{\psi}(k) = \int\limits_{-\infty}^{\infty} dx \, \psi(x,0) \, e^{-ikx} = \sqrt{2} \, \big(\pi\ell_0^2\big)^{-1/4} \, e^{-(k-k_0)^2 \ell_0^2/2} \quad . \tag{10.174}$$

We now compute $\psi(x,t)$ valid for all times t:

$$\begin{aligned} \psi(x,t) &= \sqrt{2} \, \big(\pi\ell_0^2\big)^{-1/4} \int\limits_{-\infty}^{\infty} \frac{dk}{2\pi} \, e^{ikx} \, e^{-(k-k_0)^2 \ell_0^2/2} \, e^{ikx} \, e^{-i\hbar k^2 t/2m} \\ &= \big(\pi\ell_0^2\big)^{-1/4} \, \big(1 + it/\tau\big)^{-1/2} \exp\!\left[-\frac{\big(x - \hbar k_0 t/m\big)^2}{2\,\ell_0^2 \, \big(1 + t^2/\tau^2\big)}\right] \exp\!\left[\frac{i\big(2k_0 \, \ell_0^2 \, x + x^2 \, t/\tau - k_0^2 \, \ell_0^4 \, t/\tau\big)}{2\,\ell_0^2 \, \big(1 + t^2/\tau^2\big)}\right] \quad , \end{aligned} \tag{10.175}$$

where where $\tau \equiv m\ell_0^2/\hbar$. The probability density is then the normalized Gaussian

$$\rho(x,t) = \frac{1}{\sqrt{\pi \, \ell^2(t)}} \, e^{-(x - v_0 t)^2/\ell^2(t)} \quad , \tag{10.176}$$

where $v_0 = \hbar k_0/m$ and

$$\ell(t) = \ell_0 \, \sqrt{1 + t^2/\tau^2} \quad . \tag{10.177}$$

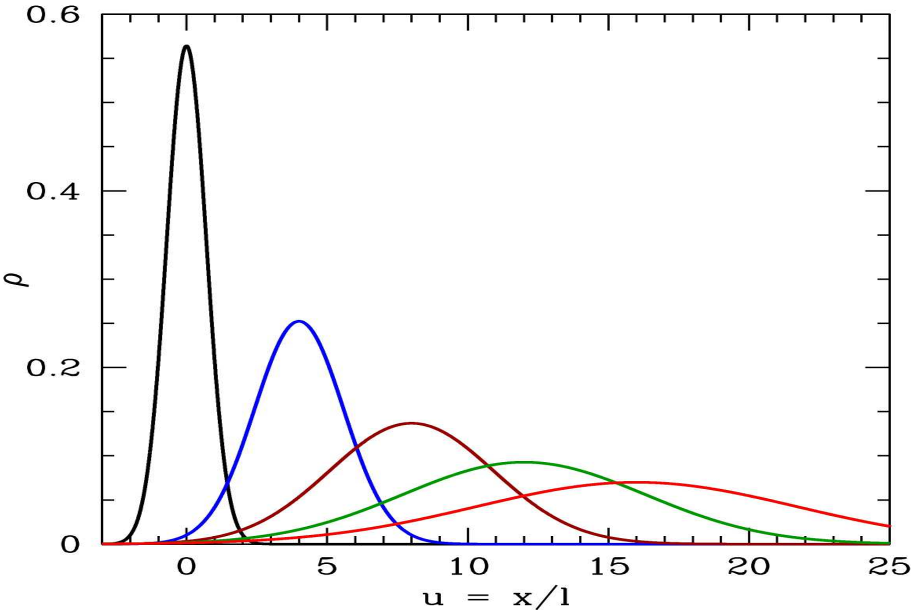


Figure 10.9: Wavepacket spreading for $k_0\,\ell_0 = 2$ with $t/\tau = 0$, 2, 4, 6, and 8.

Note that $\ell(t)$ gives the width of the wavepacket, and that this width increases as a function of time, with $\ell(t \gg \tau) \simeq \ell_0\, t/\tau$.

Unlike the case of the Helmholtz equation, the solution to the Schrödinger equation does not retain its shape as it moves. This phenomenon is known as the *spreading of the wavepacket*. In fig. 10.9, we show the motion and spreading of the wavepacket.

For a given plane wave $e^{ikx}\, e^{-i\omega(k)t}$, the wavefronts move at the *phase velocity*

$$v_{\rm p}(k) = \frac{\omega(k)}{k} \quad . \tag{10.178}$$

The center of the wavepacket, however, travels at the *group velocity*

$$v_{\rm g}(k) = \frac{d\omega}{dk}\bigg|_{k_0} \quad , \tag{10.179}$$

where $k = k_0$ is the maximum of $\big|\hat{\psi}(k)\big|^2$.

10.5 General Field Theoretic Formulation

Continuous systems possess an infinite number of degrees of freedom. They are described by a set of fields $\phi_a(\boldsymbol{x},t)$ which depend on space and time. These fields may represent local displacement, pressure, velocity, *etc.* The equations of motion of the fields are again determined by extremizing the action, which, in turn, is an integral of the *Lagrangian density* over all space and time. Extremization yields a set of (generally coupled) *partial* differential equations.

10.5.1 Euler-Lagrange equations for classical field theories

Suppose $\phi_a(\boldsymbol{x})$ depends on n independent variables, $\{x^1, x^2, \ldots, x^n\}$. Consider the functional

$$S\big[\{\phi_a(\boldsymbol{x})\big] = \int\limits_\Omega d\boldsymbol{x}\,\mathcal{L}(\phi_a\,\partial_\mu\phi_a, \boldsymbol{x}) \quad , \tag{10.180}$$

i.e. the *Lagrangian density* $\mathcal{L}$ is a function of the fields ϕ_a and their partial derivatives $\partial\phi_a/\partial x^\mu$. Here Ω is a region in $\mathbb{R}^n$. Then the first variation of S is

$$\begin{aligned}\delta S &= \int\limits_\Omega d\boldsymbol{x}\left\{\frac{\partial\mathcal{L}}{\partial\phi_a}\,\delta\phi_a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\,\frac{\partial\,\delta\phi_a}{\partial x^\mu}\right\}\\ &= \oint\limits_{\partial\Omega} d\Sigma\, n^\mu\,\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\,\delta\phi_a + \int\limits_\Omega d\boldsymbol{x}\left\{\frac{\partial\mathcal{L}}{\partial\phi_a} - \frac{\partial}{\partial x^\mu}\bigg(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\bigg)\right\}\delta\phi_a \quad ,\end{aligned} \tag{10.181}$$

where $\partial\Omega$ is the $(n-1)$-dimensional boundary of Ω, $d\Sigma$ is the differential surface area, and n^μ is the unit normal. If we demand either $\partial\mathcal{L}/\partial(\partial_\mu\phi_a)\big|_{\partial\Omega} = 0$ or $\delta\phi_a\big|_{\partial\Omega} = 0$, the surface term vanishes, and we conclude

$$\frac{\delta S}{\delta\phi_a(\boldsymbol{x})} = \left[\frac{\partial\mathcal{L}}{\partial\phi_a} - \frac{\partial}{\partial x^\mu}\bigg(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\bigg)\right]_{\boldsymbol{x}} \quad , \tag{10.182}$$

where the subscript means we are to evaluate the term in brackets at $\boldsymbol{x}$. In a mechanical system, one of the n independent variables (usually x^0), is the time t. However, we may be interested in a time-independent context in which we wish to extremize the energy functional, for example. In any case, setting the first variation of S to zero yields the Euler-Lagrange equations,

$$\delta S = 0 \quad\Rightarrow\quad \frac{\partial\mathcal{L}}{\partial\phi_a} - \frac{\partial}{\partial x^\mu}\bigg(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\bigg) = 0 \tag{10.183}$$

The stress-energy tensor is defined as

$$T^\mu_{\ \nu} = \sum_a \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\,\partial_\nu\phi_a - \delta^\mu_\nu\,\mathcal{L} \quad . \tag{10.184}$$

When $\mathcal{L} = \mathcal{L}(\phi_a, \partial_\mu\phi_a)$ is independent of the independent varlables $\boldsymbol{x}$, one has that the stress-energy tensor is conserved: $\partial_\mu\,T^\mu_{\ \nu} = 0$. (Students should check this result.)

Maxwell theory

The Lagrangian density for an electromagnetic field with sources is

$$\mathcal{L} = -\tfrac{1}{16\pi}\,F_{\mu\nu}\,F^{\mu\nu} - J_\nu A^\nu \quad . \tag{10.185}$$

The equations of motion are then

$$\frac{\partial\mathcal{L}}{\partial A_\nu} - \frac{\partial}{\partial x^\mu}\bigg(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)}\bigg) = 0 \quad\Rightarrow\quad \partial_\mu\,F^{\mu\nu} = 4\pi J^\nu \quad , \tag{10.186}$$

which are Maxwell's equations.

10.5.2 Conserved currents in field theory

Recall the result of Noether's theorem for mechanical systems:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\,\frac{\partial \tilde{q}_\sigma}{\partial \zeta}\right)_{\zeta=0} = 0 \quad , \tag{10.187}$$

where $\tilde{q}_\sigma = \tilde{q}_\sigma(q,\zeta)$ is a one-parameter (ζ) family of transformations of the generalized coordinates which leaves L invariant. We generalize to field theory by replacing

$$q_\sigma(t) \longrightarrow \phi_a(\boldsymbol{x},t) \quad , \tag{10.188}$$

where $\{\phi_a(\boldsymbol{x},t)\}$ are a set of fields, which are functions of the independent variables $\{x,y,z,t\}$. We will adopt covariant relativistic notation and write for four-vector $x^\mu = (ct,x,y,z)$. The generalization of $dQ/dt = 0$ is

$$\frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi_a\right)}\,\frac{\partial \tilde{\phi}_a}{\partial \zeta}\right)_{\zeta=0} = 0 \quad , \tag{10.189}$$

where there is an implied sum on both μ and a. We can write this as $\partial_\mu\, J^\mu = 0$, where

$$J^\mu \equiv \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi_a\right)}\,\frac{\partial \tilde{\phi}_a}{\partial \zeta}\bigg|_{\zeta=0} \quad . \tag{10.190}$$

Let $x^\mu = \{x^0, x^1, \ldots, x^d\}$, with $n = d+1$ the number of independent variables and $x^0 \equiv ct$ with c having dimensions of velocity. Here we are privileging one of the independent variables (x^0) to be the time variable. We call $Q_\Omega = c^{-1}\int\limits_\Omega d^3x\, J^0$ the *total charge* in a spatial region Ω. If we assume $\hat{\boldsymbol{n}}\cdot\boldsymbol{J} = 0$ along the spatial boundary $\partial\Omega$, where $\hat{\boldsymbol{n}}$ is the local unit normal to the surface, then integrating the conservation law $\partial_\mu\, J^\mu$ over the spatial region Ω yields

$$\frac{dQ_\Omega}{dt} = \int\limits_\Omega d^3\!x\, \partial_0\, J^0 = -\int\limits_\Omega d^3\!x\, \boldsymbol{\nabla}\cdot\boldsymbol{J} = -\oint\limits_{\partial\Omega} d\Sigma\, \hat{\boldsymbol{n}}\cdot\boldsymbol{J} = 0 \quad . \tag{10.191}$$

This tells us that the rate of change of the charge Q_Ω in spatial region Ω is the negative of the integral over the surface $\partial\Omega$ of $\hat{\boldsymbol{n}}\cdot\boldsymbol{J}$, *i.e.* $\dot{Q}_\Omega$ is minus the total integrated flux leaving the region Ω.

As an example, consider the case of a complex scalar field, with Lagrangian density[2]

$$\mathcal{L}(\psi,,\psi^*,\partial_\mu\psi,\partial_\mu\psi^*) = \tfrac{1}{2}K\,(\partial_\mu\psi^*)(\partial^\mu\psi) - U\big(\psi^*\psi\big) \quad . \tag{10.192}$$

This is invariant under the transformation $\psi \to e^{i\zeta}\,\psi$, $\psi^* \to e^{-i\zeta}\,\psi^*$. Thus,

$$\frac{\partial\tilde{\psi}}{\partial\zeta} = i\,e^{i\zeta}\,\psi \qquad , \qquad \frac{\partial\tilde{\psi}^*}{\partial\zeta} = -i\,e^{-i\zeta}\,\psi^* \quad , \tag{10.193}$$

[2]We raise and lower indices using the Minkowski metric $g_{\mu\nu} = \mathrm{diag}\,(+,-,-,-)$.

and, summing over both ψ and ψ^* fields, we have

$$J^\mu = \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \psi\right)} \cdot (i\psi) + \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \psi^*\right)} \cdot (-i\psi^*) = \frac{K}{2i}\big(\psi^* \partial^\mu \psi - \psi\, \partial^\mu \psi^*\big) \quad . \tag{10.194}$$

The potential, which depends on $|\psi|^2$, is independent of ζ. Hence, this form of conserved 4-current is valid for an entire class of potentials.

10.5.3 Gross-Pitaevskii model

As one final example of a field theory, consider the Gross-Pitaevskii model, with

$$\mathcal{L} = i\hbar\, \psi^* \frac{\partial \psi}{\partial t} - \frac{\hbar^2}{2m}\, \boldsymbol{\nabla}\psi^* \cdot \boldsymbol{\nabla}\psi - g\left(|\psi|^2 - n_0\right)^2 \quad . \tag{10.195}$$

This describes a Bose fluid with repulsive short-ranged interactions. Here $\psi(\boldsymbol{x}, t)$ is again a complex scalar field, and ψ^* is its complex conjugate. Using the Leibniz rule, we have

$$\begin{aligned}
\delta S[\psi^*, \psi] &= S[\psi^* + \delta\psi^*, \psi + \delta\psi] \\
&= \int\! dt \!\int\! d^d\!x \, \bigg\{ i\hbar\, \psi^* \frac{\partial \delta\psi}{\partial t} + i\hbar\, \delta\psi^* \frac{\partial \psi}{\partial t} - \frac{\hbar^2}{2m}\, \boldsymbol{\nabla}\psi^* \cdot \boldsymbol{\nabla}\delta\psi \\
&\qquad\qquad - \frac{\hbar^2}{2m}\, \boldsymbol{\nabla}\delta\psi^* \cdot \boldsymbol{\nabla}\psi - 2g\left(|\psi|^2 - n_0\right)\left(\psi^* \delta\psi + \psi \delta\psi^*\right) \bigg\} \\
&= \int\! dt \!\int\! d^d\!x \, \bigg\{ \Big[-i\hbar\, \frac{\partial \psi^*}{\partial t} + \frac{\hbar^2}{2m}\, \nabla^2 \psi^* - 2g\left(|\psi|^2 - n_0\right) \psi^* \Big]\, \delta\psi \\
&\qquad\qquad + \Big[i\hbar\, \frac{\partial \psi}{\partial t} + \frac{\hbar^2}{2m}\, \nabla^2 \psi - 2g\left(|\psi|^2 - n_0\right) \psi \Big]\, \delta\psi^* \bigg\} \quad ,
\end{aligned} \tag{10.196}$$

where we have integrated by parts where necessary and discarded the boundary terms. Extremizing $S[\psi^*, \psi]$ therefore results in the *nonlinear Schrödinger equation* (NLSE),

$$i\hbar\, \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\, \nabla^2 \psi + 2g\left(|\psi|^2 - n_0\right) \psi \tag{10.197}$$

as well as its complex conjugate,

$$-\, i\hbar\, \frac{\partial \psi^*}{\partial t} = -\frac{\hbar^2}{2m}\, \nabla^2 \psi^* + 2g\left(|\psi|^2 - n_0\right) \psi^* \quad . \tag{10.198}$$

Note that these equations are indeed the Euler-Lagrange equations:

$$\begin{aligned}
\frac{\delta S}{\delta \psi} &= \frac{\partial \mathcal{L}}{\partial \psi} - \frac{\partial}{\partial x^\mu} \left(\frac{\partial \mathcal{L}}{\partial\, \partial_\mu \psi} \right) \\
\frac{\delta S}{\delta \psi^*} &= \frac{\partial \mathcal{L}}{\partial \psi^*} - \frac{\partial}{\partial x^\mu} \left(\frac{\partial \mathcal{L}}{\partial\, \partial_\mu \psi^*} \right) \quad ,
\end{aligned} \tag{10.199}$$

with $x^\mu = (t, \boldsymbol{x})$ the space-time four-vector[3]. Plugging in

$$\frac{\partial \mathcal{L}}{\partial \psi} = -2g\left(|\psi|^2 - n_0\right)\psi^* \quad , \quad \frac{\partial \mathcal{L}}{\partial\, \partial_t \psi} = i\hbar\,\psi^* \quad , \quad \frac{\partial \mathcal{L}}{\partial\, \boldsymbol{\nabla}\psi} = -\frac{\hbar^2}{2m}\,\boldsymbol{\nabla}\psi^* \tag{10.200}$$

and

$$\frac{\partial \mathcal{L}}{\partial \psi^*} = i\hbar\,\psi - 2g\left(|\psi|^2 - n_0\right)\psi \quad , \quad \frac{\partial \mathcal{L}}{\partial\, \partial_t \psi^*} = 0 \quad , \quad \frac{\partial \mathcal{L}}{\partial\, \boldsymbol{\nabla}\psi^*} = -\frac{\hbar^2}{2m}\,\boldsymbol{\nabla}\psi \quad , \tag{10.201}$$

we recover the NLSE and its conjugate.

The Gross-Pitaevskii model also possesses a $\mathrm{U}(1)$ invariance, under

$$\psi(\boldsymbol{x},t) \to \tilde{\psi}(\boldsymbol{x},t) = e^{i\zeta}\,\psi(\boldsymbol{x},t) \quad , \quad \psi^*(\boldsymbol{x},t) \to \tilde{\psi}^*(\boldsymbol{x},t) = e^{-i\zeta}\,\psi^*(\boldsymbol{x},t) \quad . \tag{10.202}$$

Thus, the conserved Noether current is then

$$J^\mu = \frac{\partial \mathcal{L}}{\partial\, \partial_\mu \psi}\,\frac{\partial \tilde{\psi}}{\partial \zeta}\bigg|_{\zeta=0} + \frac{\partial \mathcal{L}}{\partial\, \partial_\mu \psi^*}\,\frac{\partial \tilde{\psi}^*}{\partial \zeta}\bigg|_{\zeta=0} \tag{10.203}$$

so that

$$J^0 = -\hbar\,|\psi|^2 \qquad , \qquad \boldsymbol{J} = -\frac{\hbar^2}{2im}\left(\psi^*\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\psi^*\right) \quad . \tag{10.204}$$

Dividing out by $\hbar$, taking $J^0 \equiv -\hbar\rho$ and $\boldsymbol{J} \equiv -\hbar\boldsymbol{j}$, we obtain the continuity equation,

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{j} = 0 \quad , \tag{10.205}$$

where

$$\rho = |\psi|^2 \quad , \quad \boldsymbol{j} = \frac{\hbar}{2im}\left(\psi^*\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\psi^*\right) \quad . \tag{10.206}$$

are the particle density and the particle current, respectively.

10.6 Appendix: Three Strings

Problem: Three identical strings are connected to a ring of mass m as shown in fig. 10.10. The linear mass density of each string is σ and each string is under identical tension τ. In equilibrium, all strings are coplanar. All motion on the string is in the $\hat{\boldsymbol{z}}$-direction, which is perpendicular to the equilibrium plane. The ring slides frictionlessly along a vertical pole.

It is convenient to describe each string as a half line $[-\infty, 0]$. We can choose coordinates x_1, x_2, and x_3 for the three strings, respectively. For each string, the ring lies at $x_i = 0$.

A pulse is sent down the first string. After a time, the pulse arrives at the ring. Transmitted waves are sent down the other two strings, and a reflected wave down the first string. The solution to the wave equation in the strings can be written as follows. In string #1, we have

$$z = f(ct - x_1) + g(ct + x_1) \quad . \tag{10.207}$$

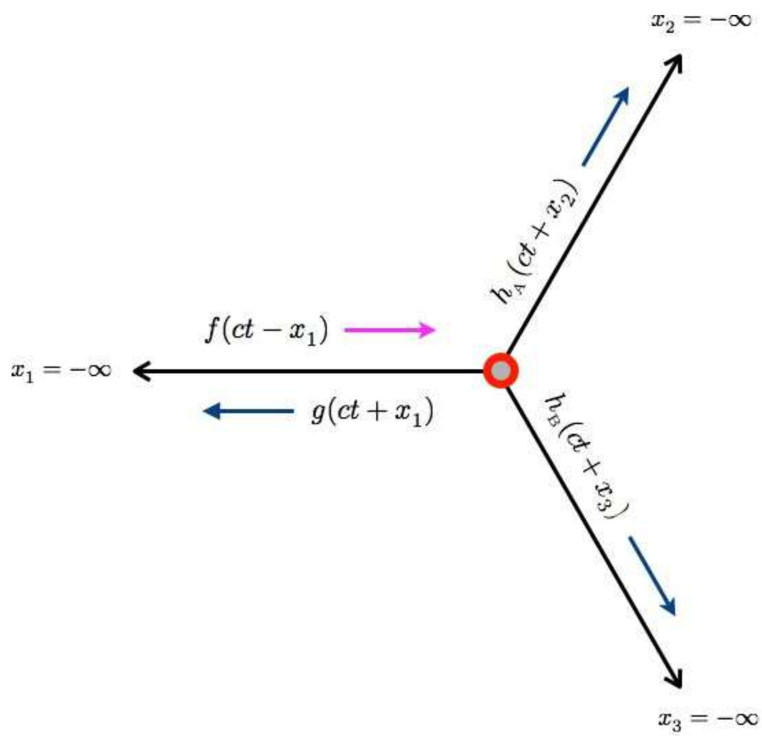


Figure 10.10: Three identical strings arranged symmetrically in a plane, attached to a common end. All motion is in the direction perpendicular to this plane. The red ring, whose mass is m, slides frictionlessly in this direction along a pole.

In the other two strings, we may write $z = h_A(ct + x_2)$ and $z = h_B(ct + x_3)$, as indicated in the figure.

(a) Write the wave equation in string #1. Define all constants.

(b) Write the equation of motion for the ring.

(c) Solve for the reflected wave $g(\xi)$ in terms of the incident wave $f(\xi)$. You may write this relation in terms of the Fourier transforms $\hat{f}(k)$ and $\hat{g}(k)$.

(d) Suppose a very long wavelength pulse of maximum amplitude A is incident on the ring. What is the maximum amplitude of the reflected pulse? What do we mean by "very long wavelength"?

Solution:

(a) The wave equation is

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{c^2}\,\frac{\partial^2 z}{\partial t^2} \quad , \tag{10.208}$$

where x is the coordinate along the string, and $c = \sqrt{\tau/\sigma}$ is the speed of wave propagation.

(b) Let Z be the vertical coordinate of the ring. Newton's second law says $m\ddot{Z} = F$, where the force on

[3] In the nonrelativistic case, there is no utility in defining $x^0 = ct$, so we simply define $x^0 = t$.

the ring is the sum of the vertical components of the tension in the three strings at $x = 0$:

$$F = -\tau\Big[- f'(ct) + g'(ct) + h'_{\rm A}(ct) + h'_{\rm B}(ct)\Big] \quad , \tag{10.209}$$

where prime denotes differentiation with respect to argument.

(c) To solve for the reflected wave, we must eliminate the unknown functions $h_{\rm A,B}$ and then obtain g in terms of f. This is much easier than it might at first seem. We start by demanding continuity at the ring. This means

$$Z(t) = f(ct) + g(ct) = h_{\rm A}(ct) = h_{\rm B}(ct) \tag{10.210}$$

for all t. We can immediately eliminate $h_{\rm A,B}$:

$$h_{\rm A}(\xi) = h_{\rm B}(\xi) = f(\xi) + g(\xi) \quad , \tag{10.211}$$

for all ξ. Newton's second law from part (b) may now be written as

$$mc^2\big[f''(\xi) + g''(\xi)\big] = -\tau\big[f'(\xi) + 3g'(\xi)\big] \quad . \tag{10.212}$$

This linear ODE becomes a simple linear algebraic equation for the Fourier transforms,

$$f(\xi) = \int\limits_{-\infty}^{\infty}\frac{dk}{2\pi}\,\hat f(k)\,e^{ik\xi} \quad , \tag{10.213}$$

etc. We readily obtain

$$\hat g(k) = -\bigg(\frac{k - iQ}{k - 3iQ}\bigg)\,\hat f(k) \quad , \tag{10.214}$$

where $Q \equiv \tau/mc^2$ has dimensions of inverse length. Since $h_{\rm A,B} = f + g$, we have

$$\hat h_{\rm A}(k) = \hat h_{\rm B}(k) = -\bigg(\frac{2iQ}{k - 3iQ}\bigg)\,\hat f(k) \quad . \tag{10.215}$$

(d) For a very long wavelength pulse, composed of plane waves for which $|k| \ll Q$, we have $\hat g(k) \approx -\frac{1}{3}\,\hat f(k)$. Thus, the reflected pulse is inverted, and is reduced by a factor $\frac{1}{3}$ in amplitude. Note that for a very *short* wavelength pulse, for which $k \gg Q$, we have perfect reflection with inversion, and no transmission. This is due to the inertia of the ring.

It is straightforward to generalize this problem to one with n strings. The transmission into each of the $(n-1)$ channels is of course identical (by symmetry). One then finds the reflection and transmission amplitudes

$$\hat r(k) = -\bigg(\frac{k - i(n-2)Q}{k - inQ}\bigg) \quad , \quad \hat t(k) = -\bigg(\frac{2iQ}{k - inQ}\bigg) \quad . \tag{10.216}$$

Conservation of energy means that the sum of the squares of the reflection amplitude and all the $(n-1)$ transmission amplitudes must be unity:

$$\big|\hat r(k)\big|^2 + (n-1)\,\big|\hat t(k)\big|^2 = 1 \quad . \tag{10.217}$$

10.7 Appendix: Green's Functions for Strings

10.7.1 Inhomogeneous Sturm-Liouville problem

Suppose we add a forcing term,

$$\mu(x)\,\frac{\partial^2 y}{\partial t^2} - \frac{\partial}{\partial x}\bigg[\tau(x)\,\frac{\partial y}{\partial x}\bigg] + v(x)\,y = \text{Re}\Big[\mu(x)\,f(x)\,e^{-i\omega t}\Big] \quad . \tag{10.218}$$

We write the solution as

$$y(x,t) = \text{Re}\Big[y(x)\,e^{-i\omega t}\Big] \quad , \tag{10.219}$$

where

$$-\frac{d}{dx}\bigg[\tau(x)\,\frac{dy(x)}{dx}\bigg] + v(x)\,y(x) - \omega^2\,\mu(x)\,y(x) = \mu(x)\,f(x) \quad , \tag{10.220}$$

or

$$\Big[K - \omega^2 \mu(x)\Big]\,y(x) = \mu(x)\,f(x) \quad , \tag{10.221}$$

where K is a differential operator,

$$K \equiv -\frac{d}{dx}\,\tau(x)\,\frac{d}{dx} + v(x) \quad . \tag{10.222}$$

Note that the eigenfunctions of K are the $\{\psi_n(x)\}$:

$$K\,\psi_n(x) = \omega_n^2\,\mu(x)\,\psi_n(x) \quad . \tag{10.223}$$

The formal solution to equation 10.221 is then

$$y(x) = \Big[K - \omega^2\mu\Big]^{-1}_{x,x'}\,\mu(x')\,f(x') = \int\limits_{x_a}^{x_b} dx'\,\mu(x')\,G_\omega(x,x')\,f(x'). \tag{10.224}$$

What do we mean by the term in brackets? If we define the *Green's function*

$$G_\omega(x,x') \equiv \Big[K - \omega^2\mu\Big]^{-1}_{x,x'} \quad , \tag{10.225}$$

what this means is

$$\Big[K - \omega^2\mu(x)\Big]\,G_\omega(x,x') = \delta(x-x') \quad . \tag{10.226}$$

Note that the Green's function may be expanded in terms of the (real) eigenfunctions, as

$$G_\omega(x,x') = \sum_n \frac{\psi_n(x)\,\psi_n(x')}{\omega_n^2 - \omega^2} \quad , \tag{10.227}$$

which follows from completeness of the eigenfunctions:

$$\mu(x)\sum_{n=1}^{\infty}\psi_n(x)\,\psi_n(x') = \delta(x-x') \quad . \tag{10.228}$$

The expansion in eqn. 10.227 is formally exact, but difficult to implement, since it requires summing over an infinite set of eigenfunctions. It is more practical to construct the Green's function from solutions to the homogeneous Sturm Liouville equation, as follows. When $x \neq x'$, we have that $(K - \omega^2 \mu)\, G_\omega(x,x') = 0$, which is a homogeneous ODE of degree two. Consider first the interval $x \in [x_a, x']$. A second order homogeneous ODE has two solutions, and further invoking the boundary condition at $x = x_a$, there is a unique solution, up to a multiplicative constant. Call this solution $y_1(x)$. Next, consider the interval $x \in [x', x_b]$. Once again, there is a unique solution to the homogeneous Sturm-Liouville equation, up to a multiplicative constant, which satisfies the boundary condition at $x = x_b$. Call this solution $y_2(x)$. We then can write

$$G_\omega(x,x') = \begin{cases} A(x')\, y_1(x) & \text{if } x_a \le x < x' \\ \\ B(x')\, y_2(x) & \text{if } x' < x \le x_b \quad . \end{cases} \tag{10.229}$$

Here, $A(x')$ and $B(x')$ are undetermined functions. We now invoke the inhomogeneous Sturm-Liouville equation,

$$-\frac{d}{dx}\left[\tau(x)\,\frac{dG_\omega(x,x')}{dx}\right] + v(x)\, G_\omega(x,x') - \omega^2 \mu(x)\, G_\omega(x,x') = \delta(x - x') \quad . \tag{10.230}$$

We integrate this from $x = x' - \epsilon$ to $x = x' + \epsilon$, where ϵ is a positive infinitesimal. This yields

$$\tau(x')\Big[A(x')\, y_1'(x') - B(x')\, y_2'(x')\Big] = 1 \quad . \tag{10.231}$$

Continuity of $G_\omega(x,x')$ itself demands

$$A(x')\, y_1(x') = B(x')\, y_2(x') \quad . \tag{10.232}$$

Solving these two equations for $A(x')$ and $B(x')$, we obtain

$$A(x') = -\frac{y_2(x')}{\tau(x')\, \mathcal{W}_{y_1,y_2}(x')} \quad , \quad B(x') = -\frac{y_1(x')}{\tau(x')\, \mathcal{W}_{y_1,y_2}(x')} \quad , \tag{10.233}$$

where $\mathcal{W}_{y_1,y_2}(x)$ is the *Wronskian*

$$\begin{aligned} \mathcal{W}_{y_1,y_2}(x) &= \det \begin{pmatrix} y_1(x) & y_2(x) \\ \\ y_1'(x) & y_2'(x) \end{pmatrix} \\ &= y_1(x)\, y_2'(x) - y_2(x)\, y_1'(x) \quad . \end{aligned} \tag{10.234}$$

Now it is easy to show that $\mathcal{W}_{y_1,y_2}(x)\, \tau(x) = \mathcal{W}\, \tau$ is a constant. This follows from

$$\begin{aligned} 0 &= y_2\, K\, y_1 - y_2\, K\, y_1 \\ &= \frac{d}{dx}\bigg\{\tau(x)\Big[y_1\, y_2' - y_2\, y_1'\Big]\bigg\} \quad . \end{aligned} \tag{10.235}$$

Thus, we have

$$G_\omega(x,x') = \begin{cases} -y_1(x)\, y_2(x')/\mathcal{W} & \text{if } x_a \le x < x' \\ \\ -y_1(x')\, y_2(x)/\mathcal{W} & \text{if } x' < x \le x_b \end{cases} \quad , \tag{10.236}$$

or, in compact form,

$$G_\omega(x,x') = -\frac{y_1(x_<)\, y_2(x_>)}{\mathcal{W}\,\tau} \quad , \tag{10.237}$$

where $x_< = \min(x,x')$ and $x_> = \max(x,x')$.

As an example, consider a uniform string (*i.e.* μ and τ constant, $v = 0$) with fixed endpoints at $x_a = 0$ and $x_b = L$. The normalized eigenfunctions are

$$\psi_n(x) = \sqrt{\frac{2}{\mu L}}\, \sin\left(\frac{n\pi x}{L}\right) \quad , \tag{10.238}$$

and the eigenvalues are $\omega_n = n\pi c/L$. The Green's function is

$$G_\omega(x,x') = \frac{2}{\mu L} \sum_{n=1}^\infty \frac{\sin(n\pi x/L)\, \sin(n\pi x'/L)}{(n\pi c/L)^2 - \omega^2} \quad . \tag{10.239}$$

Now construct the homogeneous solutions:

$$(K - \omega^2 \mu)\, y_1 = 0 \quad , \quad y_1(0) = 0 \qquad \Longrightarrow \qquad y_1(x) = \sin\left(\frac{\omega x}{c}\right) \tag{10.240}$$

$$(K - \omega^2 \mu)\, y_2 = 0 \quad , \quad y_2(L) = 0 \qquad \Longrightarrow \qquad y_2(x) = \sin\left(\frac{\omega(L-x)}{c}\right) \quad . \tag{10.241}$$

The Wronskian is

$$\mathcal{W} = y_1\, y_2' - y_2\, y_1' = -\frac{\omega}{c}\, \sin\left(\frac{\omega L}{c}\right) \quad . \tag{10.242}$$

Therefore, the Green's function is

$$G_\omega(x,x') = \frac{\sin\big(\omega x_</c\big)\, \sin\big(\omega(L - x_>)/c\big)}{(\omega\tau/c)\, \sin(\omega L/c)} \quad . \tag{10.243}$$

10.7.2 Perturbation theory

Suppose we have solved for the Green's function for the linear operator K_0 and mass density $\mu_0(x)$. *I.e.* we have

$$\big(K_0 - \omega^2 \mu_0(x)\big)\, G^0_\omega(x,x') = \delta(x-x') \quad . \tag{10.244}$$

We now imagine perturbing $\tau_0 \to \tau_0 + \lambda\tau_1$, $v_0 \to v_0 + \lambda v_2$, $\mu_0 \to \mu_0 + \lambda\mu_1$. What is the new Green's function $G_\omega(x,x')$? We must solve

$$\big(L_0 + \lambda L_1\big)\, G_\omega(x,x') = \delta(x-x') \quad , \tag{10.245}$$

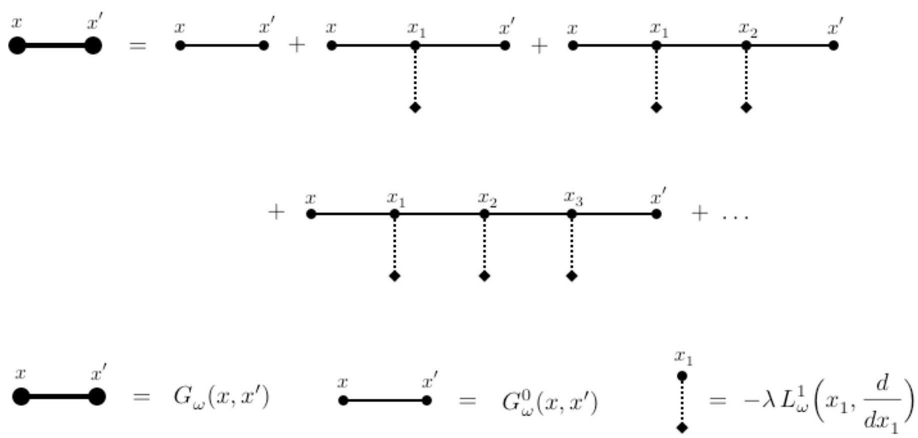


Figure 10.11: Diagrammatic representation of the perturbation expansion in eqn. 10.247.

where

$$L^0_\omega \equiv K_0 - \omega^2\,\mu_0 \qquad , \qquad L^1_\omega \equiv K_1 - \omega^2\,\mu_1 \quad . \tag{10.246}$$

Dropping the ω subscript for simplicity, the full Green's function is then given by

$$\begin{aligned} G_\omega &= \Big[L^0_\omega + \lambda L^1_\omega\Big]^{-1} = \Big[\big(G^0_\omega\big)^{-1} + \lambda L^1_\omega\Big]^{-1} = \Big[1 + \lambda\, G^0_\omega\, L^1_\omega\Big]^{-1} G^0_\omega \\ &= G^0_\omega - \lambda\, G^0_\omega\, L^1_\omega\, G^0_\omega + \lambda^2\, G^0_\omega\, L^1_\omega\, G^0_\omega\, L^1_\omega\, G^0_\omega + \ldots \quad . \end{aligned} \tag{10.247}$$

The 'matrix multiplication' is of course a convolution, *i.e.*

$$G_\omega(x,x') = G^0_\omega(x,x') - \lambda \int\limits_{x_a}^{x_b}\! dx_1\, G^0_\omega(x,x_1)\, L^1_\omega\big(x_1, \tfrac{d}{dx_1}\big)\, G^0_\omega(x_1,x') + \ldots \quad . \tag{10.248}$$

Each term in the perturbation expansion of eqn. 10.247 may be represented by a diagram, as depicted in fig. 10.11.

As an example, consider a string with $x_a = 0$ and $x_b = L$ with a mass point m affixed at the point $x = d$. Thus, $\mu_1(x) = m\,\delta(x-d)$, and $L^1_\omega = -m\omega^2\,\delta(x-d)$, with $\lambda = 1$. The perturbation expansion gives

$$\begin{aligned} G_\omega(x,x') &= G^0_\omega(x,x') + m\omega^2\, G^0_\omega(x,d)\, G^0_\omega(d,x') + m^2\omega^4\, G^0_\omega(x,d)\, G^0_\omega(d,d)\, G^0_\omega(d,x') + \ldots \\ &= G^0_\omega(x,x') + \frac{m\omega^2\, G^0_\omega(x,d)\, G^0_\omega(d,x')}{1 - m\omega^2\, G^0_\omega(d,d)} \quad . \end{aligned} \tag{10.249}$$

Note that the eigenfunction expansion,

$$G_\omega(x,x') = \sum_n \frac{\psi_n(x)\,\psi_n(x')}{\omega_n^2 - \omega^2} \quad , \tag{10.250}$$

says that the exact eigenfrequencies are poles of $G_\omega(x,x')$, and furthermore the residue at each pole is

$$\operatorname*{Res}_{\omega=\omega_n} G_\omega(x,x') = -\frac{1}{2\omega_n}\,\psi_n(x)\,\psi_n(x') \quad . \tag{10.251}$$

According to eqn. 10.249, the poles of $G_\omega(x,x')$ are located at solutions to[4]

$$m\omega^2\, G^0_\omega(d,d) = 1 \quad . \tag{10.252}$$

For simplicity let us set $d = \frac{1}{2}L$, so the mass point is in the middle of the string. Then according to eqn. 10.243,

$$G^0_\omega\big(\tfrac{1}{2}L, \tfrac{1}{2}L\big) = \frac{\sin^2(\omega L/2c)}{(\omega\tau/c)\,\sin(\omega L/c)} = \frac{c}{2\omega\tau}\,\tan\left(\frac{\omega L}{2c}\right) \quad . \tag{10.253}$$

The eigenvalue equation is therefore

$$\tan\left(\frac{\omega L}{2c}\right) = \frac{2\tau}{m\omega c} \quad , \tag{10.254}$$

which can be manipulated to yield

$$\frac{m}{M}\,\lambda = \text{ctn}\,\lambda \quad , \tag{10.255}$$

where $\lambda = \omega L/2c$ and $M = \mu L$ is the total mass of the string. When $m = 0$, the LHS vanishes, and the roots lie at $\lambda = (n + \frac{1}{2})\pi$, which gives $\omega = \omega_{2n+1}$. Why don't we see the poles at the even mode eigenfrequencies ω_{2n}? The answer is that these poles are present in the Green's function. They do not cancel for $d = \frac{1}{2}L$ because the perturbation does not couple to the even modes, which all have $\psi_{2n}(\frac{1}{2}L) = 0$. The case of general d may be instructive in this regard. One finds the eigenvalue equation

$$\frac{\sin(2\lambda)}{2\lambda\,\sin\big(2\epsilon\lambda\big)\,\sin\big(2(1-\epsilon)\lambda\big)} = \frac{m}{M} \quad , \tag{10.256}$$

where $\epsilon = d/L$. Now setting $m = 0$ we recover $2\lambda = n\pi$, which says $\omega = \omega_n$, and all the modes are recovered.

10.7.3 Perturbation theory for eigenvalues and eigenfunctions

We wish to solve

$$\big(K_0 + \lambda K_1\big)\,\psi = \omega^2\big(\mu_0 + \lambda\mu_1\big)\,\psi \quad , \tag{10.257}$$

which is equivalent to

$$L^0_\omega\,\psi = -\lambda L^1_\omega\,\psi \quad . \tag{10.258}$$

Multiplying by $\big(L^0_\omega\big)^{-1} = G^0_\omega$ on the left, we have

$$\begin{aligned}\psi(x) &= -\lambda\int\limits_{x_a}^{x_b}\!dx'\,G_\omega(x,x')\,L^1_\omega\,\psi(x')\\ &= \lambda\sum_{m=1}^\infty \frac{\psi_m(x)}{\omega^2-\omega_m^2}\int\limits_{x_a}^{x_b}\!dx'\,\psi_m(x')\,L^1_\omega\,\psi(x') \quad .\end{aligned} \tag{10.259}$$

[4]Note in particular that there is no longer any divergence at the location of the original poles of $G^0_\omega(x,x')$. These poles are cancelled.

We are free to choose any normalization we like for $\psi(x)$. We choose

$$\langle \psi \,|\, \psi_n \rangle = \int\limits_{x_a}^{x_b} \!dx\; \mu_0(x)\, \psi_n(x)\, \psi(x) = 1 \quad , \tag{10.260}$$

which entails

$$\omega^2 - \omega_n^2 = \lambda \int\limits_{x_a}^{x_b} \!dx\, \psi_n(x)\, L^1_\omega\, \psi(x) \tag{10.261}$$

as well as

$$\psi(x) = \psi_n(x) + \lambda \sum_{\substack{k \\ (k\neq n)}} \frac{\psi_k(x)}{\omega^2 - \omega_k^2} \int\limits_{x_a}^{x_b} \!dx'\, \psi_k(x')\, L^1_\omega\, \psi(x') \quad . \tag{10.262}$$

By expanding ψ and ω^2 in powers of λ, we can develop an order by order perturbation series.

To lowest order, we have

$$\omega^2 = \omega_n^2 + \lambda \int\limits_{x_a}^{x_b} \!dx\, \psi_n(x)\, L^1_{\omega_n} \psi_n(x) \quad . \tag{10.263}$$

For the case $L^1_\omega = -m\, \omega^2\, \delta(x-d)$, we have

$$\frac{\delta\omega_n}{\omega_n} = -\tfrac{1}{2} m \big[\psi_n(d)\big]^2 = -\frac{m}{M}\, \sin^2\!\Big(\frac{n\pi d}{L}\Big) \quad . \tag{10.264}$$

For $d = \frac{1}{2}L$, only the odd n modes are affected, as the even n modes have a node at $x = \frac{1}{2}L$.

Carried out to second order, one obtains for the eigenvalues,

$$\begin{aligned} \omega^2 = \omega_n^2 &+ \lambda \int\limits_{x_a}^{x_b} \!dx\, \psi_n(x)\, L^1_{\omega_n}\, \psi_n(x) + \lambda^2 \sum_{\substack{k \\ (k\neq n)}} \frac{\Big| \int_{x_a}^{x_b} dx\, \psi_k(x)\, L^1_{\omega_n}\, \psi_n(x) \Big|^2}{\omega_n^2 - \omega_k^2} \\ &- \lambda^2 \int\limits_{x_a}^{x_b} \!dx\, \psi_n(x)\, L^1_{\omega_n}\, \psi_n(x) \cdot \int\limits_{x_a}^{x_b} \!dx'\, \mu_1(x')\, \big[\psi_n(x')\big]^2 + \mathcal{O}(\lambda^3) \quad . \end{aligned} \tag{10.265}$$

Chapter 11

Special Relativity

For an extraordinarily lucid, if characteristically brief, discussion, see chs. 1 and 2 of L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (*Course of Theoretical Physics, vol. 2*).

11.1 Introduction

All distances are relative in physics. They are measured with respect to a fixed *frame of reference*. Frames of reference in which free particles move with constant velocity are called *inertial frames*. The *principle of relativity* states that the laws of Nature are identical in all inertial frames.

11.1.1 Michelson-Morley experiment

We learned how sound waves in a fluid, such as air, obey the Helmholtz equation. Let us restrict our attention for the moment to solutions of the form $\phi(x,t)$ which do not depend on y or z. We then have a one-dimensional wave equation,

$$\frac{\partial^2\phi}{\partial x^2} = \frac{1}{c^2}\,\frac{\partial^2\phi}{\partial t^2} \quad . \tag{11.1}$$

The fluid in which the sound propagates is assumed to be at rest. But suppose the fluid is not at rest. We can investigate this by shifting to a moving frame, defining $x' = x - ut$, with $y' = y$, $z' = z$ and of course $t' = t$. This is a Galilean transformation. In terms of the new variables, we have

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x'} \qquad , \qquad \frac{\partial}{\partial t} = -u\,\frac{\partial}{\partial x'} + \frac{\partial}{\partial t'} \quad . \tag{11.2}$$

The wave equation is then

$$\bigg(1 - \frac{u^2}{c^2}\bigg)\frac{\partial^2\phi}{\partial x'^2} = \frac{1}{c^2}\,\frac{\partial^2\phi}{\partial t'^2} - \frac{2u}{c^2}\,\frac{\partial^2\phi}{\partial x'\,\partial t'} \quad . \tag{11.3}$$

Clearly the wave equation acquires a different form when expressed in the new variables (x', t'), *i.e.* in a frame in which the fluid is not at rest. The general solution is then of the modified d'Alembert form,

$$\phi(x', t') = f(x' - c_{\rm R} t') + g(x' + c_{\rm L} t') \quad , \tag{11.4}$$

where $c_{\rm R} = c - u$ and $c_{\rm L} = c + u$ are the speeds of rightward and leftward propagating disturbances, respectively. Thus, there is a *preferred frame of reference* – the frame in which the fluid is at rest. In the rest frame of the fluid, sound waves travel with velocity c in either direction.

Light, as we know, is a wave phenomenon in classical physics. The propagation of light is described by Maxwell's equations,

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = 4\pi\rho \qquad\qquad \boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} \tag{11.5}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \qquad\qquad \boldsymbol{\nabla} \times \boldsymbol{B} = \frac{4\pi}{c}\boldsymbol{j} + \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} \quad , \tag{11.6}$$

where ρ and $\boldsymbol{j}$ are the local charge and current density, respectively. Taking the curl of Faraday's law, and restricting to free space where $\rho = \boldsymbol{j} = 0$, we once again have (using a Cartesian system for the fields) the wave equation,

$$\nabla^2 \boldsymbol{E} = \frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2} \quad . \tag{11.7}$$

(We shall discuss below, in section 11.8, the beautiful properties of Maxwell's equations under general coordinate transformations.)

In analogy with the theory of sound, it was assumed prior to Einstein that there was in fact a preferred reference frame for electromagnetic radiation – one in which the medium which was excited during the EM wave propagation was at rest. This notional medium was called the *lumineferous ether*. Indeed, it was generally assumed during the $19^{\rm th}$ century that light, electricity, magnetism, and heat (which was not understood until Boltzmann's work in the late $19^{\rm th}$ century) all had separate ethers. It was Maxwell who realized that light, electricity, and magnetism were all unified phenomena, and accordingly he proposed a single ether for electromagnetism. It was believed at the time that the earth's motion through the ether would result in a drag on the earth.

In 1887, Michelson and Morley set out to measure the changes in the speed of light on earth due to the earth's movement through the ether (which was generally assumed to be at rest in the frame of the Sun). The Michelson interferometer is shown in fig. 11.1, and works as follows. Suppose the apparatus is moving with velocity $u\,\hat{\boldsymbol{x}}$ through the ether. Then the time it takes a light ray to travel from the half-silvered mirror to the mirror on the right and back again is

$$t_x = \frac{\ell}{c+u} + \frac{\ell}{c-u} = \frac{2\ell c}{c^2 - u^2} \quad . \tag{11.8}$$

For motion along the other arm of the interferometer, the geometry in the inset of fig. 11.1 shows $\ell' = \sqrt{\ell^2 + \frac{1}{4}u^2 t_y^2}$, hence

$$t_y = \frac{2\ell'}{c} = \frac{2}{c}\sqrt{\ell^2 + \tfrac{1}{4}u^2 t_y^2} \qquad \Rightarrow \qquad t_y = \frac{2\ell}{\sqrt{c^2 - u^2}} \quad . \tag{11.9}$$

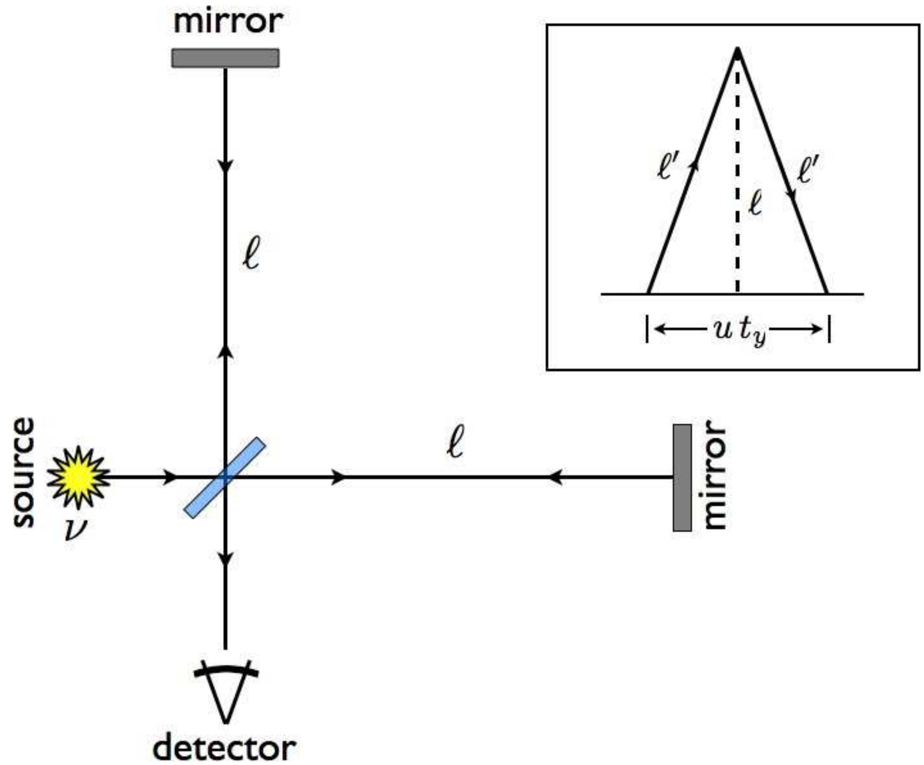


Figure 11.1: The Michelson-Morley experiment (1887) used an interferometer to effectively measure the time difference for light to travel along two different paths. Inset: analysis for the y-directed path.

Thus, the difference in times along these two paths is

$$\Delta t = t_x - t_y = \frac{2\ell c}{c^2} - \frac{2\ell}{\sqrt{c^2 - u^2}} \approx \frac{\ell}{c} \cdot \frac{u^2}{c^2} \quad . \tag{11.10}$$

Thus, the difference in phase between the two paths is

$$\frac{\Delta\phi}{2\pi} = \nu\,\Delta t \approx \frac{\ell}{\lambda} \cdot \frac{u^2}{c^2} \quad , \tag{11.11}$$

where λ is the wavelength of the light. We take $u \approx 30\,\mathrm{km/s}$, which is the earth's orbital velocity, and $\lambda \approx 5000\,\text{Å}$. From this we find that $\Delta\phi \approx 0.02 \times 2\pi$ if $\ell = 1\,\mathrm{m}$. Michelson and Morley found that the observed fringe shift $\Delta\phi/2\pi$ was approximately 0.02 times the expected value. The inescapable conclusion was that the speed of light did not depend on the motion of the source. This was very counterintuitive!

The history of the development of special relativity is quite interesting, but we shall not have time to dwell here on the many streams of scientific thought during those exciting times. Suffice it to say that the Michelson-Morley experiment, while a landmark result, was not the last word. It had been proposed that the ether could be dragged, either entirely or partially, by moving bodies. If the earth dragged the ether along with it, then there would be no ground-level 'ether wind' for the MM experiment to detect. Other experiments, however, such as stellar aberration, in which the apparent position of a distant star

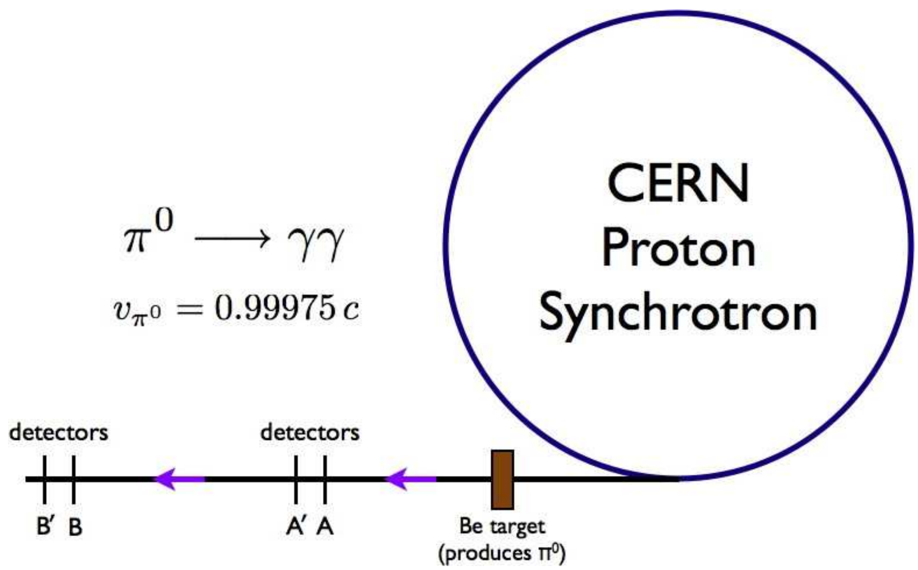


Figure 11.2: Experimental setup of Alvager *et al.* (1964), who used the decay of high energy neutral pions to test the source velocity dependence of the speed of light.

varies due to the earth's orbital velocity, rendered the "ether drag" theory untenable – the notional 'ether bubble' dragged by the earth could not reasonably be expected to extend to the distant stars.

A more recent test of the effect of a moving source on the speed of light was performed by T. Alvåger *et al.*, *Phys. Lett.* **12**, 260 (1964), who measured the velocity of γ-rays (photons) emitted from the decay of highly energetic neutral pions (π^0). The pion energies were in excess of 6 GeV, which translates to a velocity of $v = 0.99975\,c$, according to special relativity. Thus, photons emitted in the direction of the pions should be traveling at close to $2c$, if the source and photon velocities were to add. Instead, the velocity of the photons was found to be $c = 2.9977 \pm 0.0004 \times 10^{10}\,\mathrm{cm/s}$, which is within experimental error of the best accepted value.

11.1.2 Einsteinian and Galilean relativity

The *Principle of Relativity* states that the laws of nature are the same when expressed in any inertial frame. This principle can further be refined into two classes, depending on whether one takes the velocity of the propagation of interactions to be finite or infinite.

The interaction of matter in classical mechanics is described by a potential function $U(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N)$. Typically, one has two-body interactions in which case one writes $U = \sum_{i<j} U(\boldsymbol{r}_i, \boldsymbol{r}_j)$. These interactions are thus assumed to be instantaneous, which is unphysical. The interaction of particles is mediated by the exchange of gauge bosons, such as the photon (for electromagnetic interactions), gluons (for the strong interaction, at least on scales much smaller than the 'confinement length'), or the graviton (for gravity). Their velocity of propagation, according to the principle of relativity, is the same in all reference frames, and is given by the speed of light, $c = 2.998 \times 10^8\,\mathrm{m/s}$.

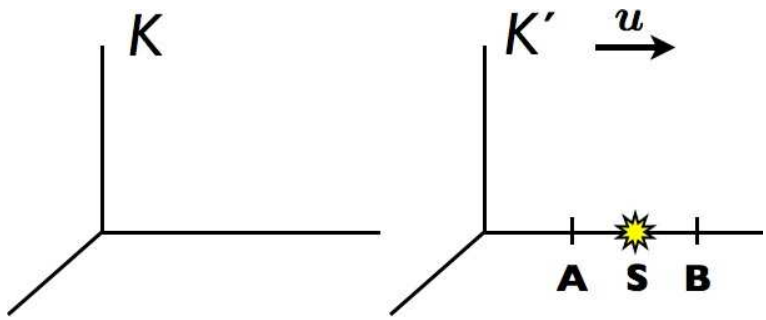


Figure 11.3: Two reference frames.

Since c is so large in comparison with terrestrial velocities, and since d/c is much shorter than all other relevant time scales for typical interparticle separations d, the assumption of an instantaneous interaction is usually quite accurate. The combination of the principle of relativity with finiteness of c is known as Einsteinian relativity. When $c = \infty$, the combination comprises Galilean relativity:

$$\begin{aligned} c < \infty \quad &: \quad \text{Einsteinian relativity} \\ c = \infty \quad &: \quad \text{Galilean relativity} \quad . \end{aligned}$$

Consider a train moving at speed u. In the rest frame of the train track, the speed of the light beam emanating from the train's headlight is $c+u$. This would contradict the principle of relativity. This leads to some very peculiar consequences, foremost among them being the fact that events which are simultaneous in one inertial frame will not in general be simultaneous in another. In Newtonian mechanics, on the other hand, time is absolute, and is independent of the frame of reference. If two events are simultaneous in one frame then they are simultaneous in all frames. This is not the case in Einsteinian relativity!

We can begin to apprehend this curious feature of simultaneity by the following *Gedankenexperiment* (a long German word meaning "thought experiment")[1]. Consider the case in fig. 11.3 in which frame K' moves with velocity $u\,\hat{\boldsymbol{x}}$ with respect to frame K. Let a source at S emit a signal (a light pulse) at $t = 0$. In the frame K' the signal's arrival at equidistant locations A and B is simultaneous. In frame K, however, A moves toward left-propagating emitted wavefront, and B moves away from the right-propagating wavefront. For classical sound, the speed of the left-moving and right-moving wavefronts is $c \mp u$, taking into account the motion of the source, and thus the relative velocities of the signal and the detectors remain at c. But according to the principle of relativity, the speed of light is c in all frames, and is so in frame K for both the left-propagating and right-propagating signals. Therefore, the relative velocity of A and the left-moving signal is $c+u$ and the relative velocity of B and the right-moving signal is $c - u$. Therefore, A 'closes in' on the signal and receives it before B, which is moving away from the signal. We might expect the arrival times to be $t^*_{\rm A} = d/(c+u)$ and $t^*_{\rm B} = d/(c-u)$, where d is the distance between the source S and either detector A or B in the K' frame. Later on we shall analyze this problem and show that

$$t^*_{\rm A} = \sqrt{\frac{c-u}{c+u}} \cdot \frac{d}{c} \qquad , \qquad t^*_{\rm B} = \sqrt{\frac{c+u}{c-u}} \cdot \frac{d}{c} \quad . \tag{11.12}$$

[1]Unfortunately, many important physicists were German and we have to put up with a legacy of long German words like *Gedankenexperiment*, *Zitterbewegung*, *Brehmsstrahlung*, *Stosszahlansatz*, *Kartoffelsalat*, *etc.*

Our naïve analysis has omitted an important detail – the *Lorentz contraction* of the distance d as seen by an observer in the K frame.

11.2 Intervals

Now let us express mathematically the constancy of c in all frames. An *event* is specified by the time and place where it occurs. Thus, an event is specified by *four* coordinates, (t, x, y, z). The four-dimensional space spanned by these coordinates is called *spacetime*. The *interval* between two events in spacetime at (t_1, x_1, y_1, z_1) and (t_2, x_2, y_2, z_2) is defined to be

$$s_{12} = \sqrt{c^2(t_1 - t_2)^2 - (x_1 - x_2)^2 - (y_1 - y_2)^2 - (z_1 - z_2)^2} \quad . \tag{11.13}$$

For two events separated by an infinitesimal amount, the interval ds is infinitesimal, with

$$ds^2 = c^2\,dt^2 - dx^2 - dy^2 - dz^2 \quad . \tag{11.14}$$

Now when the two events denote the emission and reception of an electromagnetic signal, we have $ds^2 = 0$. This must be true in any frame, owing to the invariance of c, hence since ds and ds' are differentials of the same order, we must have $ds'^2 = ds^2$. This last result requires homogeneity and isotropy of space as well. Finally, if infinitesimal intervals are invariant, then integrating we obtain $s = s'$, and we conclude that *the interval between two space-time events is the same in all inertial frames*.

When $s_{12}^2 > 0$, the interval is said to be *time-like*. For timelike intervals, we can always find a reference frame in which the two events occur at the same *locations*. As an example, consider a passenger sitting on a train. Event #1 is the passenger yawning at time t_1. Event #2 is the passenger yawning again at some later time t_2. To an observer sitting in the train station, the two events take place at different locations, but in the frame of the passenger, they occur at the same location.

When $s_{12}^2 < 0$, the interval is said to be *space-like*. Note that $s_{12} = \sqrt{s_{12}^2} \in i\mathbb{R}$ is pure imaginary, so one says that imaginary intervals are spacelike. As an example, at this moment, in the frame of the reader, the North and South poles of the earth are separated by a space-like interval. If the interval between two events is space-like, a reference frame can always be found in which the events are simultaneous.

An interval with $s_{12} = 0$ is said to be *light-like*.

This leads to the concept of the *light cone*, depicted in fig. 11.4. Consider an event E. In the frame of an inertial observer, all events with $s^2 > 0$ and $\Delta t > 0$ are in E's *forward light cone* and are part of his *absolute future*. Events with $s^2 > 0$ and $\Delta t < 0$ lie in E's *backward light cone* are are part of his *absolute past*. Events with spacelike separations $s^2 < 0$ are *causally disconnected* from E. Two events which are causally disconnected can not possible influence each other. Uniform rectilinear motion is represented by a line $t = x/v$ with constant slope. If $v < c$, this line is contained within E's light cone. E is potentially influenced by all events in its backward light cone, *i.e.* its absolute past. It is impossible to find a frame of reference which will transform past into future, or spacelike into timelike intervals.

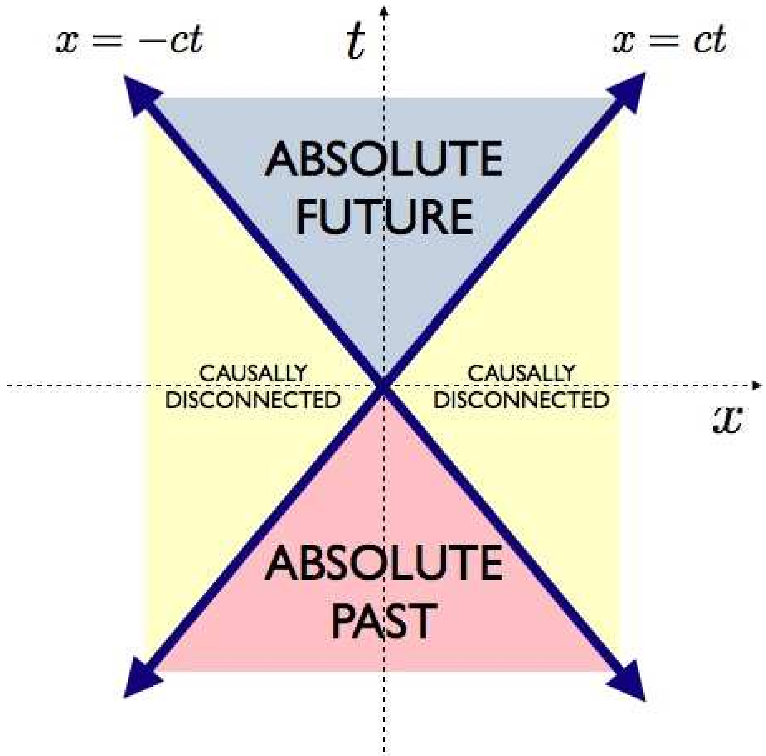


Figure 11.4: A $(1+1)$–dimensional light cone. The forward light cone consists of timelike events with $\Delta t > 0$. The backward light cone consists of timelike events with $\Delta t < 0$. The causally disconnected regions are time-like, and intervals connecting the origin to any point on the light cone itself are light-like.

11.2.1 Proper time

Proper time is the time read on a clock traveling with a moving observer. Consider two observers, one at rest and one in motion. If dt is the differential time elapsed in the rest frame, then

$$\begin{aligned} ds^2 &= c^2\,dt^2 - dx^2 - dy^2 - dz^2 \\ &= c^2\,dt'^2 \quad , \end{aligned} \tag{11.15}$$

where dt' is the differential time elapsed on the moving clock. Thus,

$$dt' = dt\,\sqrt{1 - \frac{\boldsymbol{v}^2}{c^2}} \quad , \tag{11.16}$$

and the time elapsed on the moving observer's clock is

$$t'_2 - t'_1 = \int\limits_{t_1}^{t_2} dt\,\sqrt{1 - \frac{\boldsymbol{v}^2(t)}{c^2}} \quad . \tag{11.17}$$

Thus, *moving clocks run slower*. This is an essential feature which is key to understanding many important aspects of particle physics. A particle with a brief lifetime can, by moving at speeds close to c, appear to an observer in our frame to be long-lived. It is customary to define two dimensionless measures of a particle's velocity:

$$\boldsymbol{\beta} \equiv \frac{\boldsymbol{v}}{c} \qquad , \qquad \gamma \equiv \frac{1}{\sqrt{1-\boldsymbol{\beta}^2}} \quad . \tag{11.18}$$

As $v \to c$, we have $\beta \to 1$ and $\gamma \to \infty$.

Suppose we wish to compare the elapsed time on two clocks. We keep one clock at rest in an inertial frame, while the other executes a closed path in space, returning to its initial location after some interval of time. When the clocks are compared, the moving clock will show a smaller elapsed time. This is often stated as the "twin paradox." The total elapsed time on a moving clock is given by

$$\tau = \frac{1}{c}\int_a^b ds \quad , \tag{11.19}$$

where the integral is taken over the *world line* of the moving clock. The elapsed time τ takes on a minimum value when the path from a to b is a straight line. To see this, one can express $\tau\big[\boldsymbol{x}(t)\big]$ as a functional of the path $\boldsymbol{x}(t)$ and extremize. This results in $\ddot{\boldsymbol{x}} = 0$.

11.2.2 Irreverent problem from Spring 2002 final exam

Flowers for Algernon – Bob's beloved hamster, Algernon, is very ill. He has only three hours to live. The veterinarian tells Bob that Algernon can be saved only through a gallbadder transplant. A suitable donor gallbladder is available from a hamster recently pronounced brain dead after a blender accident in New York (miraculously, the gallbladder was unscathed), but it will take Life Flight five hours to bring the precious rodent organ to San Diego.

Bob embarks on a bold plan to save Algernon's life. He places him in a cage, ties the cage to the end of a strong meter-long rope, and whirls the cage above his head while the Life Flight team is *en route*. Bob reasons that *if he can make time pass more slowly for Algernon*, the gallbladder will arrive in time to save his life.

(a) At how many revolutions per second must Bob rotate the cage in order that the gallbladder arrive in time for the life-saving surgery? What is Algernon's speed v_0?

Solution : We have $\beta(t) = \omega_0 R/c$ is constant, therefore, from eqn. 11.17,

$$\Delta t = \gamma\, \Delta t' \quad . \tag{11.20}$$

Setting $\Delta t' = 3\,\text{hr}$ and $\Delta t = 5\,\text{hr}$, we have $\gamma = \frac{5}{3}$, which entails $\beta = \sqrt{1-\gamma^{-2}} = \frac{4}{5}$. Thus, $v_0 = \frac{4}{5}\,c$, which requires a rotation frequency of $\omega_0/2\pi = 38.2\,\text{MHz}$.

(b) Bob finds that he cannot keep up the pace! Assume Algernon's speed is given by

$$v(t) = v_0\sqrt{1-\frac{t}{T}} \tag{11.21}$$

where v_0 is the speed from part (a), and $T = 5\,\mathrm{h}$. As the plane lands at the pet hospital's emergency runway, Bob peers into the cage to discover that Algernon is dead! In order to fill out his death report, the veterinarian needs to know: *when did Algernon die?* Assuming he died after his own hamster watch registered three hours, derive an expression for the elapsed time on the veterinarian's clock at the moment of Algernon's death.

Solution : ⟨Sniffle⟩. We have $\beta(t) = \frac{4}{5}\left(1 - \frac{t}{T}\right)^{1/2}$. We set

$$T' = \int\limits_0^{T^*} dt\,\sqrt{1-\beta^2(t)} \tag{11.22}$$

where $T' = 3\,\mathrm{hr}$ and T^* is the time of death in Bob's frame. We write $\beta_0 = \frac{4}{5}$ and $\gamma_0 = (1-\beta_0^2)^{-1/2} = \frac{5}{3}$. Note that $T'/T = \sqrt{1-\beta_0^2} = \gamma_0^{-1}$.

Rescaling by writing $\zeta = t/T$, we have

$$\begin{aligned}\frac{T'}{T} = \gamma_0^{-1} &= \int\limits_0^{T^*/T} d\zeta\,\sqrt{1-\beta_0^2+\beta_0^2\,\zeta} \\ &= \frac{2}{3\beta_0^2}\left[\left(1-\beta_0^2+\beta_0^2\,\frac{T^*}{T}\right)^{3/2} - (1-\beta_0^2)^{3/2}\right] \\ &= \frac{2}{3\gamma_0}\cdot\frac{1}{\gamma_0^2-1}\left[\left(1+(\gamma_0^2-1)\,\frac{T^*}{T}\right)^{3/2} - 1\right] \quad .\end{aligned} \tag{11.23}$$

Solving for T^*/T we have

$$\frac{T^*}{T} = \frac{\left(\frac{3}{2}\,\gamma_0^2 - \frac{1}{2}\right)^{2/3} - 1}{\gamma_0^2 - 1} \quad . \tag{11.24}$$

With $\gamma_0 = \frac{5}{3}$ we obtain

$$\frac{T^*}{T} = \tfrac{9}{16}\left[\left(\tfrac{11}{3}\right)^{2/3} - 1\right] = 0.77502\ldots \tag{11.25}$$

Thus, $T^* = 3.875\,\mathrm{hr} = 3$ hr 52 min 50.5 sec after Bob starts swinging.

(c) Identify at least three practical problems with Bob's scheme.

Solution : As you can imagine, student responses to this part were varied and generally sarcastic. *E.g.* "the atmosphere would ignite," or "Bob's arm would fall off," or "Algernon's remains would be found on the inside of the far wall of the cage, squashed flatter than a coat of semi-gloss paint," *etc.*

11.3 Four-Vectors and Lorentz Transformations

We have spoken thus far about different reference frames. So how precisely do the coordinates (t, x, y, z) transform between frames K and K'? In classical mechanics, we have $t = t'$ and $\boldsymbol{x} = \boldsymbol{x}' + \boldsymbol{u}\,t$, according to fig. 11.3. This yields the *Galilean transformation*,

$$\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ u_x & 1 & 0 & 0 \\ u_y & 0 & 1 & 0 \\ u_z & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} \quad . \tag{11.26}$$

Such a transformation does not leave intervals invariant.

Let us define the *four-vector* x^μ as

$$x^\mu = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \equiv \begin{pmatrix} ct \\ \boldsymbol{x} \end{pmatrix} \quad . \tag{11.27}$$

Thus, $x^0 = ct$, $x^1 = x$, $x^2 = y$, and $x^3 = z$. In order for intervals to be invariant, the transformation between x^μ in frame K and $x'^{\,\mu}$ in frame K' must be linear:

$$x^\mu = L^\mu_{\ \nu}\, x'^{\,\nu} \quad , \tag{11.28}$$

where we are using the Einstein convention of summing over repeated indices. We define the *Minkowski metric tensor* $g_{\mu\nu}$ as follows:

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad . \tag{11.29}$$

Clearly $g = g^{\rm t}$ is a symmetric matrix.

Note that the matrix $L^\alpha_{\ \beta}$ has one raised index and one lowered index. For the notation we are about to develop, it is very important to distinguish raised from lowered indices. To raise or lower an index, we use the metric tensor. For example,

$$x_\mu = g_{\mu\nu}\, x^\nu = \begin{pmatrix} ct \\ -x \\ -y \\ -z \end{pmatrix} \quad . \tag{11.30}$$

The act of summing over an identical raised and lowered index is called *index contraction*. Note that

$$g^\mu_{\ \nu} = g^{\mu\rho}\, g_{\rho\nu} = \delta^\mu_{\ \nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad . \tag{11.31}$$

Now let's investigate the invariance of the interval. We must have $x'^{\,\mu}\, x'_\mu = x^\mu\, x_\mu$. Note that

$$\begin{aligned} x^\mu\, x_\mu &= L^\mu_{\ \alpha}\, x'^{\,\alpha}\, L_\mu^{\ \beta}\, x'_\beta \\ &= \big(L^\mu_{\ \alpha}\, g_{\mu\nu}\, L^\nu_{\ \beta}\big)\, x'^{\,\alpha}\, x'^{\,\beta} \quad , \end{aligned} \tag{11.32}$$

from which we conclude

$$L^{\mu}{}_{\alpha}\, g_{\mu\nu}\, L^{\nu}{}_{\beta} = g_{\alpha\beta} \quad . \tag{11.33}$$

This result also may be written in other ways:

$$L^{\mu\alpha}\, g_{\mu\nu}\, L^{\nu\beta} = g^{\alpha\beta} \quad , \quad L^{\mathrm{t}\,\mu}_{\alpha}\; g_{\mu\nu}\, L^{\nu}{}_{\beta} = g_{\alpha\beta} \tag{11.34}$$

Another way to write this equation is $L^{\mathrm{t}}\, g\, L = g$. A rank-4 matrix which satisfies this constraint, with $g = \mathrm{diag}(+,-,-,-)$ is an element of the group $\mathrm{O}(3,1)$, known as the *Lorentz group*.

Let us now count the freedoms in L. As a 4×4 real matrix, it contains 16 elements. The matrix $L^{\mathrm{t}}\, g\, L$ is a symmetric 4×4 matrix, which contains 10 independent elements: 4 along the diagonal and 6 above the diagonal. Thus, there are 10 constraints on 16 elements of L, and we conclude that the group $\mathrm{O}(3,1)$ is 6-dimensional. This is also the dimension of the four-dimensional orthogonal group $\mathrm{O}(4)$, by the way. Three of these six parameters may be taken to be the Euler angles. That is, the group $\mathrm{O}(3)$ constitutes a three-dimensional *subgroup* of the Lorentz group $\mathrm{O}(3,1)$, with elements

$$L^{\mu}{}_{\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & R_{11} & R_{12} & R_{13} \\ 0 & R_{21} & R_{22} & R_{23} \\ 0 & R_{31} & R_{32} & R_{33} \end{pmatrix} \quad , \tag{11.35}$$

where $R^{\mathrm{t}} R = MI$, *i.e.* $R \in \mathrm{O}(3)$ is a rank-3 orthogonal matrix, parameterized by the three Euler angles (ϕ, θ, ψ). The remaining three parameters form a vector $\boldsymbol{\beta} = (\beta_x, \beta_y, \beta_z)$ and define a second class of Lorentz transformations, called boosts:[2]

$$L^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & \gamma\,\beta_x & \gamma\,\beta_y & \gamma\,\beta_z \\ \gamma\,\beta_x & 1+(\gamma-1)\,\hat{\beta}_x\,\hat{\beta}_x & (\gamma-1)\,\hat{\beta}_x\,\hat{\beta}_y & (\gamma-1)\,\hat{\beta}_x\,\hat{\beta}_z \\ \gamma\,\beta_y & (\gamma-1)\,\hat{\beta}_x\,\hat{\beta}_y & 1+(\gamma-1)\,\hat{\beta}_y\,\hat{\beta}_y & (\gamma-1)\,\hat{\beta}_y\,\hat{\beta}_z \\ \gamma\,\beta_z & (\gamma-1)\,\hat{\beta}_x\,\hat{\beta}_z & (\gamma-1)\,\hat{\beta}_y\,\hat{\beta}_z & 1+(\gamma-1)\,\hat{\beta}_z\,\hat{\beta}_z \end{pmatrix} \quad , \tag{11.36}$$

where

$$\widehat{\boldsymbol{\beta}} = \frac{\boldsymbol{\beta}}{|\boldsymbol{\beta}|} \quad , \quad \gamma = \big(1-\boldsymbol{\beta}^2\big)^{-1/2} \quad . \tag{11.37}$$

IMPORTANT : Since the components of $\boldsymbol{\beta}$ are not the spatial components of a four vector, we will only write these components with a lowered index, as β_i, with $i = 1, 2, 3$. We will not write β^i with a raised index, but if we did, we'd mean the same thing, *i.e.* $\beta^i = \beta_i$. Note that for the spatial components of a 4-vector like x^{μ}, we have $x_i = -x^i$.

Let's look at a simple example, where $\beta_x = \beta$ and $\beta_y = \beta_z = 0$. Then

$$L^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & \gamma\,\beta & 0 & 0 \\ \gamma\,\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad . \tag{11.38}$$

[2] Unlike rotations, the boosts do not themselves define a subgroup of $\mathrm{O}(3,1)$.

The effect of this Lorentz transformation $x^\mu = L^\mu{}_\nu \, x'^{\,\nu}$ is thus

$$\begin{aligned} ct &= \gamma ct' + \gamma\beta x' \\ x &= \gamma\beta ct' + \gamma x' \quad . \end{aligned} \tag{11.39}$$

How fast is the origin of K' moving in the K frame? We have $dx' = 0$ and thus

$$\frac{1}{c}\,\frac{dx}{dt} = \frac{\gamma\beta\, c\, dt'}{\gamma\, c\, dt'} = \beta \quad . \tag{11.40}$$

Thus, $u = \beta c$, *i.e.* $\beta = u/c$.

It is convenient to take advantage of the fact that $\mathrm{P}^{\boldsymbol{\beta}}_{ij} \equiv \hat\beta_i \, \hat\beta_j$ is a *projection operator*, which satisfies $\big(\mathrm{P}^{\boldsymbol{\beta}}\big)^2 = \mathrm{P}^{\boldsymbol{\beta}}$. The action of $\mathrm{P}^{\boldsymbol{\beta}}_{ij}$ on any vector $\boldsymbol{\xi}$ is to project that vector onto the $\hat{\boldsymbol{\beta}}$ direction:

$$\mathrm{P}^{\boldsymbol{\beta}}\,\boldsymbol{\xi} = (\hat{\boldsymbol{\beta}}\cdot\boldsymbol{\xi})\,\hat{\boldsymbol{\beta}} \quad . \tag{11.41}$$

We may now write the general Lorentz boost, with $\boldsymbol{\beta} = \boldsymbol{u}/c$, as

$$L = \begin{pmatrix} \gamma & \gamma\boldsymbol{\beta}^{\mathrm{t}} \\ \gamma\boldsymbol{\beta} & \mathrm{I} + (\gamma-1)\,\mathrm{P}^{\boldsymbol{\beta}} \end{pmatrix} \quad , \tag{11.42}$$

where I is the 3×3 unit matrix, and where we write column and row vectors

$$\boldsymbol{\beta} = \begin{pmatrix} \beta_x \\ \beta_y \\ \beta_z \end{pmatrix} \qquad , \qquad \boldsymbol{\beta}^{\mathrm{t}} = \begin{pmatrix} \beta_x & \beta_y & \beta_z \end{pmatrix} \tag{11.43}$$

as a mnemonic to help with matrix multiplications. We now have

$$\begin{pmatrix} ct \\ \boldsymbol{x} \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\boldsymbol{\beta}^{\mathrm{t}} \\ \gamma\boldsymbol{\beta} & \mathrm{I} + (\gamma-1)\,\mathrm{P}^{\boldsymbol{\beta}} \end{pmatrix} \begin{pmatrix} ct' \\ \boldsymbol{x}' \end{pmatrix} = \begin{pmatrix} \gamma ct' + \gamma\boldsymbol{\beta}\cdot\boldsymbol{x}' \\ \gamma\boldsymbol{\beta}ct' + \boldsymbol{x}' + (\gamma-1)\,\mathrm{P}^{\boldsymbol{\beta}}\,\boldsymbol{x}' \end{pmatrix} \quad . \tag{11.44}$$

Thus,

$$\begin{aligned} ct &= \gamma ct' + \gamma\boldsymbol{\beta}\cdot\boldsymbol{x}' \\ \boldsymbol{x} &= \gamma\boldsymbol{\beta}ct' + \boldsymbol{x}' + (\gamma-1)\,(\hat{\boldsymbol{\beta}}\cdot\boldsymbol{x}')\,\hat{\boldsymbol{\beta}} \quad . \end{aligned} \tag{11.45}$$

If we resolve $\boldsymbol{x}$ and $\boldsymbol{x}'$ into components parallel and perpendicular to $\boldsymbol{\beta}$, writing

$$x_{\|} = \hat{\boldsymbol{\beta}}\cdot\boldsymbol{x} \qquad , \qquad \boldsymbol{x}_{\perp} = \boldsymbol{x} - (\hat{\boldsymbol{\beta}}\cdot\boldsymbol{x})\,\hat{\boldsymbol{\beta}} \quad , \tag{11.46}$$

with corresponding definitions for $x'_{\|}$ and $\boldsymbol{x}'_{\perp}$, the general Lorentz boost may be written as

$$\begin{aligned} ct &= \gamma ct' + \gamma\beta x'_{\|} \\ x_{\|} &= \gamma\beta ct' + \gamma x'_{\|} \\ \boldsymbol{x}_{\perp} &= \boldsymbol{x}'_{\perp} \quad . \end{aligned} \tag{11.47}$$

Thus, the components of $\boldsymbol{x}$ and $\boldsymbol{x}'$ which are parallel to $\boldsymbol{\beta}$ enter into a one-dimensional Lorentz boost along with t and t', as described by eqn. 11.39. The components of $\boldsymbol{x}$ and $\boldsymbol{x}'$ which are perpendicular to $\boldsymbol{\beta}$ are unaffected by the boost.

Finally, the Lorentz group $\mathrm{O}(3,1)$ is a group under multiplication, which means that if L_a and L_b are elements, then so is the product $L_\mathrm{a}\,L_\mathrm{b}$. Explicitly, we have

$$(L_\mathrm{a}\,L_\mathrm{b})^\mathrm{t}\,g\,L_\mathrm{a}\,L_\mathrm{b} = L_\mathrm{b}^\mathrm{t}\,(L_\mathrm{a}^\mathrm{t}\,g\,L_\mathrm{a})\,L_\mathrm{b} = L_\mathrm{b}^\mathrm{t}\,g\,L_\mathrm{b} = g \quad . \tag{11.48}$$

11.3.1 Covariance and contravariance

Note that

$$\begin{aligned} L^{\mathrm{t}\,\mu}_{\alpha}\; g_{\mu\nu}\, L^{\nu}{}_{\beta} &= \begin{pmatrix} \gamma & \gamma\,\beta & 0 & 0 \\ \gamma\,\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \gamma & \gamma\,\beta & 0 & 0 \\ \gamma\,\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = g_{\alpha\beta} \quad , \end{aligned} \tag{11.49}$$

since $\gamma^2\,(1-\beta^2)=1$. This is in fact the general way that tensors transform under a Lorentz transformation:

$$\begin{aligned} \text{covariant vectors}: &\;\; x^\mu = L^\mu{}_\nu\, x'^{\,\nu} \\ \text{covariant tensors}: &\;\; F^{\mu\nu} = L^\mu{}_\alpha\, L^\nu{}_\beta\, F'^{\,\alpha\beta} = L^\mu{}_\alpha\, F'^{\,\alpha\beta}\, L^{\mathrm{t}\,\nu}_{\beta} \end{aligned} \tag{11.50}$$

Note how index contractions always involve one raised index and one lowered index. Raised indices are called *contravariant indices* and lowered indiced are called *covariant indices*. The transformation rules for contravariant vectors and tensors are

$$\begin{aligned} \text{contravariant vectors}: &\;\; x_\mu = L_\mu^{\;\nu}\, x'_\nu \\ \text{contravariant tensors}: &\;\; F_{\mu\nu} = L_\mu^{\;\alpha}\, L_\nu^{\;\beta}\, F'{}_{\alpha\beta} = L_\mu^{\;\alpha}\, F'{}_{\alpha\beta}\, L^{\mathrm{t}\beta}{}_{\nu} \end{aligned} \tag{11.51}$$

A *Lorentz scalar* has no indices at all. For example,

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu \quad , \tag{11.52}$$

is a Lorentz scalar. In this case, we have contracted a tensor with two four-vectors. The dot product of two four-vectors is also a Lorentz scalar:

$$\begin{aligned} a \cdot b \equiv a^\mu\, b_\mu &= g_{\mu\nu}\, a^\mu\, b^\nu \\ &= a^0\, b^0 - a^1\, b^1 - a^2\, b^2 - a^3\, b^3 \\ &= a^0\, b^0 - \boldsymbol{a}\cdot\boldsymbol{b} \quad . \end{aligned} \tag{11.53}$$

Note that the dot product $a \cdot b$ of four-vectors is invariant under a simultaneous Lorentz transformation of both a^μ and b^μ, *i.e.* $a \cdot b = a' \cdot b'$. Indeed, this invariance is the very definition of what it means for something to be a Lorentz scalar. Derivatives with respect to covariant vectors yield contravariant vectors:

$$\frac{\partial f}{\partial x^\mu} \equiv \partial_\mu f \qquad , \qquad \frac{\partial A^\mu}{\partial x^\nu} = \partial_\nu A^\mu \equiv B^\mu{}_\nu \qquad , \qquad \frac{\partial B^\mu{}_\nu}{\partial x^\lambda} = \partial_\lambda B^\mu{}_\nu \equiv C^\mu{}_{\nu\lambda} \tag{11.54}$$

et cetera. Note that differentiation with respect to the *co*variant vector x^μ is expressed by the *contra*variant differential operator ∂_μ:

$$\begin{aligned} \frac{\partial}{\partial x^\mu} &\equiv \partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}\,,\,\frac{\partial}{\partial x}\,,\,\frac{\partial}{\partial y}\,,\,\frac{\partial}{\partial z}\right) \\ \frac{\partial}{\partial x_\mu} &\equiv \partial^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}\,,\,-\frac{\partial}{\partial x}\,,\,-\frac{\partial}{\partial y}\,,\,-\frac{\partial}{\partial z}\right) \quad . \end{aligned} \tag{11.55}$$

The contraction $\Box \equiv \partial^\mu \partial_\mu$ is a Lorentz scalar differential operator, called the *D'Alembertian*:

$$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} \quad . \tag{11.56}$$

The Helmholtz equation for scalar waves propagating with speed c can thus be written in compact form as $\Box\,\phi = 0$.

11.3.2 What to do if you hate raised and lowered indices

Admittedly, this covariant and contravariant business takes some getting used to. Ultimately, it helps to keep straight which indices transform according to L (covariantly) and which transform according to $L^{\rm t}$ (contravariantly). If you find all this irksome, the raising and lowering can be safely ignored. We define the position four-vector as before, but with no difference between raised and lowered indices. In fact, we can just represent all vectors and tensors with lowered indices exclusively, writing *e.g.* $\mathrm{x}_\mu = (ct, x, y, z)$. The metric tensor is $\mathrm{g} = \mathrm{diag}(+,-,-,-)$ as before. The dot product of two four-vectors is

$$\mathrm{x} \cdot \mathrm{y} = \mathrm{g}_{\mu\nu}\, \mathrm{x}_\mu\, \mathrm{y}_\nu \quad . \tag{11.57}$$

The Lorentz transformation is

$$\mathrm{x}_\mu = \mathrm{L}_{\mu\nu}\, \mathrm{x}'_\nu \quad . \tag{11.58}$$

Since this preserves intervals, we must have

$$\begin{aligned} \mathrm{g}_{\mu\nu}\, \mathrm{x}_\mu\, \mathrm{y}_\nu &= \mathrm{g}_{\mu\nu}\, \mathrm{L}_{\mu\alpha}\, \mathrm{x}'_\alpha\, \mathrm{L}_{\nu\beta}\, \mathrm{y}'_\beta \\ &= \big(\mathrm{L}^{\rm t}_{\alpha\mu}\, \mathrm{g}_{\mu\nu}\, \mathrm{L}_{\nu\beta}\big)\, \mathrm{x}'_\alpha\, \mathrm{y}'_\beta \quad , \end{aligned} \tag{11.59}$$

which entails

$$\mathrm{L}^{\rm t}_{\alpha\mu}\, \mathrm{g}_{\mu\nu}\, \mathrm{L}_{\nu\beta} = \mathrm{g}_{\alpha\beta} \quad . \tag{11.60}$$

In terms of the quantity $L^\mu{}_\nu$ defined above, we have $\mathrm{L}_{\mu\nu} = L^\mu{}_\nu$. In this convention, we could completely avoid raised indices, or we could simply make no distinction, taking $\mathrm{x}^\mu = \mathrm{x}_\mu$ and $\mathrm{L}_{\mu\nu} = \mathrm{L}^\mu{}_\nu = \mathrm{L}^{\mu\nu}$, *etc.*

11.3.3 Comparing frames

Suppose in the K frame we have a measuring rod which is at rest. What is its length as measured in the K' frame? Recall K' moves with velocity $\boldsymbol{u} = u\,\hat{\boldsymbol{x}}$ with respect to K. From the Lorentz transformation in eqn. 11.39, we have

$$\begin{aligned} x_1 &= \gamma(x_1' + \beta c\, t_1') \\ x_2 &= \gamma(x_2' + \beta c\, t_2') \quad , \end{aligned} \tag{11.61}$$

where $x_{1,2}$ are the positions of the ends of the rod in frame K. The rod's length in any frame is the instantaneous spatial separation of its ends. Thus, we set $t_1' = t_2'$ and compute the separation $\Delta x' = x_2' - x_1'$:

$$\Delta x = \gamma\,\Delta x' \qquad \Longrightarrow \qquad \Delta x' = \gamma^{-1}\Delta x = \big(1-\beta^2\big)^{1/2}\,\Delta x \quad . \tag{11.62}$$

The *proper length* ℓ_0 of a rod is its instantaneous end-to-end separation in its rest frame. We see that

$$\ell(\beta) = \big(1-\beta^2\big)^{1/2}\,\ell_0 \quad , \tag{11.63}$$

so the length is always greatest in the rest frame. This is an example of a *Lorentz-Fitzgerald contraction*. Note that the *transverse* dimensions do not contract:

$$\Delta y' = \Delta y \qquad , \qquad \Delta z' = \Delta z \quad . \tag{11.64}$$

Thus, the *volume contraction* of a bulk object is given by its length contraction: $\mathcal{V}' = \gamma^{-1}\,\mathcal{V}$.

A striking example of relativistic issues of length, time, and simultaneity is the famous 'pole and the barn' paradox, described in the Appendix (section 11.9). Here we illustrate some essential features via two examples.

11.3.4 Example I

Next, let's analyze the situation depicted in fig. 11.3. In the K' frame, we'll denote the following space-time points:

$$A' = \begin{pmatrix} ct' \\ -d \end{pmatrix} \quad , \quad B' = \begin{pmatrix} ct' \\ +d \end{pmatrix} \quad , \quad S_-' = \begin{pmatrix} ct' \\ -ct' \end{pmatrix} \quad , \quad S_+' = \begin{pmatrix} ct' \\ +ct' \end{pmatrix} \quad . \tag{11.65}$$

Note that the origin in K' is given by $O' = (ct', 0)$. Here we are setting $y = y' = z = z' = 0$ and dealing only with one spatial dimension. The points $S_\pm'$ denote the left-moving (S_-') and right-moving (S_+') wavefronts. We see that the arrival of the signal S_1' at A' requires $S_1' = A'$, hence $ct' = d$. The same result holds when we set $S_2' = B'$ for the arrival of the right-moving wavefront at B'.

We now use the Lorentz transformation

$$L^\mu{}_\nu = \begin{pmatrix} \gamma & \gamma\,\beta \\ \gamma\,\beta & \gamma \end{pmatrix} \tag{11.66}$$

to transform to the K frame. Thus,

$$\begin{aligned} A &= \begin{pmatrix} ct^*_A \\ x^*_A \end{pmatrix} = LA' = \gamma \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix} \begin{pmatrix} d \\ -d \end{pmatrix} = \gamma(1-\beta)d \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ B &= \begin{pmatrix} ct^*_B \\ x^*_B \end{pmatrix} = LB' = \gamma \begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix} \begin{pmatrix} d \\ +d \end{pmatrix} = \gamma(1+\beta)d \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad . \end{aligned} \tag{11.67}$$

Thus, $t^*_A = \gamma(1-\beta)d/c$ and $t^*_B = \gamma(1+\beta)d/c$. Thus, the two events are *not* simultaneous in K. The arrival at A is first.

11.3.5 Example II

Consider a rod of length ℓ_0 extending from the origin to the point $\ell_0\,\hat{\boldsymbol{x}}$ at rest in frame K. In the frame K, the two ends of the rod are located at spacetime coordinates

$$A = \begin{pmatrix} ct \\ 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} ct \\ \ell_0 \end{pmatrix} \quad , \tag{11.68}$$

respectively. Now consider the origin in frame K'. Its spacetime coordinates are

$$C' = \begin{pmatrix} ct' \\ 0 \end{pmatrix} \quad . \tag{11.69}$$

To an observer in the K frame, we have

$$C = \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} ct' \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma ct' \\ \gamma\beta ct' \end{pmatrix} \quad . \tag{11.70}$$

Now consider two events. The first event is the coincidence of A with C, *i.e.* the origin of K' instantaneously coincides with the origin of K. Setting $A = C$ we obtain $t = t' = 0$. The second event is the coincidence of B with C. Setting $B = C$ we obtain $t = l_0/\beta c$ and $t' = \ell_0/\gamma\beta c$. Note that $t = \ell(\beta)/\beta c$, *i.e.* due to the Lorentz-Fitzgerald contraction of the rod as seen in the K' frame, where $\ell(\beta) = \ell_0/\gamma$.

11.3.6 Deformation of a rectangular plate

Problem: A rectangular plate of dimensions $a \times b$ moves at relativistic velocity $\boldsymbol{V} = V\hat{\boldsymbol{x}}$ as shown in fig. 11.5. In the rest frame of the rectangle, the a side makes an angle θ with respect to the $\hat{\boldsymbol{x}}$ axis. Describe in detail and sketch the shape of the plate as measured by an observer in the laboratory frame. Indicate the lengths of all sides and the values of all interior angles. Evaluate your expressions for the case $\theta = \frac{1}{4}\pi$ and $V = \sqrt{\frac{2}{3}}\,c$.

Solution: An observer in the laboratory frame will measure lengths parallel to $\hat{\boldsymbol{x}}$ to be Lorentz contracted by a factor γ^{-1}, where $\gamma = (1-\beta^2)^{-1/2}$ and $\beta = V/c$. Lengths perpendicular to $\hat{\boldsymbol{x}}$ remain unaffected. Thus, we have the situation depicted in fig. 11.6. Simple trigonometry then says

$$\tan\phi = \gamma \tan\theta \quad , \quad \tan\tilde{\phi} = \gamma^{-1} \tan\theta \quad , \tag{11.71}$$

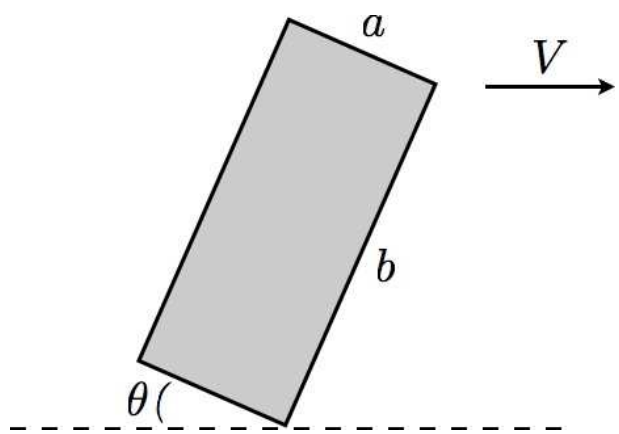


Figure 11.5: A rectangular plate moving at velocity $\boldsymbol{V} = V\,\hat{\boldsymbol{x}}$.

as well as

$$\begin{aligned} a' &= a\sqrt{\gamma^{-2}\cos^2\!\theta + \sin^2\!\theta} = a\sqrt{1-\beta^2\cos^2\!\theta} \\ b' &= b\sqrt{\gamma^{-2}\sin^2\!\theta + \cos^2\!\theta} = b\sqrt{1-\beta^2\sin^2\!\theta} \quad . \end{aligned} \tag{11.72}$$

The plate deforms to a parallelogram, with internal angles

$$\begin{aligned} \chi &= \tfrac{1}{2}\pi + \tan^{-1}(\gamma\tan\theta) - \tan^{-1}(\gamma^{-1}\tan\theta) \\ \tilde{\chi} &= \tfrac{1}{2}\pi - \tan^{-1}(\gamma\tan\theta) + \tan^{-1}(\gamma^{-1}\tan\theta) \quad . \end{aligned} \tag{11.73}$$

Note that the area of the plate as measured in the laboratory frame is

$$\begin{aligned} \varOmega' &= a'\,b'\,\sin\chi = a'\,b'\,\cos(\phi-\tilde{\phi}) \\ &= \gamma^{-1}\,\varOmega \quad , \end{aligned} \tag{11.74}$$

where $\varOmega = ab$ is the proper area. The area contraction factor is γ^{-1} and not γ^{-2} (or γ^{-3} in a three-dimensional system) because only the parallel dimension gets contracted.

Setting $V = \sqrt{\frac{2}{3}}\,c$ gives $\gamma = \sqrt{3}$, and with $\theta = \frac{1}{4}\pi$ we have $\phi = \frac{1}{3}\pi$ and $\tilde{\phi} = \frac{1}{6}\pi$. The interior angles are then $\chi = \frac{2}{3}\pi$ and $\tilde{\chi} = \frac{1}{3}\pi$. The side lengths are $a' = \sqrt{\frac{2}{3}}\,a$ and $b' = \sqrt{\frac{2}{3}}\,b$.

11.3.7 Transformation of velocities

Let K' move at velocity $\boldsymbol{u} = c\boldsymbol{\beta}$ relative to K. The transformation from K' to K is given by the Lorentz boost,

$$L^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & \gamma\,\beta_x & \gamma\,\beta_y & \gamma\,\beta_z \\ \gamma\,\beta_x & 1+(\gamma-1)\,\hat{\beta}_x\,\hat{\beta}_x & (\gamma-1)\,\hat{\beta}_x\,\hat{\beta}_y & (\gamma-1)\,\hat{\beta}_x\,\hat{\beta}_z \\ \gamma\,\beta_y & (\gamma-1)\,\hat{\beta}_x\,\hat{\beta}_y & 1+(\gamma-1)\,\hat{\beta}_y\,\hat{\beta}_y & (\gamma-1)\,\hat{\beta}_y\,\hat{\beta}_z \\ \gamma\,\beta_z & (\gamma-1)\,\hat{\beta}_x\,\hat{\beta}_z & (\gamma-1)\,\hat{\beta}_y\,\hat{\beta}_z & 1+(\gamma-1)\,\hat{\beta}_z\,\hat{\beta}_z \end{pmatrix} \quad . \tag{11.75}$$

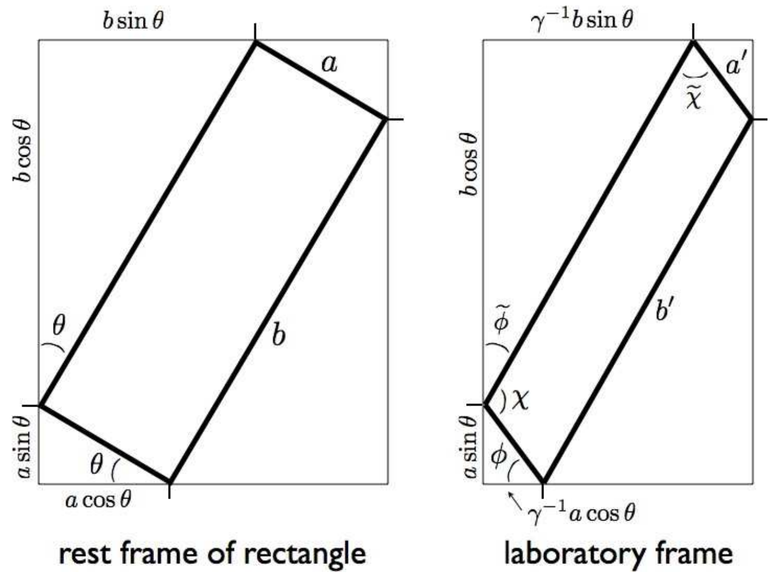


Figure 11.6: Relativistic deformation of the rectangular plate.

Applying this, we have

$$dx^\mu = L^\mu_{\ \nu}\, dx'^{\,\nu} \quad . \tag{11.76}$$

This yields

$$\begin{aligned} dx^0 &= \gamma\, dx'^{\,0} + \gamma\, \boldsymbol{\beta}\cdot d\boldsymbol{x}' \\ d\boldsymbol{x} &= \gamma\, \boldsymbol{\beta}\, dx'^{\,0} + d\boldsymbol{x}' + (\gamma - 1)\, \hat{\boldsymbol{\beta}}\, \hat{\boldsymbol{\beta}}\cdot d\boldsymbol{x}' \quad . \end{aligned} \tag{11.77}$$

We then have

$$\begin{aligned} \boldsymbol{V} = c\,\frac{d\boldsymbol{x}}{dx^0} &= \frac{c\,\gamma\,\boldsymbol{\beta}\, dx'^{\,0} + c\, d\boldsymbol{x}' + c\,(\gamma-1)\,\hat{\boldsymbol{\beta}}\,\hat{\boldsymbol{\beta}}\cdot d\boldsymbol{x}'}{\gamma\, dx'^{\,0} + \gamma\,\boldsymbol{\beta}\cdot d\boldsymbol{x}'} \\ &= \frac{\boldsymbol{u} + \gamma^{-1}\,\boldsymbol{V}' + (1-\gamma^{-1})\,\hat{\boldsymbol{u}}\,\hat{\boldsymbol{u}}\cdot\boldsymbol{V}'}{1 + \boldsymbol{u}\cdot\boldsymbol{V}'/c^2} \quad . \end{aligned} \tag{11.78}$$

The second line is obtained by dividing both numerator and denominator by $dx'^{\,0}$, and then writing $\boldsymbol{V}' = d\boldsymbol{x}'/dx'^{\,0}$. There are two special limiting cases:

$$\begin{aligned} \text{velocities parallel } \big(\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{V}}' = 1\big) \quad &\Longrightarrow \quad \boldsymbol{V} = \frac{(u + V')\,\hat{\boldsymbol{u}}}{1 + u\,V'/c^2} \\ \text{velocities perpendicular } \big(\hat{\boldsymbol{u}}\cdot\hat{\boldsymbol{V}}' = 0\big) \quad &\Longrightarrow \quad \boldsymbol{V} = \boldsymbol{u} + \gamma^{-1}\boldsymbol{V}' \quad . \end{aligned} \tag{11.79}$$

Note that if either u or V' is equal to c, the resultant expression has $|\boldsymbol{V}| = c$ as well. One can't boost the speed of light!

Let's revisit briefly the example in section 11.3.4. For an observer, in the K frame, the relative velocity of S and A is $c + u$, because even though we must boost the velocity $-c\,\hat{\boldsymbol{x}}$ of the left-moving light wave by $u\,\hat{\boldsymbol{x}}$, the result is still $-c\,\hat{\boldsymbol{x}}$, according to our velocity addition formula. The distance between the emission and detection points is $d(\beta) = d/\gamma$. Thus,

$$t_A^* = \frac{d(\beta)}{c+u} = \frac{d}{\gamma} \cdot \frac{1}{c+u} = \frac{d}{\gamma c} \cdot \frac{1-\beta}{1-\beta^2} = \gamma\,(1-\beta)\,\frac{d}{c} \quad . \tag{11.80}$$

This result is exactly as found in section 11.3.4 by other means. A corresponding analysis yields $t_B^* = \gamma\,(1+\beta)\,d/c$. again in agreement with the earlier result. Here, it is crucial to account for the Lorentz contraction of the distance between the source S and the observers A and B as measured in the K frame.

11.3.8 Four-velocity and four-acceleration

In nonrelativistic mechanics, the velocity $\boldsymbol{V} = \frac{d\boldsymbol{x}}{dt}$ is locally tangent to a particle's trajectory. In relativistic mechanics, one defines the *four-velocity*,

$$u^\alpha \equiv \frac{dx^\alpha}{ds} = \frac{dx^\alpha}{\sqrt{1-\boldsymbol{\beta}^2}\;c\,dt} = \begin{pmatrix} \gamma \\ \gamma\boldsymbol{\beta} \end{pmatrix} \quad , \tag{11.81}$$

which is locally tangent to the world line of a particle. Note that

$$g_{\alpha\beta}\,u^\alpha\,u^\beta = 1 \quad . \tag{11.82}$$

The four-acceleration is defined as

$$w^\nu \equiv \frac{du^\nu}{ds} = \frac{d^2x^\nu}{ds^2} \quad . \tag{11.83}$$

Note that $u \cdot w = 0$, so the 4-velocity and 4-acceleration are orthogonal with respect to the Minkowski metric.

11.4 Three Kinds of Relativistic Rockets

11.4.1 Constant acceleration model

Consider a rocket which undergoes constant acceleration along $\hat{\boldsymbol{x}}$. Clearly the rocket has no rest frame *per se*, because its velocity is changing. However, this poses no serious obstacle to discussing its relativistic motion. We consider a frame K' in which the rocket is *instantaneously* at rest. In such a frame, the rocket's 4-acceleration is $w'^{\,\alpha} = (0, a/c^2)$, where we suppress the transverse coordinates y and z. In an inertial frame K, we have

$$w^\alpha = \frac{d}{ds}\begin{pmatrix} \gamma \\ \gamma\beta \end{pmatrix} = \frac{\gamma}{c}\begin{pmatrix} \dot\gamma \\ \gamma\dot\beta + \dot\gamma\beta \end{pmatrix} \quad . \tag{11.84}$$

Transforming $w'^{\,\alpha}$ into the K frame, we have

$$w^\alpha = \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix}\begin{pmatrix} 0 \\ a/c^2 \end{pmatrix} = \begin{pmatrix} \gamma\beta a/c^2 \\ \gamma a/c^2 \end{pmatrix} \quad . \tag{11.85}$$

Taking the upper component, we obtain the equation

$$\dot{\gamma} = \frac{\beta a}{c} \qquad \Longrightarrow \qquad \frac{d}{dt}\left(\frac{\beta}{\sqrt{1-\beta^2}}\right) = \frac{a}{c} \quad , \tag{11.86}$$

the solution of which, with $\beta(0) = 0$, is

$$\beta(t) = \frac{at}{\sqrt{c^2 + a^2 t^2}} \qquad , \qquad \gamma(t) = \sqrt{1 + \left(\frac{at}{c}\right)^2} \quad . \tag{11.87}$$

The proper time for an observer moving with the rocket is thus

$$\tau = \int\limits_0^t \frac{c\, dt_1}{\sqrt{c^2 + a^2 t_1^2}} = \frac{c}{a} \sinh^{-1}\left(\frac{at}{c}\right) \quad . \tag{11.88}$$

For large times $t \gg c/a$, the proper time grows logarithmically in t, which is parametrically slower. To find the position of the rocket, we integrate $\dot{x} = c\beta$, and obtain, with $x(0) = 0$,

$$x(t) = \int\limits_0^t \frac{a\, ct_1\, dt_1}{\sqrt{c^2 + a^2 t_1^2}} = \frac{c}{a}\left(\sqrt{c^2 + a^2\, t^2} \, - \, c\right) \quad . \tag{11.89}$$

It is interesting to consider the situation in the frame K'. We then have

$$\beta(\tau) = \tanh(a\tau/c) \quad , \quad \gamma(\tau) = \cosh(a\tau/c) \quad . \tag{11.90}$$

For an observer in the frame K', the distance he has traveled is $\Delta x'(\tau) = \Delta x(\tau)/\gamma(\tau)$, as we found in eqn. 11.62. Now $x(\tau) = (c^2/a)\big(\cosh(a\tau/c) - 1\big)$, hence

$$\Delta x'(\tau) = \frac{c^2}{a}\Big(1 - \mathrm{sech}(a\tau/c)\Big) \quad . \tag{11.91}$$

For $\tau \ll c/a$, we expand $\mathrm{sech}(a\tau/c) \approx 1 - \frac{1}{2}(a\tau/c)^2$ and find $x'(\tau) = \frac{1}{2}a\tau^2$, which clearly is the nonrelativistic limit. For $\tau \to \infty$, however, we have $\Delta x'(\tau) \to c^2/a$ is *finite*! Thus, while the entire Universe is falling behind the accelerating observer, it all piles up at a *horizon* a distance c^2/a behind it, in the frame of the observer. The light from these receding objects is increasingly red-shifted (see section 11.6 below), until it is no longer visible. Thus, as John Baez describes it, the horizon is "a dark plane that appears to be swallowing the entire Universe!" In the frame of the inertial observer, however, nothing strange appears to be happening at all!

11.4.2 Constant force with decreasing mass

Suppose instead the rocket is subjected to a constant force F_0 in its instantaneous rest frame, and furthermore that the rocket's mass satisfies $m(\tau) = m_0(1 - \alpha\tau)$, where τ is the proper time for an observer moving with the rocket. Then from eqn. 11.86, we have

$$\begin{aligned}\frac{F_0}{m_0(1-\alpha\tau)} &= \frac{d(\gamma\beta)}{dt} = \gamma^{-1}\,\frac{d(\gamma\beta)}{d\tau} \\ &= \frac{1}{1-\beta^2}\,\frac{d\beta}{d\tau} = \frac{d}{d\tau}\,\tfrac{1}{2}\log\!\left(\frac{1+\beta}{1-\beta}\right) \quad ,\end{aligned} \tag{11.92}$$

after using the chain rule, and with $d\tau/dt = \gamma^{-1}$. Integrating, we find

$$\log\!\left(\frac{1+\beta}{1-\beta}\right) = \frac{2F_0}{\alpha m_0 c}\,\log\big(1-\alpha\tau\big) \quad\Longrightarrow\quad \beta(\tau) = \frac{1-(1-\alpha\tau)^r}{1+(1-\alpha\tau)^r} \quad , \tag{11.93}$$

with $r = 2F_0/\alpha m_0 c$. As $\tau \to \alpha^{-1}$, the rocket loses all its mass, and it asymptotically approaches the speed of light.

It is convenient to write

$$\beta(\tau) = \tanh\left[\frac{r}{2}\,\log\!\left(\frac{1}{1-\alpha\tau}\right)\right] \quad , \tag{11.94}$$

in which case

$$\begin{aligned}\gamma = \frac{dt}{d\tau} &= \cosh\left[\frac{r}{2}\,\log\!\left(\frac{1}{1-\alpha\tau}\right)\right] \\ \frac{1}{c}\,\frac{dx}{d\tau} &= \sinh\left[\frac{r}{2}\,\log\!\left(\frac{1}{1-\alpha\tau}\right)\right] \quad .\end{aligned} \tag{11.95}$$

Integrating the first of these from $\tau = 0$ to $\tau = \alpha^{-1}$, we find $t^* \equiv t\big(\tau = \alpha^{-1}\big)$ is

$$t^* = \frac{1}{2\alpha}\int\limits_0^1 \! d\sigma\,\Big(\sigma^{-r/2} + \sigma^{r/2}\Big) = \begin{cases} \Big[\alpha^2 - \big(\frac{F_0}{mc}\big)^2\Big]^{-1}\alpha & \text{if } \alpha > \frac{F_0}{mc} \\ \\ \infty & \text{if } \alpha \le \frac{F_0}{mc} \quad . \end{cases} \tag{11.96}$$

Since $\beta(\tau = \alpha^{-1}) = 1$, this is the time in the K frame when the rocket reaches the speed of light.

11.4.3 Constant *ejecta* velocity

Our third relativistic rocket model is a generalization of what is commonly known as the *rocket equation* in classical physics. The model is one of a rocket which is continually ejecting burnt fuel at a velocity $-u$ in the instantaneous rest frame of the rocket. The nonrelativistic rocket equation follows from overall momentum conservation:

$$dp_{\rm rocket} + dp_{\rm fuel} = d(mv) + (v-u)\,(-dm) = 0 \quad , \tag{11.97}$$

since if $dm < 0$ is the differential change in rocket mass, the differential *ejecta* mass is $-dm$. This immediately gives

$$m\,dv + u\,dm = 0 \quad \Longrightarrow \quad v = u\,\log\bigg(\frac{m_0}{m}\bigg) \quad , \tag{11.98}$$

where the rocket is assumed to begin at rest, and where m_0 is the initial mass of the rocket. Note that as $m \to 0$ the rocket's speed increases without bound, which of course violates special relativity.

In relativistic mechanics, as we shall see in section 11.5, the rocket's momentum, as described by an inertial observer, is $p = \gamma m v$, and its energy is $\gamma m c^2$. We now write two equations for overall conservation of momentum and energy:

$$\begin{aligned} d(\gamma m v) + \gamma_\text{e} v_\text{e}\, dm_\text{e} &= 0 \\ d(\gamma m c^2) + \gamma_\text{e} (dm_\text{e}\, c^2) &= 0 \quad , \end{aligned} \tag{11.99}$$

where v_e is the velocity of the *ejecta* in the inertial frame, dm_e is the differential mass of the *ejecta*, and $\gamma_\text{e} = \big(1 - \frac{v_\text{e}^2}{c^2}\big)^{-1/2}$. From the second of these equations, we have

$$\gamma_\text{e}\, dm_\text{e} = -d(\gamma m) \quad , \tag{11.100}$$

which we can plug into the first equation to obtain

$$(v - v_\text{e})\, d(\gamma m) + \gamma m\, dv = 0 \quad . \tag{11.101}$$

Before solving this, we remark that eqn. 11.100 implies that $dm_\text{e} < |dm|$ – the differential mass of the *ejecta* is less than the mass lost by the rocket! This is Einstein's famous equation $E = mc^2$ at work – more on this later.

To proceed, we need to use the parallel velocity addition formula of eqn. 11.79 to find v_e:

$$v_\text{e} = \frac{v - u}{1 - \frac{uv}{c^2}} \qquad \Longrightarrow \qquad v - v_\text{e} = \frac{u\big(1 - \frac{v^2}{c^2}\big)}{\big(1 - \frac{uv}{c^2}\big)} \quad . \tag{11.102}$$

We now define $\beta_u = u/c$, in which case eqn, 11.101 becomes

$$\beta_u\,(1 - \beta^2)\, d(\gamma m) + (1 - \beta\beta_u)\,\gamma m\, d\beta = 0 \quad . \tag{11.103}$$

Using $d\gamma = \gamma^3 \beta\, d\beta$, we observe a felicitous cancellation of terms, leaving

$$\beta_u\,\frac{dm}{m} + \frac{d\beta}{1 - \beta^2} = 0 \quad . \tag{11.104}$$

Integrating, we obtain

$$\beta = \tanh\bigg(\beta_u\,\log\frac{m_0}{m}\bigg) \quad . \tag{11.105}$$

Note that this agrees with the result of eqn. 11.94, if we take $\beta_u = F_0/\alpha mc$.

11.5 Relativistic Mechanics

Relativistic particle dynamics follows from an appropriately extended version of Hamilton's principle $\delta S = 0$. The action S must be a Lorentz scalar. The action for a free particle is

$$S\big[\boldsymbol{x}(t)\big] = -mc\int\limits_a^b ds = -mc^2\int\limits_{t_a}^{t_b} dt\,\sqrt{1-\frac{\boldsymbol{v}^2}{c^2}} \quad . \tag{11.106}$$

Thus, the free particle Lagrangian is

$$L = -mc^2\sqrt{1-\frac{\boldsymbol{v}^2}{c^2}} = -mc^2 + \tfrac{1}{2}m\boldsymbol{v}^2 + \tfrac{1}{8}mc^2\left(\frac{\boldsymbol{v}^2}{c^2}\right)^2 + \ldots \quad . \tag{11.107}$$

Thus, L can be written as an expansion in powers of $\boldsymbol{v}^2/c^2$. Note that $L(\boldsymbol{v}=0) = -mc^2$. We interpret this as $-U_0$, where $U_0 = mc^2$ is the *rest energy* of the particle. As a constant, it has no consequence for the equations of motion. The next term in L is the familiar nonrelativistic kinetic energy, $\frac{1}{2}m\boldsymbol{v}^2$. Higher order terms are smaller by increasing factors of $\beta^2 = v^2/c^2$.

We can add a potential $U(\boldsymbol{x},t)$ to obtain

$$L(\boldsymbol{x},\dot{\boldsymbol{x}},t) = -mc^2\sqrt{1-\frac{\dot{\boldsymbol{x}}^2}{c^2}} - U(\boldsymbol{x},t) \quad . \tag{11.108}$$

The momentum of the particle is

$$\boldsymbol{p} = \frac{\partial L}{\partial \dot{\boldsymbol{x}}} = \gamma m\dot{\boldsymbol{x}} \quad . \tag{11.109}$$

The force is $\boldsymbol{F} = -\boldsymbol{\nabla} U$ as usual, and Newton's Second Law still reads $\dot{\boldsymbol{p}} = \boldsymbol{F}$. Note that

$$\dot{\boldsymbol{p}} = \gamma m\left(\dot{\boldsymbol{v}} + \frac{v\dot{v}}{c^2}\,\gamma^2\boldsymbol{v}\right) \quad . \tag{11.110}$$

Thus, the force $\boldsymbol{F}$ is not necessarily in the direction of the acceleration $\boldsymbol{a} = \dot{\boldsymbol{v}}$. The Hamiltonian, recall, is a function of coordinates and momenta, and is given by

$$H = \boldsymbol{p}\cdot\dot{\boldsymbol{x}} - L = \sqrt{m^2c^4 + \boldsymbol{p}^2c^2} + U(\boldsymbol{x},t) \quad . \tag{11.111}$$

Since $\partial L/\partial t = 0$ for our case, H is conserved by the motion of the particle. There are two limits of note:

$$\begin{aligned} |\boldsymbol{p}| \ll mc \quad &\text{(non-relativistic)} &:&\quad H = mc^2 + \frac{\boldsymbol{p}^2}{2m} + U + \mathcal{O}(p^4/m^4c^4) \\ |\boldsymbol{p}| \gg mc \quad &\text{(ultra-relativistic)} &:&\quad H = c|\boldsymbol{p}| + U + \mathcal{O}(mc/p) \quad . \end{aligned} \tag{11.112}$$

Expressed in terms of the coordinates and velocities, we have $H = E$, the total energy, with

$$E = \gamma mc^2 + U \quad . \tag{11.113}$$

In particle physics applications, one often defines the kinetic energy T as

$$T = E - U - mc^2 = (\gamma - 1)mc^2 \quad . \tag{11.114}$$

When electromagnetic fields are included,

$$\begin{aligned} L(\boldsymbol{x}, \dot{\boldsymbol{x}}, t) &= -mc^2\sqrt{1 - \frac{\dot{\boldsymbol{x}}^2}{c^2}} - q\,\phi + \frac{q}{c}\boldsymbol{A}\cdot\dot{\boldsymbol{x}} \\ &= -\gamma mc^2 - \frac{q}{c}\,A_\mu\,\frac{dx^\mu}{dt} \quad , \end{aligned} \tag{11.115}$$

where the electromagnetic *4-potential* is $A^\mu = (\phi\,,\,\boldsymbol{A})$. Recall $A_\mu = g_{\mu\nu}\,A^\nu$ has the sign of its spatial components reversed. One the has

$$\boldsymbol{p} = \frac{\partial L}{\partial \dot{\boldsymbol{x}}} = \gamma m\dot{\boldsymbol{x}} + \frac{q}{c}\,\boldsymbol{A} \quad , \tag{11.116}$$

and the Hamiltonian is

$$H = \sqrt{m^2c^4 + \Big(\boldsymbol{p} - \frac{q}{c}\boldsymbol{A}\Big)^2} + q\,\phi \quad . \tag{11.117}$$

11.5.1 Relativistic harmonic oscillator

From $E = \gamma mc^2 + U$, we have

$$\dot{x}^2 = c^2\left[1 - \left(\frac{mc^2}{E - U(x)}\right)^2\right] \quad . \tag{11.118}$$

Consider the one-dimensional harmonic oscillator potential $U(x) = \frac{1}{2}kx^2$. We define the turning points as $x = \pm b$, satisfying

$$E - mc^2 = U(\pm b) = \tfrac{1}{2}kb^2 \quad . \tag{11.119}$$

Now define the angle θ via $x \equiv b\cos\theta$, and further define the dimensionless parameter $\epsilon = kb^2/4mc^2$. Then, after some manipulations, one obtains

$$\dot{\theta} = \omega_0\,\frac{\sqrt{1 + \epsilon\sin^2\!\theta}}{1 + 2\epsilon\sin^2\!\theta} \quad , \tag{11.120}$$

with $\omega_0 = \sqrt{k/m}$ as in the nonrelativistic case. Hence, the problem is reduced to quadratures (a quaint way of saying 'doing an an integral'):

$$t(\theta) - t_0 = \omega_0^{-1}\int\limits_{\theta_0}^{\theta} d\vartheta\,\frac{1 + 2\epsilon\sin^2\!\vartheta}{\sqrt{1 + \epsilon\sin^2\!\vartheta}} \quad . \tag{11.121}$$

While the result can be expressed in terms of elliptic integrals, such an expression is not particularly illuminating. Here we will content ourselves with computing the period $T(\epsilon)$:

$$\begin{aligned} T(\epsilon) &= \frac{4}{\omega_0}\int\limits_0^{\frac{\pi}{2}} d\vartheta\,\frac{1+2\epsilon\sin^2\!\vartheta}{\sqrt{1+\epsilon\sin^2\!\vartheta}} \\ &= \frac{4}{\omega_0}\int\limits_0^{\frac{\pi}{2}} d\vartheta\,\Big(1+\tfrac{3}{2}\epsilon\sin^2\!\vartheta-\tfrac{5}{8}\epsilon^2\sin^4\!\vartheta+\ldots\Big) \\ &= \frac{2\pi}{\omega_0}\cdot\Big\{1+\tfrac{3}{4}\epsilon-\tfrac{15}{64}\epsilon^2+\ldots\Big\} \quad . \end{aligned} \tag{11.122}$$

Thus, for the relativistic harmonic oscillator, the period does depend on the amplitude, unlike the non-relativistic case.

11.5.2 Energy-momentum 4-vector

Let's focus on the case where $U(\boldsymbol{x}) = 0$. This is in fact a realistic assumption for subatomic particles, which propagate freely between collision events.

The differential proper time for a particle is given by

$$d\tau = \frac{ds}{c} = \gamma^{-1}\,dt \quad , \tag{11.123}$$

where $x^\mu = (ct, \boldsymbol{x})$ are coordinates for the particle in an inertial frame. Thus,

$$\boldsymbol{p} = \gamma m\dot{\boldsymbol{x}} = m\,\frac{d\boldsymbol{x}}{d\tau} \qquad , \qquad \frac{E}{c} = mc\gamma = m\,\frac{dx^0}{d\tau} \quad , \tag{11.124}$$

with $x^0 = ct$. Thus, we can write the *energy-momentum 4-vector* as

$$p^\mu = m\,\frac{dx^\mu}{d\tau} = \begin{pmatrix} E/c \\ p^x \\ p^y \\ p^z \end{pmatrix} \quad . \tag{11.125}$$

Note that $p^\nu = mcu^\nu$, where u^ν is the 4-velocity of eqn. 11.81. The four-momentum satisfies the relation

$$p^\mu\,p_\mu = \frac{E^2}{c^2} - \boldsymbol{p}^2 = m^2c^2 \quad . \tag{11.126}$$

The relativistic generalization of force is

$$f^\mu = \frac{dp^\mu}{d\tau} = \big(\gamma\boldsymbol{F}\cdot\boldsymbol{v}/c\,,\,\gamma\boldsymbol{F}\big) \quad , \tag{11.127}$$

where $\boldsymbol{F} = d\boldsymbol{p}/dt$ as usual.

The energy-momentum four-vector transforms covariantly under a Lorentz transformation. This means

$$p^\mu = L^\mu_{\ \ \nu}\, p'^{\,\nu} \quad . \tag{11.128}$$

If frame K' moves with velocity $\boldsymbol{u} = c\beta\,\hat{\boldsymbol{x}}$ relative to frame K, then

$$\frac{E}{c} = \frac{c^{-1}E' + \beta\, p'^x}{\sqrt{1-\beta^2}} \quad , \quad p^x = \frac{p'^x + \beta c^{-1} E'}{\sqrt{1-\beta^2}} \quad , \quad p^y = p'^y \quad , \quad p^z = p'^z \quad . \tag{11.129}$$

In general, from eqns. 11.47, we have

$$\begin{aligned} \frac{E}{c} &= \gamma\frac{E'}{c} + \gamma\beta p'_\parallel \\ p_\parallel &= \gamma\beta\frac{E}{c} + \gamma p'_\parallel \\ \boldsymbol{p}_\perp &= \boldsymbol{p}'_\perp \end{aligned} \tag{11.130}$$

where $p_\parallel = \hat{\boldsymbol{\beta}}\cdot\boldsymbol{p}$ and $\boldsymbol{p}_\perp = \boldsymbol{p} - (\hat{\boldsymbol{\beta}}\cdot\boldsymbol{p})\,\hat{\boldsymbol{\beta}}$.

11.5.3 4-momentum for massless particles

For a massless particle, such as a photon, we have $p^\mu\, p_\mu = 0$, which means $E^2 = \boldsymbol{p}^2\, c^2$. The 4-momentum may then be written $p^\mu = \big(|\boldsymbol{p}|\,,\,\boldsymbol{p}\big)$. We define the *4-wavevector* k^μ by the relation $p^\mu = \hbar k^\mu$, where $\hbar = h/2\pi$ and h is Planck's constant. We also write $\omega = ck$, with $E = \hbar\omega$.

11.6 Relativistic Doppler Effect

The 4-wavevector $k^\mu = \big(\omega/c\,,\,\boldsymbol{k}\big)$ for electromagnetic radiation satisfies $k^\mu\, k_\mu = 0$. The energy-momentum 4-vector is $p^\mu = \hbar k^\mu$. The phase $\phi(x^\mu) = -k_\mu\, x^\mu = \boldsymbol{k}\cdot\boldsymbol{x} - \omega t$ of a plane wave is a Lorentz scalar. This means that the total number of wave crests (*i.e.* $\phi = 2\pi n$) emitted by a source will be the total number observed by a detector.

Suppose a moving source emits radiation of angular frequency ω' in its rest frame. Then

$$\begin{aligned} k'^{\,\mu} &= L^\mu_{\ \ \nu}(-\boldsymbol{\beta})\, k^\nu \\ &= \begin{pmatrix} \gamma & -\gamma\,\beta_x & -\gamma\,\beta_y & -\gamma\,\beta_z \\ -\gamma\,\beta_x & 1+(\gamma-1)\,\hat\beta_x\,\hat\beta_x & (\gamma-1)\,\hat\beta_x\,\hat\beta_y & (\gamma-1)\,\hat\beta_x\,\hat\beta_z \\ -\gamma\,\beta_y & (\gamma-1)\,\hat\beta_x\,\hat\beta_y & 1+(\gamma-1)\,\hat\beta_y\,\hat\beta_y & (\gamma-1)\,\hat\beta_y\,\hat\beta_z \\ -\gamma\,\beta_z & (\gamma-1)\,\hat\beta_x\,\hat\beta_z & (\gamma-1)\,\hat\beta_y\,\hat\beta_z & 1+(\gamma-1)\,\hat\beta_z\,\hat\beta_z \end{pmatrix} \begin{pmatrix} \omega/c \\ k^x \\ k^y \\ k^z \end{pmatrix} \quad . \end{aligned} \tag{11.131}$$

This gives

$$\frac{\omega'}{c} = \gamma\,\frac{\omega}{c} - \gamma\,\boldsymbol{\beta}\cdot\boldsymbol{k} = \gamma\,\frac{\omega}{c}\,(1-\beta\cos\theta) \quad , \tag{11.132}$$

where $\theta = \cos^{-1}(\hat{\boldsymbol{\beta}} \cdot \hat{\boldsymbol{k}})$ is the angle measured in K between $\hat{\boldsymbol{\beta}}$ and $\hat{\boldsymbol{k}}$. Solving for ω, we have

$$\omega = \frac{\sqrt{1-\beta^2}}{1-\beta\cos\theta}\,\omega_0 \quad , \tag{11.133}$$

where $\omega_0 = \omega'$ is the angular frequency in the rest frame of the moving source. Thus,

$$\begin{aligned} \theta = 0 \quad &\Rightarrow \quad \text{source approaching} \quad \Rightarrow \quad \omega = \sqrt{\frac{1+\beta}{1-\beta}}\,\omega_0 \\ \theta = \tfrac{1}{2}\pi \quad &\Rightarrow \quad \text{source perpendicular} \quad \Rightarrow \quad \omega = \sqrt{1-\beta^2}\,\omega_0 \\ \theta = \pi \quad &\Rightarrow \quad \text{source receding} \quad \Rightarrow \quad \omega = \sqrt{\frac{1-\beta}{1+\beta}}\,\omega_0 \quad . \end{aligned} \tag{11.134}$$

Recall the non-relativistic Doppler effect:

$$\omega = \frac{\omega_0}{1-(V/c)\cos\theta} \quad . \tag{11.135}$$

We see that approaching sources have their frequencies shifted higher; this is called the *blue shift*, since blue light is on the high frequency (short wavelength) end of the optical spectrum. By the same token, receding sources are *red-shifted* to lower frequencies.

11.6.1 Romantic example

Alice and Bob have a "May-December" thang going on. Bob is May and Alice December, if you get my drift. The social stigma is too much to bear! To rectify this, they decide that Alice should take a ride in a space ship. Alice's itinerary takes her along a sector of a circle of radius R and angular span of $\Theta = 1$ radian, as depicted in fig. 11.7. Define $\mathrm{O} \equiv (r=0)$, $\mathrm{P} \equiv (r=R, \phi=-\frac{1}{2}\Theta)$, and $\mathrm{Q} \equiv (r=R, \phi=\frac{1}{2}\Theta)$. Alice's speed along the first leg (straight from O to P) is $v_{\mathrm{a}} = \frac{3}{5}\,c$. Her speed along the second leg (an arc from P to Q) is $v_{\mathrm{b}} = \frac{12}{13}\,c$. The final leg (straight from Q to O) she travels at speed $v_{\mathrm{c}} = \frac{4}{5}\,c$. Remember that the length of an circular arc of radius R and angular spread α (radians) is $\ell = \alpha R$.

(a) Alice and Bob synchronize watches at the moment of Alice's departure. What is the elapsed time on Bob's watch when Alice returns? What is the elapsed time on Alice's watch? What must R be in order for them to erase their initial 30 year age difference?

Solution : In Bob's frame, Alice's trip takes a time

$$\begin{aligned} \Delta t &= \frac{R}{c\beta_{\mathrm{a}}} + \frac{R\Theta}{c\beta_{\mathrm{b}}} + \frac{R}{c\beta_{\mathrm{c}}} \\ &= \frac{R}{c}\left(\tfrac{5}{3} + \tfrac{13}{12} + \tfrac{5}{4}\right) = \frac{4R}{c} \quad . \end{aligned} \tag{11.136}$$

The elapsed time on Alice's watch is

$$\begin{aligned} \Delta t' &= \frac{R}{c\gamma_{\mathrm{a}}\beta_{\mathrm{a}}} + \frac{R\Theta}{c\gamma_{\mathrm{b}}\beta_{\mathrm{b}}} + \frac{R}{c\gamma_{\mathrm{c}}\beta_{\mathrm{c}}} \\ &= \frac{R}{c}\left(\tfrac{5}{3}\cdot\tfrac{4}{5} + \tfrac{13}{12}\cdot\tfrac{5}{13} + \tfrac{5}{4}\cdot\tfrac{3}{5}\right) = \frac{5R}{2c} \quad . \end{aligned} \tag{11.137}$$

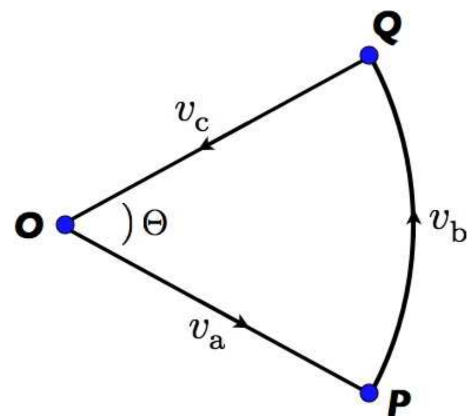


Figure 11.7: Alice's big adventure.

Thus, $\Delta T = \Delta t - \Delta t' = 3R/2c$ and setting $\Delta T = 30\,\text{yr}$, we find $R = 20\,\text{ly}$. So Bob will have aged 80 years and Alice 50 years upon her return. (Maybe this isn't such a good plan after all.)

(b) As a signal of her undying love for Bob, Alice continually shines a beacon throughout her trip. The beacon produces monochromatic light at wavelength $\lambda_0 = 6000\,\text{Å}$ (frequency $f_0 = c/\lambda_0 = 5 \times 10^{14}$ Hz). Every night, Bob peers into the sky (with a radiotelescope), hopefully looking for Alice's signal. What frequencies f_a, f_b, and f_c does Bob see?

Solution : Using the relativistic Doppler formula, we have

$$
\begin{aligned}
f_a &= \sqrt{\frac{1-\beta_a}{1+\beta_a}} \times f_0 = \tfrac{1}{2} f_0 \\
f_b &= \sqrt{1-\beta_b^2} \times f_0 = \tfrac{5}{13} f_0 \\
f_c &= \sqrt{\frac{1+\beta_c}{1-\beta_c}} \times f_0 = 3 f_0 \quad .
\end{aligned}
\tag{11.138}
$$

(c) Show that the total number of wave crests counted by Bob is the same as the number emitted by Alice, over the entire trip.

Solution : Consider first the O–P leg of Alice's trip. The proper time elapsed on Alice's watch during this leg is $\Delta t'_a = R/c\gamma_a\beta_a$, hence she emits $N'_a = Rf_0/c\gamma_a\beta_a$ wavefronts during this leg. Similar considerations hold for the P–Q and Q–O legs, so $N'_b = R\Theta f_0/c\gamma_b\beta_b$ and $N'_c = Rf_0/c\gamma_c\beta_c$.

Although the duration of the O–P segment of Alice's trip takes a time $\Delta t_a = R/c\beta_a$ in Bob's frame, he keeps receiving the signal at the Doppler-shifted frequency f_a until the wavefront emitted when Alice arrives at P makes its way back to Bob. That takes an extra time R/c, hence the number of crests emitted for Alice's O–P leg is

$$
N_a = \left(\frac{R}{c\beta_a} + \frac{R}{c}\right)\sqrt{\frac{1-\beta_a}{1+\beta_a}} \times f_0 = \frac{Rf_0}{c\gamma_a\beta_a} = N'_a \quad , \tag{11.139}
$$

since the source is receding from the observer.

During the P–Q leg, we have $\theta = \frac{1}{2}\pi$, and Alice's velocity is orthogonal to the wavevector $\boldsymbol{k}$, which is directed radially inward. Bob's first signal at frequency f_b arrives a time R/c after Alice passes P, and his last signal at this frequency arrives a time R/c after Alice passes Q. Thus, the total time during which Bob receives the signal at the Doppler-shifted frequency f_b is $\Delta t_\text{b} = R\Theta/c$, and

$$N_\text{b} = \frac{R\Theta}{c\beta_\text{b}} \cdot \sqrt{1-\beta_\text{b}^2} \times f_0 = \frac{R\Theta f_0}{c\gamma_b\beta_b} = N'_\text{b} \quad . \tag{11.140}$$

Finally, during the Q–O home stretch, Bob first starts to receive the signal at the Doppler-shifted frequency f_c a time R/c after Alice passes Q, and he continues to receive the signal until the moment Alice rushes into his open and very flabby old arms when she makes it back to O. Thus, Bob receives the frequency f_c signal for a duration $\Delta t_\text{c} - R/c$, where $\Delta t_\text{c} = R/c\beta_\text{c}$. Thus,

$$N_\text{c} = \left(\frac{R}{c\beta_\text{c}} - \frac{R}{c}\right)\sqrt{\frac{1+\beta_\text{c}}{1-\beta_\text{c}}} \times f_0 = \frac{Rf_0}{c\gamma_c\beta_c} = N'_\text{c} \quad , \tag{11.141}$$

since the source is approaching.

Therefore, the number of wavelengths emitted by Alice will be precisely equal to the number received by Bob – none of the waves gets lost.

11.7 Relativistic Kinematics of Particle Collisions

As should be expected, special relativity is essential toward the understanding of subatomic particle collisions, where the particles themselves are moving at close to the speed of light. In our analysis of the kinematics of collisions, we shall find it convenient to adopt the standard convention on units, where we set $c \equiv 1$. Energies will typically be given in GeV, where $1\,\text{GeV} = 10^9\,\text{eV} = 1.602 \times 10^{-10}\,R_\text{J}$. Momenta will then be in units of GeV/c, and masses in units of GeV/c^2. With $c \equiv 1$, it is then customary to quote masses in energy units. For example, the mass of the proton in these units is $m_\text{p} = 938\,\text{MeV}$, and $m_{\pi^-} = 140\,\text{MeV}$.

For a particle of mass M, its 4-momentum satisfies $P_\mu\,P^\mu = M^2$ (remember $c = 1$). Consider now an observer with 4-velocity U^μ. The energy of the particle, in the rest frame of the observer is $E = P^\mu\,U_\mu$. For example, if $P^\mu = (M, 0, 0, 0)$ is its rest frame, and $U^\mu = (\gamma\,,\,\gamma\boldsymbol{\beta})$, then $E = \gamma M$, as we have already seen.

Consider next the emission of a photon of 4-momentum $P^\mu = (\hbar\omega/c, \hbar\boldsymbol{k})$ from an object with 4-velocity V^μ, and detected in a frame with 4-velocity U^μ. In the frame of the detector, the photon energy is $E = P^\mu\,U_\mu$, while in the frame of the emitter its energy is $E' = P^\mu\,V_\mu$. If $U^\mu = (1, 0, 0, 0)$ and $V^\mu = (\gamma\,,\,\gamma\boldsymbol{\beta})$, then $E = \hbar\omega$ and $E' = \hbar\omega' = \gamma\hbar(\omega - \boldsymbol{\beta}\cdot\boldsymbol{k}) = \gamma\hbar\omega(1 - \beta\cos\theta)$, where $\theta = \cos^{-1}\big(\hat{\boldsymbol{\beta}}\cdot\hat{\boldsymbol{k}}\big)$. Thus, $\omega = \gamma^{-1}\omega'/(1 - \beta\cos\theta)$. This recapitulates our earlier derivation in eqn. 11.132.

Consider next the interaction of several particles. If in a given frame the 4-momenta of the reactants are P_i^μ, where n labels the reactant 'species', and the 4-momenta of the products are Q_j^μ, then if the collision is elastic, we have that total 4-momentum is conserved, *i.e.*

$$\sum_{i=1}^{N} P_i^\mu = \sum_{j=1}^{\bar{N}} Q_j^\mu \quad , \tag{11.142}$$

where there are N reactants and $\bar{N}$ products. For massive particles, we can write

$$P_i^\mu = \gamma_i \, m_i \, \big(1 \, , \, \boldsymbol{v}_i\big) \quad , \quad Q_j^\mu = \bar{\gamma}_j \, \bar{m}_j \, \big(1 \, , \, \bar{v}_j\big) \quad , \tag{11.143}$$

while for massless particles,

$$P_i^\mu = \hbar k_i \, \big(1 \, , \, \hat{\boldsymbol{k}}\big) \quad , \quad Q_j^\mu = \hbar \bar{k}_j \, \big(1 \, , \, \hat{\bar{\boldsymbol{k}}}\big) \quad . \tag{11.144}$$

11.7.1 Spontaneous particle decay into two products

Consider first the decay of a particle of mass M into two particles. We have $P^\mu = Q_1^\mu + Q_2^\mu$, hence in the rest frame of the (sole) reactant, which is also called the 'center of mass' (CM) frame since the total 3-momentum vanishes therein, we have $M = E_1 + E_2$. Since $E_i^{\rm CM} = \gamma^{\rm CM} m_i$, and $\gamma_i \geq 1$, clearly we must have $M > m_1 + m_2$, or else the decay cannot possibly conserve energy. To analyze further, write $P^\mu - Q_1^\mu = Q_2^\mu$. Squaring, we obtain

$$M^2 + m_1^2 - 2P_\mu \, Q_1^\mu = m_2^2 \quad . \tag{11.145}$$

The dot-product $P \cdot Q_1$ is a Lorentz scalar, and hence may be evaluated in any frame.

Let us first consider the CM frame, where $P^\mu = M(1,0,0,0)$, and $P_\mu \, Q_1^\mu = M E_1^{\rm CM}$, where $E_1^{\rm CM}$ is the energy of $n = 1$ product in the rest frame of the reactant. Thus,

$$E_1^{\rm CM} = \frac{M^2 + m_1^2 - m_2^2}{2M} \quad , \quad E_2^{\rm CM} = \frac{M^2 + m_2^2 - m_1^2}{2M} \quad , \tag{11.146}$$

where the second result follows merely from switching the product labels. We may now write $Q_1^\mu = (E_1^{\rm CM}, \boldsymbol{p}^{\rm CM})$ and $Q_2^\mu = (E_2^{\rm CM}, -\boldsymbol{p}^{\rm CM})$, with

$$\begin{aligned} (\boldsymbol{p}^{\rm CM})^2 &= (E_1^{\rm CM})^2 - m_1^2 = (E_2^{\rm CM})^2 - m_2^2 \\ &= \bigg(\frac{M^2 - m_1^2 - m_2^2}{2M}\bigg)^2 - \bigg(\frac{m_1 \, m_2}{M}\bigg)^2 \quad . \end{aligned} \tag{11.147}$$

In the laboratory frame, we have $P^\mu = \gamma M \, (1 \, , \, \boldsymbol{V})$ and $Q_i^\mu = \gamma_i \, m_i \, (1 \, , \, \boldsymbol{V}_i)$. Energy and momentum conservation then provide four equations for the six unknowns $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$. Thus, there is a two-parameter family of solutions, assuming we regard the reactant velocity $\boldsymbol{V}^{\rm K}$ as fixed, corresponding to the freedom to choose $\hat{\boldsymbol{p}}^{\rm CM}$ in the CM frame solution above. Clearly the three vectors $\boldsymbol{V}$, $\boldsymbol{V}_1$, and $\boldsymbol{V}_2$ must lie in the same plane, and with $\boldsymbol{V}$ fixed, only one additional parameter is required to fix this plane. The other free

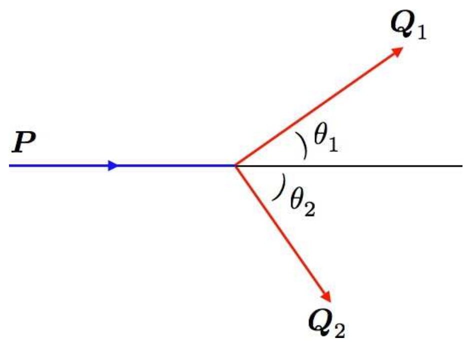


Figure 11.8: Spontaneous decay of a single reactant into two products.

parameter may be taken to be the relative angle $\theta_1 = \cos^{-1}(\hat{\boldsymbol{V}} \cdot \hat{\boldsymbol{V}}_1)$ (see fig. 11.8). The angle θ_2 as well as the speed V_2 are then completely determined. We can use eqn. 11.145 to relate θ_1 and V_1:

$$M^2 + m_1^2 - m_2^2 = 2Mm_1\gamma\gamma_1\big(1 - VV_1\cos\theta_1\big) \quad . \tag{11.148}$$

It is convenient to express both γ_1 and V_1 in terms of the energy E_1:

$$\gamma_1 = \frac{E_1}{m_1} \qquad , \qquad V_1 = \sqrt{1-\gamma_1^{-2}} = \sqrt{1 - \frac{m_1^2}{E_1^2}} \quad . \tag{11.149}$$

This results in a quadratic equation for E_1, which may be expressed as

$$(1 - V^2\cos^2\!\theta_1)E_1^2 - 2\sqrt{1-V^2}\,E_1^{\rm CM}E_1 + (1-V^2)(E_1^{\rm CM})^2 + m_1^2V^2\cos^2\!\theta_1 = 0 \quad , \tag{11.150}$$

the solutions of which are

$$E_1 = \frac{\sqrt{1-V^2}\,E_1^{\rm CM} \pm V\cos\theta_1\sqrt{(1-V^2)(E_1^{\rm CM})^2 - (1-V^2\cos^2\!\theta_1)m_1^2}}{1 - V^2\cos^2\!\theta_1} \quad . \tag{11.151}$$

The discriminant is positive provided

$$\left(\frac{E_1^{\rm CM}}{m_1}\right)^2 > \frac{1 - V^2\cos^2\!\theta_1}{1-V^2} \quad , \tag{11.152}$$

which means

$$\sin^2\!\theta_1 < \frac{V^{-2}-1}{(V_1^{\rm CM})^{-2}-1} \equiv \sin^2\!\theta_1^* \quad , \tag{11.153}$$

where

$$V_1^{\rm CM} = \sqrt{1 - \left(\frac{m_1}{E_1^{\rm CM}}\right)^2} \tag{11.154}$$

is the speed of product 1 in the CM frame. Thus, for $V < V_1^{\rm CM} < 1$, the scattering angle θ_1 may take on any value, while for larger reactant speeds $V_1^{\rm CM} < V < 1$ the quantity $\sin^2\!\theta_1$ cannot exceed a critical value.

11.7.2 Miscellaneous examples of particle decays

Let us now consider some applications of the formulae in eqn. 11.146:

- Consider the decay $\pi^0 \to \gamma\gamma$, for which $m_1 = m_2 = 0$. We then have $E_1^{\rm CM} = E_2^{\rm CM} = \frac{1}{2}M$. Thus, with $M = m_{\pi^0} = 135\,$MeV, we have $E_1^{\rm CM} = E_2^{\rm CM} = 67.5\,$MeV for the photon energies in the CM frame.
- For the reaction $K^+ \longrightarrow \mu^+ + \nu_\mu$ we have $M = m_{K^+} = 494\,$MeV and $m_1 = m_{\mu^-} = 106\,$MeV. The neutrino mass is $m_2 \approx 0$, hence $E_2^{\rm CM} = 236\,$MeV is the emitted neutrino's energy in the CM frame.
- A $\varLambda^0$ hyperon with a mass $M = m_{\varLambda^0} = 1116\,$MeV decays into a proton ($m_1 = m_{\rm p} = 938\,$MeV) and a pion $m_2 = m_{\pi^-} = 140\,$MeV). The CM energy of the emitted proton is $E_1^{\rm CM} = 943\,$MeV and that of the emitted pion is $E_2^{\rm CM} = 173\,$MeV.

11.7.3 Threshold particle production with a stationary target

Consider now a particle of mass M_1 moving with velocity $\boldsymbol{V}_1 = V_1\,\hat{\boldsymbol{x}}$, incident upon a stationary target particle of mass M_2, as indicated in fig. 11.9. Let the product masses be $m_1, m_2, \ldots, m_{N'}$. The 4-momenta of the reactants and products are

$$P_1^\mu = \big(E_1\,,\,\boldsymbol{P}_1\big) \quad , \quad P_2^\mu = M_2\,\big(1\,,\,0\big) \quad , \quad Q_j^\mu = \big(\varepsilon_j\,,\,\boldsymbol{p}_j\big) \quad . \tag{11.155}$$

Note that $E_1^2 - \boldsymbol{P}_1^2 = M_1^2$ and $\varepsilon_j^2 - \boldsymbol{p}_j^2 = m_j^2$, with $j \in \{1, 2, \ldots, N'\}$.

Conservation of momentum means that

$$P_1^\mu + P_2^\mu = \sum_{j=1}^{N'} Q_j^\mu \quad . \tag{11.156}$$

In particular, taking the $\mu = 0$ component, we have

$$E_1 + M_2 = \sum_{j=1}^{N'} \varepsilon_j \quad , \tag{11.157}$$

which certainly entails

$$E_1 \ge \sum_{j=1}^{N'} m_j - M_2 \tag{11.158}$$

since $\varepsilon_j = \gamma_j\, m_j \ge m_j$. But can the equality ever be achieved? This would only be the case if $\gamma_j = 1$ for all j, *i.e.* the final velocities are all zero. But this itself is quite impossible, since the initial state momentum is $\boldsymbol{P}$.

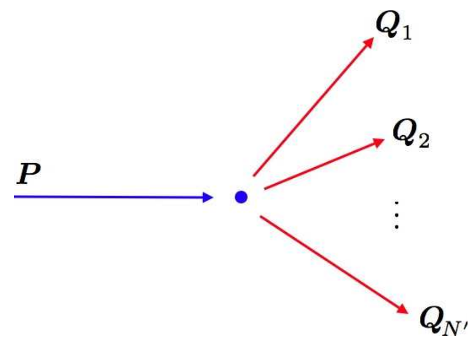


Figure 11.9: A two-particle initial state, with a stationary target in the LAB frame, and an N'-particle final state.

To determine the threshold energy $E_1^{\rm thr}$, we compare the length of the total momentum vector in the LAB and CM frames:

$$\begin{aligned}(P_1+P_2)^2 &= M_1^2+M_2^2+2E_1\,M_2 \qquad \text{(LAB)}\\ &= \bigg(\sum_{j=1}^{N'} \varepsilon_j^{\rm CM}\bigg)^2 \qquad \text{(CM)} \quad .\end{aligned} \tag{11.159}$$

Thus,

$$E_1 = \frac{\Big(\sum_{j=1}^{N'} \varepsilon_j^{\rm CM}\Big)^2 - M_1^2 - M_2^2}{2M_2} \tag{11.160}$$

and we conclude

$$E_1 \geq E_1^{\rm THR} = \frac{\Big(\sum_{j=1}^{N'} m_j\Big)^2 - M_1^2 - M_2^2}{2M_2} \quad . \tag{11.161}$$

Note that in the CM frame it *is* possible for each $\varepsilon_j^{\rm CM} = m_j$.

Finally, we must have $E_1^{\rm THR} \geq \sum_{j=1}^{N'} m_j - M_2$. This then requires

$$M_1 + M_2 \leq \sum_{j=1}^{N'} m_j \quad . \tag{11.162}$$

11.7.4 Transformation between frames

Consider a particle with 4-velocity u^μ in frame K and consider a Lorentz transformation between this frame and a frame K' moving relative to K with velocity $\boldsymbol{V}$. We may write

$$u^\mu = \big(\gamma\,,\,\gamma v\cos\theta\,,\,\gamma v\sin\theta\,\hat{\boldsymbol{n}}_\perp\big) \quad , \quad u'^\mu = \big(\gamma'\,,\,\gamma' v'\cos\theta'\,,\,\gamma' v'\sin\theta'\,\hat{\boldsymbol{n}}'_\perp\big) \quad . \tag{11.163}$$

According to the general transformation rules of eqns. 11.47, we may write

$$\begin{aligned} \gamma &= \Gamma\,\gamma' + \Gamma V\,\gamma' v' \cos\theta' \\ \gamma v \cos\theta &= \Gamma V\,\gamma' + \Gamma\,\gamma' v' \cos\theta' \\ \gamma v \sin\theta &= \gamma' v' \sin\theta' \\ \hat{\boldsymbol{n}}_\perp &= \hat{\boldsymbol{n}}'_\perp \quad , \end{aligned} \tag{11.164}$$

where the $\hat{\boldsymbol{x}}$ axis is taken to be $\hat{\boldsymbol{V}}$, and where $\Gamma \equiv (1-V^2)^{-1/2}$. Note that the last two of these equations may be written as a single vector equation for the transverse components.

Dividing the third and second of eqns. 11.164, we obtain the result

$$\tan\theta = \frac{\sin\theta'}{\Gamma\Big(\frac{V}{v'} + \cos\theta'\Big)} \quad . \tag{11.165}$$

We can then use the first of eqns. 11.164 to relate v' and $\cos\theta'$:

$$\gamma'^{-1} = \sqrt{1-v'^2} = \frac{\Gamma}{\gamma}\,\big(1 + V v' \cos\theta'\big) \quad . \tag{11.166}$$

Squaring both sides, we obtain a quadratic equation whose roots are

$$v' = \frac{-\Gamma^2\,V\cos\theta' \pm \sqrt{\gamma^4 - \Gamma^2\,\gamma^2\,(1 - V^2\cos^2\!\theta')}}{\gamma^2 + \Gamma^2\,V^2\cos^2\!\theta'} \quad . \tag{11.167}$$

CM frame mass and velocity

To find the velocity of the CM frame, simply write

$$\begin{aligned} P^\mu_{\text{tot}} = \sum_{i=1}^N P^\mu_i &= \bigg(\sum_{i=1}^N \gamma_i\,m_i\,,\,\sum_{i=1}^N \gamma_i\,m_i\,\boldsymbol{v}_i\bigg) \\ &\equiv \Gamma\,M\,(1\,,\,\boldsymbol{V}) \quad . \end{aligned} \tag{11.168}$$

Then

$$M^2 = \bigg(\sum_{i=1}^N \gamma_i\,m_i\bigg)^2 - \bigg(\sum_{i=1}^N \gamma_i\,m_i\,\boldsymbol{v}_i\bigg)^2 \tag{11.169}$$

and

$$\boldsymbol{V} = \frac{\sum_{i=1}^N \gamma_i\,m_i\,\boldsymbol{v}_i}{\sum_{i=1}^N \gamma_i\,m_i} \quad . \tag{11.170}$$

11.7.5 Compton scattering

An extremely important example of relativistic scattering occurs when a photon scatters off an electron: $e^- + \gamma \longrightarrow e^- + \gamma$ (see fig. 11.10). Let us work in the rest frame of the reactant electron. Then we have

$$P^\mu_e = m_{\text{e}}\,(1, 0) \qquad , \qquad \widetilde{P}^\mu_e = m_{\text{e}}\,(\gamma\,,\,\gamma\boldsymbol{V}) \tag{11.171}$$

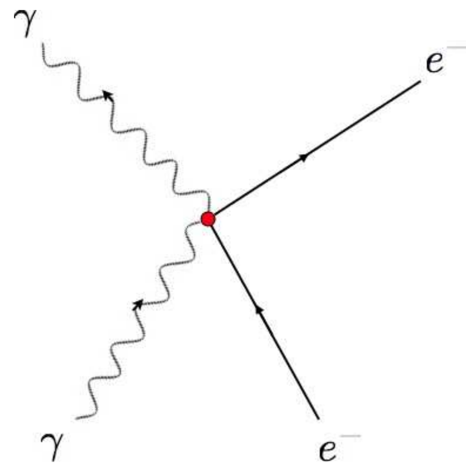


Figure 11.10: Compton scattering of a photon and an electron.

for the initial and final 4-momenta of the electron. For the photon, we have

$$P^\mu_\gamma = (\omega\,,\,\boldsymbol{k}) \qquad , \qquad \widetilde{P}^\mu_\gamma = (\widetilde{\omega}\,,\,\widetilde{\boldsymbol{k}}) \quad , \tag{11.172}$$

where we've set $\hbar = 1$ as well. Conservation of 4-momentum entails

$$P^\mu_\gamma - \widetilde{P}^\mu_\gamma = \widetilde{P}^\mu_e - P^\mu_e \quad . \tag{11.173}$$

Thus,

$$\big(\omega - \tilde{\omega}\,,\,\boldsymbol{k} - \tilde{\boldsymbol{k}}\big) = m_{\rm e}\,\big(\gamma - 1\,,\gamma\boldsymbol{V}\big) \quad . \tag{11.174}$$

Squaring each side, we obtain

$$\begin{aligned}
\big(\omega - \tilde{\omega}\big)^2 - \big(\boldsymbol{k} - \tilde{\boldsymbol{k}}\big)^2 &= 2\omega\,\tilde{\omega}\,(\cos\theta - 1) \\
&= m_{\rm e}^2\Big((\gamma-1)^2 - \gamma^2\boldsymbol{V}^2\Big) \\
&= 2m_{\rm e}^2(1-\gamma) \\
&= 2m_{\rm e}\,\big(\tilde{\omega} - \omega\big) \quad .
\end{aligned} \tag{11.175}$$

Here we have used $|\boldsymbol{k}| = \omega$ for photons, and also $(\gamma - 1)\,m_{\rm e} = \omega - \tilde{\omega}$, from eqn. 11.174.

Restoring the units $\hbar$ and c, we find the Compton formula

$$\frac{1}{\tilde{\omega}} - \frac{1}{\omega} = \frac{\hbar}{m_{\rm e}c^2}\,\big(1 - \cos\theta\big) \quad . \tag{11.176}$$

This is often expressed in terms of the photon wavelengths, as

$$\tilde{\lambda} - \lambda = \frac{4\pi\hbar}{m_{\rm e}c}\,\sin^2\!\big(\tfrac{1}{2}\theta\big) \quad , \tag{11.177}$$

showing that the wavelength of the scattered light increases with the scattering angle in the rest frame of the target electron.

11.8 Covariant Electrodynamics

We begin with the following expression for the Lagrangian density of charged particles coupled to an electromagnetic field, and then show that the Euler-Lagrange equations recapitulate Maxwell's equations. The Lagrangian density is

$$\mathcal{L} = -\frac{1}{16\pi}\, F_{\mu\nu}\, F^{\mu\nu} - \frac{1}{c}\, j_\mu\, A^\mu \quad . \tag{11.178}$$

Here, $A^\mu = (\phi\,,\boldsymbol{A})$ is the *electromagnetic 4-potential*, which combines the scalar field ϕ and the vector field $\boldsymbol{A}$ into a single 4-vector. The quantity $F_{\mu\nu}$ is the *electromagnetic field strength tensor* and is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad . \tag{11.179}$$

Note that as defined $F_{\mu\nu} = -F_{\nu\mu}$ is antisymmetric. Note that, if $i = 1, 2, 3$ is a spatial index, then

$$\begin{aligned} F_{0i} &= -\frac{1}{c}\frac{\partial A^i}{\partial t} - \frac{\partial A^0}{\partial x^i} = E_i \\ F_{ij} &= \frac{\partial A^i}{\partial x^j} - \frac{\partial A^j}{\partial x^i} = -\,\epsilon_{ijk}\, B_k \quad . \end{aligned} \tag{11.180}$$

Here we have used $A^\mu = (A^0\,,\,\boldsymbol{A})$ and $A_\mu = (A^0\,,\,-\boldsymbol{A})$, as well as $\partial_\mu = (c^{-1}\partial_t\,,\boldsymbol{\nabla})$.

IMPORTANT : Since the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are not part of a 4-vector, we do not use covariant / contravariant notation for their components. Thus, E_i is the $i^{\rm th}$ component of the vector $\boldsymbol{E}$. We will not write E^i with a raised index, but if we did, we'd mean the same thing: $E^i = E_i$. By contrast, for the spatial components of a four-vector like A^μ, we have $A_i = -A^i$.

Explicitly, then, we have

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \quad , \quad F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \quad , \tag{11.181}$$

where $F^{\mu\nu} = g^{\mu\alpha}\, g^{\nu\beta}\, F_{\alpha\beta}$. Note that when comparing $F^{\mu\nu}$ and $F_{\mu\nu}$, the components with one space and one time index differ by a minus sign. Thus,

$$-\,\frac{1}{16\pi}\, F_{\mu\nu}\, F^{\mu\nu} = \frac{\boldsymbol{E}^2 - \boldsymbol{B}^2}{8\pi} \quad , \tag{11.182}$$

which is the electromagnetic Lagrangian density. The $j \cdot A$ term accounts for the interaction between matter and electromagnetic degrees of freedom. We have

$$\frac{1}{c}\, j_\mu A^\mu = \varrho\, \phi - \frac{1}{c}\, \boldsymbol{j} \cdot \boldsymbol{A} \quad , \tag{11.183}$$

where

$$j^\mu = \begin{pmatrix} c\varrho \\ \boldsymbol{j} \end{pmatrix} \quad , \quad A^\mu = \begin{pmatrix} \phi \\ \boldsymbol{A} \end{pmatrix} \quad , \tag{11.184}$$

where ϱ is the charge density and $\boldsymbol{j}$ is the current density. Charge conservation requires

$$\partial_\mu \, j^\mu = \frac{\partial \varrho}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{j} = 0 \quad . \tag{11.185}$$

We shall have more to say about this further on below.

Let us now derive the Euler-Lagrange equations for the action functional,

$$S = -c^{-1} \int \! d^4\!x \left(\frac{1}{16\pi} \, F_{\mu\nu} \, F^{\mu\nu} + c^{-1} \, j_\mu \, A^\mu \right) \quad . \tag{11.186}$$

We first vary with respect to A_μ. Clearly

$$\delta F_{\mu\nu} = \partial_\mu \, \delta A_\nu - \partial_\nu \, \delta A_\mu \quad . \tag{11.187}$$

We then have

$$\delta \mathcal{L} = \left(\frac{1}{4\pi} \, \partial_\mu F^{\mu\nu} - c^{-1} j^\nu \right) \delta A_\nu - \partial_\mu \left(\frac{1}{4\pi} \, F^{\mu\nu} \, \delta A_\nu \right) \quad . \tag{11.188}$$

Ignoring the boundary term, we obtain Maxwell's equations,

$$\partial_\mu \, F^{\mu\nu} = 4\pi c^{-1} \, j^\nu \tag{11.189}$$

The $\nu = k$ component of these equations yields

$$\partial_0 \, F^{0k} + \partial_i \, F^{jk} = -\partial_0 \, E_k - \epsilon_{jkl} \, \partial_j \, B_l = 4\pi c^{-1} \, j^k \quad , \tag{11.190}$$

which is the k component of the Maxwell-Ampère law,

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \frac{4\pi}{c} \, \boldsymbol{j} + \frac{1}{c} \frac{\partial \boldsymbol{E}}{\partial t} \quad . \tag{11.191}$$

The $\nu = 0$ component reads

$$\partial_i \, F^{i0} = \frac{4\pi}{c} \, j^0 \quad \Rightarrow \quad \boldsymbol{\nabla} \cdot \boldsymbol{E} = 4\pi \varrho \quad , \tag{11.192}$$

which is Gauss's law. The remaining two Maxwell equations come 'for free' from the very definitions of $\boldsymbol{E}$ and $\boldsymbol{B}$:

$$\begin{aligned} \boldsymbol{E} &= -\boldsymbol{\nabla} A^0 - \frac{1}{c} \frac{\partial \boldsymbol{A}}{\partial t} \\ \boldsymbol{B} &= \boldsymbol{\nabla} \times \boldsymbol{A} \quad , \end{aligned} \tag{11.193}$$

which imply

$$\begin{aligned} \boldsymbol{\nabla} \times \boldsymbol{E} &= -\frac{1}{c} \frac{\partial \boldsymbol{B}}{\partial t} \\ \boldsymbol{\nabla} \cdot \boldsymbol{B} &= 0 \quad . \end{aligned} \tag{11.194}$$

11.8.1 Lorentz force law

This has already been worked out in chapter 7. Here we reiterate our earlier derivation. The 4-current may be written as

$$j^\mu(\boldsymbol{x},t) = c\sum_n q_n \int d\tau \, \frac{dX_n^\mu}{d\tau} \, \delta^{(4)}(x - X) \quad . \tag{11.195}$$

Thus, writing $X_n^\mu = \big(ct\,, \boldsymbol{X}_n(t)\big)$, we have

$$\begin{aligned} j^0(\boldsymbol{x},t) &= \sum_n q_n \, c \, \delta\big(\boldsymbol{x} - \boldsymbol{X}_n(t)\big) \\ \boldsymbol{j}(\boldsymbol{x},t) &= \sum_n q_n \, \dot{\boldsymbol{X}}_n(t) \, \delta\big(\boldsymbol{x} - \boldsymbol{X}_n(t)\big) \quad . \end{aligned} \tag{11.196}$$

The Lagrangian for the matter-field interaction term is then

$$\begin{aligned} L &= -c^{-1} \int d^3\!x \, \big(j^0 A^0 - \boldsymbol{j}\cdot\boldsymbol{A}\big) \\ &= -\sum_n \left[q_n \, \phi(\boldsymbol{X}_n,t) - \frac{q_n}{c} \, \boldsymbol{A}(\boldsymbol{X}_n,t)\cdot\dot{\boldsymbol{X}}_n \right] \quad , \end{aligned} \tag{11.197}$$

where $\phi = A^0$. For each charge q_n, this is equivalent to a particle with velocity-dependent potential energy

$$U(\boldsymbol{x},t) = q\,\phi(\boldsymbol{x},t) - \frac{q}{c}\,\boldsymbol{A}(\boldsymbol{r},t)\cdot\dot{\boldsymbol{x}} \quad , \tag{11.198}$$

where $\boldsymbol{x} = \boldsymbol{X}_n$.

Let's work out the equations of motion. We assume a kinetic energy $T = \frac{1}{2}m\dot{\boldsymbol{x}}^2$ for the charge. We then have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{x}}}\right) = \frac{\partial L}{\partial \boldsymbol{x}} \tag{11.199}$$

with $L = T - U$, which gives

$$m\,\ddot{\boldsymbol{x}} + \frac{q}{c}\,\frac{d\boldsymbol{A}}{dt} = -q\,\boldsymbol{\nabla}\phi + \frac{q}{c}\,\boldsymbol{\nabla}(\boldsymbol{A}\cdot\dot{\boldsymbol{x}}) \quad , \tag{11.200}$$

or, in component notation,

$$m\,\ddot{x}^i + \frac{q}{c}\,\frac{\partial A^i}{\partial x^j}\,\dot{x}^j + \frac{q}{c}\,\frac{\partial A^i}{\partial t} = -q\,\frac{\partial\phi}{\partial x^i} + \frac{q}{c}\,\frac{\partial A^j}{\partial x^i}\,\dot{x}^j \quad , \tag{11.201}$$

which is to say

$$m\,\ddot{x}^i = -q\,\frac{\partial\phi}{\partial x^i} - \frac{q}{c}\,\frac{\partial A^i}{\partial t} + \frac{q}{c}\left(\frac{\partial A^j}{\partial x^i} - \frac{\partial A^i}{\partial x^j}\right)\dot{x}^j \quad . \tag{11.202}$$

It is convenient to express the cross product in terms of the completely antisymmetric tensor of rank three, ϵ_{ijk}:

$$B_i = \epsilon_{ijk}\,\frac{\partial A^k}{\partial x^j} \quad , \tag{11.203}$$

and using the result

$$\epsilon_{ijk}\,\epsilon_{imn} = \delta_{jm}\,\delta_{kn} - \delta_{jn}\,\delta_{km} \quad , \tag{11.204}$$

we have $\epsilon_{ijk}\,B_i = \partial^j A^k - \partial^k A^j$, and

$$m\,\ddot{x}^i = -q\,\frac{\partial \phi}{\partial x^i} - \frac{q}{c}\,\frac{\partial A^i}{\partial t} + \frac{q}{c}\,\epsilon_{ijk}\,\dot{x}^j\,B_k \quad , \tag{11.205}$$

or, in vector notation,

$$\begin{aligned} m\,\ddot{\boldsymbol{x}} &= -q\,\boldsymbol{\nabla}\phi - \frac{q}{c}\,\frac{\partial \boldsymbol{A}}{\partial t} + \frac{q}{c}\,\dot{\boldsymbol{x}} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) \\ &= q\,\boldsymbol{E} + \frac{q}{c}\,\dot{\boldsymbol{x}} \times \boldsymbol{B} \quad , \end{aligned} \tag{11.206}$$

which is, of course, the Lorentz force law.

11.8.2 Gauge invariance

The action $S = c^{-1}\int d^4\!x\,\mathcal{L}$ admits a *gauge invariance*. Let $A^\mu \to A^\mu + \partial^\mu \Lambda$, where $\Lambda(\boldsymbol{x}, t)$ is an arbitrary scalar function of spacetime coordinates. Clearly

$$F_{\mu\nu} \to F_{\mu\nu} + \big(\partial_\mu \partial_\nu \Lambda - \partial_\nu \partial_\mu \Lambda\big) = F_{\mu\nu} \quad , \tag{11.207}$$

and hence the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ remain *invariant* under the gauge transformation, even though the 4-potential itself changes. What about the matter term? Clearly

$$\begin{aligned} -c^{-1}\,j^\mu\,A_\mu \to &- c^{-1}\,j^\mu\,A_\mu - c^{-1}\,j^\mu\,\partial_\mu \Lambda \\ &= -c^{-1}\,j^\mu\,A_\mu + c^{-1}\Lambda\,\partial_\mu\,j^\mu - \partial_\mu\big(c^{-1}\Lambda\,j^\mu\big) \quad . \end{aligned} \tag{11.208}$$

Once again we ignore the boundary term. We may now invoke charge conservation to write $\partial_\mu\,j^\mu = 0$, and we conclude that the action is invariant! Woo hoo! Note also the very deep connection

$$\text{gauge invariance} \quad \longleftrightarrow \quad \text{charge conservation} \quad . \tag{11.209}$$

11.8.3 Transformations of fields

One last detail remains, and that is to exhibit explicitly the Lorentz transformation properties of the electromagnetic field. For the case of vectors like A^μ, we have

$$A^\mu = L^\mu_{\ \nu}\,A'^{\,\nu} \quad . \tag{11.210}$$

The $\boldsymbol{E}$ and $\boldsymbol{B}$ fields, however, appear as elements in the field strength tensor $F^{\mu\nu}$. Clearly this must transform as a tensor:

$$F^{\mu\nu} = L^\mu_{\ \alpha}\,L^\nu_{\ \beta}\,F'^{\,\alpha\beta} = L^\mu_{\ \alpha}\,F'^{\,\alpha\beta}\,L^{\mathrm{t}\ \nu}_{\beta} \quad . \tag{11.211}$$

Figure 11.11: Homer celebrates the manifest gauge invariance of classical electromagnetic theory.

We can write a general Lorentz transformation as a product of a rotation $L_{\rm rot}$ and a boost $L_{\rm boost}$. Let's first see how rotations act on the field strength tensor. We take

$$L = L_{\rm rot} = \begin{pmatrix} 1_{1\times 1} & 0_{1\times 3} \\ 0_{3\times 1} & R_{3\times 3} \end{pmatrix} \quad , \tag{11.212}$$

where $R^{\rm t} R = \mathbb{I}$, *i.e.* $R \in {\rm O}(3)$ is an orthogonal matrix. We must compute

$$\begin{aligned} L^\mu_{\ \alpha}\, F'^{\,\alpha\beta}\, L^{{\rm t}\ \nu}_{\beta} &= \begin{pmatrix} 1 & 0 \\ 0 & R_{ij} \end{pmatrix} \begin{pmatrix} 0 & -E'_k \\ E'_j & -\,\epsilon_{jkm}\, B'_m \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & R^{\rm t}_{kl} \end{pmatrix} \\ &= \begin{pmatrix} 0 & -E'_k\, R^{\rm t}_{kl} \\ R_{ij}\, E'_j & -\,\epsilon_{jkm}\, R_{ij}\, R_{lk}\, B'_m \end{pmatrix} \quad . \end{aligned} \tag{11.213}$$

Thus, we conclude

$$\begin{aligned} E_l &= R_{lk}\, E'_k \\ \epsilon_{iln}\, B_n &= \epsilon_{jkm}\, R_{ij}\, R_{lk}\, B'_m \quad . \end{aligned} \tag{11.214}$$

Now for any 3×3 matrix R we have

$$\epsilon_{jks}\, R_{ij}\, R_{lk}\, R_{rs} = \det(R)\, \epsilon_{ilr} \quad , \tag{11.215}$$

and therefore

$$\begin{aligned} \epsilon_{jkm}\, R_{ij}\, R_{lk}\, B'_m &= \epsilon_{jkm}\, R_{ij}\, R_{lk}\, R_{nm}\, R_{ns}\, B'_s \\ &= \det(R)\, \epsilon_{iln}\, R_{ns}\, B'_s \quad , \end{aligned} \tag{11.216}$$

Therefore,

$$E_i = R_{ij}\, E'_j \qquad , \qquad B_i = \det(R) \cdot R_{ij}\, B'_j \quad . \tag{11.217}$$

For any orthogonal matrix, $R^{\rm t} R = \mathbb{I}$ gives that $\det(R) = \pm 1$. The extra factor of $\det(R)$ in the transformation properties of $\boldsymbol{B}$ is due to the fact that the electric field transforms as a *vector*, while the magnetic field transforms as a *pseudovector*. Under space inversion, for example, where $R = -\mathbb{I}$, the electric field is *odd* under this transformation ($\boldsymbol{E} \to -\boldsymbol{E}$) while the magnetic field is *even* ($\boldsymbol{B} \to +\boldsymbol{B}$). Similar considerations hold in particle mechanics for the linear momentum, $\boldsymbol{p}$ (a vector) and the angular momentum $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$ (a pseudovector). The analogy is not complete, however, because while both $\boldsymbol{p}$ and $\boldsymbol{L}$ are odd under the operation of time-reversal, $\boldsymbol{E}$ is even while $\boldsymbol{B}$ is odd.

OK, so how about boosts? We can write the general boost, from eqn. 11.36, as

$$L = \begin{pmatrix} \gamma & \gamma\hat{\boldsymbol{\beta}} \\ \gamma\hat{\boldsymbol{\beta}} & {\rm I} + (\gamma-1){\rm P}^{\boldsymbol{\beta}} \end{pmatrix} \tag{11.218}$$

where ${\rm P}^{\boldsymbol{\beta}}_{ij} = \hat{\beta}_i\,\hat{\beta}_j$ is the projector onto the direction of $\boldsymbol{\beta}$. We now compute

$$L^\mu_{\ \alpha}\, F'^{\,\alpha\beta}\, L^{{\rm t}\ \nu}_{\beta} = \begin{pmatrix} \gamma & \gamma\boldsymbol{\beta}^{\rm t} \\ \gamma\boldsymbol{\beta} & {\rm I} + (\gamma-1){\rm P} \end{pmatrix} \begin{pmatrix} 0 & -\boldsymbol{E}'^{\,\rm t} \\ \boldsymbol{E}' & -\epsilon_{jkm}\, B'_m \end{pmatrix} \begin{pmatrix} \gamma & \gamma\boldsymbol{\beta}^{\rm t} \\ \gamma\boldsymbol{\beta} & {\rm I} + (\gamma-1){\rm P} \end{pmatrix} \quad . \tag{11.219}$$

Carrying out the matrix multiplications, we obtain

$$\begin{aligned} \boldsymbol{E} &= \gamma(\boldsymbol{E}' - \boldsymbol{\beta}\times\boldsymbol{B}') - (\gamma-1)(\hat{\boldsymbol{\beta}}\cdot\boldsymbol{E}')\hat{\boldsymbol{\beta}} \\ \boldsymbol{B} &= \gamma(\boldsymbol{B}' + \boldsymbol{\beta}\times\boldsymbol{E}') - (\gamma-1)(\hat{\boldsymbol{\beta}}\cdot\boldsymbol{B}')\hat{\boldsymbol{\beta}} \quad . \end{aligned} \tag{11.220}$$

Expressed in terms of the components $E_\parallel$, $\boldsymbol{E}_\perp$, $B_\parallel$, and $\boldsymbol{B}_\perp$, one has

$$\begin{aligned} E_\parallel &= E'_\parallel \quad , \quad & \boldsymbol{E}_\perp &= \gamma\big(\boldsymbol{E}'_\perp - \boldsymbol{\beta}\times\boldsymbol{B}'_\perp\big) \\ B_\parallel &= B'_\parallel \quad , \quad & \boldsymbol{B}_\perp &= \gamma\big(\boldsymbol{B}'_\perp + \boldsymbol{\beta}\times\boldsymbol{E}'_\perp\big) \quad . \end{aligned} \tag{11.221}$$

Recall that for any vector $\boldsymbol{\xi}$, we write

$$\begin{aligned} \xi_\parallel &= \hat{\boldsymbol{\beta}}\cdot\boldsymbol{\xi} \\ \boldsymbol{\xi}_\perp &= \boldsymbol{\xi} - (\hat{\boldsymbol{\beta}}\cdot\boldsymbol{\xi})\,\hat{\boldsymbol{\beta}} \quad , \end{aligned} \tag{11.222}$$

so that $\hat{\boldsymbol{\beta}}\cdot\boldsymbol{\xi}_\perp = 0$.

11.8.4 Invariance *versus* covariance

We saw that the laws of electromagnetism were *gauge* ***in****variant*. That is, the solutions to the field equations did not change under a gauge transformation $A^\mu \to A^\mu + \partial^\mu \Lambda$. With respect to Lorentz transformations, however, the theory is *Lorentz* ***co****variant*. This means that Maxwell's equations in different inertial frames take the exact same form, $\partial_\mu F^{\mu\nu} = 4\pi c^{-1} j^\nu$, but that both the fields and the sources transform appropriately under a change in reference frames. The sources are described by the current 4-vector

$j^\mu = (c\varrho\, ,\, \boldsymbol{j})$ and transform as

$$\begin{aligned} c\varrho &= \gamma c\varrho' + \gamma\beta j'_\parallel \\ j_\parallel &= \gamma\beta c\varrho' + \gamma j'_\parallel \\ \boldsymbol{j}_\perp &= \boldsymbol{j}'_\perp \quad . \end{aligned} \tag{11.223}$$

The fields transform according to eqns. 11.221.

Consider, for example, a static point charge q located at the origin in the frame K', which moves with velocity $u\,\hat{\boldsymbol{x}}$ with respect to K. An observer in K' measures a charge density $\varrho'(\boldsymbol{x}', t') = q\,\delta(\boldsymbol{x}')$. The electric and magnetic fields in the K' frame are then $\boldsymbol{E}' = q\,\hat{\boldsymbol{r}}'/r'^{\,2}$ and $\boldsymbol{B}' = 0$. For an observer in the K frame, the coordinates transform as

$$\begin{aligned} ct &= \gamma ct' + \gamma\beta x' & \qquad ct' &= \gamma ct - \gamma\beta x \\ x &= \gamma\beta ct' + \gamma x' & \qquad x' &= -\gamma\beta ct + \gamma x \quad , \end{aligned} \tag{11.224}$$

as well as $y = y'$ and $z = z'$. The observer in the K frame sees instead a charge at $x^\mu = (ct\, ,\, ut\, ,\, 0\, ,\, 0)$ and both a charge density as well as a current density:

$$\begin{aligned} \varrho(\boldsymbol{x}, t) &= \gamma\varrho(\boldsymbol{x}', t') = q\,\delta(x-ut)\,\delta(y)\,\delta(z) \\ \boldsymbol{j}(\boldsymbol{x}, t) &= \gamma\beta c\,\varrho(\boldsymbol{x}', t')\,\hat{\boldsymbol{x}} = u\,q\,\delta(x-ut)\,\delta(y)\,\delta(z)\,\hat{\boldsymbol{x}} \quad . \end{aligned} \tag{11.225}$$

OK, so much for the sources. How about the fields? Expressed in terms of Cartesian coordinates, the electric field in K' is given by

$$\boldsymbol{E}'(\boldsymbol{x}', t') = q\,\frac{x'\hat{\boldsymbol{x}} + y'\hat{\boldsymbol{y}} + z'\hat{\boldsymbol{z}}}{\big(x'^{\,2} + y'^{\,2} + z'^{\,2}\big)^{3/2}} \quad . \tag{11.226}$$

From eqns. 11.221, we have $E_x = E'_x$ and $B_x = B'_x = 0$. Furthermore, we have $E_y = \gamma E'_y$, $E_z = \gamma E'_z$, $B_y = -\gamma\beta E'_z$, and $B_z = \gamma\beta E'_y$. Thus,

$$\begin{aligned} \boldsymbol{E}(\boldsymbol{x}, t) &= \gamma q\,\frac{(x-ut)\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}}{\big[\gamma^2(x-ut)^2 + y^2 + z^2\big]^{3/2}} \\ \boldsymbol{B}(\boldsymbol{x}, t) &= \frac{\gamma u}{c}\,q\,\frac{y\hat{\boldsymbol{z}} - z\hat{\boldsymbol{y}}}{\big[\gamma^2(x-ut)^2 + y^2 + z^2\big]^{3/2}} \quad . \end{aligned} \tag{11.227}$$

Let us define

$$\boldsymbol{R}(t) = (x-ut)\,\hat{\boldsymbol{x}} + y\,\hat{\boldsymbol{y}} + z\,\hat{\boldsymbol{z}} \quad . \tag{11.228}$$

We further define the angle $\theta \equiv \cos^{-1}\big(\hat{\boldsymbol{\beta}}\cdot\hat{\boldsymbol{R}}\big)$. We may then write

$$\begin{aligned} \boldsymbol{E}(x, t) &= \frac{q\,\boldsymbol{R}}{R^3}\cdot\frac{1-\beta^2}{\big(1-\beta^2\sin^2\!\theta\big)^{3/2}} \\ \boldsymbol{B}(x, t) &= \frac{q\,\hat{\boldsymbol{\beta}}\times\boldsymbol{R}}{R^3}\cdot\frac{1-\beta^2}{\big(1-\beta^2\sin^2\!\theta\big)^{3/2}} \quad . \end{aligned} \tag{11.229}$$

The fields are therefore enhanced in the transverse directions: $E_\perp/E_\parallel = \gamma^3$.

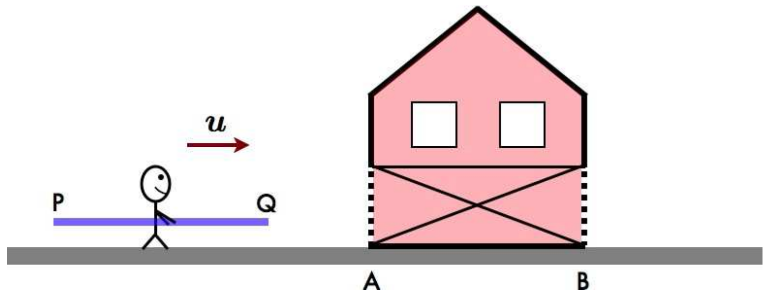


Figure 11.12: A relativistic runner carries a pole of proper length ℓ into a barn of proper length ℓ.

11.9 Appendix I: The Pole, the Barn, and *Rashoman*

Akira Kurosawa's 1950 cinematic masterpiece, *Rashoman*, describes a rape, murder, and battle from four different and often contradictory points of view. It poses deep questions regarding the nature of truth. Psychologists sometimes refer to problems of subjective perception as the *Rashoman effect*. In literature, William Faulkner's 1929 novel, *The Sound and the Fury*, which describes the tormented incestuous life of a Mississippi family, also is told from four points of view. Perhaps Faulkner would be a more apt comparison with Einstein, since time plays an essential role in his novel. For example, Quentin's watch, given to him by his father, represents time and the sweep of life's arc (*"Quentin, I give you the mausoleum of all hope and desire..."*). By breaking the watch, Quentin symbolically attempts to escape time and fate. One could draw an analogy to Einstein, inheriting a watch from those who came before him, which he too broke – and refashioned. Did Faulkner know of Einstein? But I digress.

Consider a relativistic runner carrying a pole of proper length ℓ, as depicted in fig. 11.12. He runs toward a barn of proper length ℓ at velocity $u = c\beta$. Let the frame of the barn be K and the frame of the runner be K'. Recall that the Lorentz transformations between frames K and K' are given by

$$\begin{aligned} ct &= \gamma ct' + \gamma\beta x' & \qquad ct' &= \gamma ct - \gamma\beta x \\ x &= \gamma\beta ct' + \gamma x' & \qquad x' &= -\gamma\beta ct + \gamma x \quad . \end{aligned} \tag{11.230}$$

We define the following points. Let A denote the left door of the barn and B the right door. Furthermore, let P denote the left end of the pole and Q its right end. The spacetime coordinates for these points in the two frames are clearly .

$$\begin{aligned} A &= (ct\,,\,0) & \qquad P' &= (ct'\,,\,0) \\ B &= (ct\,,\,\ell) & \qquad Q' &= (ct'\,,\,\ell) \end{aligned} \tag{11.231}$$

We now compute A' and B' in frame K', as well as P and Q in frame K:

$$\begin{aligned} A' &= (\gamma ct\,,\,-\gamma\beta ct) & \qquad B' &= (\gamma ct - \gamma\beta\ell\,,\,-\gamma\beta ct + \gamma\ell) \\ &\equiv (ct'\,,\,-\beta ct') & \qquad &\equiv (ct'\,,\,-\beta ct' + \gamma^{-1}\ell) \quad . \end{aligned} \tag{11.232}$$

Similarly,

$$\begin{aligned} P &= (\gamma ct'\,,\,\gamma\beta ct') & \qquad Q &= (\gamma ct' + \gamma\beta\ell\,,\,\gamma\beta ct' + \gamma\ell) \\ &\equiv (ct\,,\,\beta ct) & \qquad &\equiv (ct\,,\,\beta ct + \gamma^{-1}\ell) \quad . \end{aligned} \tag{11.233}$$

We now define four events, by the coincidences of A and B with P and Q:

- Event I : The right end of the pole enters the left door of the barn. This is described by $Q = A$ in frame K and by $Q' = A'$ in frame K'.
- Event II : The right end of the pole exits the right door of the barn. This is described by $Q = B$ in frame K and by $Q' = B'$ in frame K'.
- Event III : The left end of the pole enters the left door of the barn. This is described by $P = A$ in frame K and by $P' = A'$ in frame K'.
- Event IV : The left end of the pole exits the right door of the barn. This is described by $P = B$ in frame K and by $P' = B'$ in frame K'.

Mathematically, we have in frame K that

$$\begin{aligned}
\text{I}:&\quad Q = A &\quad\Rightarrow\quad& t_{\text{I}} = -\frac{\ell}{\gamma u}\\
\text{II}:&\quad Q = B &\quad\Rightarrow\quad& t_{\text{II}} = (\gamma - 1)\,\frac{\ell}{\gamma u}\\
\text{III}:&\quad P = A &\quad\Rightarrow\quad& t_{\text{III}} = 0\\
\text{IV}:&\quad P = B &\quad\Rightarrow\quad& t_{\text{IV}} = \frac{\ell}{u}
\end{aligned} \tag{11.234}$$

In frame K', however

$$\begin{aligned}
\text{I}:&\quad Q' = A' &\quad\Rightarrow\quad& t'_{\text{I}} = -\frac{\ell}{u}\\
\text{II}:&\quad Q' = B' &\quad\Rightarrow\quad& t'_{\text{II}} = -(\gamma - 1)\,\frac{\ell}{\gamma u}\\
\text{III}:&\quad P' = A' &\quad\Rightarrow\quad& t'_{\text{III}} = 0\\
\text{IV}:&\quad P' = B' &\quad\Rightarrow\quad& t'_{\text{IV}} = \frac{\ell}{\gamma u}
\end{aligned} \tag{11.235}$$

Thus, to an observer in frame K, the order of events is I, III, II, and IV, because

$$t_{\text{I}} < t_{\text{III}} < t_{\text{II}} < t_{\text{IV}} \quad . \tag{11.236}$$

For $t_{\text{III}} < t < t_{\text{II}}$, he observes that *the pole is entirely in the barn*. Indeed, the right door can start shut and the left door open, and sensors can automatically and, for the purposes of argument, instantaneously trigger the closing of the left door immediately following event III and the opening of the right door immediately prior to event II. So the pole can be inside the barn with both doors shut!

But now for the *Rashoman effect*: according to the runner, the order of events is I, II, III, and IV, because

$$t'_{\text{I}} < t'_{\text{II}} < t'_{\text{III}} < t'_{\text{IV}} \quad . \tag{11.237}$$

At no time does the runner observe the pole to be entirely within the barn. Indeed, for $t'_{\text{II}} < t' < t'_{\text{III}}$, both ends of the pole are sticking outside of the barn!

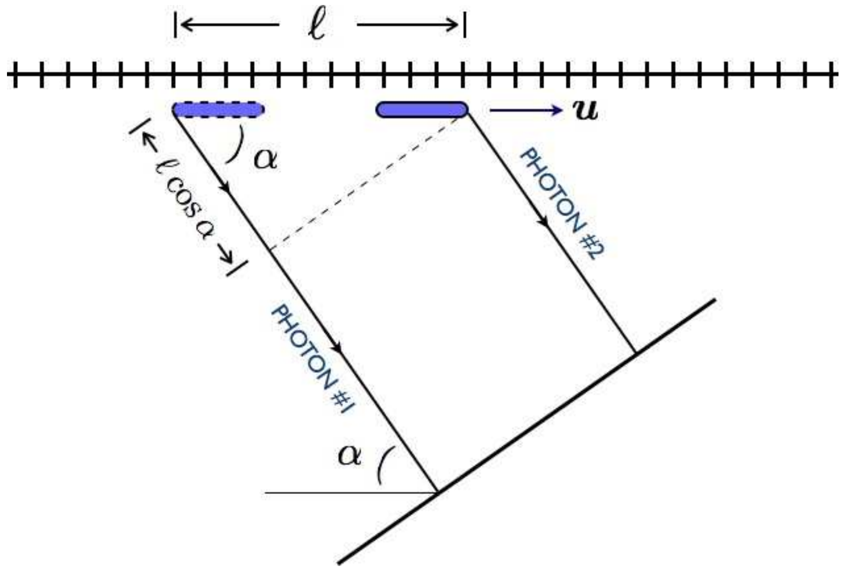


Figure 11.13: An object of proper length ℓ and moving with velocity $\boldsymbol{u}$, when photographed from an angle α, appears to have a length $\tilde{\ell}$.

11.10 Appendix II: Photographing a Moving Pole

What is the length ℓ of a moving pole of proper length ℓ_0 as measured by an observer at rest? The answer would appear to be $\gamma^{-1}\ell_0$, as we computed in eqn. 11.63. However, we should be more precise when we we speak of 'length'. The relation $\ell(\beta) = \gamma^{-1}\ell_0$ tells us the *instantaneous end-to-end distance as measured in the observer's rest frame* K. But an actual experiment might not measure this quantity.

For example, suppose a relativistic runner carrying a pole of proper length ℓ_0 runs past a measuring rod which is at rest in the rest frame K of an observer. The observer *takes a photograph* of the moving pole as it passes by. Suppose further that the angle between the observer's line of sight and the velocity $\boldsymbol{u}$ of the pole is α, as shown in fig. 11.13. What is the apparent length $\ell(\alpha, u)$ of the pole as observed in the photograph? (*I.e.* the pole will appear to cover a portion of the measuring rod which is of length ℓ.)

The point here is that the shutter of the camera is very fast (otherwise the image will appear blurry). In our analysis we will assume the shutter opens and closes instantaneously. Let's define two events:

- Event 1 : photon γ_1 is emitted by the rear end of the pole.
- Event 2 : photon γ_2 is emitted by the front end of the pole.

Both photons must arrive at the camera's lens simultaneously. Since, as shown in the figure, the path of photon #1 is longer by a distance $\ell \cos\alpha$, where ℓ is the apparent length of the pole, γ_2 must be emitted a time $\Delta t = c^{-1}\ell\cos\alpha$ after γ_1. Now if we Lorentz transform from frame K to frame K', we have

$$\Delta x' = \gamma\Delta x - \gamma\beta\Delta t \quad . \tag{11.238}$$

But $\Delta x' = \ell_0$ is the proper length of the pole, and $\Delta x = \ell$ is the apparent length. With $c\Delta t = \ell \cos\alpha$, then, we have

$$\ell = \frac{\gamma^{-1}\,\ell_0}{1 - \beta\cos\alpha} \quad . \tag{11.239}$$

When $\alpha = 90°$, we recover the familiar Lorentz-Fitzgerald contraction $\ell(\beta) = \gamma^{-1}\,\ell_0$. This is because the photons γ_1 and γ_2 are then emitted simultaneously, and the photograph measures the instantaneous end-to-end distance of the pole as measured in the observer's rest frame K. When $\cos\alpha \neq 0$, however, the two photons are not emitted simultaneously, and the apparent length is given by eqn. 11.239.

Chapter 12

Dynamical Systems

12.1 Introduction

12.1.1 Phase space and phase curves

Dynamics is the study of motion through phase space. The phase space of a given dynamical system is described as an N-dimensional manifold, $\mathcal{M}$. A (differentiable) manifold $\mathcal{M}$ is a topological space that is locally diffeomorphic to $\mathbb{R}^N$.[1] Typically in this course $\mathcal{M}$ will $\mathbb{R}^N$ itself, but other common examples include the circle $\mathbb{S}^1$, the torus $\mathbb{T}^2$, the sphere $\mathbb{S}^2$, *etc.*

Let $g_t \colon \mathcal{M} \to \mathcal{M}$ be a one-parameter family of transformations from $\mathcal{M}$ to itself, with $g_{t=0} = 1$, the identity. We call g_t the t-advance mapping. It satisfies the composition rule

$$g_t\, g_s = g_{t+s} \quad . \tag{12.1}$$

Let us choose a point $\boldsymbol{\varphi}_0 \in \mathcal{M}$. Then we write $\boldsymbol{\varphi}(t) = g_t\, \boldsymbol{\varphi}_0$, which also is in $\mathcal{M}$. The set $\big\{ g_t\, \boldsymbol{\varphi}_0 \,\big|\, t \in \mathbb{R} \,,\ \boldsymbol{\varphi}_0 \in \mathcal{M} \big\}$ is called a *phase curve*. A graph of the motion $\boldsymbol{\varphi}(t)$ in the product space $\mathbb{R} \times \mathcal{M}$ is called an *integral curve*.

12.1.2 Vector fields

The *velocity* vector $\boldsymbol{V}(\boldsymbol{\varphi})$ is given by the derivative

$$\boldsymbol{V}(\boldsymbol{\varphi}) = \frac{d}{dt}\bigg|_{t=0} g_t\, \boldsymbol{\varphi} \quad . \tag{12.2}$$

The velocity $\boldsymbol{V}(\boldsymbol{\varphi})$ is an element of the *tangent space* to $\mathcal{M}$ at $\boldsymbol{\varphi}$, abbreviated $\mathrm{T}\mathcal{M}_{\boldsymbol{\varphi}}$. If $\mathcal{M}$ is N-dimensional, then so is each $\mathrm{T}\mathcal{M}_{\boldsymbol{\varphi}}$ (for all $\boldsymbol{\varphi}$). However, $\mathcal{M}$ and $\mathrm{T}\mathcal{M}_{\boldsymbol{\varphi}}$ may differ topologically. For example, if $\mathcal{M} = \mathbb{S}^1$, the circle, the tangent space at any point is isomorphic to $\mathbb{R}$.

[1] A *diffeomorphism* $F \colon \mathcal{M} \to \mathcal{N}$ is a differentiable map with a differentiable inverse. This is a special type of *homeomorphism*, which is a continuous map with a continuous inverse.

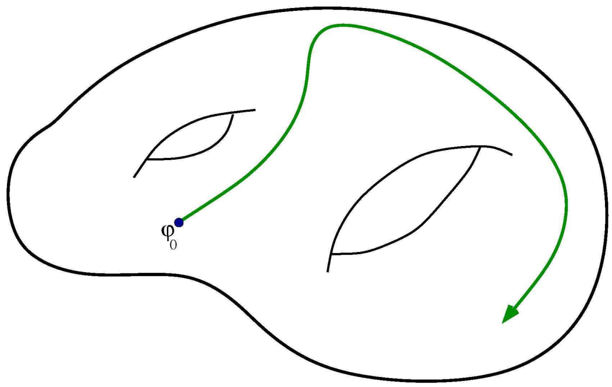


Figure 12.1: An example of a phase curve.

For our purposes, we will take $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_N)$ to be an N-tuple, *i.e.* a point in $\mathbb{R}^N$. The equation of motion is then

$$\frac{d}{dt}\,\boldsymbol{\varphi}(t) = \boldsymbol{V}\big(\boldsymbol{\varphi}(t)\big) \quad . \tag{12.3}$$

Note that any N^{th} order ODE, of the general form

$$\frac{d^N x}{dt^N} = F\bigg(x, \frac{dx}{dt}, \ldots, \frac{d^{N-1}x}{dt^{N-1}}\bigg) \quad , \tag{12.4}$$

may be represented by the first order system $\dot{\boldsymbol{\varphi}} = \boldsymbol{V}(\boldsymbol{\varphi})$. To see this, define $\varphi_k = d^{k-1}x/dt^{k-1}$, with $k = 1, \ldots, N$. Thus, for $j < N$ we have $\dot{\varphi}_j = \varphi_{j+1}$, and $\dot{\varphi}_N = f$. In other words,

$$\overbrace{\frac{d}{dt}\begin{pmatrix}\varphi_1\\ \vdots\\ \varphi_{N-1}\\ \varphi_N\end{pmatrix}}^{\dot{\boldsymbol{\varphi}}} = \overbrace{\begin{pmatrix}\varphi_2\\ \vdots\\ \varphi_N\\ F\big(\varphi_1, \ldots, \varphi_N\big)\end{pmatrix}}^{\boldsymbol{V}(\boldsymbol{\varphi})} \quad . \tag{12.5}$$

12.1.3 Existence / uniqueness / extension theorems

Theorem : Given $\dot{\boldsymbol{\varphi}} = \boldsymbol{V}(\boldsymbol{\varphi})$ and $\boldsymbol{\varphi}(0)$, if each $\boldsymbol{V}(\boldsymbol{\varphi})$ is a smooth vector field over some open set $\mathcal{D} \in \mathcal{M}$, then for $\boldsymbol{\varphi}(0) \in \mathcal{D}$ the initial value problem has a solution on some finite time interval $(-\tau, +\tau)$ and the solution is unique. Furthermore, the solution has a unique extension forward or backward in time, either indefinitely or until $\boldsymbol{\varphi}(t)$ reaches the boundary of $\mathcal{D}$.

Corollary : *Different trajectories never intersect!*

12.1.4 Linear differential equations

A homogeneous linear $N^{\rm th}$ order ODE,

$$\frac{d^N x}{dt^N} + c_{N-1} \frac{d^{N-1} x}{dt^{N-1}} + \ldots + c_1 \frac{dx}{dt} + c_0 \, x = 0 \tag{12.6}$$

may be written in matrix form, as

$$\frac{d}{dt} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_N \end{pmatrix} = \overbrace{\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -c_0 & -c_1 & -c_2 & \cdots & -c_{N-1} \end{pmatrix}}^{M} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_N \end{pmatrix} \quad . \tag{12.7}$$

Thus,

$$\dot{\boldsymbol{\varphi}} = M \boldsymbol{\varphi} \quad , \tag{12.8}$$

and if the coefficients c_k are time-independent, *i.e.* the ODE is *autonomous*, the solution is obtained by exponentiating the constant matrix Q:

$$\boldsymbol{\varphi}(t) = \exp(Mt) \, \boldsymbol{\varphi}(0) \ ; \tag{12.9}$$

the exponential of a matrix may be given meaning by its Taylor series expansion. If the ODE is not autonomous, then $M = M(t)$ is time-dependent, and the solution is given by the path-ordered exponential,

$$\boldsymbol{\varphi}(t) = \mathcal{P} \exp \left\{ \int\limits_0^t dt' \, M(t') \right\} \boldsymbol{\varphi}(0) \quad , \tag{12.10}$$

As defined, the equation $\dot{\boldsymbol{\varphi}} = \boldsymbol{V}(\boldsymbol{\varphi})$ is autonomous, since g_t depends only on t and on no other time variable. However, by extending the phase space from $\mathcal{M}$ to $\mathbb{R} \times \mathcal{M}$, which is of dimension $(N+1)$, one can describe arbitrary time-dependent ODEs.

12.1.5 Lyapunov functions

For a general dynamical system $\dot{\boldsymbol{\varphi}} = \boldsymbol{V}(\boldsymbol{\varphi})$, a *Lyapunov function* $L(\boldsymbol{\varphi})$ is a function which satisfies

$$\boldsymbol{\nabla} L(\boldsymbol{\varphi}) \cdot \boldsymbol{V}(\boldsymbol{\varphi}) \le 0 \quad . \tag{12.11}$$

There is no simple way to determine whether a Lyapunov function exists for a given dynamical system, or, if it does exist, what the Lyapunov function is. However, if a Lyapunov function can be found, then this severely limits the possible behavior of the system. This is because $L\big(\boldsymbol{\varphi}(t)\big)$ must be a monotonic function of time:

$$\frac{d}{dt} L\big(\boldsymbol{\varphi}(t)\big) = \boldsymbol{\nabla} L \cdot \frac{d\boldsymbol{\varphi}}{dt} = \boldsymbol{\nabla} L(\boldsymbol{\varphi}) \cdot \boldsymbol{V}(\boldsymbol{\varphi}) \le 0 \quad . \tag{12.12}$$

Thus, the system evolves toward a local minimum of the Lyapunov function. In general this means that oscillations are impossible in systems for which a Lyapunov function exists. For example, the relaxational dynamics of the magnetization M of a system are sometimes modeled by the equation

$$\frac{dM}{dt} = -\Gamma\,\frac{\partial F}{\partial M} \quad , \tag{12.13}$$

where $F(M,T)$ is the *free energy* of the system. In this model, assuming constant temperature T, $\dot{F} = F'(M)\,\dot{M} = -\Gamma\,\big[F'(M)\big]^2 \le 0$. So the free energy $F(M)$ itself is a Lyapunov function, and it monotonically decreases during the evolution of the system. We shall meet up with this example again in the next chapter when we discuss imperfect bifurcations.

12.2 $N = 1$ Systems

We now study phase flows in a one-dimensional phase space, governed by the equation

$$\frac{du}{dt} = f(u) \quad . \tag{12.14}$$

Again, the equation $\dot{u} = h(u,t)$ is first order, but not autonomous, and it corresponds to the $N = 2$ system,

$$\frac{d}{dt}\begin{pmatrix} u \\ t \end{pmatrix} = \begin{pmatrix} h(u,t) \\ 1 \end{pmatrix} \quad . \tag{12.15}$$

The equation 12.14 is easily integrated:

$$\frac{du}{f(u)} = dt \qquad \Longrightarrow \qquad t - t_0 = \int\limits_{u_0}^{u} \frac{du'}{f(u')} \quad . \tag{12.16}$$

This gives $t(u)$; we must then invert this relationship to obtain $u(t)$.

Example : Suppose $f(u) = a - bu$, with a and b constant. Then

$$dt = \frac{du}{a - bu} = -b^{-1}\, d\log(a - bu) \tag{12.17}$$

whence

$$t = \frac{1}{b}\log\!\left(\frac{a - b\,u(0)}{a - b\,u(t)}\right) \quad \Longrightarrow \quad u(t) = \frac{a}{b} + \Big(u(0) - \frac{a}{b}\Big)\,\exp(-bt) \quad . \tag{12.18}$$

Even if one cannot analytically obtain $u(t)$, the behavior is very simple, and easily obtained by graphical analysis. Sketch the function $f(u)$. Then note that

$$\dot{u} = f(u) \quad \Longrightarrow \quad \begin{cases} f(u) > 0 & \dot{u} > 0 \quad \Rightarrow \quad \text{move to right} \\ f(u) < 0 & \dot{u} < 0 \quad \Rightarrow \quad \text{move to left} \\ f(u) = 0 & \dot{u} = 0 \quad \Rightarrow \quad \text{fixed point} \end{cases} \tag{12.19}$$

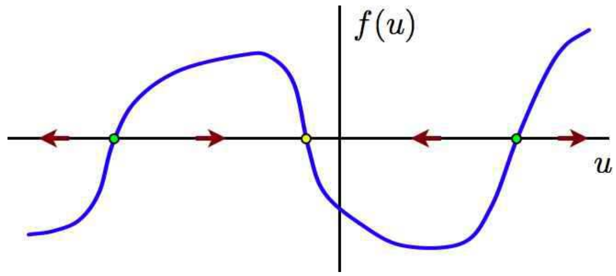


Figure 12.2: Phase flow for an $N = 1$ system.

The behavior of $N = 1$ systems is particularly simple: $u(t)$ flows to the first stable fixed point encountered, where it then (after a logarithmically infinite time) stops. The motion is monotonic – the velocity $\dot{u}$ never changes sign. Thus, *oscillations never occur for $N = 1$ phase flows*.[2]

12.2.1 Classification of fixed points ($N = 1$)

A *fixed point* u^* satisfies $f(u^*) = 0$. Generically, $f'(u^*) \neq 0$ at a fixed point.[3] Suppose $f'(u^*) < 0$. Then to the left of the fixed point, the function $f(u < u^*)$ is positive, and the flow is to the right, *i.e.* toward u^*. To the right of the fixed point, the function $f(u > u^*)$ is negative, and the flow is to the left, *i.e.* again toward u^*. Thus, when $f'(u^*) < 0$ the fixed point is said to be *stable*, since the flow in the vicinity of u^* is to u^*. Conversely, when $f'(u^*) > 0$, the flow is always away from u^*, and the fixed point is then said to be *unstable*. Indeed, if we linearize about the fixed point, and let $\epsilon \equiv u - u^*$, then

$$\dot{\epsilon} = f'(u^*)\,\epsilon + \tfrac{1}{2}\,f''(u^*)\,\epsilon^2 + \mathcal{O}(\epsilon^3) \quad , \tag{12.20}$$

and dropping all terms past the first on the RHS gives

$$\epsilon(t) = \exp\Big[f'(u^*)\,t\Big]\,\epsilon(0) \quad . \tag{12.21}$$

The deviation decreases exponentially for $f'(u^*) < 0$ and increases exponentially for $f(u^*) > 0$. Note that

$$t(\epsilon) = \frac{1}{f'(u^*)}\,\log\bigg(\frac{\epsilon}{\epsilon(0)}\bigg) \quad , \tag{12.22}$$

so the approach to a stable fixed point takes a logarithmically infinite time. For the unstable case, the deviation grows exponentially, until eventually the linearization itself fails.

12.2.2 Logistic equation

This model for population growth was first proposed by Verhulst in 1838. Let N denote the population in question. The dynamics are modeled by the first order ODE,

$$\frac{dN}{dt} = rN\Big(1 - \frac{N}{K}\Big) \quad , \tag{12.23}$$

[2]When I say 'never' I mean 'sometimes' – see the section 12.3.
[3]The system $f(u^*) = 0$ and $f'(u^*) = 0$ is overdetermined, with two equations for the single variable u^*.

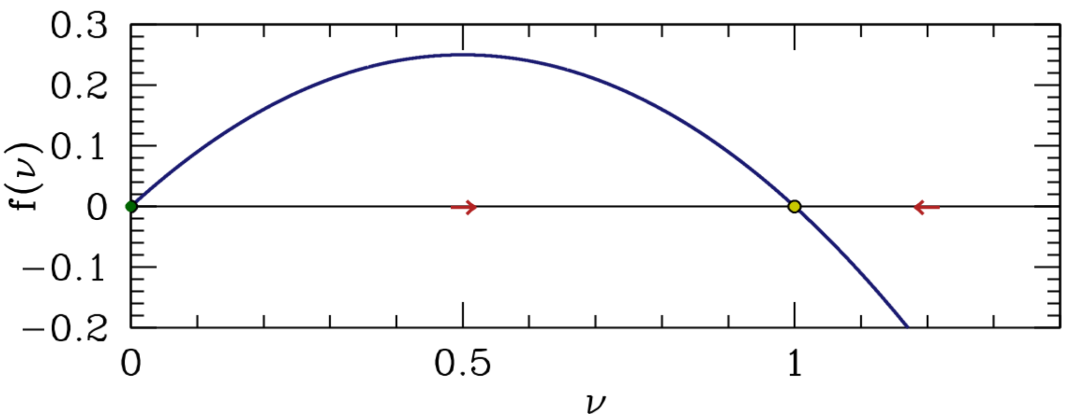


Figure 12.3: Flow diagram for the logistic equation.

where N, r, and K are all positive. For $N \ll K$ the growth rate is r, but as N increases a quadratic nonlinearity kicks in and the rate vanishes for $N = K$ and is negative for $N > K$. The nonlinearity models the effects of competition between the organisms for food, shelter, or other resources. Or maybe they crap all over each other and get sick. Whatever.

There are two fixed points, one at $N^* = 0$, which is unstable ($f'(0) = r > 0$). The other, at $N^* = K$, is stable ($f'(K) = -r$). The equation is adimensionalized by defining $\nu = N/K$ and $s = rt$, whence

$$\dot{\nu} = \nu(1 - \nu) \quad . \tag{12.24}$$

Integrating,

$$\frac{d\nu}{\nu(1-\nu)} = d\log\Big(\frac{\nu}{1-\nu}\Big) = ds \qquad \Longrightarrow \qquad \nu(s) = \frac{\nu_0}{\nu_0 + \big(1 - \nu_0\big)\exp(-s)} \quad . \tag{12.25}$$

As $s \to \infty$, $\nu(s) = 1 - \big(\nu_0^{-1} - 1\big)\, e^{-s} + \mathcal{O}(e^{-2s})$, and the relaxation to equilibrium ($\nu^* = 1$) is exponential, as usual.

Another application of this model is to a simple autocatalytic reaction, such as

$$A + X \rightleftharpoons 2X \quad , \tag{12.26}$$

i.e. X catalyses the reaction $A \longrightarrow X$. Assuming a fixed concentration of A, we have

$$\dot{x} = \kappa_+\, a\, x - \kappa_-\, x^2 \quad , \tag{12.27}$$

where x is the concentration of X, and $\kappa_\pm$ are the forward and backward reaction rates.

12.2.3 Singular $f(u)$

Suppose that in the vicinity of a fixed point we have $f(u) = A\,\big|u - u^*\big|^\alpha$, with $A > 0$. We now analyze both sides of the fixed point.

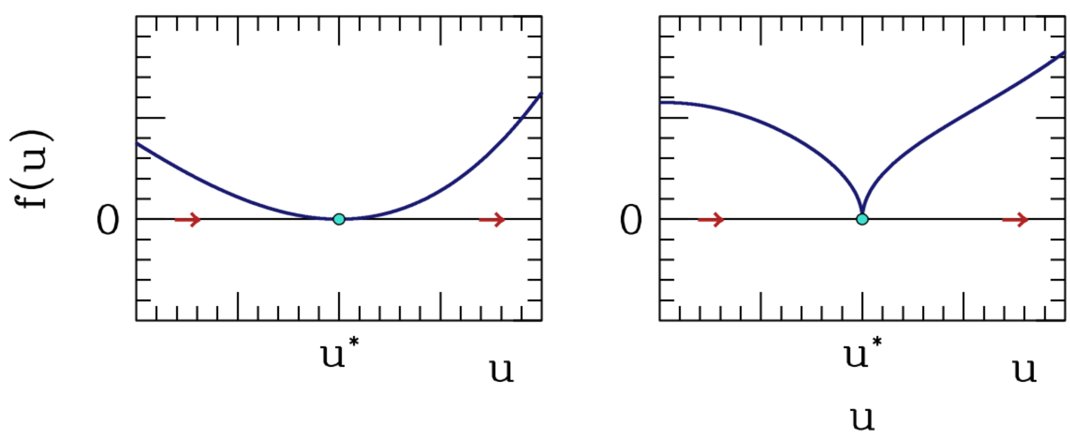


Figure 12.4: $f(u) = A\left|u - u^*\right|^\alpha$, for $\alpha > 1$ and $\alpha < 1$.

$u < u^*$: Let $\epsilon = u^* - u$. Then

$$\dot{\epsilon} = -A\,\epsilon^\alpha \quad \Longrightarrow \quad \frac{\epsilon^{1-\alpha}}{1-\alpha} = \frac{\epsilon_0^{1-\alpha}}{1-\alpha} - At \quad , \tag{12.28}$$

hence

$$\epsilon(t) = \Big[\epsilon_0^{1-\alpha} + (\alpha - 1)A\,t\Big]^{\frac{1}{1-\alpha}} \quad . \tag{12.29}$$

This, for $\alpha < 1$ the fixed point $\epsilon = 0$ is reached in a finite time: $\epsilon(t_c) = 0$, with

$$t_c = \frac{\epsilon_0^{1-\alpha}}{(1-\alpha)\,A} \quad . \tag{12.30}$$

For $\alpha > 1$, we have $\lim_{t\to\infty} \epsilon(t) = 0$, but $\epsilon(t) > 0 \;\forall\, t < \infty$.

The fixed point $u = u^*$ is now *half-stable* – the flow from the left is toward u^* but from the right is away from u^*. Let's analyze the flow on either side of u^*.

$u > u^*$: Let $\epsilon = u - u^*$. Then $\dot{\epsilon} = A\,\epsilon^\alpha$, and

$$\epsilon(t) = \Big[\epsilon_0^{1-\alpha} + (1 - \alpha)A\,t\Big]^{\frac{1}{1-\alpha}} \quad . \tag{12.31}$$

For $\alpha < 1$, $\epsilon(t)$ escapes to $\epsilon = \infty$ only after an infinite time. For $\alpha > 1$, the escape to infinity takes a finite time: $\epsilon(t_c) = \infty$, with

$$t_c = \frac{\epsilon_0^{1-\alpha}}{(\alpha-1)\,A} \quad . \tag{12.32}$$

In both cases, higher order terms in the (nonanalytic) expansion of $f(u)$ about $u = u^*$ will eventually come into play.

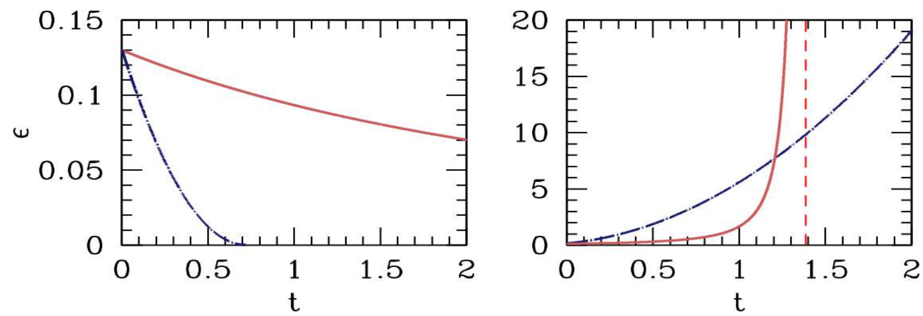


Figure 12.5: Solutions to $\dot{\epsilon} = \mp A\,\epsilon^\alpha$. Left panel: $\epsilon = u^* - u$, with $\alpha = 1.5$ (solid red) and $\alpha = 0.5$ (dot-dashed blue); $A = 1$ in both cases. Right panel: $\epsilon = u - u^*$, $\alpha = 1.5$ (solid red) and $\alpha = 0.5$ (dot-dashed blue); $A = 4$ in both cases

12.2.4 Recommended exercises

It is constructive to sketch the phase flows for the following examples:

$$\begin{aligned} \dot{v} &= -g & \dot{u} &= A\sin(u) \\ m\dot{v} &= -mg - \gamma v & \dot{u} &= A\,(u-a)(u-b)(u-c) \\ m\dot{v} &= -mg - cv^2\,\mathrm{sgn}(v) & \dot{u} &= au^2 - bu^3 \quad . \end{aligned}$$

In each case, identify all the fixed points and assess their stability. Assume all constants A, a, b, c, γ, *etc.* are positive.

12.2.5 Non-autonomous ODEs

Non-autonomous ODEs of the form $\dot{u} = h(u,t)$ are in general impossible to solve by quadratures. One can always go to the computer, but it is worth noting that in the *separable* case, $h(u,t) = f(u)\,g(t)$, one can obtain the solution

$$\frac{du}{f(u)} = g(t)\,dt \quad \Longrightarrow \quad \int\limits_{u_0}^{u} \frac{du'}{f(u')} = \int\limits_0^t dt'\,g(t') \quad , \tag{12.33}$$

which implicitly gives $u(t)$. Note that $\dot{u}$ may now change sign, and $u(t)$ may even oscillate. For an explicit example, consider the equation

$$\dot{u} = A\,(u+1)\,\sin(\beta t) \quad , \tag{12.34}$$

the solution of which is

$$u(t) = -1 + \big(u_0 + 1\big)\,\exp\left\{\frac{A}{\beta}\,\big[1 - \cos(\beta t)\big]\right\} \quad . \tag{12.35}$$

In general, the non-autonomous case defies analytic solution. Many have been studied, such as the Riccati equation,

$$\frac{du}{dt} = P(t)u^2 + Q(t)u + R(t) \quad . \tag{12.36}$$

Riccati equations have the special and remarkable property that one can generate *all* solutions (*i.e.* with arbitrary boundary condition $u(0) = u_0$) from *any* given solution (*i.e.* with any boundary condition).

12.3 Flows on the Circle

We had remarked that oscillations are impossible for the equation $\dot{u} = f(u)$ because the flow is to the first stable fixed point encountered. If there are no stable fixed points, the flow is unbounded. However, suppose phase space itself is bounded, *e.g.* a circle $\mathbb{S}^1$ rather than the real line $\mathbb{R}$. Thus,

$$\dot{\theta} = f(\theta) \quad , \tag{12.37}$$

with $f(\theta + 2\pi) = f(\theta)$. Now if there are no fixed points, $\theta(t)$ endlessly winds around the circle, and in this sense we can have oscillations.

12.3.1 Nonuniform oscillator

A particularly common example is that of the nonuniform oscillator,

$$\dot{\theta} = \omega - \sin\theta \quad , \tag{12.38}$$

which has applications to electronics, biology, classical mechanics, and condensed matter physics. Note that the general equation $\dot{\theta} = \omega - A\sin\theta$ may be rescaled to the above form. A simple application is to the dynamics of a driven, overdamped pendulum. The equation of motion is

$$I\ddot{\theta} + b\,\dot{\theta} + I\omega_0^2 \sin\theta = N \quad , \tag{12.39}$$

where I is the moment of inertia, b is the damping parameter, N is the external torque (presumed constant), and ω_0 is the frequency of small oscillations when $b = N = 0$. When b is large, the inertial term $I\ddot{\theta}$ may be neglected, and after rescaling we arrive at eqn. 12.38.

The book by Strogatz provides a biological example of the nonuniform oscillator: fireflies. An individual firefly will on its own flash at some frequency f. This can be modeled by the equation $\dot{\phi} = \beta$, where $\beta = 2\pi f$ is the angular frequency. A flash occurs when $\phi = 2\pi n$ for $n \in \mathbb{Z}$. When subjected to a periodic stimulus, fireflies will attempt to synchronize their flash to the flash of the stimulus. Suppose the stimulus is periodic with angular frequency Ω. The firefly synchronization is then modeled by the equation

$$\dot{\phi} = \beta - A\sin(\phi - \Omega t) \quad . \tag{12.40}$$

Here, A is a measure of the firefly's ability to modify its natural frequency in response to the stimulus. Note that when $0 < \phi - \Omega t < \pi$, *i.e.* when the firefly is leading the stimulus, the dynamics tell the firefly to slow down. Conversely, when $-\pi < \phi - \Omega t < 0$, the firefly is lagging the stimulus, the the dynamics tell it to speed up. Now focus on the difference $\theta \equiv \phi - \Omega t$. We have

$$\dot{\theta} = \beta - \Omega - A\sin\theta \quad , \tag{12.41}$$

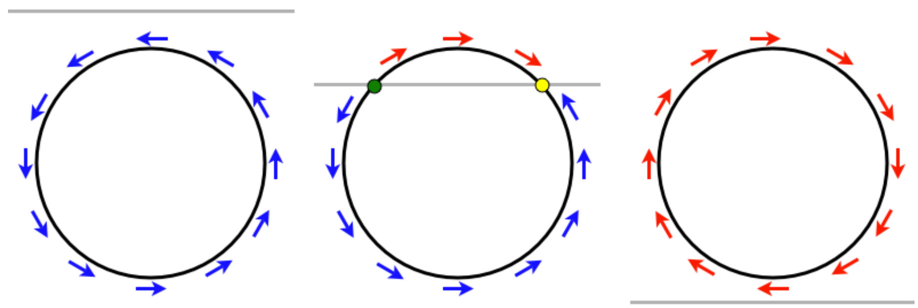


Figure 12.6: Flow for the nonuniform oscillator $\dot{\theta} = \omega - \sin\theta$ for three characteristic values of ω.

which is the nonuniform oscillator. We can adimensionalize by defining

$$s \equiv At \quad , \quad \omega \equiv \frac{\beta - \Omega}{A} \quad , \tag{12.42}$$

yielding $\frac{d\theta}{ds} = f(\theta) = \omega - \sin\theta$.

Fixed points θ^* occur only for $|\omega| < 1$, at $\sin\theta^* = \omega$, in which case $f'(\theta) = -\cos\theta^*$. As we have seen above, stability requires $f'(\theta^*) < 0$, which means $\theta^* \in \big(-\frac{\pi}{2}\,,\,\frac{\pi}{2}\big)$, *i.e.* θ^* must lie on the right half of the circle. For $|\omega| > 1$, the angular velocity never vanishes anywhere along the circle, and there are no fixed points. In this case the motion is eternally clockwise ($\omega < -1$) or counterclockwise ($\omega > +1$). The situation is depicted in Fig. 12.6.

To integrate, set $z = \exp(i\theta)$, in which case

$$\frac{dz}{ds} = -\tfrac{1}{2}(z^2 - 2i\omega z - 1) = -\tfrac{1}{2}(z - \xi_-)(z - \xi_+) \quad , \tag{12.43}$$

where $\xi_\pm = i\omega \pm \sqrt{1-\omega^2} \equiv i\omega \pm \nu$ with $\nu \equiv \sqrt{1-\omega^2}$. This yields

$$d\log\!\left(\frac{z - \xi_+}{z - \xi_-}\right) = \tfrac{1}{2}\big(\xi_- - \xi_+\big)\, ds = -\nu\, ds \tag{12.44}$$

which integrates to

$$z(s) = \frac{(\xi_+ - e^{-\nu s}\,\xi_-)\, z(0) + (e^{-\nu s} - 1)\,\xi_+\xi_-}{(1 - e^{-\nu s})\, z(0) + (\xi_+\, e^{-\nu s} - \xi_-)} \quad . \tag{12.45}$$

When $\omega^2 > 1$, ν is pure imaginary and $\exp(-\nu s)$ continually winds about the unit circle. When $\omega^2 < 1$, ν is real and positive. We then have that $z(s \to \infty) = \xi_+$ while $z(s \to -\infty) = \xi_-$. Note that $\xi_\pm$ lie on the appropriate halves of the circle as depicted in fig. 12.6.

For $|\omega| > 1$, the motion is periodic, with period

$$T = \int_0^{2\pi} \frac{d\theta}{|\omega| - \sin\theta} = \frac{2\pi}{\sqrt{\omega^2 - 1}} \quad . \tag{12.46}$$

12.4 Appendix I: Evolution of Phase Space Volumes

Recall the general form of a dynamical system, $\dot{\boldsymbol{\varphi}} = \boldsymbol{V}(\boldsymbol{\varphi})$. Usually we are interested in finding integral curves $\boldsymbol{\varphi}(t)$. However, consider for the moment a collection of points in phase space comprising a region $\mathcal{R}$. As the dynamical system evolves, this region will also evolve, so that $\mathcal{R} = \mathcal{R}(t)$. We now ask: how does the volume of $\mathcal{R}(t)$,

$$\text{vol}\big[\mathcal{R}(t)\big] = \int\limits_{\mathcal{R}(t)} d\mu \quad , \tag{12.47}$$

where $d\mu = d\varphi_1 d\varphi_2 \cdots d\varphi_N$ is the phase space measure, change with time. We have, explicitly,

$$\begin{aligned} \text{vol}\big[\mathcal{R}(t+dt)\big] &= \int\limits_{\mathcal{R}(t+dt)} d\mu \\ &= \int\limits_{\mathcal{R}(t)} d\mu \left\| \frac{\partial\varphi_i(t+dt)}{\partial\varphi_j(t)} \right\| \\ &= \int\limits_{\mathcal{R}(t)} d\mu \,\Big\{ 1 + \boldsymbol{\nabla}\!\cdot\!\boldsymbol{V}\, dt + \mathcal{O}\big((dt)^2\big) \Big\} \quad , \end{aligned} \tag{12.48}$$

since

$$\frac{\partial\varphi_i(t+dt)}{\partial\varphi_j(t)} = \delta_{ij} + \frac{\partial V_i}{\partial\varphi_j}\bigg|_{\boldsymbol{\varphi}(t)} dt + \mathcal{O}\big((dt)^2\big) \quad , \tag{12.49}$$

and, using $\log \mathsf{det}\, M = \mathsf{Tr} \log M$,

$$\det(1+\epsilon A) = 1 + \epsilon \,\text{Tr} A + \mathcal{O}(\epsilon^2) \quad . \tag{12.50}$$

Thus,

$$\frac{d}{dt}\,\text{vol}\big[\mathcal{R}(t)\big] = \int\limits_{\mathcal{R}(t)} d\mu \,\boldsymbol{\nabla}\!\cdot\!\boldsymbol{V} = \int\limits_{\partial\mathcal{R}(t)} d\Sigma \,\hat{\boldsymbol{n}}\cdot\boldsymbol{V} \quad , \tag{12.51}$$

where in the last line we have used Stokes' theorem to convert the volume integral over $\mathcal{R}$ to a surface integral over its boundary $\partial\mathcal{R}$.

12.5 Appendix II: Lyapunov Characteristic Exponents

Suppose $\boldsymbol{\varphi}(t)$ is an integral curve – *i.e.* a solution of $\dot{\boldsymbol{\varphi}} = \boldsymbol{V}(\boldsymbol{\varphi})$. We now ask: how do nearby trajectories behave? Do they always remain close to $\boldsymbol{\varphi}(t)$ for all t? To answer this, we write $\widetilde{\boldsymbol{\varphi}}(t) \equiv \boldsymbol{\varphi}(t) + \boldsymbol{\eta}(t)$, in which case

$$\frac{d}{dt}\,\eta_i(t) = M_{ij}(t)\,\eta_j(t) + \mathcal{O}\big(\eta^2\big) \quad , \tag{12.52}$$

where

$$M_{ij}(t) = \frac{\partial V_i}{\partial\varphi_j}\bigg|_{\boldsymbol{\varphi}(t)} \quad . \tag{12.53}$$

The solution, valid to first order in $\delta\varphi$, is

$$\eta_i(t) = Q_{ij}(t,t_0)\,\eta_j(t_0) \quad , \tag{12.54}$$

where the matrix $Q(t,t_0)$ is given by the *path ordered exponential*,

$$\begin{aligned} Q(t,t_0) &= \mathcal{P}\exp\left\{ \int\limits_{t_0}^{t} dt'\, M(t') \right\} \\ &\equiv \lim_{N\to\infty} \Big(1 + \frac{\Delta t}{N}\, M(t_{N-1})\Big) \cdots \Big(1 + \frac{\Delta t}{N}\, M(t_1)\Big) \Big(1 + \frac{\Delta t}{N}\, M(t_0)\Big) \quad , \end{aligned} \tag{12.55}$$

with $\Delta t = t - t_0$ and $t_j = t_0 + (j/N)\Delta t$. $\mathcal{P}$ is the *path ordering operator*, which places earlier times to the right:

$$\mathcal{P}\, A(t)\, B(t') = \begin{cases} A(t)\, B(t') & \text{if } t > t' \\ B(t')\, A(t) & \text{if } t < t' \quad . \end{cases} \tag{12.56}$$

The distinction is important if $\big[A(t), B(t')\big] \neq 0$. Note that Q satisfies the composition property,

$$Q(t,t_0) = Q(t,t_1)\, Q(t_1,t_0) \tag{12.57}$$

for any $t_1 \in [t_0, t]$. When M is time-independent, as in the case of a *fixed point* where $\boldsymbol{V}(\boldsymbol{\varphi}^*) = 0$, the path ordered exponential reduces to the ordinary exponential, and $Q(t,t_0) = \exp\big(M(t-t_0)\big)$.

Generally it is impossible to analytically compute path-ordered exponentials. However, the following example may be instructive. Suppose

$$M(t) = \begin{cases} M_1 & \text{if } t/T \in \big[2j, 2j+1\big] \\ \\ M_2 & \text{if } t/T \in \big[2j+1, 2j+2\big] \quad , \end{cases} \tag{12.58}$$

for all integer j. $M(t)$ is a 'matrix-valued square wave', with period $2T$. Then, integrating over one period, from $t=0$ to $t=2T$, we have

$$\begin{aligned} A &\equiv \exp\left\{ \int\limits_0^{2T} dt\, M(t) \right\} = e^{(M_1+M_2)\,T} \\ A_{\mathcal{P}} &\equiv \mathcal{P}\exp\left\{ \int\limits_0^{2T} dt\, M(t) \right\} = e^{M_2\,T}\, e^{M_1\,T} \quad . \end{aligned} \tag{12.59}$$

In general, $A \neq A_{\mathcal{P}}$, so the path ordering has a nontrivial effect[4].

The Lyapunov exponents are defined in the following manner. Let $\hat{e}$ be an N-dimensional unit vector. Define

$$\Lambda(\boldsymbol{\varphi}_0, \hat{\boldsymbol{e}}) \equiv \lim_{t\to\infty} \lim_{b\to 0} \frac{1}{t-t_0}\, \log\left(\frac{\big\|\boldsymbol{\eta}(t)\big\|}{\big\|\boldsymbol{\eta}(t_0)\big\|} \right)_{\boldsymbol{\eta}(t_0) = b\,\hat{\boldsymbol{e}}} \quad , \tag{12.60}$$

[4]If $\big[M_1, M_2\big] = 0$ then $A = A_{\mathcal{P}}$.

where $\|\cdot\|$ denotes the Euclidean norm of a vector, and where $\boldsymbol{\varphi}_0 = \boldsymbol{\varphi}(t_0)$. A theorem due to Oseledec guarantees that there are N such values $\Lambda_i(\boldsymbol{\varphi}_0)$, depending on the choice of $\hat{e}$, for a given $\boldsymbol{\varphi}_0$. Specifically, the theorem guarantees that the matrix

$$W \equiv \big(Q^{\mathrm{t}}Q\big)^{1/(t-t_0)} \tag{12.61}$$

converges in the limit $t \to \infty$ for almost all $\boldsymbol{\varphi}_0$. The eigenvalues Λ_i correspond to the different eigenspaces of W. Oseledec's theorem (also called the 'multiplicative ergodic theorem') guarantees that the eigenspaces of W either grow ($\Lambda_i > 1$) or shrink ($\Lambda_i < 1$) *exponentially* fast. That is, the norm any vector lying in the i^{th} eigenspace of W will behave as $\Lambda_i^t = \exp(t \log \Lambda_i)$ as $t \to \infty$.

Note that while $W = W^{\mathrm{t}}$ is symmetric by construction, Q is simply a general real-valued $N \times N$ matrix. The left and right eigenvectors of a matrix $M \in \mathsf{GL}(N,\mathbb{R})$ will in general be different. The set of eigenvalues λ_α is, however, common to both sets of eigenvectors. Let $\{\psi_\alpha\}$ be the right eigenvectors and $\{\chi^*_\alpha\}$ the left eigenvectors, such that

$$\begin{aligned} M_{ij}\,\psi_{\alpha,j} &= \lambda_\alpha\,\psi_{\alpha,i} \\ \chi^*_{\alpha,i}\,M_{ij} &= \lambda_\alpha\,\chi^*_{\alpha,j} \quad . \end{aligned} \tag{12.62}$$

We can always choose the left and right eigenvectors to be orthonormal, *viz.*

$$\big\langle\, \chi_\alpha \,\big|\, \psi_\beta \,\big\rangle = \chi^*_{\alpha,i}\,\psi_{\beta,j} = \delta_{\alpha\beta} \quad . \tag{12.63}$$

Indeed, we can define the matrix $S_{i\alpha} = \psi_{\alpha,i}$, in which case $S^{-1}_{\alpha j} = \chi^*_{\alpha,j}$, and

$$S^{-1}\,M\,S = \mathrm{diag}\big(\lambda_1, \ldots, \lambda_N\big) \quad . \tag{12.64}$$

The matrix M can always be decomposed into its eigenvectors, as

$$M_{ij} = \sum_\alpha \lambda_\alpha\,\psi_{\alpha,i}\,\chi^*_{\alpha,j} \quad . \tag{12.65}$$

If we expand $\boldsymbol{u}$ in terms of the right eigenvectors,

$$\boldsymbol{\eta}(t) = \sum_\beta C_\beta(t)\,\boldsymbol{\psi}_\beta(t) \quad , \tag{12.66}$$

then upon taking the inner product with $\boldsymbol{\chi}_\alpha$, we find that C_α obeys

$$\dot{C}_\alpha + \big\langle\, \chi_\alpha \,\big|\, \dot{\psi}_\beta \,\big\rangle\, C_\beta = \lambda_\alpha\, C_\alpha \quad . \tag{12.67}$$

If $\dot{\boldsymbol{\psi}}_\beta = 0$, *e.g.* if M is time-independent, then $C_\alpha(t) = C_\alpha(0)\, e^{\lambda_\alpha t}$, and

$$\eta_i(t) = \sum_\alpha \overbrace{\sum_j \eta_j(0)\,\chi^*_{\alpha,j}}^{C_\alpha(0)}\; e^{\lambda_\alpha t}\,\psi_{\alpha,i} \quad . \tag{12.68}$$

Thus, the component of $\boldsymbol{\eta}(t)$ along $\boldsymbol{\psi}_\alpha$ increases exponentially with time if $\mathrm{Re}(\lambda_\alpha) > 0$, and decreases exponentially if $\mathrm{Re}(\lambda_\alpha) < 0$.

Nota bene: If $M \in \mathsf{GL}(N,\mathbb{R})$ is not symmetric – more generally if it does not commute with its transpose M^{t} – then it may be that the right eigenvectors of M do not span $\mathbb{R}^N$. In this case, the canonical decomposition of M contains one or more *Jordan blocks*. See §12.6 below.

12.6 Appendix III: Normal Matrices, Non-Normal Matrices, and Jordan Blocks

Normal matrices and eigenspectra : Quantum mechanical Hamiltonians can be represented as Hermitian matrices. In elementary school linear algebra class, we all learned that any Hermitian matrix H is diagonalizable by a unitary transformation, its eigenvalues are real, and eigenvectors corresponding to different eigenvalues are necessarily orthogonal. In the case of degenerate eigenvalues, their associated eigenvectors may be chosen to be mutually orthogonal via the Gram-Schmidt process. In the following discussion, we will assume our matrices are in general complex, but we can of course restrict to the real case, as is appropriate for real linear dynamical systems.

Any complex square matrix A which satisfies $A^\dagger A = AA^\dagger$ is called *normal*. Hermitian matrices are normal, but so are antihermitian and unitary matrices[5]. Real symmetric, antisymmetric, and orthogonal matrices satisfy $A^{\mathrm{t}}A = AA^{\mathrm{t}}$. The *Schur decomposition theorem* guarantees that any $n \times n$ matrix A may be decomposed as $A = VTV^\dagger$, where $V \in \mathsf{U}(n)$ and T is upper triangular. Now if A is normal, $[A, A^\dagger] = V\,[T, T^\dagger]\,V^\dagger = 0$, hence T is normal. However, it is easy to show that any normal upper triangular matrix must be diagonal[6], so $A = VDV^\dagger$, which means $D = V^\dagger AV$ is the diagonal matrix of eigenvalues of A. Conversely, if $A = VDV^\dagger$ is unitarily equivalent to a diagonal matrix, it is trivial to show that A is normal. Thus *any $n \times n$ matrix A is diagonalizable by a unitary transformation if and only if A is normal.*

There is a real version of Schur decomposition whereby a real matrix B satisfying $B^{\mathrm{t}}B = BB^{\mathrm{t}}$ may be decomposed as $B = RSR^{\mathrm{t}}$, where R is a real orthogonal matrix, and S is *block upper triangular*. The diagonal blocks of S are either 1×1, corresponding to real eigenvalues, or 2×2, corresponding to complex eigenvalues. One eventually concludes that real symmetric matrices have real eigenvalues, real antisymmetric matrices have pure imaginary (or zero) eigenvalues, and real orthogonal matrices have unimodular complex eigenvalues.

Now let's set $A = VDV^\dagger$ and consider different classes of matrix A. If A is Hermitian, $A = A^\dagger$ immediately yields $D = D^\dagger$, which says that all the eigenvalues of A must be real. If $A^\dagger = -A$, then $D^\dagger = -D$ and all the eigenvalues are purely imaginary. And if $A^\dagger = A^{-1}$, then $D^\dagger = D^{-1}$ and we conclude that all the eigenvalues are unimodular, *i.e.* of the form $e^{i\omega_j}$. This analysis also tells us that any unitary matrix U can be written in the form $U = \exp(iH)$ for some Hermitian matrix H.

Jordan blocks : What happens when an $n \times n$ matrix A is not normal? In this case A is not diagonalizable by a unitary transformation, and while the sum of the dimensions of its eigenspaces is generically equal to the matrix dimension $\mathsf{dim}(A) = n$, this is not guaranteed; it may be less than n. For example, consider the matrix

$$A = \begin{pmatrix} r & 1 \\ 0 & r \end{pmatrix} \quad . \tag{12.69}$$

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[5]There are many examples of normal matrices which are neither Hermitian, antihermitian, nor unitary. For example, any diagonal matrix with arbitrary complex diagonal entries is normal.

[6]$T^\dagger T = TT^\dagger$ says that $\sum_j |T_{ij}|^2 = \sum_j |T_{ji}|^2$, *i.e.* the sum of the square moduli of the elements in the i^{th} row is the same as that for the i^{th} column. Starting with $i = 1$, the only possible nonzero entry in the first column is $T_{1,1}$, hence all the remaining entries in the first row must vanish. Filling in all these zeros, proceed to $i = 2$. Since we just showed $T_{1,2} = 0$, we conclude that the only possible nonzero entry in the second column is $T_{2,2}$, hence all remaining entries in the second row must vanish. Continuing in this manner, we conclude that T is diagonal if it is both normal and upper triangular.

The eigenvalues are solutions to $\text{det}(\lambda I - A) = 0$, hence $\lambda = r$, but there is only one eigenvector, $\psi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. What is always true for any complex matrix A is that it can be brought to *Jordan canonical form* by a similarity transformation $J = P^{-1}AP$, where P is invertible, and

$$J = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_b \end{pmatrix} \quad , \tag{12.70}$$

where b is the number of *Jordan blocks* and where each block J_s is of the form

$$J_s = \begin{pmatrix} \lambda_s & 1 & & \\ & \lambda_s & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_s \end{pmatrix} \quad . \tag{12.71}$$

Thus each J_s is tridiagonal, with diagonal elements all given by λ_s and each element directly above the diagonal equal to one. Clearly J_s has only one eigenvalue, λ_s, and writing the corresponding *right* eigenvector as $\vec{\psi}$, the condition $J_s\vec{\psi} = \lambda_s\vec{\psi}$ yields the equations

$$\lambda_s\,\psi_1 + \psi_2 = \lambda_s\,\psi_1 \quad , \quad \lambda_s\,\psi_2 + \psi_3 = \lambda_s\,\psi_2 \quad \ldots \quad \lambda_s\,\psi_{n-1} + \psi_{n_s} = \lambda_s\,\psi_{n_s-1} \quad , \tag{12.72}$$

where $n_s = \text{dim}(J_s)$. These equations entail $\psi_2 = \psi_3 = \cdots = \psi_{n_s} = 0$, which says that there is only one such eigenvector, whose components are $\psi_j = \delta_{j,1}$. Note that the corresponding *left* eigenvector $\vec{\chi}^{\,\text{t}}$ is then $\chi_j = \delta_{j,n_s}$. If $n_s > 1$ we then have $\vec{\chi} \cdot \vec{\psi} = 0$, which means that the left and right eigenvectors of A which correspond to the Jordan blocks with $n_s > 1$ are orthogonal. *Nota bene* : It may be the case that there are degeneracies among the eigenvalues $\{\lambda_s\}$.

To summarize[7], for every general complex $n \times n$ matrix A,

- A may be brought to Jordan canonical form by a similarity transformation $J = P^{-1}AP$, where $J = \text{bdiag}(J_1, \ldots, J_b)$ is block diagonal, with each $(J_s)_{ij} = \lambda_s\,\delta_{ij} + \delta_{i,j-1}$ for $s \in \{1, \ldots, b\}$.

- There are $b \le n$ eigenvalues $\{\lambda_1, \ldots, \lambda_b\}$ (again, not necessarily all distinct) and b corresponding eigenvectors $\{\vec{\psi}_1, \ldots, \vec{\psi}_b\}$. If $b = n$ then the matrix is diagonalizable.

- The dimension n of the matrix A satisfies $n = n_1 + \ldots + n_b$, *i.e.* it is the sum of the dimensions of all its Jordan blocks.

- Let $\lambda \in \{\lambda_1, \ldots, \lambda_b\}$ be an eigenvalue, and define

  $$\text{t}_k(\lambda) = \text{dim ker}(A - \lambda)^k \quad , \tag{12.73}$$

  which is the *dimension of the null space* of the matrix $A - \lambda I$. Then

  ◇ $t_k(\lambda)$ is the number of Jordan blocks corresponding to the eigenvalue λ.

[7]See https://en.wikipedia.org/wiki/Jordan_normal_form.

- ⋄ The number of Jordan blocks of size greater than k is $\mathsf{t}_{k+1}(\lambda) - \mathsf{t}_k(\lambda)$. Thus the number of Jordan blocks of size k for the eigenvalue λ is

$$\mathsf{N}_k(\lambda) = 2\,\mathsf{t}_k(\lambda) - \mathsf{t}_{k+1}(\lambda) - \mathsf{t}_{k-1}(\lambda) \quad . \tag{12.74}$$

Singular value decomposition : Note the difference between the decomposition into Jordan canonical form and singular value decomposition (SVD), in which we write an $m \times n$ matrix A as $A = US\,V^\dagger$, where U is $m \times k$, V is $n \times k$ (hence $V^\dagger$ is $k \times n$), $U^\dagger U = V^\dagger V = \mathbb{I}_{k\times k}$, and $S = \mathsf{diag}(s_1, \ldots, s_k)$ is $k \times k$ with $k \leq \mathsf{min}(m, n)$ and each $s_j > 0$. The elements s_j are the singular values and the rows of U and V are the singular vectors. Note that $A^\dagger A = VS^2\,V^\dagger$ is $n \times n$ and $AA^\dagger = US^2\,U^\dagger$ is $m \times m$. If we define

$$R(\lambda) = \prod_{j=1}^{k} \big(\lambda - s_j^2\big) \quad , \tag{12.75}$$

Then

$$P(\lambda) \equiv \mathsf{det}(\lambda - A^\dagger A) = \lambda^{n-k} R(\lambda) \qquad , \qquad Q(\lambda) \equiv \mathsf{det}(\lambda - AA^\dagger) = \lambda^{m-k} R(\lambda) \quad . \tag{12.76}$$

For any square $n \times n$ complex matrix A we therefore have two decompositions, via JCF and SVD, *viz.*

$$A = PJ\,P^{-1} = US\,V^\dagger \quad , \tag{12.77}$$

where J is the Jordan canonical form of A. When A is normal, $k = n$ and $U = V = P$, *i.e.* the two decompositions are equivalent.

Chapter 13

Bifurcations

13.1 Types of Bifurcations

13.1.1 Saddle-node bifurcation

We remarked above how $f'(u)$ is in general nonzero when $f(u)$ itself vanishes, since two equations in a single unknown is an overdetermined set. However, consider the function $F(x,\alpha)$, where α is a control parameter. If we demand $F(x,\alpha) = 0$ and $\partial_x F(x,\alpha) = 0$, we have two equations in two unknowns, and in general there will be a zero-dimensional solution set consisting of points (x_c, α_c). The situation is depicted in Fig. 13.1.

Let's expand $F(x,\alpha)$ in the vicinity of such a point (x_c, α_c):

$$\begin{aligned} F(x,\alpha) = F(x_c,\alpha_c) &+ \frac{\partial F}{\partial x}\bigg|_{(x_c,\alpha_c)} (x - x_c) + \frac{\partial F}{\partial \alpha}\bigg|_{(x_c,\alpha_c)} (\alpha - \alpha_c) + \frac{1}{2}\,\frac{\partial^2 F}{\partial x^2}\bigg|_{(x_c,\alpha_c)} (x - x_c)^2 \\ &+ \frac{\partial^2 F}{\partial x\,\partial\alpha}\bigg|_{(x_c,\alpha_c)} (x - x_c)\,(\alpha - \alpha_c) + \frac{1}{2}\,\frac{\partial^2 F}{\partial \alpha^2}\bigg|_{(x_c,\alpha_c)} (\alpha - \alpha_c)^2 + \ldots \\ = A\,(\alpha - \alpha_c) &+ B\,(x - x_c)^2 + \ldots \quad , \end{aligned} \tag{13.1}$$

where we keep terms of lowest order in the deviations δx and $\delta\alpha$. Note that we can separately change the signs of A and B by redefining $\alpha \to -\alpha$ and/or $x \to -x$, so without loss of generality we may assume both A and B are positive. If we now rescale $u \equiv \sqrt{B/A}\,(x - x_c)$, $r \equiv \alpha - \alpha_c$, and $\tau = \sqrt{AB}\,t$, we have, neglecting the higher order terms, we obtain the 'normal form' of the saddle-node bifurcation,

$$\frac{du}{d\tau} = r + u^2 \quad . \tag{13.2}$$

The evolution of the flow is depicted in Fig. 13.2. For $r < 0$ there are two fixed points – one stable ($u^* = -\sqrt{-r}$) and one unstable ($u = +\sqrt{-r}$). At $r = 0$ these two nodes coalesce and annihilate each other. (The point $u^* = 0$ is half-stable precisely at $r = 0$.) For $r > 0$ there are no longer any fixed points

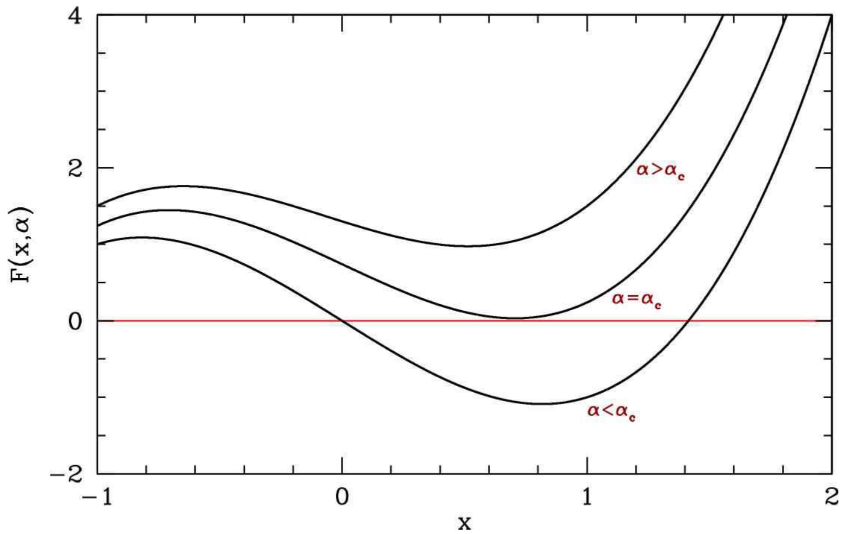


Figure 13.1: Evolution of $F(x,\alpha)$ as a function of the control parameter α.

in the vicinity of $u = 0$. In the left panel of Fig. 13.3 we show the flow in the extended (r, u) plane. The unstable and stable nodes annihilate at $r = 0$.

13.1.2 Transcritical bifurcation

Another situation which arises frequently is the *transcritical bifurcation*. Consider the equation $\dot{x} = f(x)$ in the vicinity of a fixed point x^*.

$$\frac{dx}{dt} = f'(x^*)\,(x - x^*) + \tfrac{1}{2}\,f''(x^*)(x - x^*)^2 + \ldots \quad . \tag{13.3}$$

We rescale $u \equiv \beta\,(x - x^*)$ with $\beta = -\frac{1}{2}\,f''(x^*)$ and define $r \equiv f'(x^*)$ as the control parameter, to obtain, to order u^2,

$$\frac{du}{dt} = ru - u^2 \quad . \tag{13.4}$$

Note that the sign of the u^2 term can be reversed relative to the others by sending $u \to -u$.

Consider a crude model of a laser threshold. Let n be the number of photons in the laser cavity, and N the number of excited atoms in the cavity. The dynamics of the laser are approximated by the equations

$$\begin{aligned} \dot{n} &= GNn - kn \\ N &= N_0 - \alpha n \quad . \end{aligned} \tag{13.5}$$

Here G is the gain coefficient and k the photon decay rate. N_0 is the pump strength, and α is a numerical factor. The first equation tells us that the number of photons in the cavity grows with a rate $GN - k$; gain is proportional to the number of excited atoms, and the loss rate is a constant cavity-dependent quantity (typically through the ends, which are semi-transparent). The second equation says that the number of

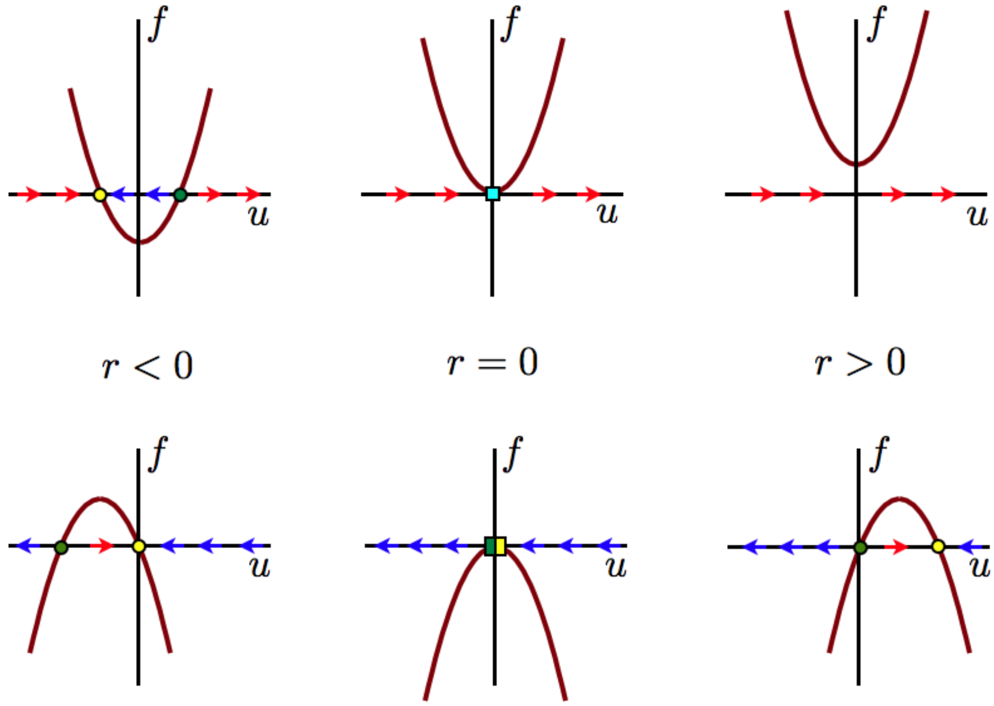


Figure 13.2: Flow diagrams for the saddle-node bifurcation $\dot{u} = r + u^2$ (top) and the transcritical bifurcation $\dot{u} = ru - u^2$ (bottom).

excited atoms is equal to the pump strength minus a term proportional to the number of photons (since the presence of a photon means an excited atom has decayed). Putting them together,

$$\dot{n} = (GN_0 - k)\, n - \alpha G n^2 \quad , \tag{13.6}$$

which exhibits a transcritical bifurcation at pump strength $N_0 = k/G$. For $N_0 < k/G$ the system acts as a lamp; for $N_0 > k/G$ the system acts as a laser.

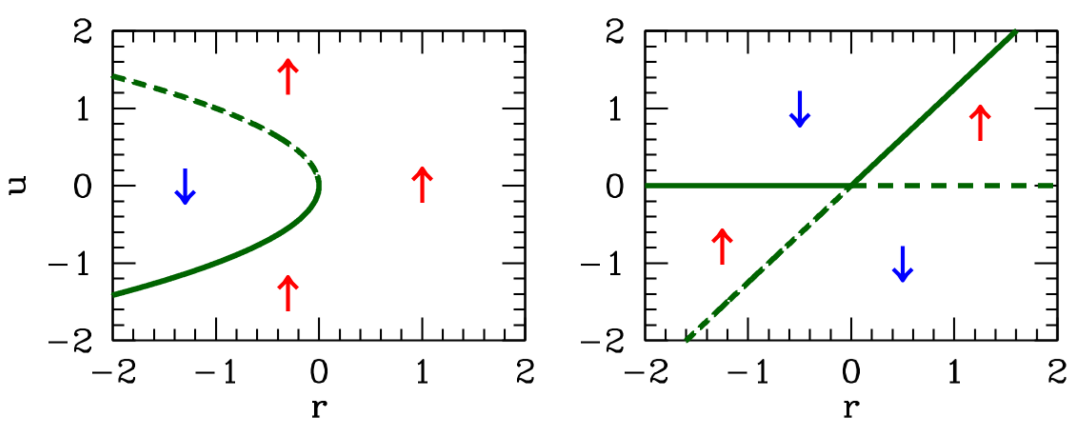


Figure 13.3: Extended phase space (r, u) flow diagrams for the saddle-node bifurcation $\dot{u} = r + u^2$ (left) and the transcritical bifurcation $\dot{u} = ru - u^2$ (right).

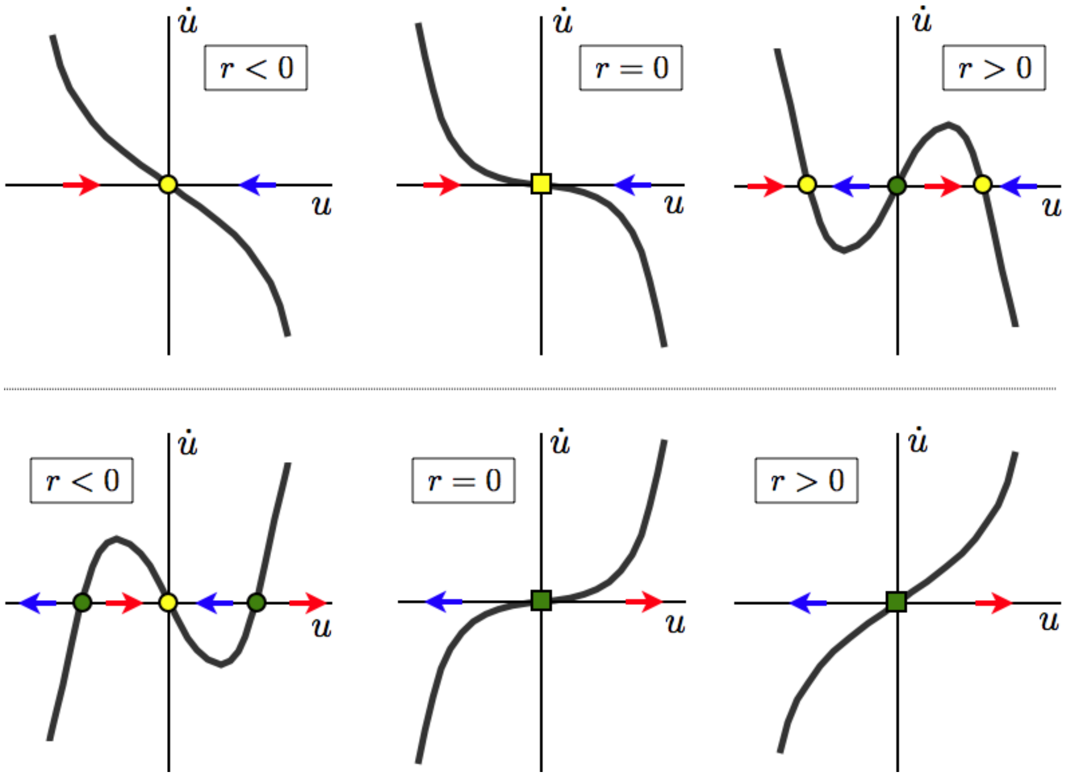


Figure 13.4: Top: supercritical pitchfork bifurcation $\dot{u} = ru - u^3$. Bottom: subcritical pitchfork bifurcation $\dot{u} = ru + u^3$.

What happens in the transcritical bifurcation is an exchange of stability of the fixed points at $u^* = 0$ and $u^* = r$ as r passes through zero. This is depicted graphically in the bottom panel of Fig. 13.2.

13.1.3 Pitchfork bifurcation

The pitchfork bifurcation is commonly encountered in systems in which there is an overall parity symmetry ($u \to -u$). There are two classes of pitchfork: supercritical and subcritical. The normal form of the supercritical bifurcation is

$$\dot{u} = ru - u^3 \quad , \tag{13.7}$$

which has fixed points at $u^* = 0$ and $u^* = \pm\sqrt{r}$. Thus, the situation is as depicted in fig. 13.4 (top panel). For $r < 0$ there is a single stable fixed point at $u^* = 0$. For $r > 0$, $u^* = 0$ is unstable, and flanked by two stable fixed points at $u^* = \pm\sqrt{r}$.

If we send $u \to -u$, $r \to -r$, and $t \to -t$, we obtain the *subcritical pitchfork*, depicted in the bottom panel of fig. 13.4. The normal form of the subcritical pitchfork bifurcation is

$$\dot{u} = ru + u^3 \quad . \tag{13.8}$$

The fixed point structure in both supercritical and subcritical cases is shown in Fig. 13.5.

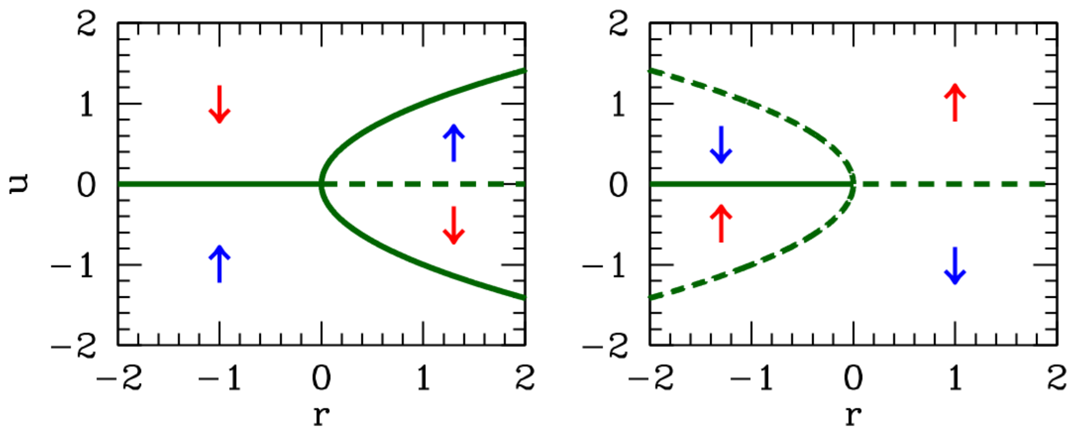


Figure 13.5: Extended phase space (r, u) flow diagrams for the supercritical pitchfork bifurcation $\dot{u} = ru - u^3$ (left), and subcritical pitchfork bifurcation $\dot{u} = ru + u^3$ (right).

13.1.4 Imperfect bifurcation

The imperfect bifurcation occurs when a symmetry-breaking term is added to the pitchfork. The normal form contains two control parameters:

$$\dot{u} = h + ru - u^3 \quad . \tag{13.9}$$

Here, the constant h breaks the parity symmetry if $u \to -u$.

This equation arises from a crude model of magnetization dynamics. Let M be the magnetization of a sample, and $F(M)$ the free energy. Assuming M is small, we can expand $F(M)$ as

$$F(M) = -HM + \tfrac{1}{2}aM^2 + \tfrac{1}{4}bM^4 + \ldots \quad , \tag{13.10}$$

where H is the external magnetic field, and a and b are temperature-dependent constants. This is called the *Landau expansion* of the free energy. We assume $b > 0$ in order that the minimum of $F(M)$ not lie at infinity. The dynamics of $M(t)$ are modeled by

$$\frac{dM}{dt} = -\Gamma \, \frac{\partial F}{\partial M} \quad , \tag{13.11}$$

with $\Gamma > 0$. Thus, the magnetization evolves toward a local minimum in the free energy. Note that the free energy is a decreasing function of time:

$$\frac{dF}{dt} = \frac{\partial F}{\partial M} \, \frac{dM}{dt} = -\Gamma \left(\frac{\partial F}{\partial M}\right)^2 \quad . \tag{13.12}$$

By rescaling $M \equiv uM_0$ with $M_0 = (b\,\Gamma)^{-1/2}$ and defining $r \equiv -a\Gamma$ and $h \equiv (\Gamma^3\, b)^{1/2}\, H$, we obtain the normal form

$$\begin{aligned} \dot{u} &= h + ru - u^3 = -\frac{\partial f}{\partial u} \\ f(u) &= -\tfrac{1}{2}ru^2 + \tfrac{1}{4}u^4 - hu \quad . \end{aligned} \tag{13.13}$$

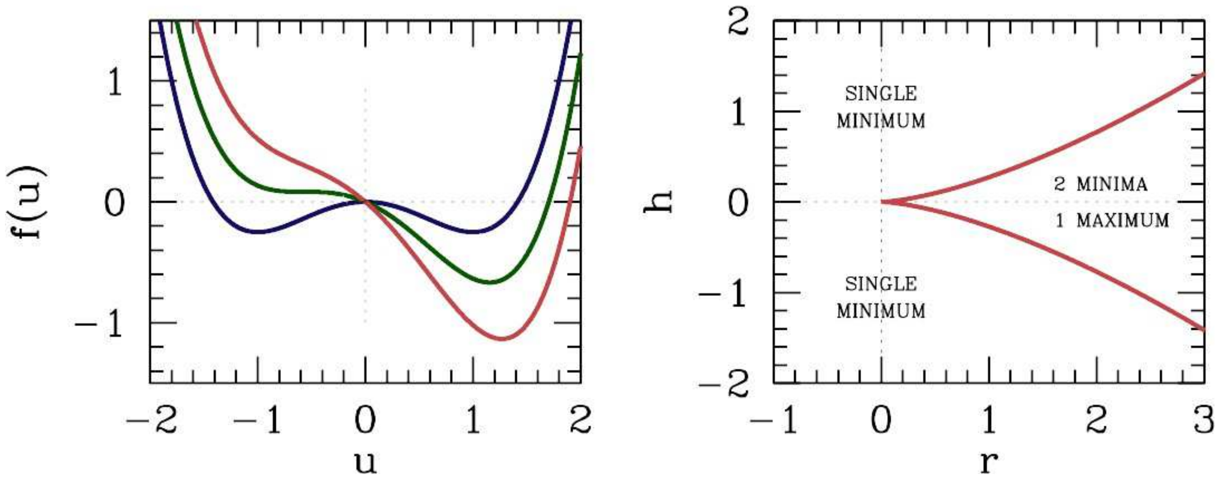


Figure 13.6: Left: scaled free energy $f(u) = -\frac{1}{2}ru^2 + \frac{1}{4}u^4 - hu$, with $h = 0$ (blue), $h = h_c$ (green), and $h = 2\,h_c$ (red), where $h_c = \frac{2}{3\sqrt{3}}\,r^{3/2}$. Right: phase diagram for the imperfect bifurcation $\dot{u} = -f'(u) = h + ru - u^3$ in the (r, h) plane.

Here, $f(u)$ is a scaled version of the free energy.

Fixed points satisfy the equation

$$u^3 - ru - h = 0 \quad , \tag{13.14}$$

and correspond to extrema in $f(u)$. By the fundamental theorem of algebra, this cubic polynomial may be uniquely factorized over the complex plane. Since the coefficients are real, the complex conjugate $\bar{u}$ satisfies the same equation as u, hence there are two possibilities for the roots: either (i) all three roots are real, or (ii) one root is real and the other two are a complex conjugate pair. Clearly for $r < 0$ we are in situation (ii) since $u^3 - ru$ is then monotonically increasing for $u \in \mathbb{R}$, and therefore takes the value h precisely once for u real. For $r > 0$, there is a region $h \in \big[-h_c(r), h_c(r)\big]$ over which there are three real roots. To find $h_c(r)$, we demand $f''(u) = 0$ as well as $f'(u) = 0$, which says that two roots have merged in a saddle-node bifurcation, forming an inflection point. Thus $f''(u) = 3u^2 - r = 0$ yields $u = \pm(r/3)^{1/2}$, which requires $r > 0$, and using this to eliminate u from the equation $f(u) = 0$ yields the critical value of h as a function of r, *viz.* $h_c(r) = \frac{2}{3\sqrt{3}}\,r^{3/2}\,\Theta(r)$.

Examples of the function $f(u)$ for $r > 0$ are shown in the left panel of Fig. 13.6 for three different values of h. For $|h| < h_c(r)$ there are three extrema satisfying $f'(u^*) = 0$: $u_1^* < u_2^* < 0 < u_3^*$, assuming (without loss of generality) that $h > 0$. Clearly u_1^* is a local minimum, u_2^* a local maximum, and u_3^* the global minimum of the function $f(u)$. The 'phase diagram' for this system, plotted in the (r, h) control parameter space, is shown in the right panel of Fig. 13.6.

In Fig. 13.7 we plot the fixed points $u^*(r)$ for fixed h. A saddle-node bifurcation occurs at $r = r_c(h) = \frac{3}{2^{2/3}}\,|h|^{2/3}$. For $h = 0$ this reduces to the supercritical pitchfork; for finite h the pitchfork is deformed and even changed topologically. Finally, in Fig. 13.7 we show the behavior of $u^*(h)$ for fixed r. When $r < 0$ the curve retraces itself as h is ramped up and down, but for $r > 0$ the system exhibits the phenomenon of *hysteresis*, *i.e.* there is an irreversible aspect to the behavior. Fig, 13.7 shows a *hysteresis loop* when $r > 0$.

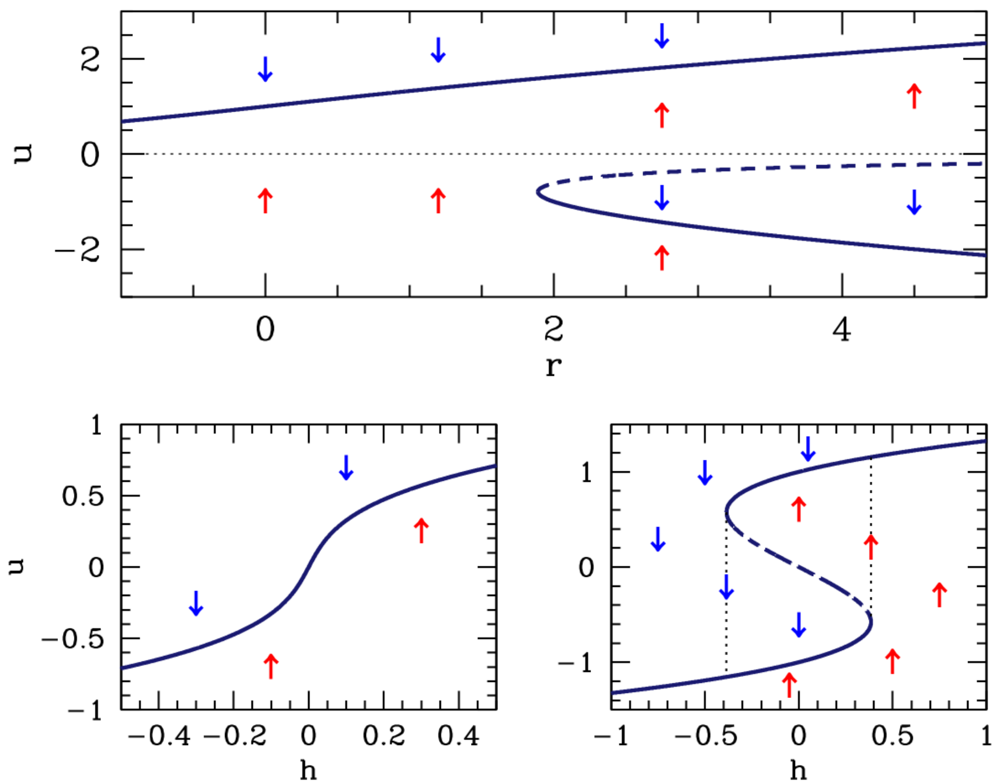


Figure 13.7: Top: extended phase space (r, u) flow diagram for $\dot{u} = h + ru - u^3$, the imperfect pitchfork bifurcation, at $h = 1$. This is in a sense a deformed supercritical pitchfork. The saddle-node bifurcation occurs at $r_c = (3/2^{2/3})|h|^{2/3} = 1.8899$. Bottom: extended phase space (h, u) flow diagram for the imperfect pitchfork bifurcation $r = -0.2$ (left panel) and $r = 1$ (right panel). For $r < 0$ the behavior is completely reversible. For $r > 0$, a regime of irreversibility sets in between $-h_c$ and $+h_c$, where $h_c = 2(r/3)^{3/2}$. The system then exhibits the phenomenon of hysteresis. The dotted vertical lines show the boundaries of the hysteresis loop.

13.2 Examples

13.2.1 Population dynamics

Consider the dynamics of a harvested population,

$$\dot{N} = rN\left(1 - \frac{N}{K}\right) - H(N) \quad , \tag{13.15}$$

where $r, K > 0$, and where $H(N)$ is the *harvesting rate*.

(a) Suppose $H(N) = H_0$ is a constant. Sketch the phase flow, and identify and classify all fixed points.

Solution: We examing $\dot{N} = f(N)$ with

$$f(N) = rN - \frac{r}{K}N^2 - H_0 \quad . \tag{13.16}$$

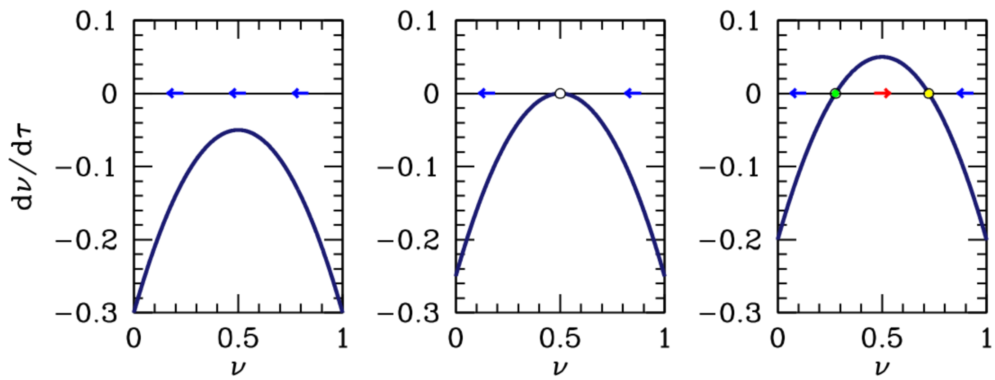


Figure 13.8: Phase flow for the constantly harvested population, $\dot{\nu} = \nu(1-\nu) - h$, for $h = 0.30$ (left), $h = 0.25$ (center), and $h = 0.20$ (right). The critical harvesting rate is $h_c = \frac{1}{4}$.

Setting $f'(N) = 0$ yields $N = \frac{1}{2}K$. $f(N)$ is a downward-opening parabola whose maximum value is $f\big(\frac{1}{2}K\big) = \frac{1}{4}rK - H_0$. Thus, if $H_0 > \frac{1}{4}rK$, the harvesting rate is too large and the population always shrinks. A saddle-node bifurcation occurs at this value of H_0, and for larger harvesting rates, there are fixed points at

$$N_\pm = \tfrac{1}{2}K \pm \tfrac{1}{2}K\sqrt{1 - \frac{4H_0}{rK}} \quad , \tag{13.17}$$

with N_- unstable and N_+ stable. By rescaling the population $\nu = N/K$, time $\tau = rt$ and harvesting rate $h = H_0/rK$, we arrive at the equation

$$\dot{\nu} = \nu(1-\nu) - h \quad . \tag{13.18}$$

The critical harvesting rate is then $h_c = \frac{1}{4}$. See fig. 13.8.

(b) One defect of the constant harvesting rate model is that $N = 0$ is not a fixed point. To remedy this, consider the following model for $H(N)$[1]:

$$H(N) = \frac{B\,N^2}{N^2 + A^2} \quad , \tag{13.19}$$

where A and B are (positive) constants. Show that one can rescale (N,t) to (n,τ), such that

$$\frac{dn}{d\tau} = \gamma\, n\left(1 - \frac{n}{c}\right) - \frac{n^2}{n^2+1} \quad , \tag{13.20}$$

where γ and c are positive constants. Provide expressions for n, τ, γ, and c.

Solution: Examining the denominator of $H(N)$, we must take $N = An$. Dividing both sides of $\dot{N} = f(N)$ by B, we obtain

$$\frac{A}{B}\frac{dN}{dt} = \frac{rA}{B}\,n\Big(1 - \frac{A}{K}\,n\Big) - \frac{n^2}{n^2+1} \quad , \tag{13.21}$$

[1]This is a model for the dynamics of the spruce budworm population, taken from ch. 1 of J. D. Murray, *Mathematical Biology* (2^{nd} edition, Springer, 1993).

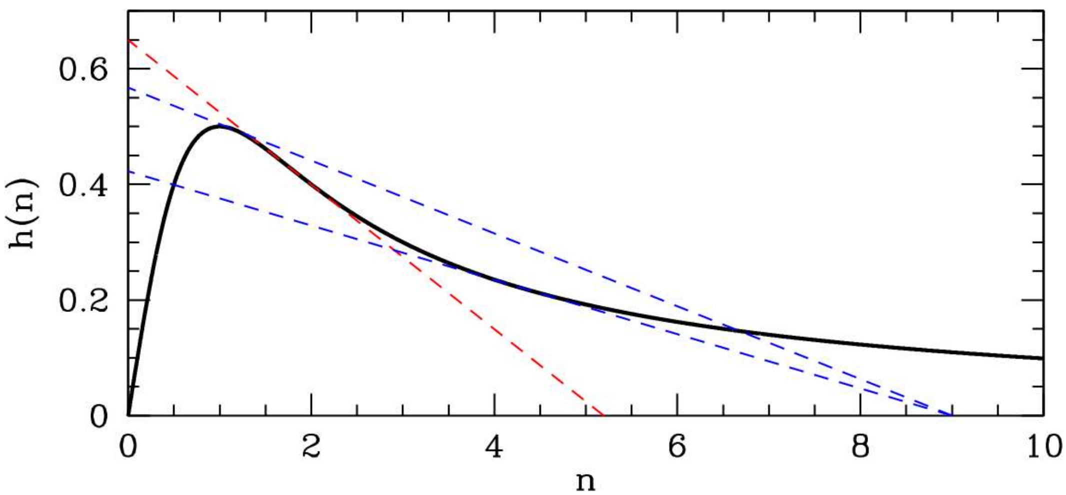


Figure 13.9: Plot of $h(n) = n/(n^2+1)$ (thick black curve). Straight lines show the function $y(n) = \gamma\big(1-\frac{n}{c}\big)$ for different values of c and γ. The red line is tangent to the inflection point of $h(n)$ and determines the minimum value $c^* = 3\sqrt{3}$ for a bifurcation. The blue lines show the construction for determining the location of the two bifurcations for $c > c^*$ (in this case, $c = 9$). See the analysis in the text.

from which we glean $\tau = Bt/A$, $\gamma = rA/B$, and $c = K/A$.

`(c)` Show that for c sufficiently small that there is a unique asymptotic ($\tau \to \infty$) value for the (scaled) population n, for any given value of γ. Thus, there are no bifurcations as a function of the control parameter γ for c fixed and $c < c^*$.

`(d)` Show that for $c > c^*$, there are two bifurcations as a function of γ, and that for $\gamma_1^* < \gamma < \gamma_2^*$ the asymptotic solution is bistable, *i.e.* there are two stable values for $n(\tau \to \infty)$. Sketch the solution set 'phase diagram' in the (c, γ) plane. *Hint: Sketch the functions* $\gamma(1-n/c)$ *and* $n/(n^2+1)$. *The* $n \neq 0$ *fixed points are given by the intersections of these two curves. Determine the boundary of the bistable region in the* (c, γ) *plane parametrically in terms of* n. Find c^* and $\gamma_1^*(c) = \gamma_2^*(c)$.

Solution `(c)` *and* `(d)` : We examine

$$\frac{dn}{d\tau} = g(n) = \left\{\gamma\Big(1 - \frac{n}{c}\Big) - \frac{n}{n^2+1}\right\} n \quad . \tag{13.22}$$

There is an unstable fixed point at $n = 0$, where $g'(0) = \gamma > 0$. The other fixed points occur when the term in the curvy brackets vanishes. In fig. 13.9 we plot the function $h(n) \equiv n/(n^2+1)$ *versus* n. We seek the intersection of this function with a two-parameter family of straight lines, given by $y(n) = \gamma\,(1-n/c)$. The n-intercept is c and the y-intercept is γ. Provided $c > c^*$ is large enough, there are two bifurcations as a function of γ, which we call $\gamma_\pm(c)$. These are shown as the dashed blue lines in figure 13.9 for $c = 9$.

Both bifurcations are of the saddle-node type. We determine the curves $\gamma_\pm(c)$ by requiring that $h(n)$ is

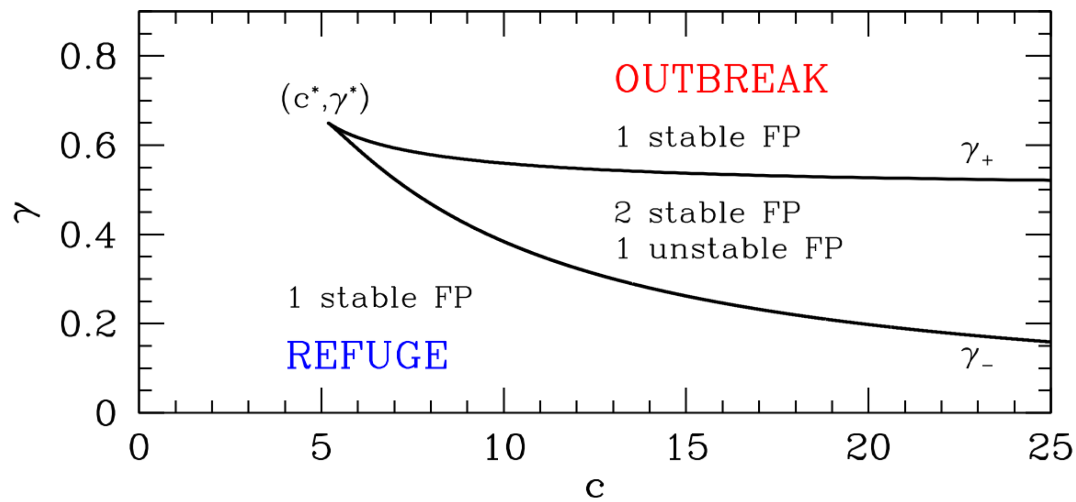


Figure 13.10: Phase diagram for the equation $\dot{n} = \gamma(1-n/c)\,n - n^2/(n^2+1)$, labeling $n \neq 0$ fixed points. (The point $n=0$ is always unstable.)

tangent to $y(n)$, which gives two equations:

$$\begin{aligned} h(n) &= \frac{n}{n^2+1} = \gamma\Big(1 - \frac{n}{c}\Big) = y(n) \\ h'(n) &= \frac{1-n^2}{(n^2+1)^2} = -\frac{\gamma}{c} = y'(n) \quad . \end{aligned} \tag{13.23}$$

Together, these give $\gamma(c)$ parametrically, *i.e.* as $\gamma(n)$ and $c(n)$:

$$\gamma(n) = \frac{2n^3}{(n^2+1)^2} \qquad , \qquad c(n) = \frac{2n^3}{(n^2-1)} \quad . \tag{13.24}$$

Since $h(n)$ is maximized for $n=1$, where $h(1) = \frac{1}{2}$, there is no bifurcation occurring at values $n<1$. If we plot $\gamma(n)$ *versus* $c(n)$ over the allowed range of n, we obtain the phase diagram in fig. 13.10. The cusp occurs at (c^*, γ^*), and is determined by the requirement that the two bifurcations coincide. This supplies a third condition, namely that $h'(n) = 0$, where

$$h''(n) = \frac{2n\,(n^2-3)}{(n^2+1)^3} \quad . \tag{13.25}$$

Hence $n = \sqrt{3}$, which gives $c^* = 3\sqrt{3}$ and $\gamma^* = \frac{3\sqrt{3}}{8}$. For $c < c^*$, there are no bifurcations at any value of γ.

13.3 Appendix I: The Bletch

Problem: The bletch is a disgusting animal native to the Forest of Jkroo on the planet Barney. The bletch population obeys the equation

$$\frac{dN}{dt} = aN^2 - bN^3 \quad , \tag{13.26}$$

Figure 13.11: Phase flow for the scaled bletch population, $\dot{n} = n^2 - n^3$.

where N is the number of bletches, and a and b are constants. (Bletches reproduce asexually, but only when another bletch is watching. However, when there are three bletches around, they beat the `@!!*$&*` out of each other.)

(a) Sketch the phase flow for N. (Strange as the bletch is, you can still rule out $N < 0$.) Identify and classify all fixed points.

(b) The bletch population is now *harvested* (they make nice shoes). To model this, we add an extra term to the dynamics:

$$\frac{dN}{dt} = -hN + aN^2 - bN^3 \quad , \tag{13.27}$$

where h is the harvesting rate. Show that the phase flow now depends crucially on h, in that there are two qualitatively different flows, depending on whether $h < h_c(a,b)$ or $h > h_c(a,b)$. Find the critical harvesting rate $h_c(a,b)$ and sketch the phase flows for the two different regimes.

(c) In equilibrium, the rate at which bletches are harvested is $R = hN^*$, where N^* is the equilibrium bletch population. Suppose we start with $h = 0$, in which case N^* is given by the value of N at the stable fixed point you found in part (a). Now let h be increased very slowly from zero. As h is increased, the equilibrium population changes. Sketch R *versus* h. What value of h achieves the biggest bletch harvest? What is the corresponding value of R_{max}?

Solution:

(a) Setting the RHS of eqn. 13.26 to zero suggests the rescaling

$$N = \frac{a}{b}\, n \qquad , \qquad t = \frac{b}{a^2}\, \tau \quad . \tag{13.28}$$

This results in

$$\frac{dn}{d\tau} = n^2 - n^3 \quad . \tag{13.29}$$

The point $n = 0$ is a (nonlinearly) repulsive fixed point, and $n = 1$, corresponding to $N = a/b$, is attractive. The flow is shown in fig. 13.11.

By the way, the dynamics can be integrated, using the method of partial fractions, to yield

$$\frac{1}{n_0} - \frac{1}{n} + \log\left(\frac{n}{n_0} \cdot \frac{1 - n_0}{1 - n}\right) = \tau \quad . \tag{13.30}$$

(b) Upon rescaling, the harvested bletch dynamics obeys the equation

$$\frac{dn}{d\tau} = -\nu n + n^2 - n^3 \quad , \tag{13.31}$$

where $\nu = bh/a^2$ is the dimensionless harvesting rate. Setting the RHS to zero yields $n(n^2 - n + \nu) = 0$, with solutions $n^* = 0$ and

$$n^*_\pm = \tfrac{1}{2} \pm \sqrt{\tfrac{1}{4} - \nu} \quad . \tag{13.32}$$

At $\nu = \frac{1}{4}$ there is a saddle-node bifurcation, and for $\nu > \frac{1}{4}$ the only fixed point (for real n) is at $n^* = 0$ (stable) – the bletch population is then *overharvested*. For $\nu > \frac{1}{4}$, there are three solutions: a stable fixed point at $n^* = 0$, an unstable fixed point at $n^* = \frac{1}{2} - \sqrt{\frac{1}{4} - \nu}$, and a stable fixed point at $n^* = \frac{1}{2} + \sqrt{\frac{1}{4} - \nu}$. The critical harvesting rate is $\nu_c = \frac{1}{4}$, which means $h_c = a^2/4b$.

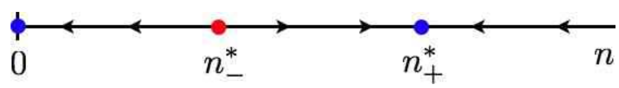


Figure 13.12: Phase flow for the harvested bletch population, $\dot{n} = -\nu n + n^2 - n^3$.

(c) The scaled bletch harvest is given by $r = \nu \, n^*_+(\nu)$. Note $R = h \, N^*_+ = \frac{a^3}{b^2} \, r$. The optimal harvest occurs when $\nu \, n^*$ is a maximum, which means we set

$$\frac{d}{d\nu}\left\{ \tfrac{1}{2}\nu + \nu\sqrt{\tfrac{1}{4} - \nu} \right\} = 0 \quad \Longrightarrow \quad \nu_{\text{opt}} = \tfrac{2}{9} \quad . \tag{13.33}$$

Thus, $n^*_+(\nu_{\text{opt}}) = \frac{2}{3}$ and $r_{\text{opt}} = \frac{4}{27}$, meaning $R = 4a^3/27\,b^2$. Note that at $\nu = \nu_c = \frac{1}{4}$ that $n^*_+(\nu_c) = \frac{1}{2}$, hence $r(\nu_c) = \frac{1}{8}$, which is smaller than $(\nu_{\text{opt}}) = \frac{2}{3}$. The harvest $r(\nu)$ discontinuously drops to zero at $\nu = \nu_c$, since for $\nu > \nu_c$ the flow is to the only stable fixed point at $n^* = 0$.

13.4 Appendix II: Landau Theory of Phase Transitions

Landau's theory of phase transitions is based on an expansion of the free energy of a thermodynamic system in terms of an *order parameter*, which is nonzero in an ordered phase and zero in a disordered phase. For example, the magnetization M of a ferromagnet in zero external field but at finite temperature typically vanishes for temperatures $T > T_c$, where T_c is the *critical temperature*, also called the *Curie temperature* in a ferromagnet. A low order expansion in powers of the order parameter is appropriate sufficiently close to T_c, *i.e.* at temperatures such that the order parameter, if nonzero, is still small.

The simplest example is the quartic free energy,

$$f(m) = f_0 + \tfrac{1}{2}am^2 + \tfrac{1}{4}bm^4 \quad , \tag{13.34}$$

where m is a dimensionless measure of the magnetization density, and where f_0, a, and b are all functions of the dimensionless temperature θ, which in a ferromagnet is the ratio $\theta = k_B T/\mathcal{J}$, where $\mathcal{J} = \sum_j J_{ij}$

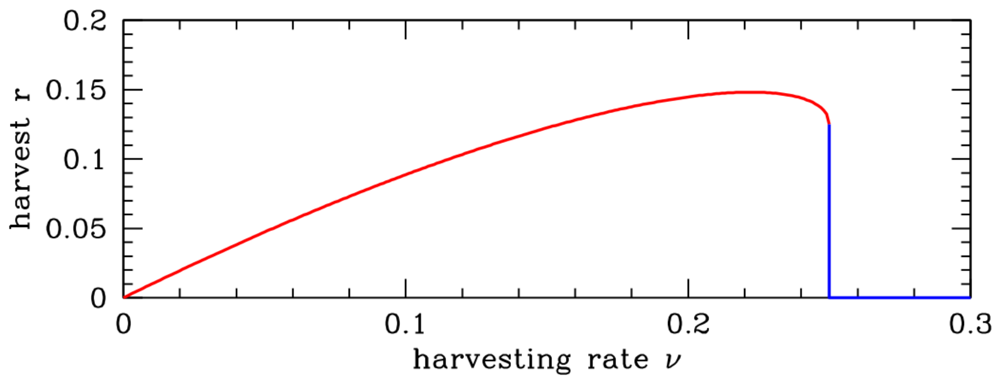


Figure 13.13: Scaled bletch harvest r *versus* scaled harvesting rate ν. Optimal harvesting occurs for $\nu_{\rm opt} = \frac{2}{9}$. The critical harvesting rate is $\nu_{\rm c} = \frac{1}{4}$, at which point the harvest discontinuously drops to zero.

is a sum over the couplings. Let us assume $b > 0$, which is necessary if the free energy is to be bounded from below[2]. The equation of state ,

$$\frac{\partial f}{\partial m} = 0 = am + bm^3 \quad , \tag{13.35}$$

has three solutions in the complex m plane: (i) $m = 0$, (ii) $m = \sqrt{-a/b}$, and (iii) $m = -\sqrt{-a/b}$. The latter two solutions lie along the (physical) real axis if $a < 0$. We assume that $a(\theta$ is monotonically increasing, and that there exists a unique temperature $\theta_{\rm c}$ where $a(\theta_{\rm c}) = 0$. Minimizing f, we find

$$\begin{aligned} \theta < \theta_{\rm c} \quad &: \quad f = f_0 - \frac{a^2}{4b} \\ \theta > \theta_{\rm c} \quad &: \quad f = f_0 \quad . \end{aligned} \tag{13.36}$$

The free energy is continuous at $\theta_{\rm c}$ since $a(\theta_{\rm c}) = 0$. The specific heat, however, is discontinuous across the transition, with

$$c(\theta_{\rm c}^+) - c(\theta_{\rm c}^-) = -\theta_{\rm c} \frac{\partial^2}{\partial \theta^2}\bigg|_{\theta=\theta_{\rm c}} \left(\frac{a^2}{4b}\right) = -\frac{\theta_{\rm c} \left[a'(\theta_{\rm c})\right]^2}{2b(\theta_{\rm c})} \quad . \tag{13.37}$$

The presence of a magnetic field h breaks the $\mathbb{Z}_2$ symmetry of $m \to -m$. The free energy becomes

$$f(m) = f_0 + \tfrac{1}{2} am^2 + \tfrac{1}{4} bm^4 - hm \quad , \tag{13.38}$$

and the mean field equation is

$$bm^3 + am - h = 0 \quad . \tag{13.39}$$

This is a cubic equation for m with real coefficients, and as such it can either have three real solutions or one real solution and two complex solutions related by complex conjugation. Clearly we must have $a < 0$ in order to have three real roots, since $bm^3 + am$ is monotonically increasing otherwise. The

[2]It is always the case that f is bounded from below, on physical grounds. Were b negative, we'd have to consider higher order terms in the Landau expansion.

boundary between these two classes of solution sets occurs when two roots coincide, which means $f''(m) = 0$ as well as $f'(m) = 0$. Simultaneously solving these two equations, we find

$$h^*(a) = \pm \frac{2}{3^{3/2}} \frac{(-a)^{3/2}}{b^{1/2}} \quad , \tag{13.40}$$

or, equivalently,

$$a^*(h) = -\frac{3}{2^{2/3}}\, b^{1/3}\, |h|^{2/3}. \tag{13.41}$$

If, for fixed h, we have $a < a^*(h)$, then there will be three real solutions to the mean field equation $f'(m) = 0$, one of which is a global minimum (the one for which $m \cdot h > 0$). For $a > a^*(h)$ there is only a single global minimum, at which m also has the same sign as h. If we solve the mean field equation perturbatively in h/a, we find

$$\begin{aligned} m(a,h) &= \frac{h}{a} - \frac{b\,h^3}{a^4} + \mathcal{O}(h^5) && (a > 0) \\ &= \frac{h}{2\,|a|} - \frac{3\,b^{1/2}\,h^2}{8\,|a|^{5/2}} + \mathcal{O}(h^3) && (a < 0) \quad . \end{aligned} \tag{13.42}$$

13.4.1 Landau coefficients from mean field theory

A simple variational density matrix for the Ising ferromagnet yields the dimensionless free energy density

$$f(m,h,\theta) = -\tfrac{1}{2}\, m^2 - hm + \theta \left\{ \left(\frac{1+m}{2}\right) \log\left(\frac{1+m}{2}\right) + \left(\frac{1-m}{2}\right) \log\left(\frac{1-m}{2}\right) \right\} \quad . \tag{13.43}$$

When m is small, it is appropriate to expand $f(m,h,\theta)$, obtaining

$$f(m,h,\theta) = -\theta\, \log 2 - hm + \tfrac{1}{2}(\theta - 1)\, m^2 + \tfrac{\theta}{12}\, m^4 + \tfrac{\theta}{30}\, m^6 + \tfrac{\theta}{56}\, m^8 + \ldots \quad . \tag{13.44}$$

Thus, we identify

$$a(\theta) = \theta - 1 \qquad , \qquad b(\theta) = \tfrac{1}{3}\theta \quad . \tag{13.45}$$

We see that $a(\theta) = 0$ at a critical temperature $\theta_c = 1$.

The free energy of eqn. 13.43 behaves qualitatively just like it does for the simple Landau expansion, where one stops at order m^4. Consider without loss of generality the case $h > 0$. The minimum of the free energy $f(m,h,\theta)$ then lies at $m > 0$ for any θ. At low temperatures, the double well structure we found in the $h = 0$ case is tilted so that the right well lies lower in energy than the left well. This is depicted in fig. 13.15. As the temperature is raised, the local minimum at $m < 0$ vanishes, annihilating with the local maximum in a saddle-node bifurcation. To find where this happens, one sets $\frac{\partial f}{\partial m} = 0$ and $\frac{\partial^2 f}{\partial m^2} = 0$ simultaneously, resulting in

$$h^*(\theta) = \sqrt{1-\theta} - \frac{\theta}{2} \log\left(\frac{1+\sqrt{1-\theta}}{1-\sqrt{1-\theta}}\right). \tag{13.46}$$

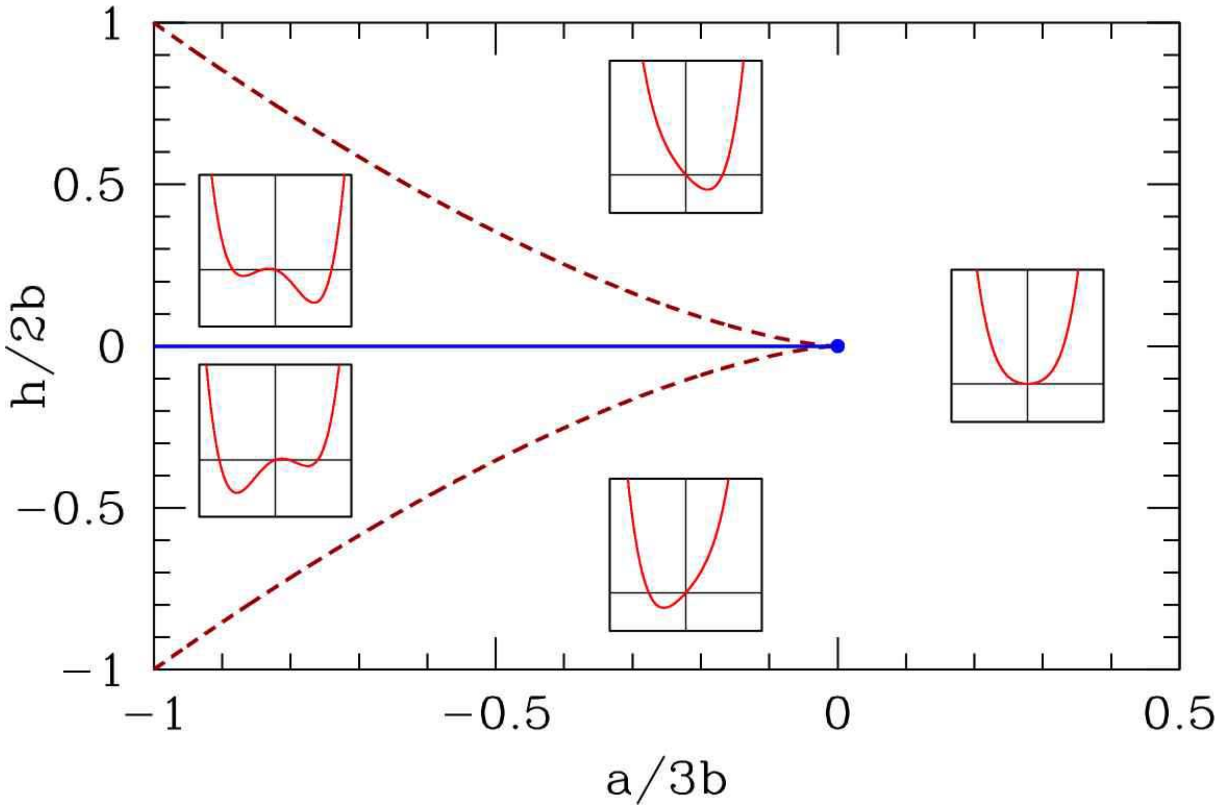


Figure 13.14: Phase diagram for the quartic mean field theory $f = f_0 + \frac{1}{2}am^2 + \frac{1}{4}bm^4 - hm$, with $b > 0$. There is a first order line at $h = 0$ extending from $a = -\infty$ and terminating in a critical point at $a = 0$. For $|h| < h^*(a)$ (dashed red line) there are three solutions to the mean field equation, corresponding to one global minimum, one local minimum, and one local maximum. Insets show behavior of the free energy $f(m)$.

The solutions lie at $h = \pm h^*(\theta)$. For $\theta < \theta_c = 1$ and $h \in \big[-h^*(\theta)\,,\,+h^*(\theta)\big]$, there are three solutions to the mean field equation. Equivalently we could in principle invert the above expression to obtain $\theta^*(h)$. For $\theta > \theta^*(h)$, there is only a single global minimum in the free energy $f(m)$ and there is no local minimum. Note $\theta^*(h = 0) = 1$.

13.4.2 Magnetization dynamics

Dissipative processes drive physical systems to minimum energy states. We can crudely model the dissipative dynamics of a magnet by writing the phenomenological equation

$$\frac{dm}{dt} = -\Gamma\,\frac{\partial f}{\partial m} \quad . \tag{13.47}$$

This drives the free energy f to smaller and smaller values:

$$\frac{df}{dt} = \frac{\partial f}{\partial m}\,\frac{dm}{dt} = -\Gamma\bigg(\frac{\partial f}{\partial m}\bigg)^2 \le 0 \quad . \tag{13.48}$$

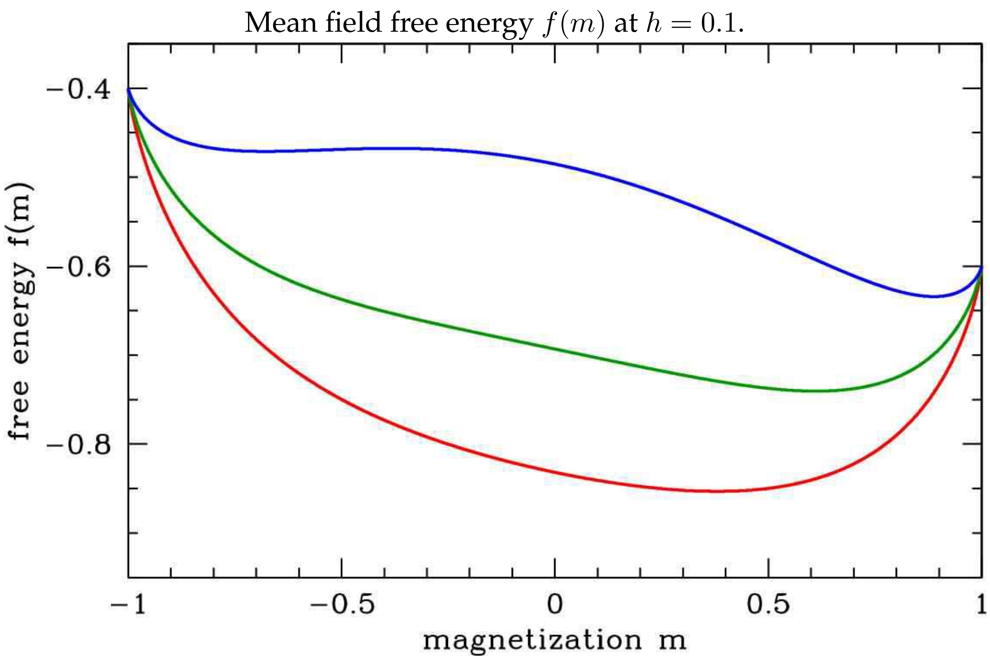


Figure 13.15: Mean field free energy $f(m)$ at $h = 0.1$. Temperatures shown: $\theta = 1.2$ (red), $\theta = 1.0$ (dark green), and $\theta = 0.7$ (blue).

Clearly the *fixed point* of these dynamics, where $\dot{m} = 0$, is a solution to the mean field equation $\frac{\partial f}{\partial m} = 0$. At the solution to the mean field equation, one has

$$\frac{\partial f}{\partial m} = 0 \qquad \Rightarrow \qquad m = \tanh\left(\frac{m+h}{\theta}\right) \quad . \tag{13.49}$$

The phase flow for the equation $\dot{m} = -\Gamma f'(m)$ is shown in fig. 13.16. As we have seen, for any value of h there is a temperature θ^* below which the free energy $f(m)$ has two local minima and one local maximum. When $h = 0$ the minima are degenerate, but at finite h one of the minima is a global minimum. Thus, for $\theta < \theta^*(h)$ there are three solutions to the mean field equations. In the language of dynamical systems, under the dynamics of eqn. 13.47, minima of $f(m)$ correspond to attractive fixed points and maxima to repulsive fixed points. If $h > 0$, the rightmost of these fixed points corresponds to the global minimum of the free energy. As θ is increased, this fixed point evolves smoothly. At $\theta = \theta^*$, the (metastable) local minimum and the local maximum coalesce and annihilate in a saddle-note bifurcation. However at $h = 0$ all three fixed points coalesce at $\theta = \theta_c$ and the bifurcation is a supercritical pitchfork. As a function of t at finite h, the dynamics are said to exhibit an *imperfect bifurcation*, which is a deformed supercritical pitchfork.

The solution set for the mean field equation is simply expressed by inverting the $\tanh$ function to obtain $h(\theta, m)$. One readily finds

$$h(\theta, m) = \frac{\theta}{2}\log\left(\frac{1+m}{1-m}\right) - m \quad . \tag{13.50}$$

As we see in the bottom panel of fig. 13.17, $m(h)$ becomes multivalued for field values $h \in \big[-h^*(\theta)\,,\,+h^*(\theta)\big]$, where $h^*(\theta)$ is given in eqn. 13.46. Now imagine that $\theta < \theta_c$ and we slowly ramp

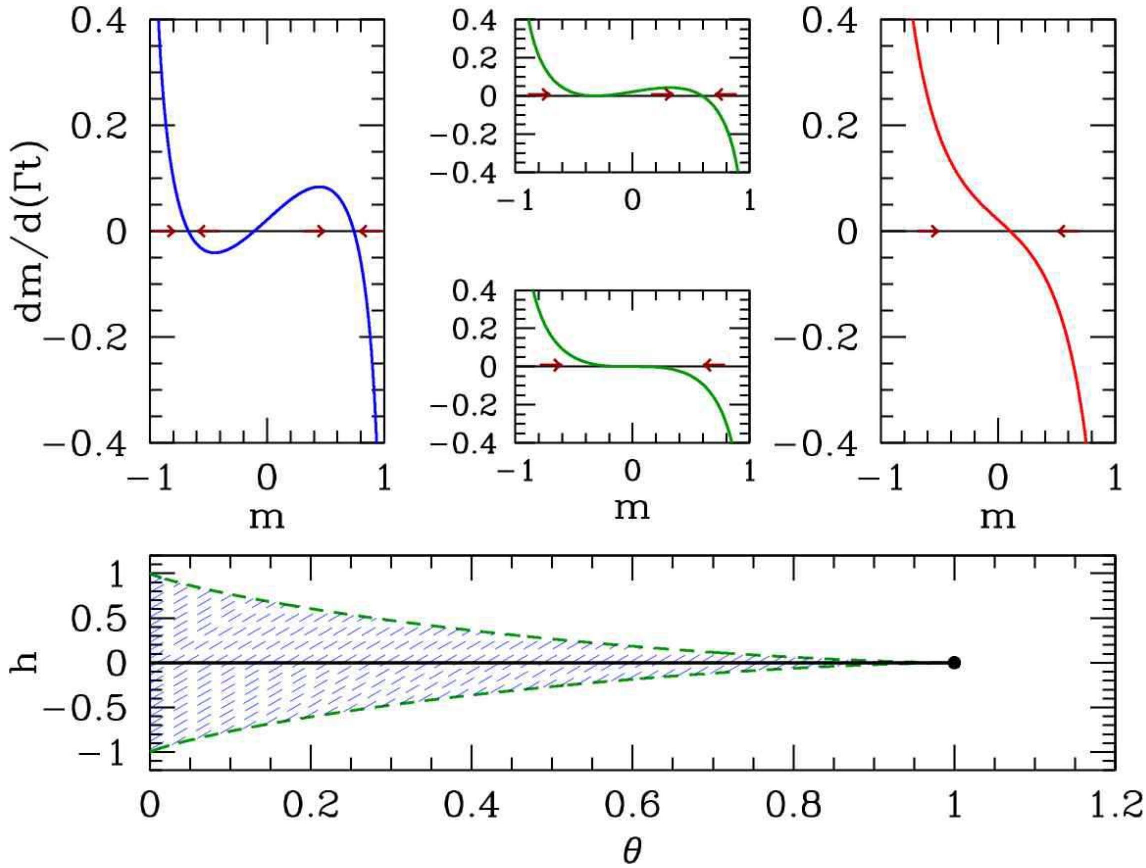


Figure 13.16: Dissipative magnetization dynamics $\dot{m} = -f'(m)$. Bottom panel shows $h^*(\theta)$ from eqn. 13.46. For (θ, h) within the blue shaded region, the free energy $f(m)$ has a global minimum plus a local minimum and a local maximum. Otherwise $f(m)$ has only a single global maximum. Top panels show an imperfect bifurcation in the magnetization dynamics at $h = 0.0215$, for which $\theta^* = 0.90$ Temperatures shown: $\theta = 0.80$ (blue), $\theta = \theta^*(h) = 0.90$ (green), and $\theta = 1.2$. The rightmost stable fixed point corresponds to the global minimum of the free energy. The bottom of the middle two upper panels shows $h = 0$, where both of the attractive fixed points and the repulsive fixed point coalesce into a single attractive fixed point (supercritical pitchfork bifurcation).

the field h from a large negative value to a large positive value, and then slowly back down to its original value. On the time scale of the magnetization dynamics, we can regard $h(t)$ as a constant. Thus, $m(t)$ will flow to the nearest stable fixed point. Initially the system starts with $m = -1$ and h large and negative, and there is only one fixed point, at $m^* \approx -1$. As h slowly increases, the fixed point value m^* also slowly increases. As h exceeds $-h^*(\theta)$, a saddle-node bifurcation occurs, and two new fixed points are created at positive m, one stable and one unstable. The global minimum of the free energy still lies at the fixed point with $m^* < 0$. However, when h crosses $h = 0$, the global minimum of the free energy lies at the most positive fixed point m^*. The dynamics, however, keep the system stuck in what is a metastable phase. This persists until $h = +h^*(\theta)$, at which point another saddle-note bifurcation occurs, and the attractive fixed point at $m^* < 0$ annihilates with the repulsive fixed point. The dynamics then act quickly to drive m to the only remaining fixed point. This process is depicted in the top panel of fig.

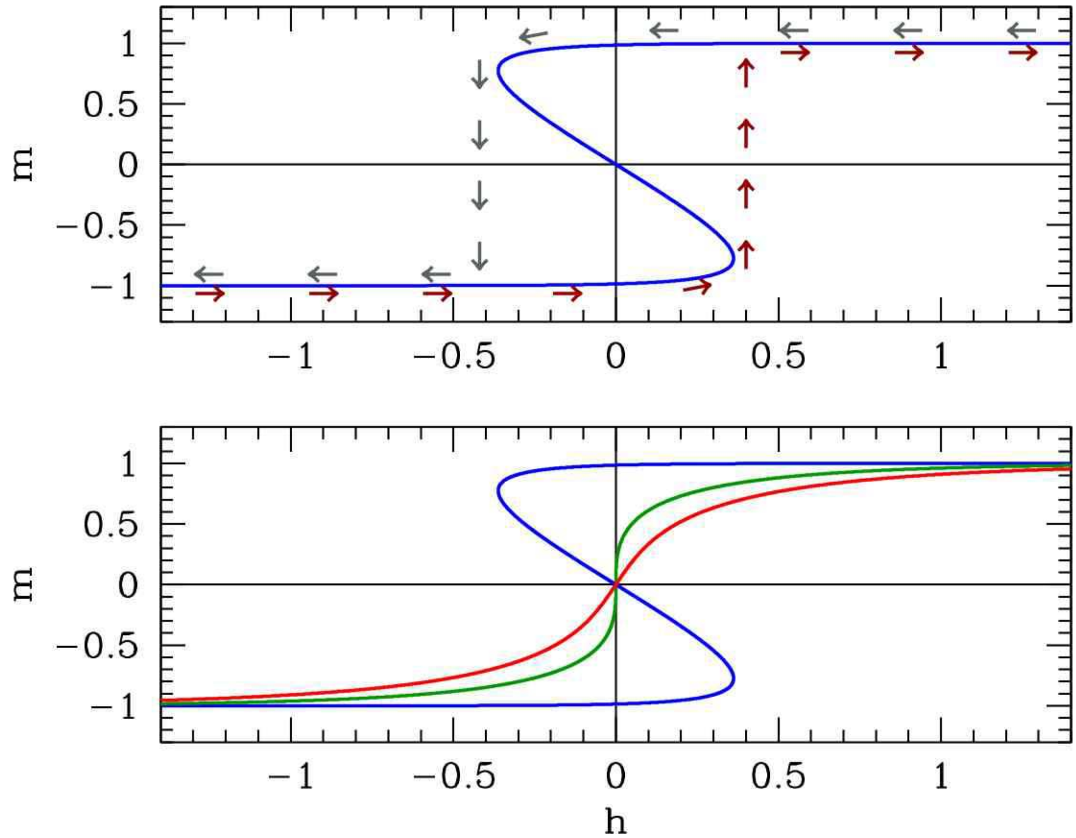


Figure 13.17: Top panel : hysteresis as a function of ramping the dimensionless magnetic field h at $\theta = 0.40$. Dark red arrows below the curve follow evolution of the magnetization on slow increase of h. Dark grey arrows above the curve follow evolution of the magnetization on slow decrease of h. Bottom panel : solution set for $m(\theta, h)$ as a function of h at temperatures $\theta = 0.40$ (blue), $\theta = \theta_c = 1.0$ (dark green), and $t = 1.25$ (red).

13.17. As one can see from the figure, the the system follows a stable fixed point until the fixed point disappears, even though that fixed point may not always correspond to a global minimum of the free energy. The resulting $m(h)$ curve is then not reversible as a function of time, and it possesses a characteristic shape known as a *hysteresis loop*. Etymologically, the word *hysteresis* derives from the Greek $\upsilon\sigma\tau\varepsilon\rho\eta\sigma\iota\varsigma$, which means 'lagging behind'. Systems which are hysteretic exhibit a *history-dependence* to their status, which is not uniquely determined by external conditions. Hysteresis may be exhibited with respect to changes in applied magnetic field, changes in temperature, or changes in other externally determined parameters.

13.4.3 Cubic terms in Landau theory : first order transitions

Next, consider a free energy with a cubic term,

$$f = f_0 + \tfrac{1}{2}am^2 - \tfrac{1}{3}ym^3 + \tfrac{1}{4}bm^4 \quad , \tag{13.51}$$

with $b > 0$ for stability. Without loss of generality, we may assume $y > 0$ (else send $m \to -m$). Note that we no longer have $m \to -m$ (*i.e.* $\mathbb{Z}_2$) symmetry. The cubic term favors positive m. What is the phase diagram in the (a, y) plane?

Extremizing the free energy with respect to m, we obtain

$$\frac{\partial f}{\partial m} = 0 = am - ym^2 + bm^3 \quad . \tag{13.52}$$

This cubic equation factorizes into a linear and quadratic piece, and hence may be solved simply. The three solutions are $m = 0$ and

$$m = m_\pm \equiv \frac{y}{2b} \pm \sqrt{\left(\frac{y}{2b}\right)^2 - \frac{a}{b}} \quad . \tag{13.53}$$

We now see that for $y^2 < 4ab$ there is only one real solution, at $m = 0$, while for $y^2 > 4ab$ there are three real solutions. Which solution has lowest free energy? To find out, we compare the energy $f(0)$ with $f(m_+)$[3]. Thus, we set

$$f(m) = f(0) \quad \Longrightarrow \quad \tfrac{1}{2}am^2 - \tfrac{1}{3}ym^3 + \tfrac{1}{4}bm^4 = 0 \quad , \tag{13.54}$$

and we now have two quadratic equations to solve simultaneously:

$$\begin{aligned} 0 &= a - ym + bm^2 \\ 0 &= \tfrac{1}{2}a - \tfrac{1}{3}ym + \tfrac{1}{4}bm^2 = 0 \quad . \end{aligned} \tag{13.55}$$

Eliminating the quadratic term gives $m = 3a/y$. Finally, substituting $m = m_+$ gives us a relation between a, b, and y:

$$y^2 = \tfrac{9}{2}\,ab \quad . \tag{13.56}$$

Thus, we have the following:

$$\begin{aligned} a > \frac{y^2}{4b} \quad &: \quad \text{1 real root } m = 0 \\ \frac{y^2}{4b} > a > \frac{2y^2}{9b} \quad &: \quad \text{3 real roots; minimum at } m = 0 \\ \frac{2y^2}{9b} > a \quad &: \quad \text{3 real roots; minimum at } m = \frac{y}{2b} + \sqrt{\left(\frac{y}{2b}\right)^2 - \frac{a}{b}} \end{aligned} \tag{13.57}$$

The solution $m = 0$ lies at a local minimum of the free energy for $a > 0$ and at a local maximum for $a < 0$. Over the range $\frac{y^2}{4b} > a > \frac{2y^2}{9b}$, then, there is a global minimum at $m = 0$, a local minimum at $m = m_+$, and a local maximum at $m = m_-$, with $m_+ > m_- > 0$. For $\frac{2y^2}{9b} > a > 0$, there is a local minimum at $a = 0$, a global minimum at $m = m_+$, and a local maximum at $m = m_-$, again with $m_+ > m_- > 0$. For $a < 0$, there is a local maximum at $m = 0$, a local minimum at $m = m_-$, and a global minimum at $m = m_+$, with $m_+ > 0 > m_-$. See fig. 13.18.

[3] We needn't waste our time considering the $m = m_-$ solution, since the cubic term prefers positive m.

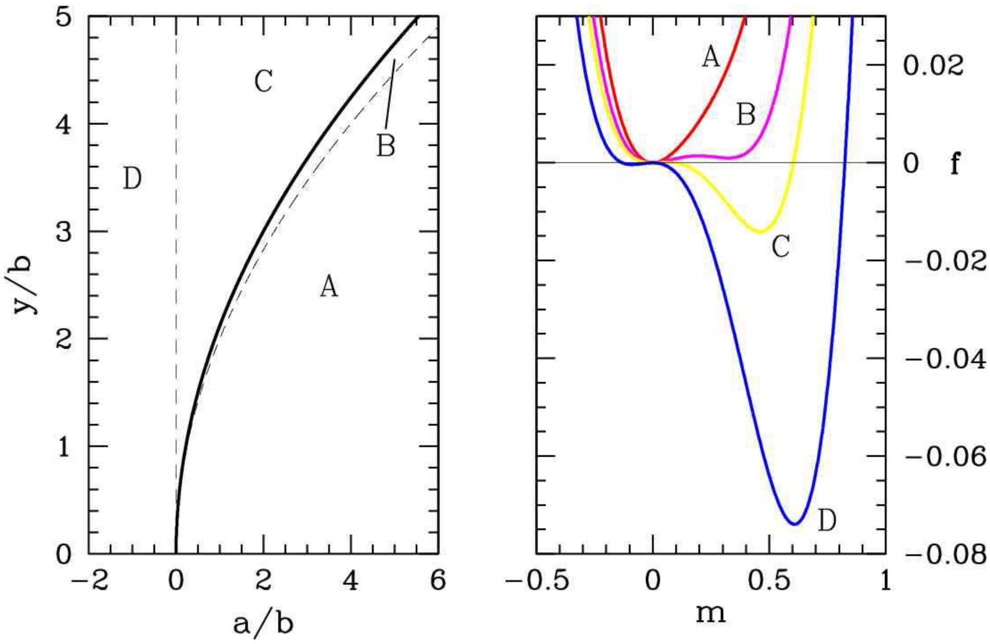


Figure 13.18: Behavior of the quartic free energy $f(m) = \frac{1}{2}am^2 - \frac{1}{3}ym^3 + \frac{1}{4}bm^4$. A: $y^2 < 4ab$; B: $4ab < y^2 < \frac{9}{2}ab$; C and D: $y^2 > \frac{9}{2}ab$. The thick black line denotes a line of first order transitions, where the order parameter is discontinuous across the transition.

13.4.4 Magnetization dynamics

Suppose we now impose some dynamics on the system, of the simple relaxational type

$$\frac{dm}{dt} = -\Gamma\,\frac{\partial f}{\partial m} \quad , \tag{13.58}$$

where Γ is a phenomenological kinetic coefficient. Assuming $y > 0$ and $b > 0$, it is convenient to adimensionalize by writing

$$m \equiv \frac{y}{b}\cdot u \qquad , \qquad a \equiv \frac{y^2}{b}\cdot \bar{r} \qquad , \qquad t \equiv \frac{b}{\Gamma y^2}\cdot s \quad . \tag{13.59}$$

Then we obtain

$$\frac{\partial u}{\partial s} = -\frac{\partial \varphi}{\partial u} \quad , \tag{13.60}$$

where the dimensionless free energy function is

$$\varphi(u) = \tfrac{1}{2}\bar{r}u^2 - \tfrac{1}{3}u^3 + \tfrac{1}{4}u^4 \quad . \tag{13.61}$$

We see that there is a single control parameter, $\bar{r}$. The fixed points of the dynamics are then the stationary points of $\varphi(u)$, where $\varphi'(u) = 0$, with

$$\varphi'(u) = u\,(\bar{r} - u + u^2) \quad . \tag{13.62}$$

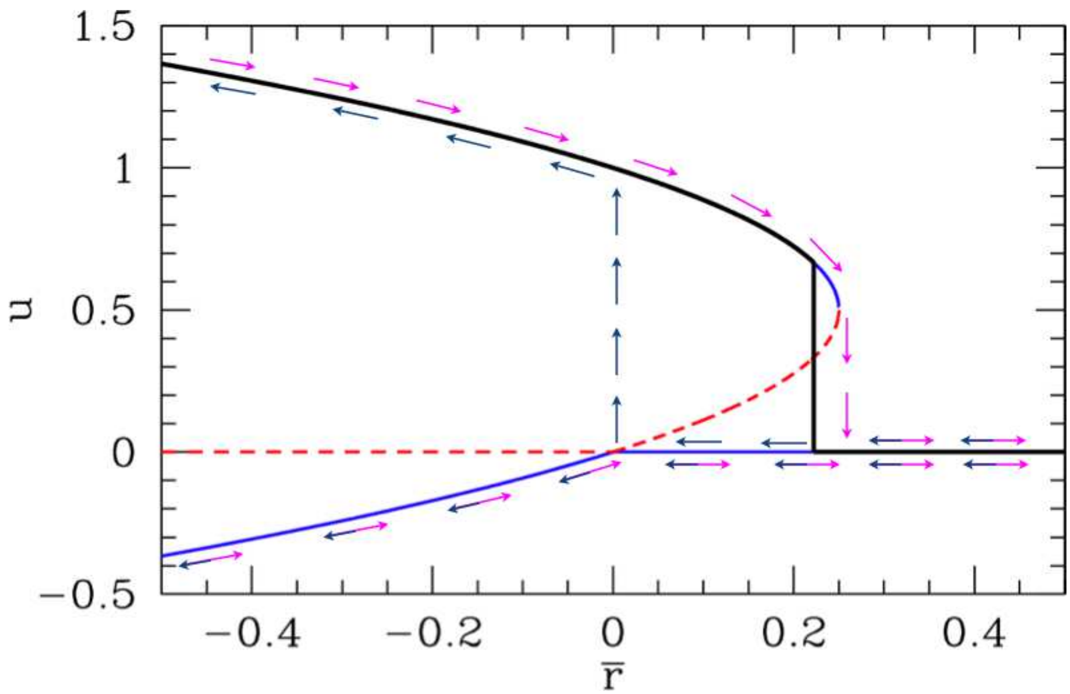


Figure 13.19: Fixed points for $\varphi(u) = \frac{1}{2}\bar{r}u^2 - \frac{1}{3}u^3 + \frac{1}{4}u^4$ and flow under the dynamics $\dot{u} = -\varphi'(u)$. Solid curves represent stable fixed points and dashed curves unstable fixed points. Magenta arrows show behavior under slowly increasing control parameter $\bar{r}$ and dark blue arrows show behavior under slowly decreasing $\bar{r}$. For $u > 0$ there is a hysteresis loop. The thick black curve shows the equilibrium thermodynamic value of $u(\bar{r})$, *i.e.* that value which minimizes the free energy $\varphi(u)$. There is a first order phase transition at $\bar{r} = \frac{2}{9}$, where the thermodynamic value of u jumps from $u = 0$ to $u = \frac{2}{3}$.

The solutions to $\varphi'(u) = 0$ are then given by

$$u^* = 0 \qquad , \qquad u^* = \tfrac{1}{2} \pm \sqrt{\tfrac{1}{4} - \bar{r}} \quad . \tag{13.63}$$

For $r > \frac{1}{4}$ there is one fixed point at $u = 0$, which is attractive under the dynamics $\dot{u} = -\varphi'(u)$ since $\varphi''(0) = \bar{r}$. At $\bar{r} = \frac{1}{4}$ there occurs a saddle-node bifurcation and a pair of fixed points is generated, one stable and one unstable. As we see from fig. 13.14, the interior fixed point is always unstable and the two exterior fixed points are always stable. At $r = 0$ there is a transcritical bifurcation where two fixed points of opposite stability collide and bounce off one another (metaphorically speaking).

At the saddle-node bifurcation, $\bar{r} = \frac{1}{4}$ and $u = \frac{1}{2}$, and we find $\varphi(u = \frac{1}{2}; \bar{r} = \frac{1}{4}) = \frac{1}{192}$, which is positive. Thus, the thermodynamic state of the system remains at $u = 0$ until the value of $\varphi(u_+)$ crosses zero. This occurs when $\varphi(u) = 0$ and $\varphi'(u) = 0$, the simultaneous solution of which yields $\bar{r} = \frac{2}{9}$ and $u = \frac{2}{3}$.

Suppose we slowly ramp the control parameter $\bar{r}$ up and down as a function of the dimensionless time s. Under the dynamics of eqn. 13.60, $u(s)$ flows to the first stable fixed point encountered – this is always the case for a dynamical system with a one-dimensional phase space. Then as $\bar{r}$ is further varied, u follows the position of whatever locally stable fixed point it initially encountered. Thus, $u\big(\bar{r}(s)\big)$ evolves smoothly until a bifurcation is encountered. The situation is depicted by the arrows in fig. 13.19. The

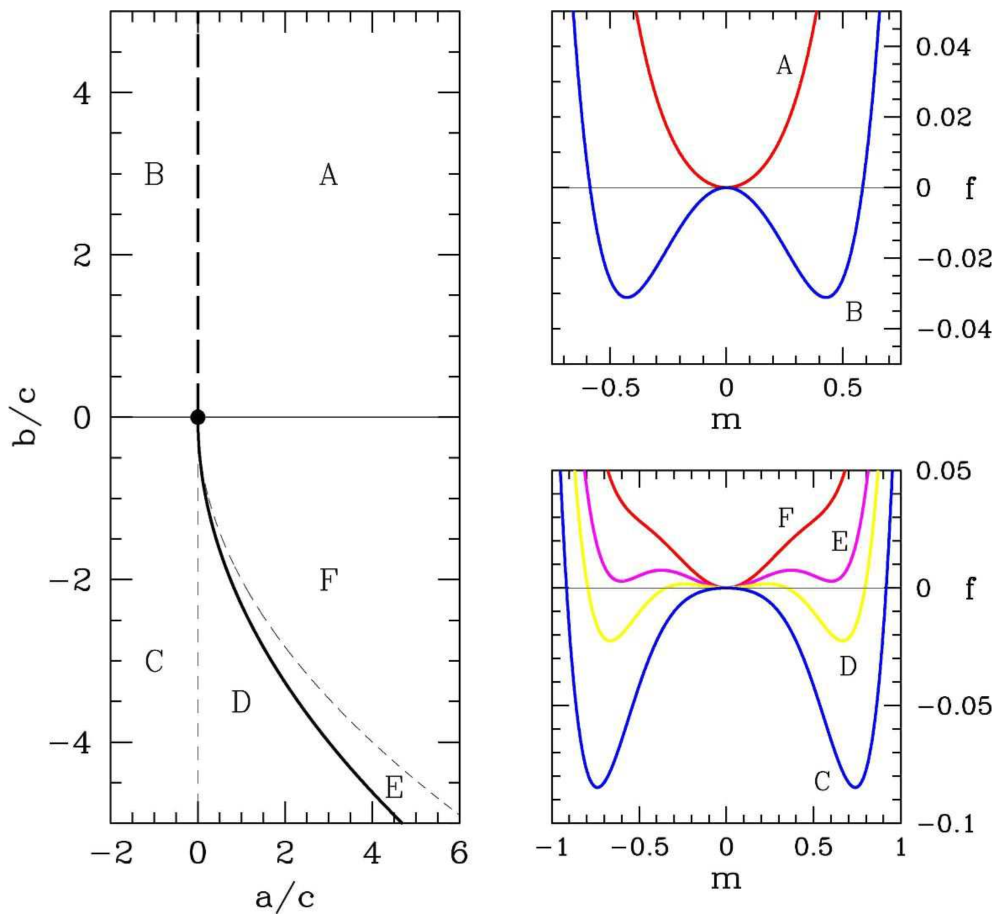


Figure 13.20: Behavior of the sextic free energy $f(m) = \frac{1}{2}am^2 + \frac{1}{4}bm^4 + \frac{1}{6}cm^6$. A: $a > 0$ and $b > 0$; B: $a < 0$ and $b > 0$; C: $a < 0$ and $b < 0$; D: $a > 0$ and $b < -\frac{4}{\sqrt{3}}\sqrt{ac}$; E: $a > 0$ and $-\frac{4}{\sqrt{3}}\sqrt{ac} < b < -2\sqrt{ac}$; F: $a > 0$ and $-2\sqrt{ac} < b < 0$. The thick dashed line is a line of second order transitions, which meets the thick solid line of first order transitions at the tricritical point, $(a, b) = (0, 0)$.

equilibrium thermodynamic value for $u(\bar{r})$ is discontinuous; there is a first order phase transition at $\bar{r} = \frac{2}{9}$, as we've already seen. As r is increased, $u(\bar{r})$ follows a trajectory indicated by the magenta arrows. For an negative initial value of u, the evolution as a function of $\bar{r}$ will be *reversible*. However, if $u(0)$ is initially positive, then the system exhibits *hysteresis*, as shown. Starting with a large positive value of $\bar{r}$, $u(s)$ quickly evolves to $u = 0^+$, which means a positive infinitesimal value. Then as r is decreased, the system remains at $u = 0^+$ even through the first order transition, because $u = 0$ is an attractive fixed point. However, once r begins to go negative, the $u = 0$ fixed point becomes repulsive, and $u(s)$ quickly flows to the stable fixed point $u_+ = \frac{1}{2} + \sqrt{\frac{1}{4} - \bar{r}}$. Further decreasing r, the system remains on this branch. If $\bar{r}$ is later increased, then $u(s)$ remains on the upper branch past $r = 0$, until the u_+ fixed point annihilates with the unstable fixed point at $u_- = \frac{1}{2} - \sqrt{\frac{1}{4} - \bar{r}}$, at which time $u(s)$ quickly flows down to $u = 0^+$ again.

13.4.5 Sixth order Landau theory : tricritical point

Finally, consider a model with $\mathbb{Z}_2$ symmetry, with the Landau free energy

$$f = f_0 + \tfrac{1}{2}am^2 + \tfrac{1}{4}bm^4 + \tfrac{1}{6}cm^6 \quad , \tag{13.64}$$

with $c > 0$ for stability. We seek the phase diagram in the (a, b) plane. Extremizing f with respect to m, we obtain

$$\frac{\partial f}{\partial m} = 0 = m\,(a + bm^2 + cm^4) \quad , \tag{13.65}$$

which is a quintic with five solutions over the complex m plane. One solution is obviously $m = 0$. The other four are

$$m = \pm\sqrt{-\frac{b}{2c} \pm \sqrt{\left(\frac{b}{2c}\right)^2 - \frac{a}{c}}} \quad . \tag{13.66}$$

For each $\pm$ symbol in the above equation, there are two options, hence four roots in all.

If $a > 0$ and $b > 0$, then four of the roots are imaginary and there is a unique minimum at $m = 0$.

For $a < 0$, there are only three solutions to $f'(m) = 0$ for real m, since the $-$ choice for the $\pm$ sign under the radical leads to imaginary roots. One of the solutions is $m = 0$. The other two are

$$m = \pm\sqrt{-\frac{b}{2c} + \sqrt{\left(\frac{b}{2c}\right)^2 - \frac{a}{c}}} \quad . \tag{13.67}$$

The most interesting situation is $a > 0$ and $b < 0$. If $a > 0$ and $b < -2\sqrt{ac}$, all five roots are real. There must be three minima, separated by two local maxima. Clearly if m^* is a solution, then so is $-m^*$. Thus, the only question is whether the outer minima are of lower energy than the minimum at $m = 0$. We assess this by demanding $f(m^*) = f(0)$, where m^* is the position of the largest root (*i.e.* the rightmost minimum). This gives a second quadratic equation,

$$0 = \tfrac{1}{2}a + \tfrac{1}{4}bm^2 + \tfrac{1}{6}cm^4 \quad , \tag{13.68}$$

which together with equation 13.65 gives

$$b = -\tfrac{4}{\sqrt{3}}\,\sqrt{ac} \quad . \tag{13.69}$$

Thus, we have the following, for fixed $a > 0$:

$$\begin{aligned} b > -2\sqrt{ac} \quad &: \quad \text{1 real root } m = 0 \\ -2\sqrt{ac} > b > -\tfrac{4}{\sqrt{3}}\,\sqrt{ac} \quad &: \quad \text{5 real roots; minimum at } m = 0 \\ -\tfrac{4}{\sqrt{3}}\,\sqrt{ac} > b \quad &: \quad \text{5 real roots; minima at } m = \pm\sqrt{-\frac{b}{2c} + \sqrt{\left(\frac{b}{2c}\right)^2 - \frac{a}{c}}} \end{aligned} \tag{13.70}$$

The point $(a, b) = (0, 0)$, which lies at the confluence of a first order line and a second order line, is known as a *tricritical point*.

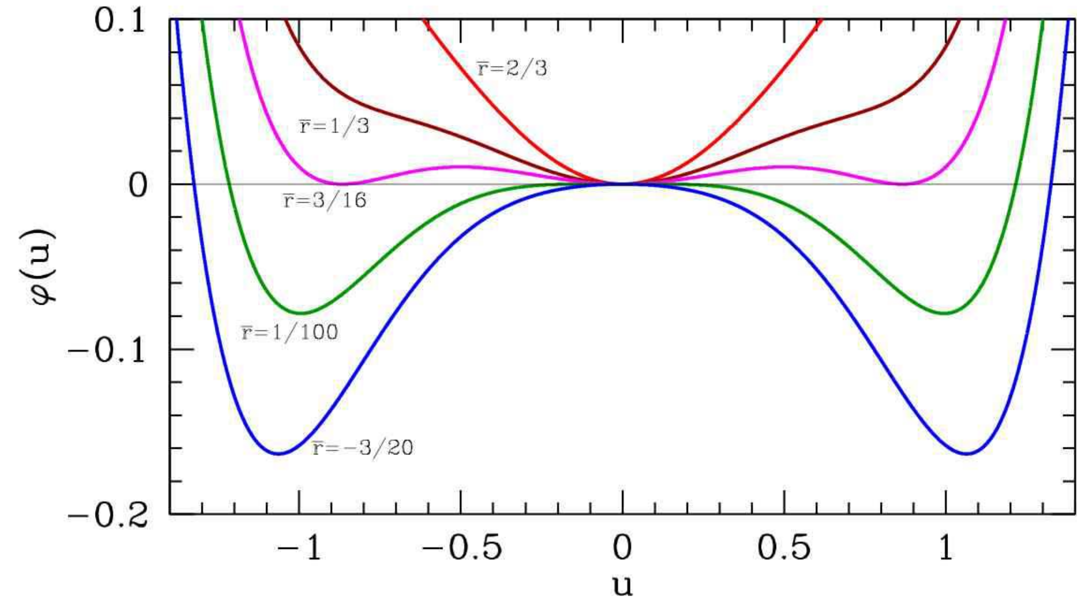


Figure 13.21: Free energy $\varphi(u) = \frac{1}{2}\bar{r}u^2 - \frac{1}{4}u^4 + \frac{1}{6}u^6$ for several different values of the control parameter $\bar{r}$.

13.4.6 Hysteresis for the sextic potential

Once again, we consider the dissipative dynamics $\dot{m} = -\Gamma\, f'(m)$. We adimensionalize by writing

$$m \equiv \sqrt{\frac{|b|}{c}} \cdot u \qquad , \qquad a \equiv \frac{b^2}{c} \cdot \bar{r} \qquad , \qquad t \equiv \frac{c}{\Gamma\, b^2} \cdot s \quad . \tag{13.71}$$

Then we obtain once again the dimensionless equation

$$\frac{\partial u}{\partial s} = -\frac{\partial \varphi}{\partial u} \quad , \tag{13.72}$$

where

$$\varphi(u) = \tfrac{1}{2}\bar{r}u^2 \pm \tfrac{1}{4}u^4 + \tfrac{1}{6}u^6 \quad . \tag{13.73}$$

In the above equation, the coefficient of the quartic term is positive if $b > 0$ and negative if $b < 0$. That is, the coefficient is $\mathrm{sgn}(b)$. When $b > 0$ we can ignore the sextic term for sufficiently small u, and we recover the quartic free energy studied earlier. There is then a second order transition at $r = 0$. .

New and interesting behavior occurs for $b > 0$. The fixed points of the dynamics are obtained by setting $\varphi'(u) = 0$. We have

$$\begin{aligned} \varphi(u) &= \tfrac{1}{2}\bar{r}u^2 - \tfrac{1}{4}u^4 + \tfrac{1}{6}u^6 \\ \varphi'(u) &= u\,(\bar{r} - u^2 + u^4) \quad . \end{aligned} \tag{13.74}$$

Thus, the equation $\varphi'(u) = 0$ factorizes into a linear factor u and a quartic factor $u^4 - u^2 + \bar{r}$ which is

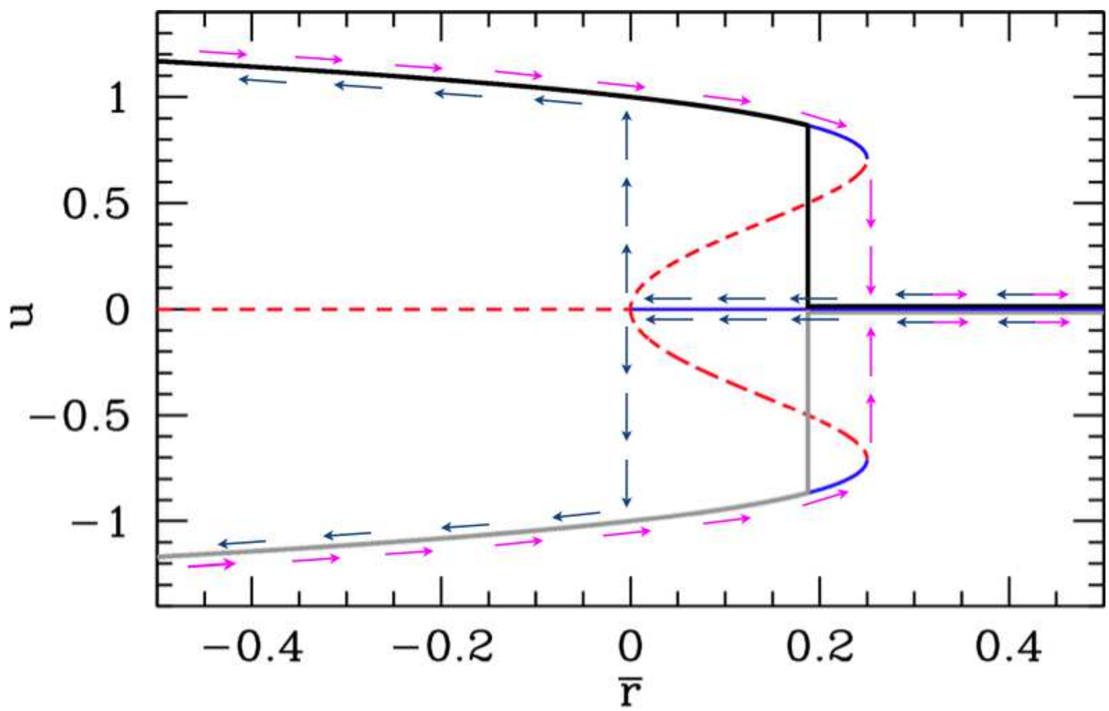


Figure 13.22: Fixed points $\varphi'(u^*) = 0$ for the sextic potential $\varphi(u) = \frac{1}{2}\bar{r}u^2 - \frac{1}{4}u^4 + \frac{1}{6}u^6$, and corresponding dynamical flow (arrows) under $\dot{u} = -\varphi'(u)$. Solid curves show stable fixed points and dashed curves show unstable fixed points. The thick solid black and solid grey curves indicate the equilibrium thermodynamic values for u; note the overall $u \to -u$ symmetry. Within the region $\bar{r} \in [0, \frac{1}{4}]$ the dynamics are irreversible and the system exhibits the phenomenon of hysteresis. There is a first order phase transition at $\bar{r} = \frac{3}{16}$.

quadratic in u^2. Thus, we can easily obtain the roots:

$$\begin{aligned} \bar{r} < 0 \quad &: \quad u^* = 0 \quad , \ u^* = \pm\sqrt{\tfrac{1}{2} + \sqrt{\tfrac{1}{4} - \bar{r}}} \\ 0 < \bar{r} < \tfrac{1}{4} \quad &: \quad u^* = 0 \quad , \ u^* = \pm\sqrt{\tfrac{1}{2} + \sqrt{\tfrac{1}{4} - \bar{r}}} \quad , \ u^* = \pm\sqrt{\tfrac{1}{2} - \sqrt{\tfrac{1}{4} - \bar{r}}} \\ \bar{r} > \tfrac{1}{4} \quad &: \quad u^* = 0 \quad . \end{aligned} \tag{13.75}$$

In fig. 13.22, we plot the fixed points and the hysteresis loops for this system. At $\bar{r} = \frac{1}{4}$, there are two symmetrically located saddle-node bifurcations at $u = \pm\frac{1}{\sqrt{2}}$. We find $\varphi(u = \pm\frac{1}{\sqrt{2}}, \bar{r} = \frac{1}{4}) = \frac{1}{48}$, which is positive, indicating that the stable fixed point $u^* = 0$ remains the thermodynamic minimum for the free energy $\varphi(u)$ as $\bar{r}$ is decreased through $\bar{r} = \frac{1}{4}$. Setting $\varphi(u) = 0$ and $\varphi'(u) = 0$ simultaneously, we obtain $\bar{r} = \frac{3}{16}$ and $u = \pm\frac{\sqrt{3}}{2}$. The thermodynamic value for u therefore jumps discontinuously from $u = 0$ to $u = \pm\frac{\sqrt{3}}{2}$ (either branch) at $\bar{r} = \frac{3}{16}$; this is a first order transition.

Under the dissipative dynamics considered here, the system exhibits hysteresis, as indicated in the figure, where the arrows show the evolution of $u(s)$ for very slowly varying $\bar{r}(s)$. When the control parameter $\bar{r}$ is large and positive, the flow is toward the sole fixed point at $u^* = 0$. At $\bar{r} = \frac{1}{4}$, two simultaneous saddle-node bifurcations take place at $u^* = \pm\frac{1}{\sqrt{2}}$; the outer branch is stable and the inner branch unstable in both cases. At $r = 0$ there is a subcritical pitchfork bifurcation, and the fixed point at $u^* = 0$ becomes unstable.

Suppose one starts off with $\bar{r} \gg \frac{1}{4}$ with some value $u > 0$. The flow $\dot{u} = -\varphi'(u)$ then rapidly results in $u \to 0^+$. This is the 'high temperature phase' in which there is no magnetization. Now let r increase slowly, using s as the dimensionless time variable. The scaled magnetization $u(s) = u^*\big(\bar{r}(s)\big)$ will remain pinned at the fixed point $u^* = 0^+$. As $\bar{r}$ passes through $\bar{r} = \frac{1}{4}$, two new stable values of u^* appear, but our system remains at $u = 0^+$, since $u^* = 0$ is a stable fixed point. But after the subcritical pitchfork, $u^* = 0$ becomes unstable. The magnetization $u(s)$ then flows rapidly to the stable fixed point at $u^* = \frac{1}{\sqrt{2}}$, and follows the curve $u^*(\bar{r}) = \big(\frac{1}{2} + (\frac{1}{4} - \bar{r})^{1/2}\big)^{1/2}$ for all $r < 0$.

Now suppose we start increasing r (*i.e.* increasing temperature). The magnetization follows the stable fixed point $u^*(\bar{r}) = \big(\frac{1}{2} + (\frac{1}{4} - \bar{r})^{1/2}\big)^{1/2}$ past $\bar{r} = 0$, beyond the first order phase transition point at $\bar{r} = \frac{3}{16}$, and all the way up to $\bar{r} = \frac{1}{4}$, at which point this fixed point is annihilated at a saddle-node bifurcation. The flow then rapidly takes $u \to u^* = 0^+$, where it remains as r continues to be increased further.

Within the region $\bar{r} \in \big[0, \frac{1}{4}\big]$ of control parameter space, the dynamics are said to be *irreversible* and the behavior of $u(s)$ is said to be *hysteretic*.

Chapter 14

Two-Dimensional Phase Flows

We've seen how, for one-dimensional dynamical systems $\dot{u} = f(u)$, the possibilities in terms of the behavior of the system are in fact quite limited. Starting from an arbitrary initial condition $u(0)$, the phase flow is monotonically toward the first stable fixed point encountered. (That point may lie at infinity.) No oscillations are possible[1]. For $N = 2$ phase flows, a richer set of possibilities arises, as we shall now see.

14.1 Harmonic Oscillator and Pendulum

14.1.1 Simple harmonic oscillator

A one-dimensional harmonic oscillator obeys the equation of motion,

$$m\frac{d^2x}{dt^2} = -kx \quad , \tag{14.1}$$

where m is the mass and k the force constant (of a spring). If we define $v = \dot{x}$, this may be written as the $N = 2$ system,

$$\frac{d}{dt}\begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\Omega^2 & 0 \end{pmatrix}\begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} v \\ -\Omega^2\, x \end{pmatrix} \quad , \tag{14.2}$$

where $\Omega = \sqrt{k/m}$ has the dimensions of frequency (inverse time). The solution is well known:

$$\begin{aligned} x(t) &= x_0\, \cos(\Omega t) + \frac{v_0}{\Omega}\, \sin(\Omega t) \\ v(t) &= v_0\, \cos(\Omega t) - \Omega\, x_0\, \sin(\Omega t) \quad . \end{aligned} \tag{14.3}$$

The phase curves are ellipses:

$$\Omega\, x^2(t) + \Omega^{-1}\, v^2(t) = C \quad , \tag{14.4}$$

[1]If phase space itself is multiply connected, *e.g.* a circle, then the system can oscillate by moving around the circle.

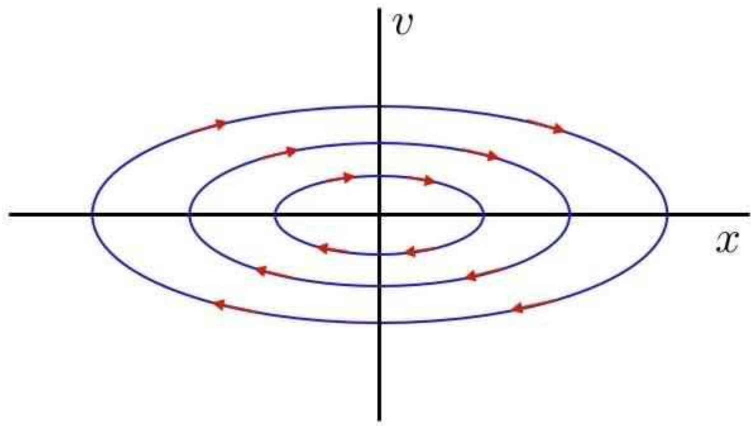


Figure 14.1: Phase curves for the harmonic oscillator.

where the constant $C = \Omega\, x_0^2 + \Omega^{-1}\, v_0^2$. A sketch of the phase curves and of the phase flow is shown in Fig. 14.1. Note that the x and v axes have different dimensions. Note also that the origin is a fixed point, however, unlike the $N = 1$ systems studied in the first lecture, here the phase flow can avoid the fixed points, and oscillations can occur.

Incidentally, eqn. 14.2 is linear, and may be solved by the following method. Write the equation as $\dot{\boldsymbol{\varphi}} = M\boldsymbol{\varphi}$, with

$$\boldsymbol{\varphi} = \begin{pmatrix} x \\ \dot{x} \end{pmatrix} \qquad \text{and} \qquad M = \begin{pmatrix} 0 & 1 \\ -\Omega^2 & 0 \end{pmatrix} \tag{14.5}$$

The formal solution to $\dot{\boldsymbol{\varphi}} = M\boldsymbol{\varphi}$ is

$$\boldsymbol{\varphi}(t) = e^{Mt}\, \boldsymbol{\varphi}(0) \quad . \tag{14.6}$$

What do we mean by the exponential of a matrix? We mean its Taylor series expansion:

$$e^{Mt} = \mathbb{I} + Mt + \tfrac{1}{2!}\, M^2\, t^2 + \tfrac{1}{3!}\, M^3\, t^3 + \ldots \quad . \tag{14.7}$$

Note that

$$\begin{aligned} M^2 &= \begin{pmatrix} 0 & 1 \\ -\Omega^2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -\Omega^2 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -\Omega^2 & 0 \\ 0 & -\Omega^2 \end{pmatrix} = -\Omega^2\, \mathbb{I} \quad , \end{aligned} \tag{14.8}$$

hence

$$M^{2k} = (-\Omega^2)^k\, \mathbb{I} \qquad , \qquad M^{2k+1} = (-\Omega^2)^k\, M \quad . \tag{14.9}$$

Thus,

$$
\begin{aligned}
e^{Mt} &= \sum_{k=0}^{\infty} \frac{1}{(2k)!}\,(-\Omega^2 t^2)^k \cdot \mathbb{I} + \sum_{k=0}^{\infty} \frac{1}{(2k+1)!}\,(-\Omega^2 t^2)^k \cdot Mt \\
&= \cos(\Omega t)\cdot \mathbb{I} + \Omega^{-1}\sin(\Omega t)\cdot M \qquad (14.10) \\
&= \begin{pmatrix} \cos(\Omega t) & \Omega^{-1}\sin(\Omega t) \\ -\Omega\,\sin(\Omega t) & \cos(\Omega t) \end{pmatrix} \quad .
\end{aligned}
$$

Plugging this into eqn. 14.6, we obtain the desired solution.

For the damped harmonic oscillator, we have

$$\ddot{x} + 2\beta\dot{x} + \Omega^2 x = 0 \qquad \Longrightarrow \qquad M = \begin{pmatrix} 0 & 1 \\ -\Omega^2 & -2\beta \end{pmatrix} \quad . \tag{14.11}$$

The phase curves then spiral inward to the fixed point at $(0, 0)$.

14.1.2 Pendulum

Next, consider the simple pendulum, composed of a mass point m affixed to a massless rigid rod of length ℓ.

$$m\ell^2\,\ddot{\theta} = -mg\ell\sin\theta \quad . \tag{14.12}$$

This is equivalent to

$$\frac{d}{dt}\begin{pmatrix}\theta \\ \omega\end{pmatrix} = \begin{pmatrix}\omega \\ -\Omega^2\,\sin\theta\end{pmatrix} \quad , \tag{14.13}$$

where $\omega = \dot{\theta}$ is the angular velocity, and where $\Omega = \sqrt{g/\ell}$ is the natural frequency of small oscillations.

The phase curves for the pendulum are shown in Fig. 14.2. The small oscillations of the pendulum are essentially the same as those of a harmonic oscillator. Indeed, within the small angle approximation, $\sin\theta \approx \theta$, and the pendulum equations of motion are exactly those of the harmonic oscillator. These oscillations are called *librations*. They involve a back-and-forth motion in real space, and the phase space motion is contractable to a point, in the topological sense. However, if the initial angular velocity is large enough, a qualitatively different kind of motion is observed, whose phase curves are *rotations*. In this case, the pendulum bob keeps swinging around in the same direction, because, as we'll see in a later lecture, the total energy is sufficiently large. The phase curve which separates these two topologically distinct motions is called a *separatrix*.

14.2 General $N = 2$ Systems

The general form to be studied is

$$\frac{d}{dt}\begin{pmatrix}x \\ y\end{pmatrix} = \begin{pmatrix}V_x(x,y) \\ V_y(x,y)\end{pmatrix} \quad . \tag{14.14}$$

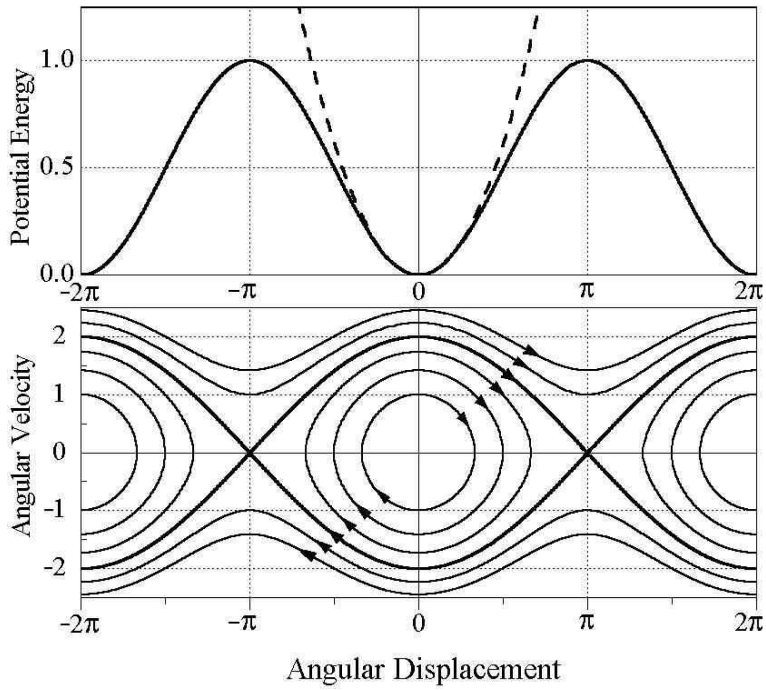


Figure 14.2: Phase curves for the simple pendulum. The *separatrix* divides phase space into regions of vibration and libration.

Special cases include autonomous second order ODEs, *viz.*

$$\ddot{x} = f(x, \dot{x}) \qquad \Longrightarrow \qquad \frac{d}{dt}\begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} v \\ f(x,v) \end{pmatrix} \quad , \tag{14.15}$$

of the type which occur in one-dimensional mechanics.

14.2.1 The damped driven pendulum

Another example is that of the damped and driven harmonic oscillator,

$$\frac{d^2\phi}{ds^2} + \gamma\,\frac{d\phi}{ds} + \sin\phi = j \quad . \tag{14.16}$$

This is equivalent to a model of a resistively and capacitively shunted Josephson junction, depicted in fig. 14.3. If ϕ is the superconducting phase difference across the junction, the current through the junction is given by $I_J = I_c \sin\phi$, where I_c is the *critical current*. The current carried by the resistor is $I_R = V/R$ from Ohm's law, and the current from the capacitor is $I_C = \dot{Q}$. Finally, the *Josephson relation* relates the voltage V across the junction to the superconducting phase difference ϕ: $V = (\hbar/2e)\,\dot{\phi}$. Summing up the parallel currents, we have that the total current I is given by

$$I = \frac{\hbar C}{2e}\,\ddot{\phi} + \frac{\hbar}{2eR}\,\dot{\phi} + I_c\,\sin\phi \quad , \tag{14.17}$$

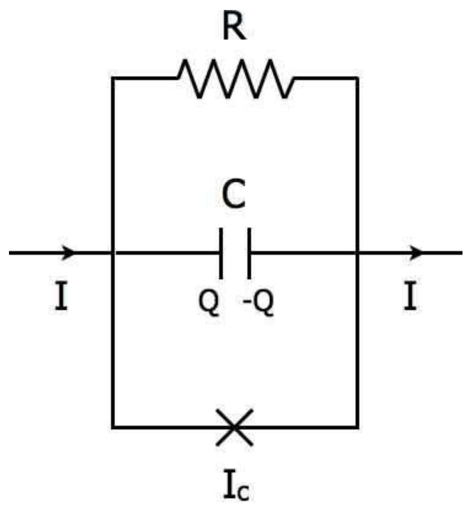


Figure 14.3: . The resistively and capacitively shunted Josephson junction. The Josephson junction is the X element at the bottom of the figure.

which, again, is equivalent to a damped, driven pendulum.

This system also has a mechanical analog. Define the 'potential'

$$U(\phi) = -I_c \cos\phi - I\phi \quad . \tag{14.18}$$

The equation of motion is then

$$\frac{\hbar C}{2e}\,\ddot{\phi} + \frac{\hbar}{2eR}\,\dot{\phi} = -\frac{\partial U}{\partial \phi} \quad . \tag{14.19}$$

Thus, the combination $\hbar C/2e$ plays the role of the inertial term (mass, or moment of inertia), while the combination $\hbar/2eR$ plays the role of a damping coefficient. The potential $U(\phi)$ is known as the *tilted washboard potential*, for obvious reasons. (Though many of you have perhaps never seen a washboard.)

The model is adimensionalized by defining the Josephson plasma frequency ω_p and the RC time constant τ:

$$\omega_p \equiv \sqrt{\frac{2eI_c}{\hbar C}} \qquad , \qquad \tau \equiv RC \quad . \tag{14.20}$$

The dimensionless combination $\omega_p\tau$ then enters the adimensionalized equation as the sole control parameter:

$$\frac{I}{I_c} = \frac{d^2\phi}{ds^2} + \frac{1}{\omega_p\tau}\,\frac{d\phi}{ds} + \sin\phi \quad , \tag{14.21}$$

where $s = \omega_p t$. In the Josephson junction literature, the quantity $\beta \equiv 2eI_cR^2C/\hbar = (\omega_p\tau)^2$, known as the *McCumber-Stewart* parameter, is a dimensionless measure of the damping (large β means small damping). In terms of eqn. 14.16, we have $\gamma = (\omega_p\tau)^{-1}$ and $j = I/I_c$.

We can write the second order ODE of eqn. 14.16 as two coupled first order ODEs:

$$\frac{d}{dt}\begin{pmatrix}\phi\\ \omega\end{pmatrix} = \begin{pmatrix}\omega\\ j - \sin\phi - \gamma\,\omega\end{pmatrix} \quad , \tag{14.22}$$

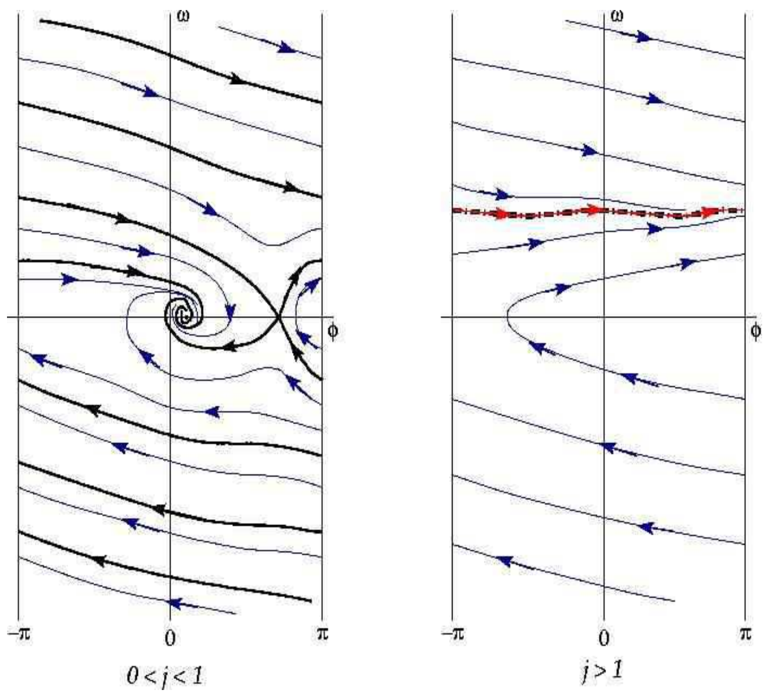


Figure 14.4: Phase flows for the equation $\ddot{\phi} + \gamma^{-1}\dot{\phi} + \sin\phi = j$. Left panel: $0 < j < 1$; note the separatrix (in black), which flows into the stable and unstable fixed points. Right panel: $j > 1$. The red curve overlying the thick black dot-dash curve is a *limit cycle*.

where $\omega = \dot{\phi}$. Phase space is a cylinder, $\mathbb{S}^1 \times \mathbb{R}^1$.

The quantity $\omega_\text{p}\tau$ typically ranges from 10^{-3} to 10^3 in Josephson junction applications. If $\omega_\text{p}\tau$ is small, then the system is heavily damped, and the inertial term $d^2\phi/ds^2$ can be neglected. One then obtains the $N = 1$ system

$$\gamma \, \frac{d\phi}{ds} = j - \sin\phi \quad . \tag{14.23}$$

If $|j| < 1$, then $\phi(s)$ evolves to the first stable fixed point encountered, where $\phi^* = \sin^{-1}(j)$ and $\cos\phi^* = \sqrt{1 - j^2}$. Since $\phi(s) \to \phi^*$ is asymptotically a constant, the voltage drop V must then vanish, as a consequence of the Josephson relation $V = (\hbar/2e)\,\dot{\phi}$. This, there is current flowing with no voltage drop!

If $|j| > 1$, the RHS never vanishes, in which case $\phi(s)$ is monotonic. We then can integrate the differential equation

$$dt = \frac{\hbar}{2eR} \cdot \frac{d\phi}{I - I_\text{c}\,\sin\phi} \quad . \tag{14.24}$$

Asymptotically the motion is periodic, with the period T obtained by integrating over the interval $\phi \in [0, 2\pi]$. One finds

$$T = \frac{\hbar}{2eR} \cdot \frac{2\pi}{\sqrt{I^2 - I_\text{c}^2}} \quad . \tag{14.25}$$

The time-averaged voltage drop is then

$$\langle V \rangle = \frac{\hbar}{2e}\,\langle \dot{\phi} \rangle = \frac{\hbar}{2e}\cdot\frac{2\pi}{T} = R\sqrt{I^2 - I_c^2} \quad . \tag{14.26}$$

This is the physics of the *current-biased resistively and capacitively shunted Josephson junction* in the strong damping limit. It is 'current-biased' because we are specifying the current I. Note that Ohm's law is recovered at large values of I.

For general $\omega_p\tau$, we can still say quite a bit. At a fixed point, both components of the vector field $\boldsymbol{V}(\phi,\omega)$ must vanish. This requires $\omega = 0$ and $j = \sin\phi$. Therefore, there are two fixed points for $|j| < 1$, one a saddle point and the other a stable spiral. For $|j| > 1$ there are no fixed points, and asymptotically the function $\phi(t)$ tends to a periodic *limit cycle* $\phi_{LC}(t)$. The flow is sketched for two representative values of j in Fig. 14.4.

14.2.2 Classification of $N = 2$ fixed points

Suppose we have solved the fixed point equations $V_x(x^*, y^*) = 0$ and $V_y(x^*, y^*) = 0$. Let us now expand about the fixed point, writing

$$\begin{aligned}\dot{x} &= \frac{\partial V_x}{\partial x}\bigg|_{(x^*,y^*)} (x - x^*) + \frac{\partial V_x}{\partial y}\bigg|_{(x^*,y^*)} (y - y^*) + \ldots \\ \dot{y} &= \frac{\partial V_y}{\partial x}\bigg|_{(x^*,y^*)} (x - x^*) + \frac{\partial V_y}{\partial y}\bigg|_{(x^*,y^*)} (y - y^*) + \ldots \quad .\end{aligned} \tag{14.27}$$

We define

$$u_1 = x - x^* \quad , \qquad u_2 = y - y^* \quad , \tag{14.28}$$

which, to linear order, satisfy

$$\frac{d}{dt}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \overbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}^{M} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \mathcal{O}(u^2) \quad . \tag{14.29}$$

The formal solution to $\dot{\boldsymbol{u}} = M\boldsymbol{u}$ is

$$\boldsymbol{u}(t) = \exp(Mt)\,\boldsymbol{u}(0) \quad , \tag{14.30}$$

where $\exp(Mt) = \sum_{n=0}^{\infty} \frac{1}{n!}\,(Mt)^n$ is the exponential of the matrix Mt.

The behavior of the system is determined by the eigenvalues of M, which are roots of the characteristic equation $P(\lambda) = 0$, where

$$\begin{aligned}P(\lambda) &= \det(\lambda\mathbb{I} - M) \\ &= \lambda^2 - T\lambda + D \quad ,\end{aligned} \tag{14.31}$$

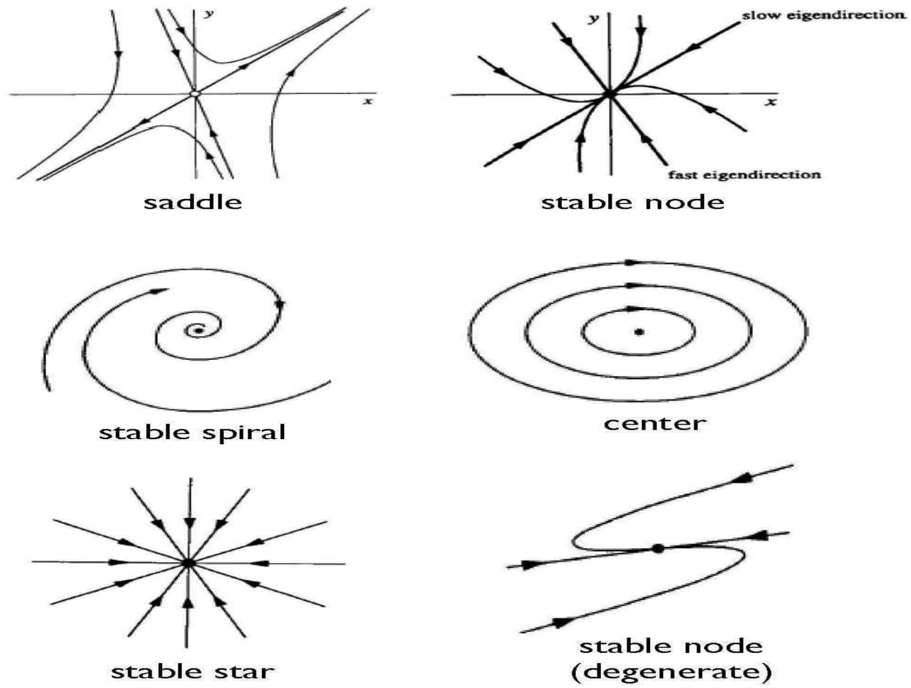


Figure 14.5: Fixed point zoo for $N = 2$ systems. Not shown: unstable versions of node, spiral, and star (reverse direction of arrows to turn stable into unstable).

with $T = a + d = \mathrm{Tr}(M)$ and $D = ad - bc = \det(M)$. The two eigenvalues are therefore

$$\lambda_{\pm} = \tfrac{1}{2}\Big(T \pm \sqrt{T^2 - 4D}\Big) \quad . \tag{14.32}$$

To see why the eigenvalues control the behavior, let us expand $\boldsymbol{u}(0)$ in terms of the eigenvectors of M. Since M is not necessarily symmetric, we should emphasize that we expand $\boldsymbol{u}(0)$ in terms of the *right* eigenvectors of M, which satisfy

$$M\boldsymbol{\psi}_a = \lambda_a \boldsymbol{\psi}_a \quad , \tag{14.33}$$

where the label a runs over the symbols $+$ and $-$, as in (14.32). We write

$$\boldsymbol{u}(t) = \sum_a C_a(t)\, \boldsymbol{\psi}_a \quad . \tag{14.34}$$

Since (we assume) the eigenvectors are *linearly independent*, the equation $\dot{\boldsymbol{u}} = M\boldsymbol{u}$ becomes

$$\dot{C}_a = \lambda_a\, C_a \quad , \tag{14.35}$$

with solution

$$C_a(t) = e^{\lambda_a t}\, C_a(0) \quad . \tag{14.36}$$

Thus, the coefficients of the eigenvectors $\boldsymbol{\psi}_a$ will *grow* in magnitude if $|\lambda_a| > 1$, and will *shrink* if $|\lambda_a| < 1$.

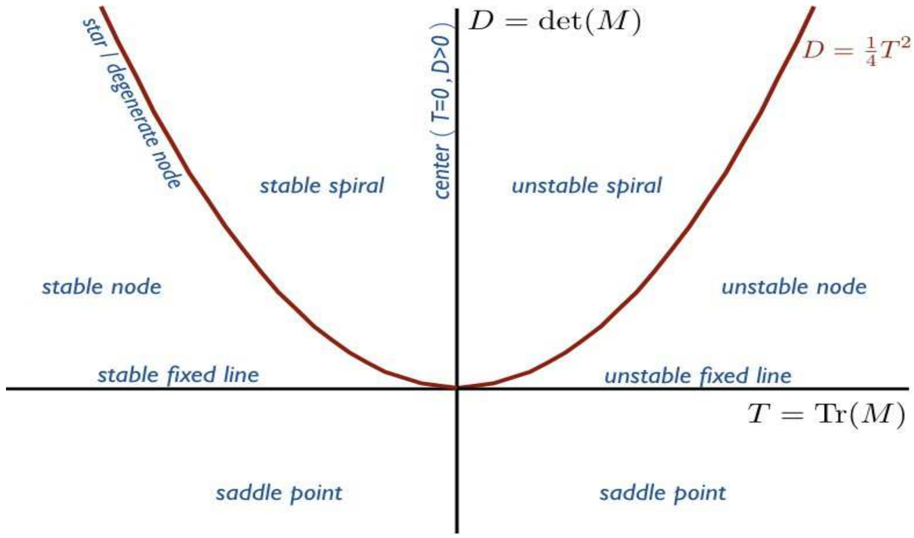


Figure 14.6: Complete classification of fixed points for the $N = 2$ system.

14.2.3 The fixed point zoo

- **Saddles** : When $D < 0$, both eigenvalues are real; one is positive and one is negative, *i.e.* $\lambda_+ > 0$ and $\lambda_- < 0$. The right eigenvector $\boldsymbol{\psi}_-$ is thus the *stable direction* while $\boldsymbol{\psi}_+$ is the *unstable direction*.

- **Nodes** : When $0 < D < \frac{1}{4}T^2$, both eigenvalues are real and of the same sign. Thus, both right eigenvectors correspond to stable or to unstable directions, depending on whether $T < 0$ (stable; $\lambda_- < \lambda_+ < 0$) or $T < 0$ (unstable; $\lambda_+ > \lambda_- > 0$). If $\lambda_\pm$ are distinct, one can distinguish *fast* and *slow* eigendirections, based on the magnitude of the eigenvalues.

- **Spirals** : When $D > \frac{1}{4}T^2$, the discriminant $T^2 - 4D$ is negative, and the eigenvalues come in a complex conjugate pair: $\lambda_- = \lambda_+^*$. The real parts are given by $\mathrm{Re}(\lambda_\pm) = \frac{1}{2}T$, so the motion is stable (*i.e.* collapsing to the fixed point) if $T < 0$ and unstable (*i.e.* diverging from the fixed point) if $T > 0$. The motion is easily shown to correspond to a spiral. One can check that the spiral rotates counterclockwise for $a > d$ and clockwise for $a < d$.

- **Degenerate Cases** : When $T = 0$ we have $\lambda_\pm = \pm\sqrt{-D}$. For $D < 0$ we have a saddle, but for $D > 0$ both eigenvalues are imaginary: $\lambda_\pm = \pm i\sqrt{D}$. The orbits do not collapse to a point, nor do they diverge to infinity, in the $t \to \infty$ limit, as they do in the case of the stable and unstable spiral. The fixed point is called a *center*, and it is surrounded by closed trajectories.

  When $D = \frac{1}{4}T^2$, the discriminant vanishes and the eigenvalues are degenerate. If the rank of M is two, the fixed point is a stable ($T < 0$) or unstable ($T > 0$) *star*. If M is degenerate and of rank one, the fixed point is a *degenerate node*.

  When $D = 0$, one of the eigenvalues vanishes. This indicates a *fixed line* in phase space, since any point on that line will not move. The fixed line can be stable or unstable, depending on whether the remaining eigenvalue is negative (stable, $T < 0$), or positive (unstable, $T > 0$).

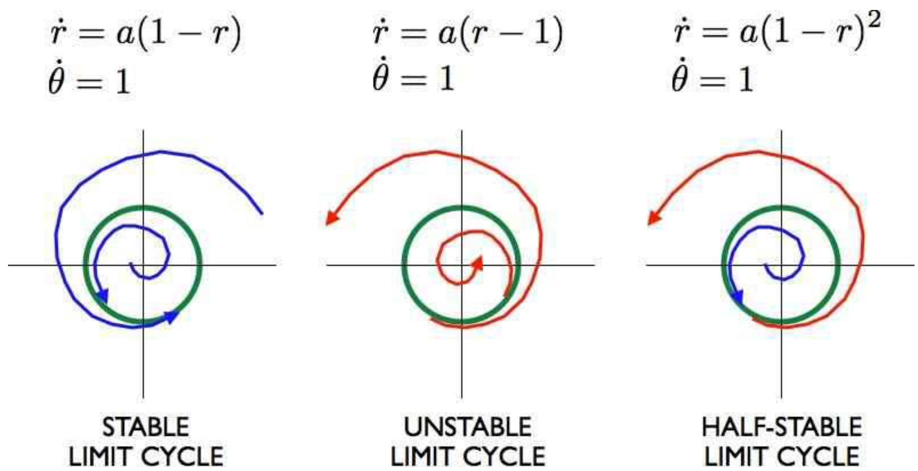


Figure 14.7: Stable, unstable, and half-stable limit cycles.

Putting it all together, an example of a phase portrait is shown in Fig. 14.8. Note the presence of an *isolated, closed trajectory*, which is called a *limit cycle*. Many self-sustained physical oscillations, *i.e.* oscillations with no external forcing, exhibit limit cycle behavior. Limit cycles, like fixed points, can be stable or unstable, or partially stable. Limit cycles are inherently nonlinear. While the linear equation $\dot{\boldsymbol{\varphi}} = M\,\boldsymbol{\varphi}$ can have periodic solutions if M has purely imaginary eigenvalues, these periodic trajectories are not *isolated*, because $\lambda\,\boldsymbol{\varphi}(t)$ is also a solution. The amplitude of these linear oscillations is fixed by the initial conditions, whereas for limit cycles, the amplitude is inherent from the dynamics itself, and the initial conditions are irrelevant (for a stable limit cycle).

In fig. 14.7 we show simple examples of stable, unstable, and half-stable limit cycles. As we shall see when we study nonlinear oscillations, the Van der Pol oscillator,

$$\ddot{x} + \mu(x^2 - 1)\,\dot{x} + x = 0 \quad , \tag{14.37}$$

with $\mu > 0$ has a stable limit cycle. The physics is easy to apprehend. The coefficient of the $\dot{x}$ term in the equation of motion is positive for $|x| > 1$ and negative for $|x| < 1$. Interpreting this as a coefficient of friction, we see that the friction is positive, *i.e.* dissipating energy, when $|x| > 1$ but *negative*, *i.e.* accumulating energy, for $|x| < 1$. Thus, any small motion with $|x| < 1$ is *amplified* due to the negative friction, and would increase without bound were it not for the fact that the friction term reverses its sign and becomes dissipative for $|x| > 1$. The limit cycle for $\mu \gg 1$ is shown in fig. 14.9.

14.2.4 Fixed points for $N = 3$ systems

For an $N = 2$ system, there are five generic types of fixed points. They are classified according to the eigenvalues of the linearized dynamics at the fixed point. For a real 2×2 matrix, the eigenvalues must

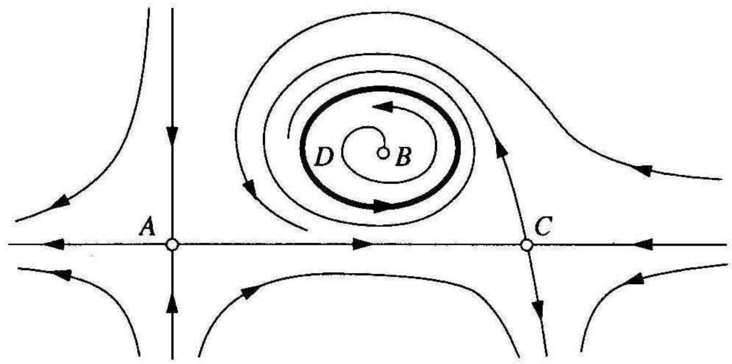


Figure 14.8: Phase portrait for an $N = 2$ flow including saddles (A,C), unstable spiral (B), and limit cycle (D).

be real or else must be a complex conjugate pair. The five types of fixed points are then

$$\begin{aligned} \lambda_1 > 0 \quad &, \lambda_2 > 0 \quad : \quad \text{(1) unstable node} \\ \lambda_1 > 0 \quad &, \lambda_2 < 0 \quad : \quad \text{(2) saddle point} \\ \lambda_1 < 0 \quad &, \lambda_2 < 0 \quad : \quad \text{(3) stable node} \\ \text{Re}\,\lambda_1 > 0 \quad &, \lambda_2 = \lambda_1^* \quad : \quad \text{(4) unstable spiral} \\ \text{Re}\,\lambda_1 < 0 \quad &, \lambda_2 = \lambda_1^* \quad : \quad \text{(5) stable spiral} \end{aligned} \tag{14.38}$$

How many possible generic fixed points are there for an $N = 3$ system?

For a general real 3×3 matrix M, the characteristic polynomial $P(\lambda) = \det(\lambda - M)$ satisfies $P(\lambda^*) = P(\lambda)$. Thus, if λ is a root then so is λ^*. This means that the eigenvalues are either real or else come in complex conjugate pairs. There are then ten generic possibilities for the three eigenvalues:

$$\begin{aligned}
&(1) & \text{unstable node} \quad &: \quad \lambda_1 > \lambda_2 > \lambda_3 > 0 \\
&(2) & (+\,+\,-)\text{ saddle} \quad &: \quad \lambda_1 > \lambda_2 > 0 > \lambda_3 \\
&(3) & (+\,-\,-)\text{ saddle} \quad &: \quad \lambda_1 > 0 > \lambda_2 > \lambda_3 \\
&(4) & \text{stable node} \quad &: \quad 0 > \lambda_1 > \lambda_2 > \lambda_3 \\
&(5) & \text{unstable spiral-node} \quad &: \quad \lambda_1 > \text{Re}\,\lambda_{2,3} > 0\,;\ \text{Im}\,\lambda_2 = -\text{Im}\,\lambda_3 \\
&(6) & \text{unstable spiral-node} \quad &: \quad \text{Re}\,\lambda_{1,2} > \lambda_3 > 0\,;\ \text{Im}\,\lambda_1 = -\text{Im}\,\lambda_2 \\
&(7) & \text{stable spiral-node} \quad &: \quad 0 > \lambda_1 > \text{Re}\,\lambda_{2,3}\,;\ \text{Im}\,\lambda_2 = -\text{Im}\,\lambda_3 \\
&(8) & \text{stable spiral-node} \quad &: \quad 0 > \text{Re}\,\lambda_{1,2} > \lambda_3\,;\ \text{Im}\,\lambda_1 = -\text{Im}\,\lambda_2 \\
&(9) & (+\,-\,-)\text{ spiral-saddle} \quad &: \quad \lambda_1 > 0 > \text{Re}\,\lambda_{2,3}\,;\ \text{Im}\,\lambda_2 = -\text{Im}\,\lambda_3 \\
&(10) & (+\,+\,-)\text{ spiral-saddle} \quad &: \quad \text{Re}\,\lambda_{1,2} > 0 > \lambda_3\,;\ \text{Im}\,\lambda_1 = -\text{Im}\,\lambda_2 \quad .
\end{aligned}$$

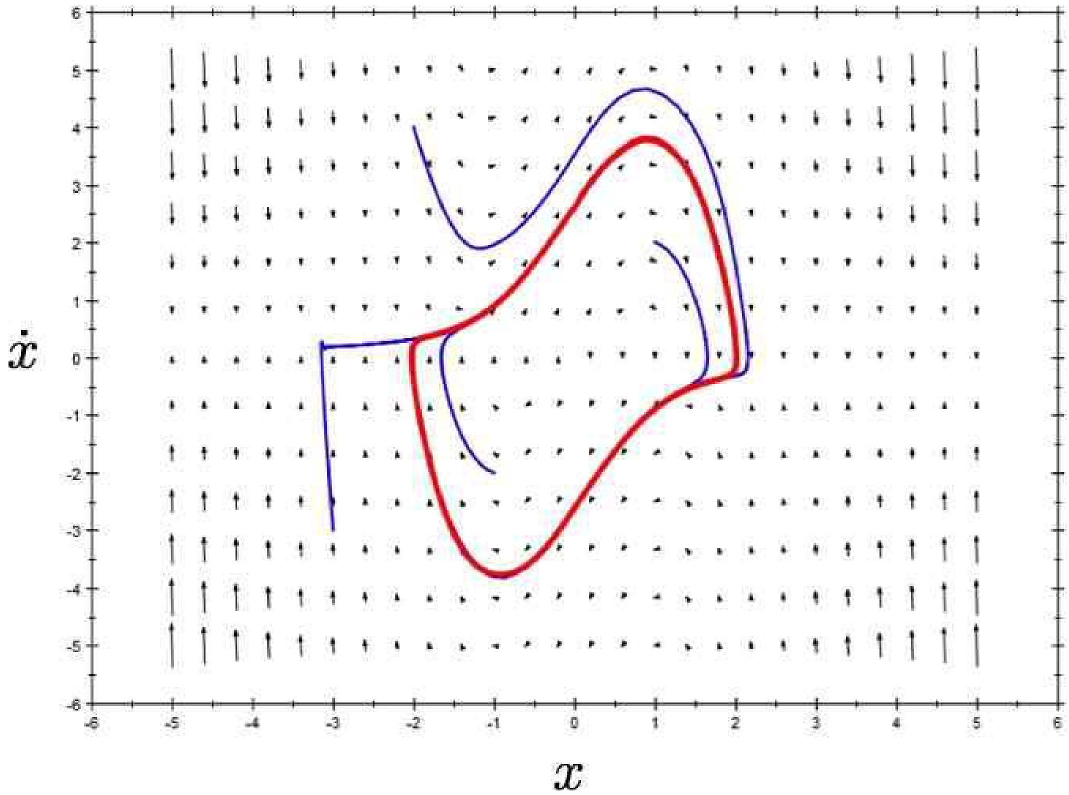


Figure 14.9: Limit cycle of the Van der Pol oscillator for $\mu \gg 1$. (Source: Wikipedia)

14.3 Andronov-Hopf Bifurcation

A bifurcation between a spiral and a limit cycle is known as an *Andronov-Hopf bifurcation*. As a simple example, consider the $N = 2$ system,

$$\begin{aligned}\dot{x} &= ax - by - C(x^2 + y^2)\,x \\ \dot{y} &= bx + ay - C(x^2 + y^2)\,y \quad ,\end{aligned} \tag{14.39}$$

where a, b, and C are real. Clearly the origin is a fixed point, at which one finds the eigenvalues $\lambda = a \pm ib$. Thus, the fixed point is a stable spiral if $a < 0$ and an unstable spiral if $a > 0$.

Written in terms of the complex variable $z = x + iy$, these two equations collapse to the single equation

$$\dot{z} = (a + ib)\,z - C\,|z|^2\,z \quad . \tag{14.40}$$

The dynamics are also simple in polar coordinates $r = |z|$, $\theta = \arg(z)$:

$$\begin{aligned}\dot{r} &= ar - Cr^3 \\ \dot{\theta} &= b \quad .\end{aligned} \tag{14.41}$$

The phase diagram, for fixed $b > 0$, is depicted in Fig. 14.10. For positive a/C, there is a limit cycle at $r = \sqrt{a/C}$. In both cases, the limit cycle disappears as a crosses the value $a^* = 0$ and is replaced by a stable $(a < 0, C > 0)$ or unstable $(a > 0, C < 0)$ spiral.

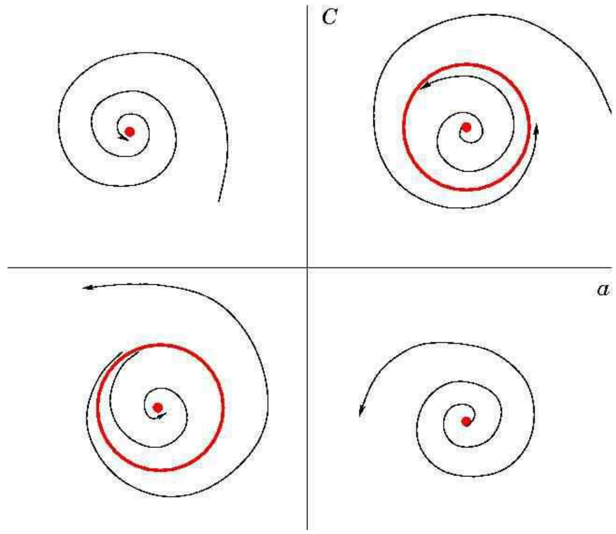


Figure 14.10: Hopf bifurcation: for $C > 0$ the bifurcation is supercritical, between stable spiral and stable limit cycle. For $C < 0$ the bifurcation is subcritical, between unstable spiral and unstable limit cycle. The bifurcation occurs at $a = 0$ in both cases.

This example also underscores the following interesting point. Adding a small nonlinear term C has no fundamental effect on the fixed point behavior so long as $a \neq 0$, when the fixed point is a stable or unstable spiral. In general, fixed points which are attractors (stable spirals or nodes), repellers (unstable spirals or nodes), or saddles are *robust* with respect to the addition of a small nonlinearity. But the fixed point behavior in the marginal cases – centers, stars, degenerate nodes, and fixed lines – is strongly affected by the presence of even a small nonlinearity. In this example, the FP is a center when $a = 0$. But as the (r, θ) dynamics shows, a small nonlinearity will destroy the center and turn the FP into an attractor $(C > 0)$ or a repeller $(C < 0)$.

14.4 Population Biology: Lotka-Volterra Models

Consider two species with populations N_1 and N_2, respectively[2]. We model the evolution of these populations by the coupled ODEs

$$\begin{aligned} \frac{dN_1}{dt} &= aN_1 + bN_1N_2 + cN_1^2 \\ \frac{dN_2}{dt} &= dN_2 + eN_1N_2 + fN_2^2 \quad , \end{aligned} \tag{14.42}$$

[2]This discussion is adapted from S. Strogatz, *Nonlinear Dynamics and Chaos*.

where $\{a, b, c, d, e, f\}$ are constants. We can eliminate some constants by rescaling $N_{1,2}$. This results in the following:

$$\begin{aligned}\dot{x} &= x\big(r - \mu x - ky\big) \\ \dot{y} &= y\big(r' - \mu' y - k' x\big) \quad ,\end{aligned} \tag{14.43}$$

where μ, and μ' can each take on one of three possible values $\{0, \pm 1\}$. By rescaling time, we can eliminate the scale of either of r or r' as well. Typically, intra-species competition guarantees $\mu = \mu' = +1$. The remaining coefficients (r, k, k') are real may also be of either sign. The values and especially the signs of the various coefficients have a physical (or biological) significance. For example, if $k < 0$ it means that x grows due to the presence of y. The effect of y on x may be of the same sign ($kk' > 0$) or of opposite sign ($kk' < 0$).

14.4.1 Rabbits and foxes

As an example, consider the model

$$\begin{aligned}\dot{x} &= x - xy \\ \dot{y} &= -\beta y + xy \quad .\end{aligned} \tag{14.44}$$

The quantity x might represent the (scaled) population of rabbits and y the population of foxes in an ecosystem. There are two fixed points: at $(0, 0)$ and at $(\beta, 1)$. Linearizing the dynamics about these fixed points, one finds that $(0, 0)$ is a saddle while $(\beta, 1)$ is a center. Let's do this explicitly.

The first step is to find the fixed points (x^*, y^*). To do this, we set $\dot{x} = 0$ and $\dot{y} = 0$. From $\dot{x} = x(1-y) = 0$ we have that $x = 0$ or $y = 1$. Suppose $x = 0$. The second equation, $\dot{y} = (x - \beta)y$ then requires $y = 0$. So $\boldsymbol{P}_1 = (0, 0)$ is a fixed point. The other possibility is that $y = 1$, which then requires $x = \beta$. So $\boldsymbol{P}_2 = (\beta, 1)$ is the second fixed point. Those are the only possibilities.

We now compute the linearized dynamics at these fixed points. The linearized dynamics are given by $\dot{\boldsymbol{\varphi}} = M\boldsymbol{\varphi}$, with

$$M = \begin{pmatrix} \partial\dot{x}/\partial x & \partial\dot{x}/\partial y \\ & \\ \partial\dot{y}/\partial x & \partial\dot{y}/\partial y \end{pmatrix} = \begin{pmatrix} 1 - y & -x \\ & \\ y & x - \beta \end{pmatrix} \quad . \tag{14.45}$$

Evaluating M at $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$, we find

$$M_1 = \begin{pmatrix} 1 & 0 \\ & \\ 0 & -\beta \end{pmatrix} \quad , \quad M_2 = \begin{pmatrix} 0 & -\beta \\ & \\ 1 & 0 \end{pmatrix} \quad . \tag{14.46}$$

The eigenvalues are easily found:

$$\begin{aligned}\boldsymbol{P}_1 &: \lambda_+ = 1 \quad , \quad \lambda_- = -\beta \\ \boldsymbol{P}_2 &: \lambda_+ = i\sqrt{\beta} \quad , \quad \lambda_- = -i\sqrt{\beta} \quad .\end{aligned} \tag{14.47}$$

Thus $\boldsymbol{P}_1$ is a saddle point and $\boldsymbol{P}_2$ is a center.

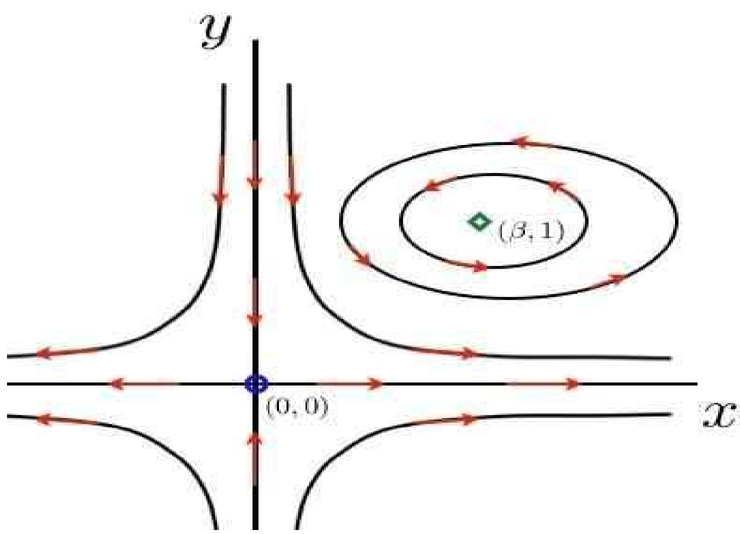


Figure 14.11: Phase flow for the rabbits *vs.* foxes Lotka-Volterra model of eqs. 14.44.

As we saw earlier, generally speaking we expect nonlinear terms to transform centers to stable or unstable spirals, possibly with a limit cycle. However for the Lotka-Volterra system there is a conserved quantity. Consider the general predator-prey system

$$\begin{aligned}\dot{x} &= (a - b\,y)\,x\\ \dot{y} &= -(c - d\,x)\,y \quad ,\end{aligned} \tag{14.48}$$

where a, b, c, and d are all positive constants. Now consider the function

$$H \equiv d\,x + b\,y - c\,\log x - a\,\log y \quad . \tag{14.49}$$

Then

$$\frac{\partial H}{\partial x} = d - \frac{c}{x} \qquad , \qquad \frac{\partial H}{\partial y} = b - \frac{a}{y} \quad . \tag{14.50}$$

Thus, we have $\dot{x} = -xy\,\frac{\partial H}{\partial y}$ and $\dot{y} = xy\,\frac{\partial H}{\partial x}$. If we define $p \equiv \log x$ and $q \equiv \log y$, then we have

$$\dot{q} = \frac{\partial H}{\partial p} \qquad , \qquad \dot{p} = -\frac{\partial H}{\partial q} \tag{14.51}$$

with

$$H(q,p) = d\,e^p + b\,e^q - c\,p - a\,q \quad . \tag{14.52}$$

So the system is a Hamiltonian system in disguise, and we know that for Hamiltonian systems the only possible fixed points are saddles and centers. The phase curves are level sets of the function H.

14.4.2 Rabbits and sheep

In the rabbits and foxes model of eqs. 14.44, the rabbits are the food for the foxes. This means $k = 1$ but $k' = -1$, *i.e.* the fox population is enhanced by the presence of rabbits, but the rabbit population is diminished by the presence of foxes. Consider now a model in which the two species (rabbits and sheep, say) compete for food:

$$\begin{aligned}\dot{x} &= x\,(r - x - ky) \\ \dot{y} &= y\,(1 - y - k'x) \quad ,\end{aligned} \tag{14.53}$$

with r, k, and k' all positive. Note that when either population x or y vanishes, the remaining population is governed by the logistic equation, *i.e.* it will flow to a nonzero fixed point.

The matrix of derivatives, which is to be evaluated at each fixed point in order to assess its stability, is

$$M = \begin{pmatrix} \partial\dot{x}/\partial x & \partial\dot{x}/\partial y \\ \partial\dot{y}/\partial x & \partial\dot{y}/\partial y \end{pmatrix} = \begin{pmatrix} r - 2x - ky & -kx \\ -k'y & 1 - 2y - k'x \end{pmatrix} \quad . \tag{14.54}$$

At each fixed point, we must evaluate $D = \mathsf{det}(M)$ and $T = \mathsf{Tr}\,(M)$ and apply the classification scheme of Fig. 14.6.

- $\boldsymbol{P}_1 = (0,0)$: This is the trivial state with no rabbits $(x=0)$ and no sheep $(y=0)$. The linearized dynamics gives $M_1 = \begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix}$, which corresponds to an unstable node.

- $\boldsymbol{P}_2 = (r,0)$: Here we have rabbits but no sheep. The linearized dynamics gives $M_2 = \begin{pmatrix} -r & -rk \\ 0 & 1 - rk' \end{pmatrix}$. For $rk' > 1$ this is a stable node; for $rk' < 1$ it is a saddle point.

- $\boldsymbol{P}_3 = (0,1)$: Here we have sheep but no rabbits. The linearized dynamics gives $M_3 = \begin{pmatrix} r - k & 0 \\ -k' & -1 \end{pmatrix}$. For $k > r$ this is a stable node; for $k < r$ it is a saddle.

- There is one remaining fixed point – a nontrivial one where both x and y are nonzero. To find it, we set $\dot{x} = \dot{y} = 0$, and divide out by x and y respectively, to get

$$\begin{aligned} x + ky &= r \\ kx' + y &= 1 \quad . \end{aligned} \tag{14.55}$$

  This is a simple rank 2 inhomogeneous linear system. If the fixed point $\boldsymbol{P}_4$ is to lie in the physical quadrant $(x > 0, y > 0)$, then either (i) $k > r$ and $k' > r^{-1}$ or (ii) $k < r$ and $k' < r^{-1}$. The solution is

$$\boldsymbol{P}_4 = \begin{pmatrix} 1 & k \\ k' & 1 \end{pmatrix}^{-1} \begin{pmatrix} r \\ 1 \end{pmatrix} = \frac{1}{1 - kk'} \begin{pmatrix} r - k \\ 1 - rk' \end{pmatrix} \quad . \tag{14.56}$$

  The linearized dynamics then gives

$$M_4 = \frac{1}{1 - kk'} \begin{pmatrix} k - r & k(k - r) \\ k'(rk' - 1) & rk' - 1 \end{pmatrix} \quad , \tag{14.57}$$

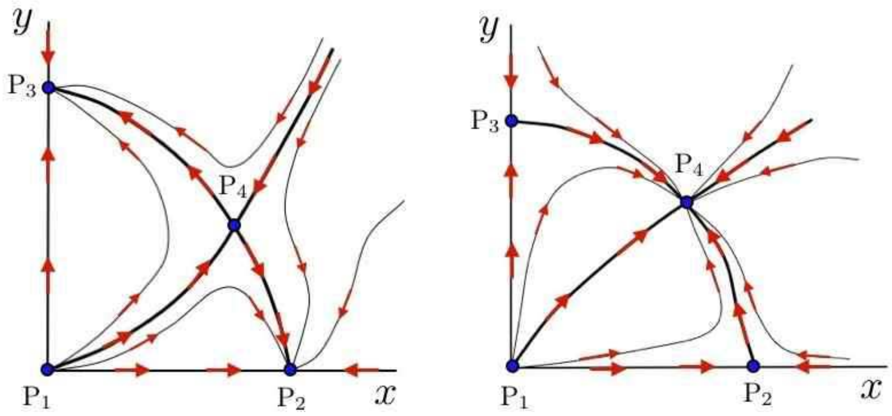


Figure 14.12: Two possible phase flows for the rabbits *vs.* sheep model of eqs. 14.53. Left panel: $k > r > k'^{-1}$. Right panel: $k < r < k'^{-1}$.

yielding

$$T = \frac{rk' - 1 + k - r}{1 - kk'}$$
$$D = \frac{(k-r)(rk'-1)}{1-kk'} \quad . \tag{14.58}$$

The classification of this fixed point can vary with parameters. Consider the case $r = 1$. If $k = k' = 2$ then both $\boldsymbol{P}_2$ and $\boldsymbol{P}_3$ are stable nodes. At $\boldsymbol{P}_4$, one finds $T = -\frac{2}{3}$ and $D = -\frac{1}{3}$, corresponding to a saddle point. In this case it is the fate of one population to die out at the expense of the other, and which one survives depends on initial conditions. If instead we took $k = k' = \frac{1}{2}$, then $T = -\frac{4}{3}$ and $D = \frac{1}{3}$, corresponding to a stable node (node $D < \frac{1}{4}T^2$ in this case). The situation is depicted in Fig. 14.12.

14.5 Poincaré-Bendixson Theorem

Although $N = 2$ systems are much richer than $N = 1$ systems, they are still ultimately rather impoverished in terms of their long-time behavior. If an orbit does not flow off to infinity or asymptotically approach a stable fixed point (node or spiral or nongeneric example), the only remaining possibility is limit cycle behavior. This is the content of the *Poincaré-Bendixson theorem*, which states:

- *IF* Ω is a compact (*i.e.* closed and bounded) subset of phase space,
- *AND* $\dot{\boldsymbol{\varphi}} = \boldsymbol{V}(\boldsymbol{\varphi})$ is continuously differentiable on Ω,
- *AND* Ω contains no fixed points (*i.e.* $\boldsymbol{V}(\boldsymbol{\varphi})$ never vanishes in Ω),
- *AND* a phase curve $\boldsymbol{\varphi}(t)$ is always confined to Ω,

◇ *THEN* $\boldsymbol{\varphi}(t)$ is either closed or approaches a closed trajectory in the limit $t \to \infty$.

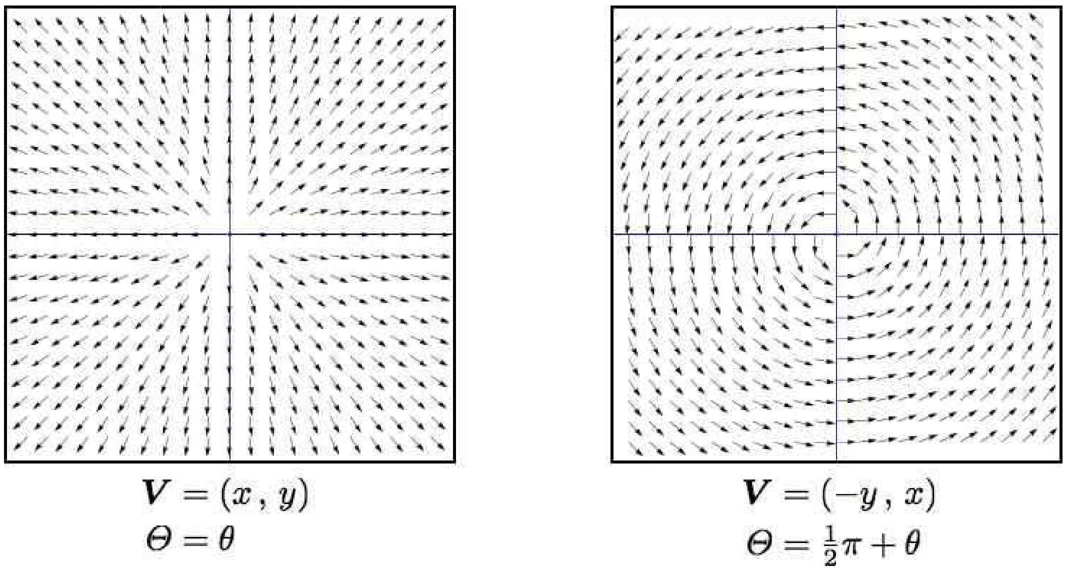


Figure 14.13: Two singularities with index $+1$. The direction field $\hat{\boldsymbol{V}} = \boldsymbol{V}/\big|\boldsymbol{V}\big|$ is shown in both cases.

Thus, under the conditions of the theorem, Ω must contain a closed orbit.

One way to prove that $\boldsymbol{\varphi}(t)$ is confined to Ω is to establish that $\boldsymbol{V}\cdot\hat{\boldsymbol{n}} \le 0$ everywhere on the boundary $\partial\Omega$, which means that the phase flow is always directed inward (or tangent) along the boundary. Let's analyze an example from the book by Strogatz. Consider the system

$$\begin{aligned}\dot{r} &= r(1-r^2) + \lambda\, r\, \cos\theta \\ \dot{\theta} &= 1 \quad ,\end{aligned} \tag{14.59}$$

with $0 < \lambda < 1$. Then define

$$a \equiv \sqrt{1-\lambda} \quad , \quad b \equiv \sqrt{1+\lambda} \tag{14.60}$$

and

$$\Omega \equiv \Big\{ (r,\theta) \;\big|\; a < r < b \Big\} \quad . \tag{14.61}$$

On the boundaries of Ω, we have

$$\begin{aligned} r = a \qquad &\Rightarrow \qquad \dot{r} = \lambda\, a\,\big(1+\cos\theta\big) \\ r = b \qquad &\Rightarrow \qquad \dot{r} = -\lambda\, b\,\big(1-\cos\theta\big) \quad . \end{aligned} \tag{14.62}$$

We see that the radial component of the flow is inward along both $r = a$ and $r = b$. Thus, any trajectory which starts inside Ω can never escape. The Poincaré-Bendixson theorem tells us that the trajectory will approach a stable limit cycle in the limit $t \to \infty$.

It is only with $N \ge 3$ systems that the interesting possibility of chaotic behavior emerges.

14.6 Index Theory

Consider a smooth two-dimensional vector field $\boldsymbol{V}(\boldsymbol{\varphi})$. The angle that the vector $\boldsymbol{V}$ makes with respect to the $\hat{\boldsymbol{\varphi}}_1$ and $\hat{\boldsymbol{\varphi}}_2$ axes is a scalar field,

$$\Theta(\boldsymbol{\varphi}) = \tan^{-1}\!\left(\frac{V_2(\boldsymbol{\varphi})}{V_1(\boldsymbol{\varphi})}\right) \quad . \tag{14.63}$$

So long as $\boldsymbol{V}$ has finite length, the angle Θ is well-defined. In particular, we expect that we can integrate $\boldsymbol{\nabla}\Theta$ over a closed curve $\mathcal{C}$ in phase space to get

$$\oint\limits_{\mathcal{C}} d\boldsymbol{\varphi}\cdot\boldsymbol{\nabla}\Theta = 0 \quad . \tag{14.64}$$

However, this can fail if $\boldsymbol{V}(\boldsymbol{\varphi})$ vanishes (or diverges) at one or more points in the interior of $\mathcal{C}$. In general, if we define

$$\mathsf{W}_{\mathcal{C}}(\boldsymbol{V}) = \frac{1}{2\pi}\oint\limits_{\mathcal{C}} d\boldsymbol{\varphi}\cdot\boldsymbol{\nabla}\Theta \quad , \tag{14.65}$$

then $\mathsf{W}_{\mathcal{C}}(\boldsymbol{V}) \in \mathbb{Z}$ is an integer valued function of $\mathcal{C}$, which is the change in Θ around the curve $\mathcal{C}$. This must be an integer, because Θ is well-defined only up to multiples of 2π. Note that *differential changes* of Θ are in general well-defined.

Thus, if $\boldsymbol{V}(\boldsymbol{\varphi})$ is finite, meaning neither infinite nor infinitesimal, *i.e.* $\boldsymbol{V}$ neither diverges nor vanishes anywhere in $\mathrm{int}(\mathcal{C})$, then $\mathsf{W}_{\mathcal{C}}(\boldsymbol{V}) = 0$. Assuming that $\boldsymbol{V}$ never diverges, any singularities in Θ must arise from points where $\boldsymbol{V} = 0$, which in general occurs at isolated points, since it entails two equations in the two variables $(\varphi_1\,,\,\varphi_2)$.

The index of a two-dimensional vector field $\boldsymbol{V}(\boldsymbol{\varphi})$ at a *point* $\boldsymbol{\varphi}$ is the integer-valued winding of $\boldsymbol{V}$ about that point:

$$\begin{aligned}\underset{\boldsymbol{\varphi}_0}{\mathsf{ind}}\,(\boldsymbol{V}) &= \lim_{a\to 0}\frac{1}{2\pi}\oint\limits_{\mathcal{C}_a(\boldsymbol{\varphi}_0)} d\boldsymbol{\varphi}\cdot\boldsymbol{\nabla}\Theta \\ &= \lim_{a\to 0}\frac{1}{2\pi}\oint\limits_{\mathcal{C}_a(\boldsymbol{\varphi}_0)} d\boldsymbol{\varphi}\cdot\frac{V_1\boldsymbol{\nabla}V_2 - V_2\boldsymbol{\nabla}V_1}{V_1^2+V_2^2} \quad ,\end{aligned} \tag{14.66}$$

where $\mathcal{C}_a(\boldsymbol{\varphi}_0)$ is a circle of radius a surrounding the point $\boldsymbol{\varphi}_0$. The index of a closed curve $\mathcal{C}$ is given by the sum of the indices at all the singularities enclosed by the curve:[3]

$$\mathsf{W}_{\mathcal{C}}(\boldsymbol{V}) = \sum_{\boldsymbol{\varphi}_i\,\in\,\mathrm{int}(\mathcal{C})} \underset{\boldsymbol{\varphi}_i}{\mathsf{ind}}\,(\boldsymbol{V}) \quad . \tag{14.67}$$

[3]Technically, we should weight the index at each enclosed singularity by the signed number of times the curve $\mathcal{C}$ encloses that singularity. For simplicity and clarity, we assume that the curve $\mathcal{C}$ is homeomorphic to the circle $\mathbb{S}^1$.

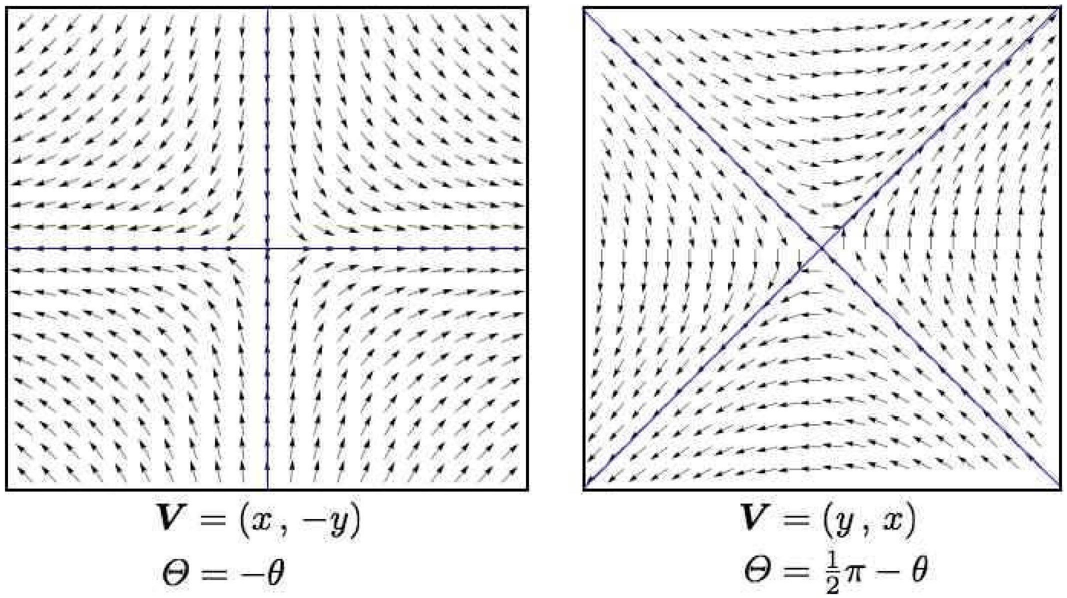


Figure 14.14: Two singularities with index -1.

As an example, consider the vector fields plotted in fig. 14.13. We have:

$$\begin{aligned} \boldsymbol{V} &= (x\,,\,y) \qquad &\Rightarrow \qquad \Theta &= \theta \\ \boldsymbol{V} &= (-y\,,\,x) \qquad &\Rightarrow \qquad \Theta &= \theta + \tfrac{1}{2}\pi \quad . \end{aligned} \tag{14.68}$$

The index is the same, $+1$, in both cases, even though the first corresponds to an unstable node and the second to a center. Any $N = 2$ fixed point with $\det M > 0$ has index $+1$.

Fig. 14.14 shows two vector fields, each with index -1:

$$\begin{aligned} \boldsymbol{V} &= (x\,,\,-y) \qquad &\Rightarrow \qquad \Theta &= -\theta \\ \boldsymbol{V} &= (y\,,\,x) \qquad &\Rightarrow \qquad \Theta &= -\theta + \tfrac{1}{2}\pi \quad . \end{aligned} \tag{14.69}$$

In both cases, the fixed point is a saddle.

As an example of the content of eqn. 14.67, consider the vector fields in eqn. 14.15. The left panel shows the vector field $\boldsymbol{V} = (x^2 - y^2\,,\,2xy)$, which has a single fixed point, at the origin $(0\,,\,0)$, of index $+2$. The right panel shows the vector field $\boldsymbol{V} = (1 + x^2 - y^2\,,\,x + 2xy)$, which has fixed points $(x^*\,,\,y^*)$ at $(0\,,\,1)$ and $(0\,,\,-1)$. The linearized dynamics is given by the matrix

$$M = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\[2ex] \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{pmatrix} = \begin{pmatrix} 2x & -2y \\[2ex] 1 + 2y & 2x \end{pmatrix} \quad . \tag{14.70}$$

Thus,

$$M_{(0,1)} = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} \qquad , \qquad M_{(0,-1)} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \quad . \tag{14.71}$$

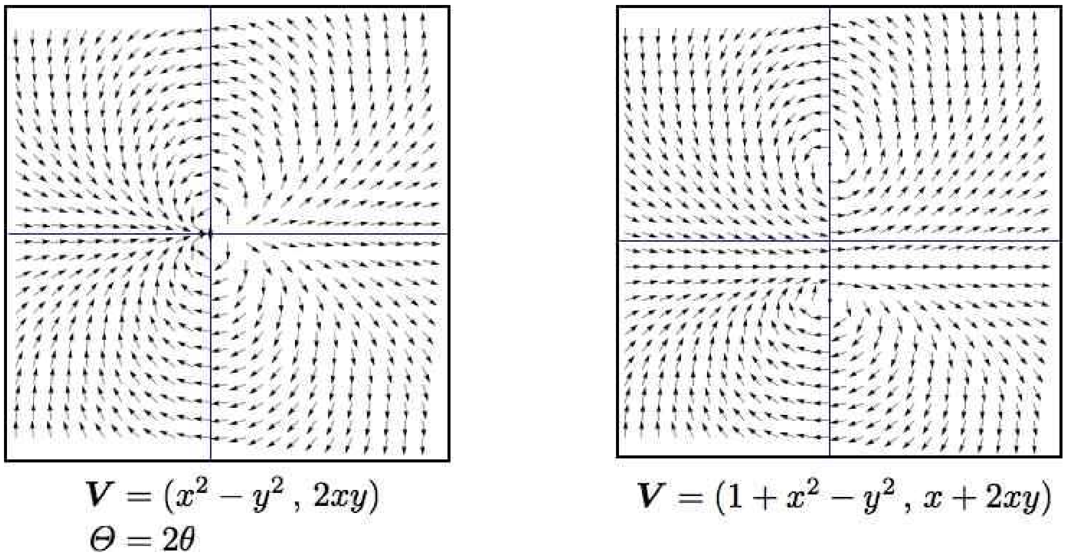


Figure 14.15: Left panel: a singularity with index $+2$. Right panel: two singularities each with index $+1$. Note that the long distance behavior of $\boldsymbol{V}$ is the same in both cases.

At each of these fixed points, we have $T = 0$ and $D = 4$, corresponding to a center, with index $+1$. If we consider a square-ish curve $\mathcal{C}$around the periphery of each figure, the vector field is almost the same along such a curve for both the left and right panels, and the winding number is $\mathsf{W}_{\mathcal{C}}(\boldsymbol{V}) = +2$.

Finally, consider the vector field shown in fig. 14.16, with $\boldsymbol{V} = (x^2 - y^2\,,\,-2xy)$. Clearly $\Theta = -2\theta$, and the index of the singularity at $(0\,,\,0)$ is -2.

To recapitulate some properties of the index / winding number:

- The index $\mathsf{ind}_{\varphi_0}(\boldsymbol{V})$ of an $N = 2$ vector field $\boldsymbol{V}$ at a point φ_0 is the winding number of $\boldsymbol{V}$ about that point.

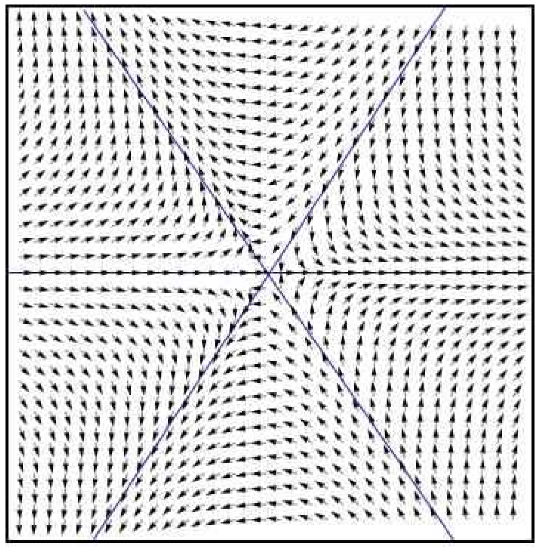
- The winding number $\mathsf{W}_{\mathcal{C}}(\boldsymbol{V})$ of a curve $\mathcal{C}$ is the sum of the indices of the singularities enclosed by that curve.

- Smooth deformations of $\mathcal{C}$ do not change its winding number. One must instead "stretch" $\mathcal{C}$ over a fixed point singularity in order to change $\mathsf{W}_{\mathcal{C}}(\boldsymbol{V})$.

- Uniformly rotating each vector in the vector field by an angle β has the effect of sending $\Theta \to \Theta + \beta$; this leaves all indices and winding numbers invariant.

- Nodes and spirals, whether stable or unstable, have index $+1$ (ss do the special cases of centers, stars, and degenerate nodes). Saddle points have index -1.

- Clearly any closed orbit must lie on a curve $\mathcal{C}$ of index $+1$.



$\boldsymbol{V} = (x^2 - y^2\,,\, -2xy)$

$\Theta = -2\theta$

Figure 14.16: A vector field with index -2.

14.6.1 Gauss-Bonnet theorem

There is a deep result in mathematics, the Gauss-Bonnet theorem, which connects the local *geometry* of a two-dimensional manifold to its global *topological structure*. The content of the theorem is as follows:

$$\int\limits_{\mathcal{M}} dA\, K = 2\pi\,\chi(\mathcal{M}) = 2\pi \sum_i \underset{\varphi_i}{\mathsf{ind}}\,(\boldsymbol{V}) \quad , \tag{14.72}$$

where $\mathcal{M}$ is a 2-manifold (a topological space locally homeomorphic to $\mathbb{R}^2$), κ is the local *Gaussian curvature* of $\mathcal{M}$, which is given by $K = (R_1\,R_2)^{-1}$, where $R_{1,2}$ are the principal radii of curvature at a given point, and dA is the differential area element. The quantity $\chi(\mathcal{M})$ is called the *Euler characteristic* of $\mathcal{M}$ and is given by $\chi(\mathcal{M}) = 2 - 2g$, where g is the *genus* of $\mathcal{M}$, which is the number of holes (or handles) of $\mathcal{M}$. Furthermore, $\boldsymbol{V}(\boldsymbol{\varphi})$ is *any* smooth vector field on $\mathcal{M}$, and $\boldsymbol{\varphi}_i$ are the singularity points of that vector field, which are fixed points of the dynamics $\dot{\boldsymbol{\varphi}} = \boldsymbol{V}(\boldsymbol{\varphi})$.

To apprehend the content of the Gauss-Bonnet theorem, it is helpful to consider an example. Let $\mathcal{M} = \mathbb{S}^2$ be the unit 2-sphere, as depicted in fig. 14.17. At any point on the unit 2-sphere, the radii of curvature are degenerate and both equal to $R = 1$, hence $K = 1$. If we integrate the Gaussian curvature over the sphere, we thus get $4\pi = 2\pi\,\chi\big(\mathbb{S}^2\big)$, which says $\chi(\mathbb{S}^2) = 2 - 2g = 2$, which agrees with $g = 0$ for the sphere. Furthermore, the Gauss-Bonnet theorem says that *any* smooth vector field on $\mathbb{S}^2$ *must* have a singularity or singularities, with the total index summed over the singularities equal to $+2$. The vector field sketched in the left panel of fig. 14.17 has two index $+1$ singularities, which could be taken at the north and south poles, but which could be anywhere. Another possibility, depicted in the right panel of fig. 14.17, is that there is a one singularity with index $+2$.

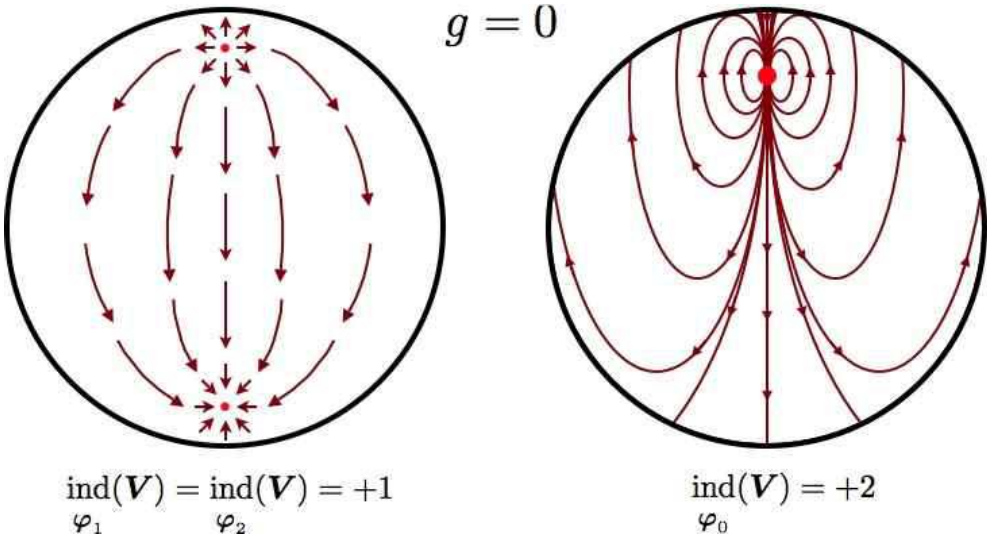


Figure 14.17: Two smooth vector fields on the sphere $\mathbb{S}^2$, which has genus $g = 0$. Left panel: two index $+1$ singularities. Right panel: one index $+2$ singularity.

In fig. 14.18 we show examples of manifolds with genii $g = 1$ and $g = 2$. The case $g = 1$ is the familiar 2-torus, which is topologically equivalent to a product of circles: $\mathbb{T}^2 \simeq \mathbb{S}^1 \times \mathbb{S}^1$, and is thus coordinatized by two angles θ_1 and θ_2. A smooth vector field pointing in the direction of increasing θ_1 never vanishes, and thus has no singularities, consistent with $g = 1$ and $\chi(\mathbb{T}^2) = 0$. Topologically, one can define a torus as the quotient space $\mathbb{R}^2/\mathbb{Z}^2$, or as a square with opposite sides identified. This is what mathematicians call a 'flat torus' – one with curvature $K = 0$ everywhere. Of course, such a torus cannot be embedded in three-dimensional Euclidean space; a two-dimensional figure embedded in a three-dimensional Euclidean space inherits a metric due to the embedding, and for a physical torus, like the surface of a bagel, the Gaussian curvature is only zero *on average*.

The $g = 2$ surface $\mathcal{M}$ shown in the right panel of fig. 14.18 has Euler characteristic $\chi(\mathcal{M}) = -2$, which means that any smooth vector field on $\mathcal{M}$ must have singularities with indices totalling -2. One possibility, depicted in the figure, is to have two saddle points with index -1; one of these singularities is shown in the figure (the other would be on the opposite side).

14.6.2 Singularities and topology

For any $N = 1$ system $\dot{x} = f(x)$, we can identify a 'charge' Q with any generic fixed point x^* by setting

$$Q = \mathsf{sgn}\Big[f'(x^*)\Big] \quad , \tag{14.73}$$

where $f(x^*) = 0$. The total charge contained in a region $\big[x_1, x_2\big]$ is then

$$Q_{12} = \tfrac{1}{2}\,\mathsf{sgn}\Big[f(x_2)\Big] - \tfrac{1}{2}\,\mathsf{sgn}\Big[f(x_1)\Big] \quad . \tag{14.74}$$

It is easy to see that Q_{12} is the sum of the charges of all the fixed points lying within the interval $\big[x_1, x_2\big]$.

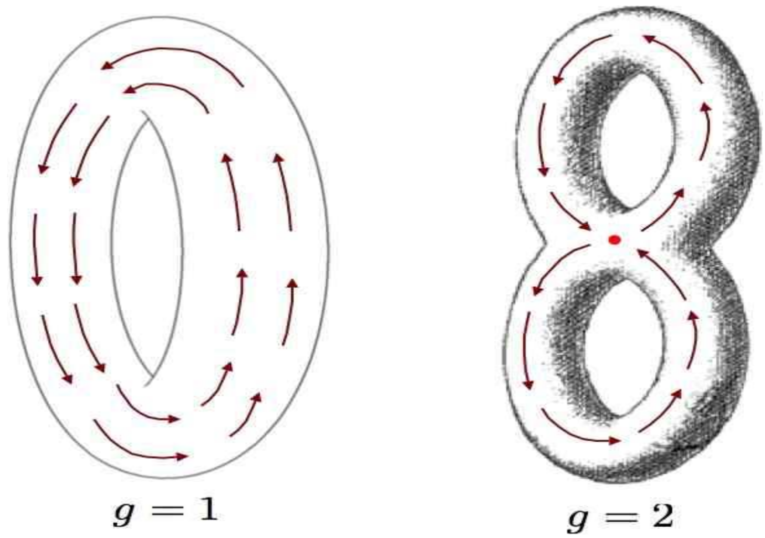


Figure 14.18: Smooth vector fields on the torus $\mathbb{T}^2$ ($g = 1$), and on a 2-manifold $\mathcal{M}$ of genus $g = 2$.

In higher dimensions, we have the following general construction. Consider an N-dimensional dynamical system $\dot{\boldsymbol{x}} = \boldsymbol{V}(\boldsymbol{x})$, and let $\hat{\boldsymbol{n}}(\boldsymbol{x})$ be the unit vector field defined by

$$\hat{\boldsymbol{n}}(\boldsymbol{x}) = \frac{\boldsymbol{V}(\boldsymbol{x})}{|\boldsymbol{V}(\boldsymbol{x})|} \quad . \tag{14.75}$$

Consider now a unit sphere in $\hat{\boldsymbol{n}}$ space, which is of dimension $(N-1)$. If we integrate over this surface, we obtain

$$\Omega_N = \oint d\sigma_a \, n^a = \frac{2\pi^{(N-1)/2}}{\Gamma\big(\frac{N-1}{2}\big)} \quad , \tag{14.76}$$

which is the surface area of the unit sphere $\mathbb{S}^{N-1}$. Thus, $\Omega_2 = 2\pi$, $\Omega_3 = 4\pi$, $\Omega_4 = 2\pi^2$, *etc.*

Now consider a change of variables over the surface of the sphere, to the set $(\xi_1, \ldots, \xi_{N-1})$. We then have

$$\Omega_N = \oint\limits_{\mathbb{S}^{N-1}} d\sigma_a \, n^a = \oint d^{N-1}\xi \; \epsilon_{a_1 \cdots a_N} \, n^{a_1} \, \frac{\partial n^{a_2}}{\partial \xi_1} \cdots \frac{\partial n^{a_N}}{\partial \xi_{N-1}} \tag{14.77}$$

The topological charge is then

$$Q = \frac{1}{\Omega_N} \oint d^{N-1}\xi \; \epsilon_{a_1 \cdots a_N} \, n^{a_1} \, \frac{\partial n^{a_2}}{\partial \xi_1} \cdots \frac{\partial n^{a_N}}{\partial \xi_{N-1}} \tag{14.78}$$

The quantity Q is an *integer topological invariant* which characterizes the map from the surface $(\xi_1, \ldots, \xi_{N-1})$ to the unit sphere $|\hat{\boldsymbol{n}}| = 1$. In mathematical parlance, Q is known as the *Pontrjagin index* of this map.

This analytical development recapitulates some basic topology. Let $\mathcal{M}$ be a topological space and consider a map from the circle $\mathbb{S}^1$ to $\mathcal{M}$. We can compose two such maps by merging the two circles, as shown in fig. 14.19. Two maps are said to be *homotopic* if they can be smoothly deformed into each other. Any two homotopic maps are said to belong to the same *equivalence class* or *homotopy class*. For general

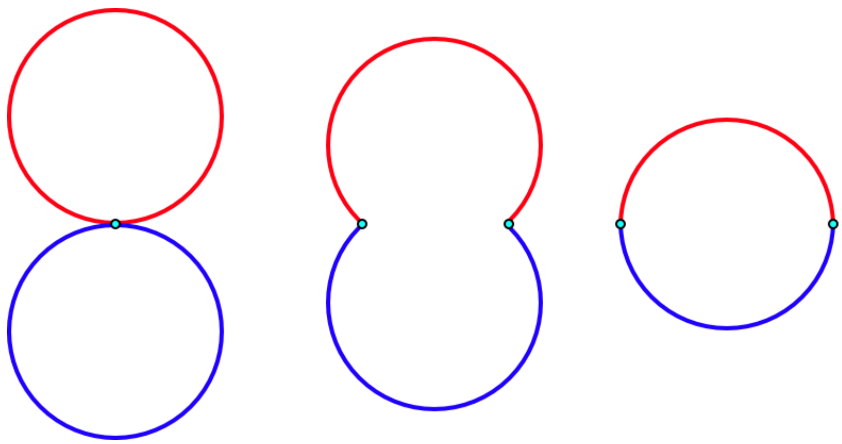


Figure 14.19: Composition of two circles. The same general construction applies to the merging of n-spheres $\mathbb{S}^n$, called the *wedge sum*.

$\mathcal{M}$, the homotopy classes may be multiplied using the composition law, resulting in a group structure. The group is called the *fundamental group* of the manifold $\mathcal{M}$, and is abbreviated $\pi_1(\mathcal{M})$. If $\mathcal{M} = \mathbb{S}^2$, then any such map can be smoothly contracted to a point on the 2-sphere, which is to say a trivial map. We then have $\pi_1(\mathcal{M}) = 0$. If $\mathcal{M} = \mathbb{S}^1$, the maps can wind nontrivially, and the homotopy classes are labeled by a single integer winding number: $\pi_1(\mathbb{S}^1) = \mathbb{Z}$. The winding number of the composition of two such maps is the sum of their individual winding numbers. If $\mathcal{M} = \mathbb{T}^2$, the maps can wind nontrivially around either of the two cycles of the 2-torus. We then have $\pi_1(\mathbb{T}^2) = \mathbb{Z}^2$, and in general $\pi_1(\mathbb{T}^n) = \mathbb{Z}^n$. This makes good sense, since an n-torus is topologically equivalent to a product of n circles. In some cases, $\pi_1(\mathcal{M})$ can be nonabelian, as is the case when $\mathcal{M}$ is the genus $g = 2$ structure shown in the right hand panel of fig. 14.18.

In general we define the n^{th} *homotopy group* $\pi_n(\mathcal{M})$ as the group under composition of maps from $\mathbb{S}^n$ to $\mathcal{M}$. For $n \geq 2$, $\pi_n(\mathcal{M})$ is abelian. If $\dim(\mathcal{M}) < n$, then $\pi_n(\mathcal{M}) = 0$. In general, $\pi_n(\mathbb{S}^n) = \mathbb{Z}$. These n^{th} homotopy classes of the n-sphere are labeled by their Pontrjagin index Q.

Finally, we ask what is Q in terms of the eigenvalues and eigenvectors of the linearized map

$$M_{ij} = \frac{\partial V_i}{\partial x_j}\bigg|_{\boldsymbol{x}^*} \quad . \tag{14.79}$$

For simple cases where all the λ_i are nonzero, we have

$$Q = \mathsf{sgn}\left(\prod_{i=1}^{N} \lambda_i\right) \quad . \tag{14.80}$$

14.7 Appendix: Example Problem

Consider the two-dimensional phase flow,

$$\begin{aligned}\dot{x} &= \tfrac{1}{2}x + xy - 2x^3 \\ \dot{y} &= \tfrac{5}{2}y + xy - y^2 \quad .\end{aligned} \tag{14.81}$$

(a) Find and classify all fixed points.

Solution : We have

$$\begin{aligned}\dot{x} &= x\left(\tfrac{1}{2} + y - 2x^2\right) \\ \dot{y} &= y\left(\tfrac{5}{2} + x - y\right) \quad .\end{aligned} \tag{14.82}$$

The matrix of first derivatives is

$$M = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} + y - 6x^2 & x \\ y & \frac{5}{2} + x - 2y \end{pmatrix} \quad . \tag{14.83}$$

There are six fixed points.

$(x, y) = (0, 0)$: The derivative matrix is

$$M = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{5}{2} \end{pmatrix} \quad . \tag{14.84}$$

The determinant is $D = \frac{5}{4}$ and the trace is $T = 3$. Since $D < \frac{1}{4}T^2$ and $T > 0$, this is an unstable node. (Duh! One can read off both eigenvalues are real and positive.) Eigenvalues: $\lambda_1 = \frac{1}{2}$, $\lambda_2 = \frac{5}{2}$.

$(x, y) = (0, \frac{5}{2})$: The derivative matrix is

$$M = \begin{pmatrix} 3 & 0 \\ \frac{5}{2} & -\frac{5}{2} \end{pmatrix} \quad , \tag{14.85}$$

for which $D = -\frac{15}{2}$ and $T = \frac{1}{2}$. The determinant is negative, so this is a saddle. Eigenvalues: $\lambda_1 = -\frac{5}{2}$, $\lambda_2 = 3$.

$(x, y) = (-\frac{1}{2}, 0)$: The derivative matrix is

$$M = \begin{pmatrix} -1 & -\frac{1}{2} \\ 0 & 2 \end{pmatrix} \quad , \tag{14.86}$$

for which $D = -2$ and $T = +1$. The determinant is negative, so this is a saddle. Eigenvalues: $\lambda_1 = -1$, $\lambda_2 = 2$.

$(x, y) = (\frac{1}{2}, 0)$: The derivative matrix is

$$M = \begin{pmatrix} -1 & \frac{1}{2} \\ 0 & 3 \end{pmatrix} \quad , \tag{14.87}$$

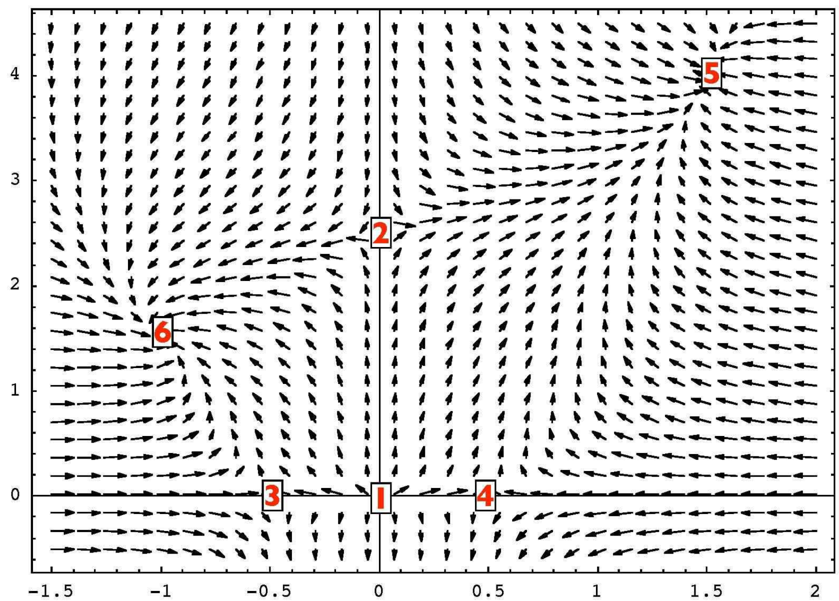


Figure 14.20: Sketch of phase flow for $\dot{x} = \frac{1}{2}x + xy - 2x^3$, $\dot{y} = \frac{5}{2}y + xy - y^2$. Fixed point classifications are in the text.

for which $D = -3$ and $T = +2$. The determinant is negative, so this is a saddle. Eigenvalues: $\lambda_1 = -1$, $\lambda_2 = 3$.

$(x, y) = (\frac{3}{2}, 4)$: This is one root obtained by setting $y = x + \frac{5}{2}$ and the solving $\frac{1}{2} + y - 2x^2 = 3 + x - 2x^2 = 0$, giving $x = -1$ and $x = +\frac{3}{2}$. The derivative matrix is

$$M = \begin{pmatrix} -9 & \frac{3}{2} \\ 4 & -4 \end{pmatrix} \quad , \tag{14.88}$$

for which $D = 30$ and $T = -13$. Since $D < \frac{1}{4}T^2$ and $T < 0$, this corresponds to a stable node. Eigenvalues: $\lambda_1 = -10$, $\lambda_2 = -3$.

$(x, y) = (-1, \frac{3}{2})$: This is the second root obtained by setting $y = x + \frac{5}{2}$ and the solving $\frac{1}{2} + y - 2x^2 = 3 + x - 2x^2 = 0$, giving $x = -1$ and $x = +\frac{3}{2}$. The derivative matrix is

$$M = \begin{pmatrix} -4 & -1 \\ \frac{3}{2} & -\frac{3}{2} \end{pmatrix} \quad , \tag{14.89}$$

for which $D = \frac{15}{2}$ and $T = -\frac{11}{2}$. Since $D < \frac{1}{4}T^2$ and $T < 0$, this corresponds to a stable node. Eigenvalues: $\lambda_1 = -3$, $\lambda_2 = -\frac{5}{2}$.

`(b)` Sketch the phase flow.

Solution : The flow is sketched in fig. 14.20. Thanks to Evan Bierman for providing the `Mathematica` code.

Chapter 15

Nonlinear Oscillators

15.1 Weakly Perturbed Linear Oscillators

Consider a nonlinear oscillator described by the equation of motion

$$\ddot{x} + \Omega_0^2\, x = \epsilon\, h(x) \quad . \tag{15.1}$$

Here, ϵ is a dimensionless parameter, assumed to be small, and $h(x)$ is a nonlinear function of x. In general, we might consider equations of the form

$$\ddot{x} + \Omega_0^2\, x = \epsilon\, h(x, \dot{x}) \quad , \tag{15.2}$$

such as the van der Pol oscillator,

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + \Omega_0^2\, x = 0 \quad . \tag{15.3}$$

First, we will focus on nondissipative systems, *i.e.* where we may write $m\ddot{x} = -\partial_x V$, with $V(x)$ some potential.

As an example, consider the simple pendulum, which obeys

$$\ddot{\theta} + \Omega_0^2 \sin\theta = 0 \quad , \tag{15.4}$$

where $\Omega_0^2 = g/\ell$, with ℓ the length of the pendulum. We may rewrite his equation as

$$\begin{aligned} \ddot{\theta} + \Omega_0^2\, \theta &= \Omega_0^2\, (\theta - \sin\theta) \\ &= \tfrac{1}{6}\, \Omega_0^2\, \theta^3 - \tfrac{1}{120}\, \Omega_0^2\, \theta^5 + \ldots \end{aligned} \tag{15.5}$$

The RHS above is a nonlinear function of θ. We can define this to be $h(\theta)$, and take $\epsilon = 1$.

15.1.1 Naïve Perturbation theory and its failure

Let's assume though that ϵ is small, and write a formal power series expansion of the solution $x(t)$ to equation 15.1 as

$$x = x_0 + \epsilon\, x_1 + \epsilon^2\, x_2 + \ldots \quad . \tag{15.6}$$

We now plug this into 15.1. We need to use Taylor's theorem,

$$h(x_0+\eta) = h(x_0) + h'(x_0)\,\eta + \tfrac{1}{2}\,h''(x_0)\,\eta^2 + \ldots \tag{15.7}$$

with

$$\eta = \epsilon\, x_1 + \epsilon^2\, x_2 + \ldots \quad . \tag{15.8}$$

Working out the resulting expansion in powers of ϵ is tedious. One finds

$$h(x) = h(x_0) + \epsilon\, h'(x_0)\, x_1 + \epsilon^2 \Big\{ h'(x_0)\, x_2 + \tfrac{1}{2}\, h''(x_0)\, x_1^2 \Big\} + \ldots \quad . \tag{15.9}$$

Equating terms of the same order in ϵ, we obtain a hierarchical set of equations,

$$\begin{aligned}
\ddot{x}_0 + \Omega_0^2\, x_0 &= 0 \\
\ddot{x}_1 + \Omega_0^2\, x_1 &= h(x_0) \\
\ddot{x}_2 + \Omega_0^2\, x_2 &= h'(x_0)\, x_1 \\
\ddot{x}_3 + \Omega_0^2\, x_3 &= h'(x_0)\, x_2 + \tfrac{1}{2}\, h''(x_0)\, x_1^2
\end{aligned} \tag{15.10}$$

et cetera, where prime denotes differentiation with respect to argument. The first of these is easily solved: $x_0(t) = A\,\cos(\Omega_0 t + \varphi)$, where A and φ are constants. This solution then is plugged in at the next order, to obtain an inhomogeneous equation for $x_1(t)$. Solve for $x_1(t)$ and insert into the following equation for $x_2(t)$, *etc.* It looks straightforward enough.

The problem is that resonant forcing terms generally appear in the RHS of each equation of the hierarchy past the first. Define $\theta \equiv \Omega_0 t + \varphi$. Then $x_0(\theta)$ is an even periodic function of θ with period 2π, hence so is $h(x_0)$. We may then expand $h\big(x_0(\theta)\big)$ in a Fourier series:

$$h\big(A\cos\theta\big) = \sum_{n=0}^{\infty} h_n(A)\,\cos(n\theta) \quad . \tag{15.11}$$

The $n=1$ term leads to resonant forcing. Thus, the solution for $x_1(t)$ is

$$x_1(t) = \frac{1}{\Omega_0^2} \sum_{\substack{n=0 \\ (n\neq 1)}}^{\infty} \frac{h_n(A)}{1-n^2}\,\cos(n\Omega_0 t + n\varphi) + \frac{h_1(A)}{2\Omega_0}\, t\,\sin(\Omega_0 t + \varphi) \quad , \tag{15.12}$$

which increases linearly with time. As an example, consider a cubic nonlinearity with $h(x) = r\,x^3$, where r is a constant. Then using

$$\cos^3\!\theta = \tfrac{3}{4}\cos\theta + \tfrac{1}{4}\cos(3\theta) \quad , \tag{15.13}$$

we have $h_1 = \frac{3}{4}\,rA^3$ and $h_3 = \frac{1}{4}\,rA^3$.

15.1.2 Poincaré-Lindstedt method

The problem here is that the nonlinear oscillator has a different frequency than its linear counterpart. Indeed, if we assume the frequency Ω is a function of ϵ, with

$$\Omega(\epsilon) = \Omega_0 + \epsilon\,\Omega_1 + \epsilon^2\Omega_2 + \ldots \quad , \tag{15.14}$$

then subtracting the unperturbed solution from the perturbed one and expanding in ϵ yields

$$\begin{aligned}\cos(\Omega t) - \cos(\Omega_0 t) &= -\sin(\Omega_0 t)\,(\Omega - \Omega_0)\,t - \tfrac{1}{2}\cos(\Omega_0 t)\,(\Omega - \Omega_0)^2\,t^2 + \ldots \\ &= -\epsilon\,\sin(\Omega_0 t)\,\Omega_1 t - \epsilon^2\Big\{\sin(\Omega_0 t)\,\Omega_2 t + \tfrac{1}{2}\cos(\Omega_0 t)\,\Omega_1^2 t^2\Big\} + \mathcal{O}(\epsilon^3) \quad .\end{aligned} \tag{15.15}$$

What perturbation theory can do for us is to provide a good solution *up to a given time*, provided that ϵ is *sufficiently small*. It *will not* give us a solution that is close to the true answer for *all* time. We see above that in order to do that, and to recover the shifted frequency $\Omega(\epsilon)$, we would have to resum perturbation theory to all orders, which is a daunting task.

The Poincaré-Lindstedt method obviates this difficulty by assuming $\Omega = \Omega(\epsilon)$ from the outset. Define a dimensionless time $s \equiv \Omega t$ and write 15.1 as

$$\Omega^2\,\frac{d^2x}{ds^2} + \Omega_0^2\,x = \epsilon\,h(x) \quad , \tag{15.16}$$

where

$$\begin{aligned} x &= x_0 + \epsilon\,x_1 + \epsilon^2\,x_2 + \ldots \\ \Omega^2 &= a_0 + \epsilon\,a_1 + \epsilon^2\,a_2 + \ldots \quad .\end{aligned} \tag{15.17}$$

We now plug the above expansions into 15.16:

$$\begin{aligned}&\big(a_0 + \epsilon\,a_1 + \epsilon^2\,a_2 + \ldots\big)\bigg(\frac{d^2x_0}{ds^2} + \epsilon\,\frac{d^2x_1}{ds^2} + \epsilon^2\,\frac{d^2x_2}{ds^2} + \ldots\bigg) + \Omega_0^2\,\big(x_0 + \epsilon\,x_1 + \epsilon^2\,x_2 + \ldots\big) \\ &\qquad\qquad \epsilon\,h(x_0) + \epsilon^2\,h'(x_0)\,x_1 + \epsilon^3\,\Big\{h'(x_0)\,x_2 + \tfrac{1}{2}\,h''(x_0)\,x_1^2\Big\} + \ldots\end{aligned} \tag{15.18}$$

Now let's write down equalities at each order in ϵ:

$$\begin{aligned} &a_0\,\frac{d^2x_0}{ds^2} + \Omega_0^2\,x_0 = 0 \\ &a_0\,\frac{d^2x_1}{ds^2} + \Omega_0^2\,x_1 = h(x_0) - a_1\,\frac{d^2x_0}{ds^2} \\ &a_0\,\frac{d^2x_2}{ds^2} + \Omega_0^2\,x_2 = h'(x_0)\,x_1 - a_2\,\frac{d^2x_0}{ds^2} - a_1\,\frac{d^2x_1}{ds^2} \quad ,\end{aligned} \tag{15.19}$$

et cetera.

The first equation of the hierarchy is immediately solved by

$$a_0 = \Omega_0^2 \qquad , \qquad x_0(s) = A\,\cos(s+\varphi) \quad . \tag{15.20}$$

At $\mathcal{O}(\epsilon)$, then, we have

$$\frac{d^2x_1}{ds^2} + x_1 = \Omega_0^{-2}\,h\big(A\cos(s+\varphi)\big) + \Omega_0^{-2}\,a_1\,A\cos(s+\varphi) \quad . \tag{15.21}$$

The LHS of the above equation has a natural frequency of unity (in terms of the dimensionless time s). We expect $h(x_0)$ to contain resonant forcing terms, per 15.11. However, we now have the freedom to adjust the undetermined coefficient a_1 to *cancel* any such resonant term. Clearly we must choose

$$a_1 = -\frac{h_1(A)}{A} \quad . \tag{15.22}$$

The solution for $x_1(s)$ is then

$$x_1(s) = \frac{1}{\Omega_0^2} \sum_{\substack{n=0 \\ (n\neq 1)}}^{\infty} \frac{h_n(A)}{1-n^2}\,\cos(ns+n\varphi) \quad , \tag{15.23}$$

which is periodic and hence does not increase in magnitude without bound, as does 15.12. The perturbed frequency is then obtained from

$$\Omega^2 = \Omega_0^2 - \frac{h_1(A)}{A}\,\epsilon + \mathcal{O}(\epsilon^2) \quad\Longrightarrow\quad \Omega(\epsilon) = \Omega_0 - \frac{h_1(A)}{2A\Omega_0}\,\epsilon + \mathcal{O}(\epsilon^2) \quad . \tag{15.24}$$

Note that Ω depends on the amplitude of the oscillations.

As an example, consider an oscillator with a quartic nonlinearity in the potential, *i.e.* $h(x) = r\,x^3$. Then

$$h\big(A\cos\theta\big) = \tfrac{3}{4} r A^3\,\cos\theta + \tfrac{1}{4} r A^3\,\cos(3\theta) \quad . \tag{15.25}$$

We then obtain, setting $\epsilon = 1$ at the end of the calculation,

$$\Omega = \Omega_0 - \frac{3\,r A^2}{8\,\Omega_0} + \ldots \tag{15.26}$$

where the remainder is higher order in the amplitude A. In the case of the pendulum,

$$\ddot\theta + \Omega_0^2\,\theta = \tfrac{1}{6}\Omega_0^2\,\theta^3 + \mathcal{O}\big(\theta^5\big) \quad , \tag{15.27}$$

and with $r = \frac{1}{6}\,\Omega_0^2$ and $\theta_0(t) = \theta_0\sin(\Omega t)$, we find

$$T(\theta_0) = \frac{2\pi}{\Omega} = \frac{2\pi}{\Omega_0}\cdot\Big\{1 + \tfrac{1}{16}\,\theta_0^2 + \ldots\Big\} \quad . \tag{15.28}$$

One can check that this is correct to lowest nontrivial order in the amplitude, using the exact result for the period,

$$T(\theta_0) = \frac{4}{\Omega_0}\,\mathbb{K}\big(\sin^2\tfrac{1}{2}\theta_0\big) \quad , \tag{15.29}$$

where $\mathbb{K}(x)$ is the complete elliptic integral.

The procedure can be continued to the next order, where the free parameter a_2 is used to eliminate resonant forcing terms on the RHS.

A good exercise to test one's command of the method is to work out the lowest order nontrivial corrections to the frequency of an oscillator with a quadratic nonlinearity, such as $h(x) = rx^2$. One finds that there are no resonant forcing terms at first order in ϵ, hence one must proceed to second order to find the first nontrivial corrections to the frequency.

15.2 Multiple Time Scale Method

Another method of eliminating secular terms (*i.e.* driving terms which oscillate at the resonant frequency of the unperturbed oscillator), and one which has applicability beyond periodic motion alone, is that of multiple time scale analysis. Consider the equation

$$\ddot{x} + x = \epsilon\, h(x, \dot{x}) \quad , \tag{15.30}$$

where ϵ is presumed small, and $h(x, \dot{x})$ is a nonlinear function of position and/or velocity. We define a hierarchy of time scales: $T_n \equiv \epsilon^n\, t$. There is a normal time scale $T_0 = t$, slow time scale $T_1 = \epsilon t$, a 'superslow' time scale $T_2 = \epsilon^2 t$, *etc.* Thus,

$$\begin{aligned} \frac{d}{dt} &= \frac{\partial}{\partial T_0} + \epsilon\, \frac{\partial}{\partial T_1} + \epsilon^2\, \frac{\partial}{\partial T_2} + \ldots \\ &= \sum_{n=0}^{\infty} \epsilon^n\, \frac{\partial}{\partial T_n} \quad . \end{aligned} \tag{15.31}$$

Next, we expand

$$x(t) = \sum_{n=0}^{\infty} \epsilon^n\, x_n(T_0\, , \, T_1, \ldots) \quad . \tag{15.32}$$

Thus, we have

$$\bigg(\sum_{n=0}^{\infty} \epsilon^n\, \frac{\partial}{\partial T_n}\bigg)^2 \bigg(\sum_{k=0}^{\infty} \epsilon^k\, x_k\bigg) + \sum_{k=0}^{\infty} \epsilon^k\, x_k = \epsilon\, h\Bigg(\sum_{k=0}^{\infty} \epsilon^k\, x_k \quad , \sum_{n=0}^{\infty} \epsilon^n\, \frac{\partial}{\partial T_n}\bigg(\sum_{k=0}^{\infty} \epsilon^k\, x_k\bigg)\Bigg) \quad .$$

We now evaluate this order by order in ϵ:

$$\begin{aligned} \mathcal{O}(\epsilon^0) \ &: \ \bigg(\frac{\partial^2}{\partial T_0^2} + 1\bigg)\, x_0 = 0 \\ \mathcal{O}(\epsilon^1) \ &: \ \bigg(\frac{\partial^2}{\partial T_0^2} + 1\bigg)\, x_1 = -2\, \frac{\partial^2 x_0}{\partial T_0\, \partial T_1} + h\bigg(x_0\, , \, \frac{\partial x_0}{\partial T_0}\bigg) \\ \mathcal{O}(\epsilon^2) \ &: \ \bigg(\frac{\partial^2}{\partial T_0^2} + 1\bigg)\, x_2 = -2\, \frac{\partial^2 x_1}{\partial T_0\, \partial T_1} - 2\, \frac{\partial^2 x_0}{\partial T_0\, \partial T_2} - \frac{\partial^2 x_0}{\partial T_1^2} + \frac{\partial h}{\partial x}\bigg|_{\{x_0, \dot{x}_0\}} x_1 + \frac{\partial h}{\partial \dot{x}}\bigg|_{\{x_0, \dot{x}_0\}} \bigg(\frac{\partial x_1}{\partial T_0} + \frac{\partial x_0}{\partial T_1}\bigg) \quad , \end{aligned} \tag{15.33}$$

et cetera. The expansion gets more and more tedious with increasing order in ϵ.

Let's carry this procedure out to first order in ϵ. To order ϵ^0,

$$x_0 = A\,\cos\big(T_0 + \phi\big) \quad , \tag{15.34}$$

where A and ϕ are arbitrary (at this point) functions of $\big\{T_1\,,\,T_2\,,\,\ldots\big\}$. Now we solve the next equation in the hierarchy, for x_1. Let $\theta \equiv T_0 + \phi$. Then $\frac{\partial}{\partial T_0} = \frac{\partial}{\partial \theta}$ and we have

$$\bigg(\frac{\partial^2}{\partial\theta^2} + 1\bigg)\,x_1 = 2\,\frac{\partial A}{\partial T_1}\,\sin\theta + 2\,A\,\frac{\partial \phi}{\partial T_1}\,\cos\theta + h\big(A\cos\theta, -A\sin\theta\big) \quad . \tag{15.35}$$

Since the arguments of h are periodic under $\theta \to \theta + 2\pi$, we may expand h in a Fourier series:

$$h(\theta) \equiv h\big(A\cos\theta, -A\sin\theta\big) = \sum_{k=1}^{\infty} \alpha_k(A)\sin(k\theta) + \sum_{k=0}^{\infty} \beta_k(A)\cos(k\theta) \quad . \tag{15.36}$$

The inverse of this relation is

$$\begin{aligned} \alpha_k(A) &= \int\limits_0^{2\pi} \frac{d\theta}{\pi}\,h(\theta)\,\sin(k\theta) \qquad , \qquad k > 0 \\ \beta_0(A) &= \int\limits_0^{2\pi} \frac{d\theta}{2\pi}\,h(\theta) \\ \beta_k(A) &= \int\limits_0^{2\pi} \frac{d\theta}{\pi}\,h(\theta)\,\cos(k\theta) \qquad , \qquad k > 0 \quad . \end{aligned} \tag{15.37}$$

We now demand that the secular terms on the RHS – those terms proportional to $\cos\theta$ and $\sin\theta$ – must vanish. This means

$$\begin{aligned} 2\,\frac{\partial A}{\partial T_1} + \alpha_1(A) &= 0 \\ 2A\,\frac{\partial \phi}{\partial T_1} + \beta_1(A) &= 0 \quad . \end{aligned} \tag{15.38}$$

These two first order equations require two initial conditions, which is sensible since our initial equation $\ddot{x} + x = \epsilon\,h(x,\dot{x})$ is second order in time.

With the secular terms eliminated, we may solve for x_1:

$$x_1 = \sum_{k\neq 1}^{\infty} \bigg\{ \frac{\alpha_k(A)}{1-k^2}\,\sin(k\theta) + \frac{\beta_k(A)}{1-k^2}\,\cos(k\theta)\bigg\} + C_0\cos\theta + D_0\sin\theta \quad . \tag{15.39}$$

Note: (i) the $k=1$ terms are excluded from the sum, and (ii) an arbitrary solution to the homogeneous equation, *i.e.* eqn. 15.35 with the right hand side set to zero, is included. The constants C_0 and D_0 are arbitrary functions of T_1, T_2, *etc.* .

The equations for A and ϕ are both first order in T_1. They will therefore involve two constants of integration – call them A_0 and ϕ_0. At second order, these constants are taken as dependent upon the superslow time scale T_2. *The method itself may break down at this order.* (See if you can find out why.)

Let's apply this to the nonlinear oscillator $\ddot{x} + \sin x = 0$, also known as the simple pendulum. We'll expand the sine function to include only the lowest order nonlinear term, and consider

$$\ddot{x} + x = \tfrac{1}{6}\,\epsilon\, x^3 \quad . \tag{15.40}$$

We'll assume ϵ is small and take $\epsilon = 1$ at the end of the calculation. This will work provided the amplitude of the oscillation is itself small. To zeroth order, we have $x_0 = A\,\cos(t+\phi)$, as always. At first order, we must solve

$$\begin{aligned}\bigg(\frac{\partial^2}{\partial\theta^2}+1\bigg)\,x_1 &= 2\,\frac{\partial A}{\partial T_1}\,\sin\theta + 2\,A\,\frac{\partial \phi}{\partial T_1}\,\cos\theta + \tfrac{1}{6}\,A^3\cos^3\theta \\ &= 2\,\frac{\partial A}{\partial T_1}\,\sin\theta + 2\,A\,\frac{\partial \phi}{\partial T_1}\,\cos\theta + \tfrac{1}{24}\,A^3\cos(3\theta) + \tfrac{1}{8}\,A^3\cos\theta \quad .\end{aligned} \tag{15.41}$$

We eliminate the secular terms by demanding

$$\frac{\partial A}{\partial T_1} = 0 \qquad , \qquad \frac{\partial \phi}{\partial T_1} = -\tfrac{1}{16}\,A^2 \quad , \tag{15.42}$$

hence $A = A_0$ and $\phi = -\frac{1}{16}\,A_0^2\,T_1 + \phi_0$, and

$$x(t) = A_0\cos\big(t - \tfrac{1}{16}\,\epsilon\,A_0^2\,t + \phi_0\big) - \tfrac{1}{192}\,\epsilon\,A_0^3\,\cos\big(3t - \tfrac{3}{16}\,\epsilon A_0^2\,t + 3\phi_0\big) + \ldots \quad , \tag{15.43}$$

which reproduces the result obtained from the Poincaré-Lindstedt method.

15.2.1 Duffing oscillator

Consider the equation

$$\ddot{x} + 2\epsilon\mu\dot{x} + x + \epsilon x^3 = 0 \quad . \tag{15.44}$$

This describes a damped nonlinear oscillator. Here we assume both the damping coefficient $\tilde{\mu} \equiv \epsilon\mu$ as well as the nonlinearity both depend linearly on the small parameter ϵ. We may write this equation in our standard form $\ddot{x} + x = \epsilon\, h(x,\dot{x})$, with $h(x,\dot{x}) = -2\mu\dot{x} - x^3$.

For $\epsilon > 0$, which we henceforth assume, it is easy to see that the only fixed point is $(x,\dot{x}) = (0,0)$. The linearized flow in the vicinity of the fixed point is given by

$$\frac{d}{dt}\begin{pmatrix} x \\ \dot{x}\end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -2\epsilon\mu\end{pmatrix}\begin{pmatrix} x \\ \dot{x}\end{pmatrix} + \mathcal{O}(x^3) \quad . \tag{15.45}$$

The determinant is $D = 1$ and the trace is $T = -2\epsilon\mu$. Thus, provided $\epsilon\mu < 1$, the fixed point is a stable spiral; for $\epsilon\mu > 1$ the fixed point becomes a stable node.

We employ the multiple time scale method to order ϵ. We have $x_0 = A\cos(T_0 + \phi)$ to zeroth order, as usual. The nonlinearity is expanded in a Fourier series in $\theta = T_0 + \phi$:

$$\begin{aligned} h\Big(x_0\,,\,\tfrac{\partial x_0}{\partial T_0}\Big) &= 2\mu A\sin\theta - A^3\cos^3\theta \\ &= 2\mu A\sin\theta - \tfrac{3}{4}A^3\cos\theta - \tfrac{1}{4}A^3\cos 3\theta \quad . \end{aligned} \tag{15.46}$$

Thus, $\alpha_1(A) = 2\mu A$ and $\beta_1(A) = -\frac{3}{4}A^3$. We now solve the first order equations,

$$\frac{\partial A}{\partial T_1} = -\tfrac{1}{2}\,\alpha_1(A) = -\mu A \qquad \Longrightarrow \qquad A(T) = A_0\,e^{-\mu T_1} \tag{15.47}$$

as well as

$$\frac{\partial\phi}{\partial T_1} = -\frac{\beta_1(A)}{2A} = \tfrac{3}{8}A_0^2\,e^{-2\mu T_1} \qquad \Longrightarrow \qquad \phi(T_1) = \phi_0 + \frac{3A_0^2}{16\mu}\big(1 - e^{-2\mu T_1}\big) \quad . \tag{15.48}$$

After elimination of the secular terms, we may read off

$$x_1(T_0\,,\,T_1) = \tfrac{1}{32}A^3(T_1)\cos\big(3T_0 + 3\phi(T_1)\big) \quad . \tag{15.49}$$

Finally, we have

$$\begin{aligned} x(t) &= A_0\,e^{-\epsilon\mu t}\,\cos\Big(t + \frac{3A_0^2}{16\mu}\big(1 - e^{-2\epsilon\mu t}\big) + \phi_0\Big) \\ &\quad + \tfrac{1}{32}\epsilon A_0^3\,e^{-3\epsilon\mu t}\,\cos\Big(3t + \frac{9A_0^2}{16\mu}\big(1 - e^{-2\epsilon\mu t}\big) + 3\phi_0\Big) \quad . \end{aligned} \tag{15.50}$$

15.2.2 Van der Pol oscillator

Let's apply this method to another problem, that of the van der Pol oscillator,

$$\ddot{x} + \epsilon\,(x^2 - 1)\,\dot{x} + x = 0 \quad , \tag{15.51}$$

with $\epsilon > 0$. The nonlinear term acts as a frictional drag for $x > 1$, and as a 'negative friction' (*i.e.* increasing the amplitude) for $x < 1$. Note that the linearized equation at the fixed point $(x = 0, \dot{x} = 0)$ corresponds to an unstable spiral for $\epsilon < 2$.

For the van der Pol oscillator, we have $h(x, \dot{x}) = (1 - x^2)\,\dot{x}$, and plugging in the zeroth order solution $x_0 = A\,\cos(t + \phi)$ gives

$$\begin{aligned} h\Big(x_0\,,\,\frac{\partial x_0}{\partial T_0}\Big) &= \big(1 - A^2\cos^2\theta\big)\big(-A\sin\theta\big) \\ &= \big(-A + \tfrac{1}{4}A^3\big)\,\sin\theta + \tfrac{1}{4}\,A^3\,\sin(3\theta) \quad , \end{aligned} \tag{15.52}$$

with $\theta \equiv t + \phi$. Thus, $\alpha_1 = -A + \frac{1}{4}A^3$ and $\beta_1 = 0$, which gives $\phi = \phi_0$ and

$$2\,\frac{\partial A}{\partial T_1} = A - \tfrac{1}{4}A^3 \quad . \tag{15.53}$$

The equation for A is easily integrated:

$$dT_1 = -\frac{8\,dA}{A\,(A^2-4)} = \left(\frac{2}{A} - \frac{1}{A-2} - \frac{1}{A+2}\right) dA = d\log\left(\frac{A}{A^2-4}\right)$$
$$\Longrightarrow \qquad A(T_1) = \frac{2}{\sqrt{1-\big(1-\frac{4}{A_0^2}\big)\exp(-T_1)}} \quad . \tag{15.54}$$

Thus,

$$x_0(t) = \frac{2\cos(t+\phi_0)}{\sqrt{1-\big(1-\frac{4}{A_0^2}\big)\exp(-\epsilon t)}} \quad . \tag{15.55}$$

This behavior describes the approach to the limit cycle $2\cos(t+\phi_0)$. With the elimination of the secular terms, we have

$$x_1(t) = -\tfrac{1}{32}A^3\sin(3\theta) = -\frac{\frac{1}{4}\sin\big(3t+3\phi_0\big)}{\Big[1-\big(1-\frac{4}{A_0^2}\big)\exp(-\epsilon t)\Big]^{3/2}} \quad . \tag{15.56}$$

15.3 Forced Nonlinear Oscillations

The forced, damped linear oscillator,

$$\ddot{x} + 2\mu\dot{x} + x = f_0\cos\mathit{\Omega} t \tag{15.57}$$

has the solution

$$x(t) = x_{\rm h}(t) + C(\mathit{\Omega})\,\cos\big(\mathit{\Omega} t + \delta(\mathit{\Omega})\big) \quad , \tag{15.58}$$

where

$$x_{\rm h}(t) = A_+\,e^{\lambda_+ t} + A_-\,e^{\lambda_- t} \quad , \tag{15.59}$$

where $\lambda_\pm = -\mu \pm \sqrt{\mu^2-1}$ are the roots of $\lambda^2 + 2\mu\lambda + 1 = 0$. The 'susceptibility' C and phase shift δ are given by

$$C(\mathit{\Omega}) = \frac{1}{\sqrt{(\mathit{\Omega}^2-1)^2 + 4\mu^2\mathit{\Omega}^2}} \quad , \qquad \delta(\mathit{\Omega}) = \tan^{-1}\left(\frac{2\mu\mathit{\Omega}}{1-\mathit{\Omega}^2}\right) \quad . \tag{15.60}$$

The homogeneous solution, $x_{\rm h}(t)$, is a transient and decays exponentially with time, since ${\rm Re}(\lambda_\pm) < 0$. The asymptotic behavior is a phase-shifted oscillation at the driving frequency $\mathit{\Omega}$.

Now let's add a nonlinearity. We study the equation

$$\ddot{x} + x = \epsilon\,h(x,\dot{x}) + \epsilon\,f_0\,\cos(t + \epsilon\nu t) \quad . \tag{15.61}$$

Note that amplitude of the driving term, $\epsilon f_0\cos(\mathit{\Omega} t)$, is assumed to be small, *i.e.* proportional to ϵ, and the driving frequency $\mathit{\Omega} = 1 + \epsilon\nu$ is assumed to be close to resonance. (The resonance frequency of the unperturbed oscillator is $\omega_{\rm res} = 1$.) Were the driving frequency far from resonance, it could be dealt with in the same manner as the non-secular terms encountered thus far. The situation when Ω is close to resonance deserves our special attention.

At order ϵ^0, we still have $x_0 = A\cos(T_0 + \phi)$. We write

$$\Omega t = t + \epsilon\nu t = T_0 + \nu T_1 \equiv \theta - \psi \quad , \tag{15.62}$$

where $\theta = T_0 + \phi(T_1)$ as before, and $\psi(T_1) \equiv \phi(T_1) - \nu T_1$. At order ϵ^1, we must then solve

$$\begin{aligned}\bigg(\frac{\partial^2}{\partial\theta^2} + 1\bigg)x_1 &= 2A'\sin\theta + 2A\phi'\cos\theta + h\big(A\cos\theta\,,\,-A\sin\theta\big) + f_0\cos(\theta-\psi)\\ &= \sum_{k\neq 1}\Big(\alpha_k\sin(k\theta) + \beta_k\cos(k\theta)\Big) + \Big(2A' + \alpha_1 + f_0\sin\psi\Big)\sin\theta\\ &\qquad + \Big(2A\psi' + 2A\nu + \beta_1 + f_0\cos\psi\Big)\cos\theta \quad ,\end{aligned} \tag{15.63}$$

where the prime denotes differentiation with respect to T_1. We thus have the $N = 2$ dynamical system

$$\begin{aligned}\frac{dA}{dT_1} &= -\tfrac{1}{2}\alpha_1(A) - \tfrac{1}{2}f_0\sin\psi\\ \frac{d\psi}{dT_1} &= -\nu - \frac{\beta_1(A)}{2A} - \frac{f_0}{2A}\cos\psi \quad .\end{aligned} \tag{15.64}$$

If we assume that $\{A,\psi\}$ approaches a fixed point of these dynamics, then at the fixed point these equations provide a relation between the amplitude A, the 'detuning' parameter ν, and the drive f_0:

$$\Big[\alpha_1(A)\Big]^2 + \Big[2\nu A + \beta_1(A)\Big]^2 = f_0^2 \quad . \tag{15.65}$$

In general this is a nonlinear equation for $A(f_0,\nu)$. The linearized (A,ψ) dynamics in the vicinity of a fixed point is governed by the matrix

$$M = \begin{pmatrix} \partial\dot{A}/\partial A & \partial\dot{A}/\partial\psi \\ \partial\dot{\psi}/\partial A & \partial\dot{\psi}/\partial\psi \end{pmatrix} = \begin{pmatrix} -\tfrac{1}{2}\alpha_1'(A) & \nu A + \tfrac{1}{2}\beta_1(A) \\ -\frac{\beta_1'(A)}{2A} - \frac{\nu}{A} & -\frac{\alpha_1(A)}{2A} \end{pmatrix} \quad . \tag{15.66}$$

If the (A,ψ) dynamics exhibits a stable fixed point (A^*,ψ^*), then one has

$$x_0(t) = A^*\cos(T_0 + \nu T_1 + \psi^*) = A^*\cos(\Omega t + \psi^*) \quad . \tag{15.67}$$

The oscillator's frequency is then the forcing frequency $\Omega = 1 + \epsilon\nu$, in which case the oscillator is said to be *entrained*, or *synchronized*, with the forcing. Note that

$$\det M = \frac{F'(A^*)}{8A^*} \quad .$$

15.3.1 Forced Duffing oscillator

Thus far our approach has been completely general. We now restrict our attention to the Duffing equation, for which

$$\alpha_1(A) = 2\mu A \quad , \quad \beta_1(A) = -\tfrac{3}{4}A^3 \quad , \tag{15.68}$$

which yields the cubic equation

$$A^6 - \tfrac{16}{3}\nu A^4 + \tfrac{64}{9}(\mu^2+\nu^2)A^2 - \tfrac{16}{9}f_0^2 = 0 \quad . \tag{15.69}$$

Analyzing the cubic is a good exercise. Setting $y = A^2$, we define

$$G(y) \equiv y^3 - \tfrac{16}{3}\nu\, y^2 + \tfrac{64}{9}(\mu^2+\nu^2)\, y \quad , \tag{15.70}$$

and we seek a solution to $G(y) = \frac{16}{9}f_0^2$. Setting $G'(y) = 0$, we find roots at

$$y_\pm = \tfrac{16}{9}\nu \pm \tfrac{8}{9}\sqrt{\nu^2 - 3\mu^2} \quad . \tag{15.71}$$

If $\nu^2 < 3\mu^2$ the roots are imaginary, which tells us that $G(y)$ is monotonically increasing for real y. There is then a unique solution to $G(y) = \frac{16}{9}f_0^2$.

If $\nu^2 > 3\mu^2$, then the cubic $G(y)$ has a local maximum at $y = y_-$ and a local minimum at $y = y_+$. For $\nu < -\sqrt{3}\,\mu$, we have $y_- < y_+ < 0$, and since $y = A^2$ must be positive, this means that once more there is a unique solution to $G(y) = \frac{16}{9}f_0^2$.

For $\nu > \sqrt{3}\,\mu$, we have $y_+ > y_- > 0$. There are then three solutions for $y(\nu)$ for $f_0 \in \big[f_0^-, f_0^+\big]$, where $f_0^\pm = \frac{3}{4}\sqrt{G(y_\mp)}$. If we define $\kappa \equiv \nu/\mu$, then

$$f_0^\pm = \tfrac{8}{9}\,\mu^{3/2}\sqrt{\kappa^3 + 9\kappa \pm \sqrt{\kappa^2-3}} \quad . \tag{15.72}$$

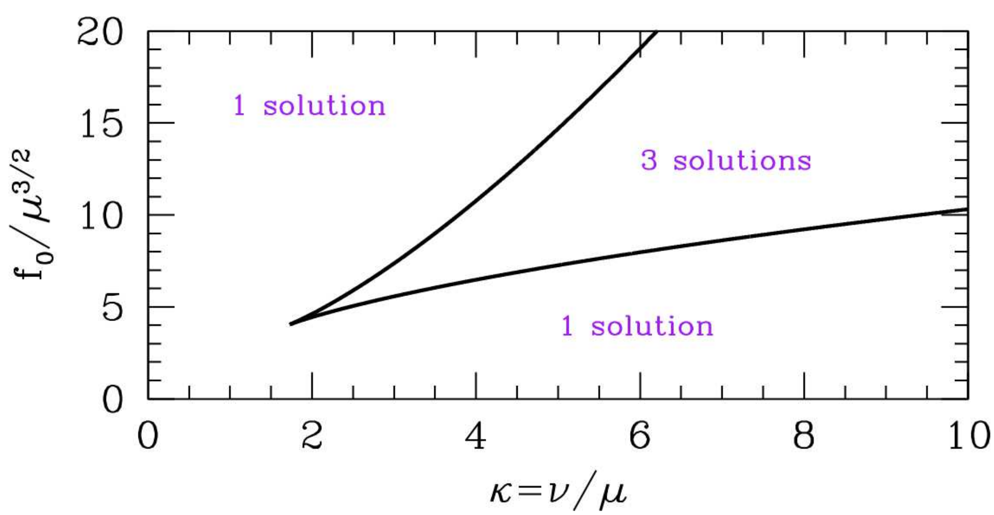


Figure 15.1: Phase diagram for the forced Duffing oscillator.

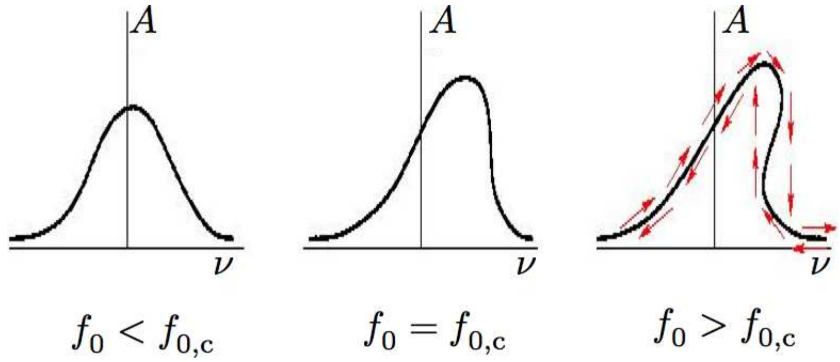


Figure 15.2: Amplitude A *versus* detuning ν for the forced Duffing oscillator for three values of the drive f_0. The critical drive is $f_{0,\mathrm{c}} = \frac{16}{3^{5/4}}\,\mu^{3/2}$. For $f_0 > f_{0,\mathrm{c}}$, there is hysteresis as a function of the detuning.

The phase diagram is shown in Fig. 15.1. The minimum value for f_0 is $f_{0,\mathrm{c}} = \frac{16}{3^{5/4}}\,\mu^{3/2}$, which occurs at $\kappa = \sqrt{3}$.

Thus far we have assumed that the (A,ψ) dynamics evolves to a fixed point. We should check to make sure that this fixed point is in fact stable. To do so, we evaluate the linearized dynamics at the fixed point. Writing $A = A^* + \delta A$ and $\psi = \psi^* + \delta\psi$, we have

$$\frac{d}{dT_1}\begin{pmatrix}\delta A\\ \delta\psi\end{pmatrix} = M\begin{pmatrix}\delta A\\ \delta\psi\end{pmatrix} \quad , \tag{15.73}$$

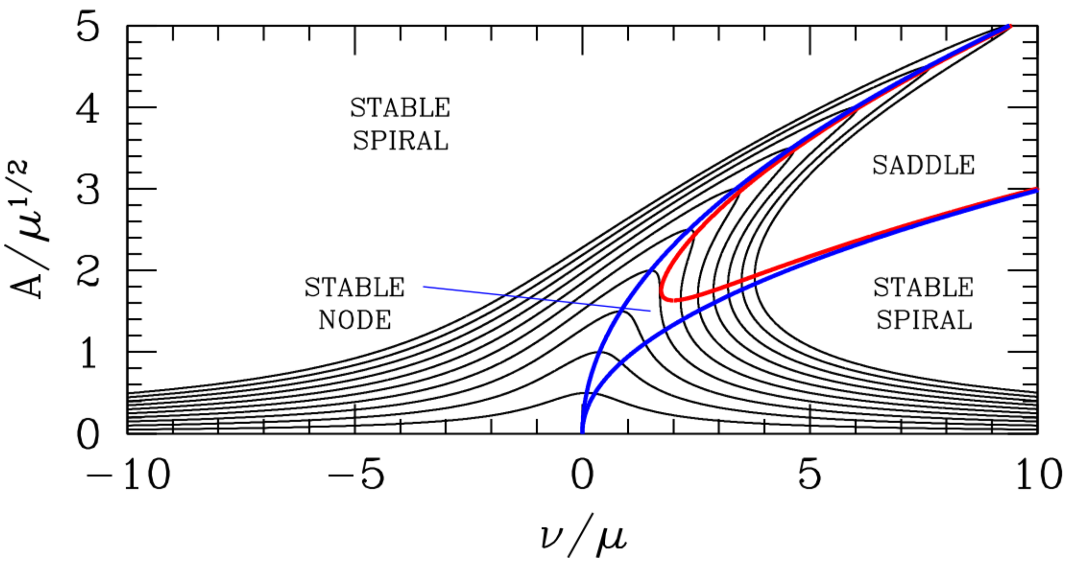


Figure 15.3: Amplitude *versus* detuning ν for the forced Duffing oscillator for ten equally spaced values of f_0 between $\mu^{3/2}$ and $10\,\mu^{3/2}$. The critical value is $f_{0,\mathrm{c}} = 4.0525\,\mu^{3/2}$. The red and blue curves are boundaries for the fixed point classification.

with

$$M = \begin{pmatrix} \frac{\partial \dot{A}}{\partial A} & \frac{\partial \dot{A}}{\partial \psi} \\ \\ \frac{\partial \dot{\psi}}{\partial A} & \frac{\partial \dot{\psi}}{\partial \psi} \end{pmatrix} = \begin{pmatrix} -\mu & -\frac{1}{2} f_0 \, \cos\psi \\ \\ \frac{3}{4} A + \frac{f_0}{2A^2} \, \cos\psi & \frac{f_0}{2A} \, \sin\psi \end{pmatrix} = \begin{pmatrix} -\mu & \nu A - \frac{3}{8} A^3 \\ \\ \frac{9}{8} A - \frac{\nu}{A} & -\mu \end{pmatrix} \quad . \tag{15.74}$$

One then has $T = -2\mu$ and

$$D = \mu^2 + \big(\nu - \tfrac{3}{8} A^2\big)\big(\nu - \tfrac{9}{8} A^2\big) \quad . \tag{15.75}$$

Setting $D = \frac{1}{4} T^2 = \mu^2$ sets the boundary between stable spiral and stable node. Setting $D = 0$ sets the boundary between stable node and saddle. The fixed point structure is as shown in Fig. 15.3. Though the amplitude exhibits hysteresis, the oscillator frequency is always synchronized with the forcing as one varies the detuning.

15.3.2 Forced van der Pol oscillator

Consider now a weakly dissipative, weakly forced van der Pol oscillator, governed by the equation

$$\ddot{x} + \epsilon \, (x^2 - 1) \, \dot{x} + x = \epsilon \, f_0 \, \cos(t + \epsilon \nu t) \quad , \tag{15.76}$$

where the forcing frequency is $\Omega = 1 + \epsilon\nu$, which is close to the natural frequency $\omega_0 = 1$. We apply the multiple time scale method, with $h(x, \dot{x}) = (1 - x^2) \, \dot{x}$. As usual, the lowest order solution is $x_0 = A(T_1) \cos\big(T_0 + \phi(T_1)\big)$, where $T_0 = t$ and $T_1 = \epsilon t$. Again, we define $\theta \equiv T_0 + \phi(T_1)$ and $\psi(T_1) \equiv \phi(T_1) - \nu T_1$. From

$$h(A\cos\theta, -A\sin\theta) = \big(\tfrac{1}{4} A^3 - A\big) \, \sin\theta + \tfrac{1}{4} A^3 \sin(3\theta) \quad , \tag{15.77}$$

we arrive at

$$\begin{aligned} \bigg(\frac{\partial^2}{\partial\theta^2} + 1\bigg) x_1 &= -2 \frac{\partial^2 x_0}{\partial T_0 \, \partial T_1} + h\bigg(x_0 \, , \, \frac{\partial x_0}{\partial T_0}\bigg) \\ &= \big(\tfrac{1}{4} A^3 - A + 2A' + f_0 \sin\psi\big) \sin\theta \\ &\qquad + \big(2A \, \psi' + 2\nu A + f_0 \cos\psi\big) \cos\theta + \tfrac{1}{4} A^3 \sin(3\theta) \quad . \end{aligned} \tag{15.78}$$

We eliminate the secular terms, proportional to $\sin\theta$ and $\cos\theta$, by demanding

$$\begin{aligned} \frac{dA}{dT_1} &= \tfrac{1}{2} A - \tfrac{1}{8} A^3 - \tfrac{1}{2} f_0 \sin\psi \\ \frac{d\psi}{dT_1} &= -\nu - \frac{f_0}{2A} \cos\psi \quad . \end{aligned} \tag{15.79}$$

Stationary solutions have $A' = \psi' = 0$, hence $\cos\psi = -2\nu A / f_0$, and hence

$$\begin{aligned} f_0^2 &= 4\nu^2 A^2 + \big(1 - \tfrac{1}{4} A^2\big)^2 A^2 \\ &= \tfrac{1}{16} A^6 - \tfrac{1}{2} A^4 + (1 + 4\nu^2) A^2 \quad . \end{aligned} \tag{15.80}$$

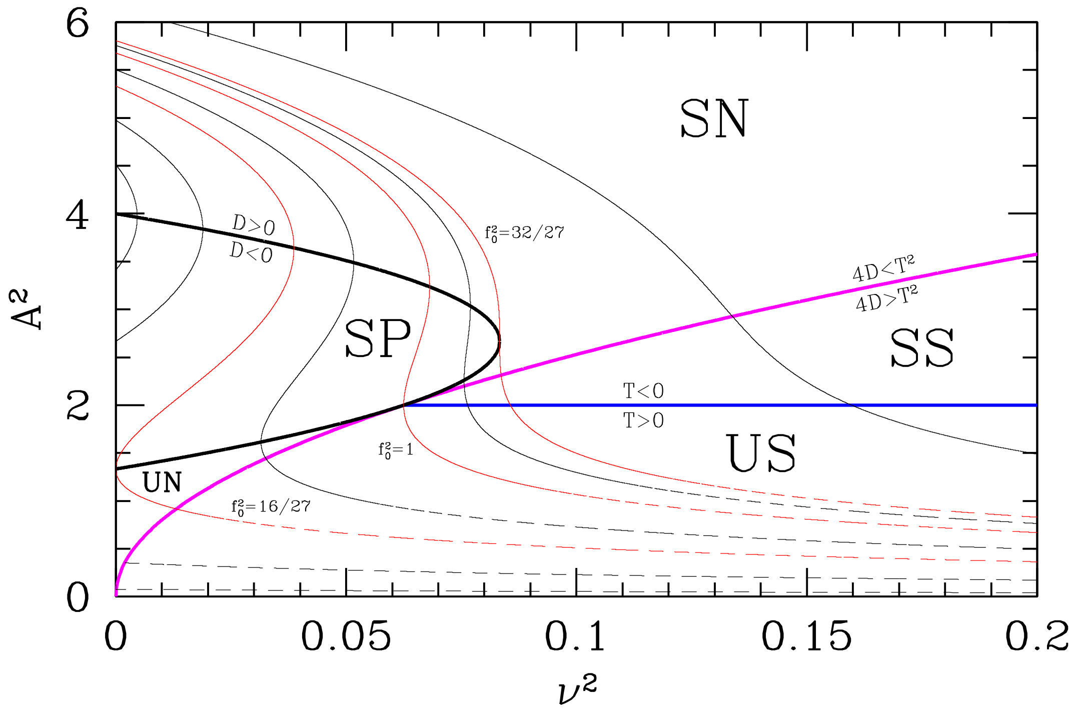


Figure 15.4: Amplitude *versus* detuning for the forced van der Pol oscillator. Fixed point classifications are abbreviated SN (stable node), SS (stable spiral), UN (unstable node), US (unstable spiral), and SP (saddle point). The dot-dashed red curves mark the boundaries of the region in which hysteresis occurs.

For this solution, we have

$$x_0 = A^* \cos(T_0 + \nu T_1 + \psi^*) \quad , \tag{15.81}$$

and the oscillator's frequency is the forcing frequency $\Omega = 1 + \varepsilon\nu$.

To proceed further, let $y = A^2$, and consider the cubic equation

$$G(y) = \tfrac{1}{16} y^3 - \tfrac{1}{2} y^2 + (1 + 4\nu^2)\, y = f_0^2 \quad . \tag{15.82}$$

Setting $G'(y) = 0$, we find the roots of $G'(y)$ lie at $y_\pm = \frac{4}{3}\,(2 \pm u)$, where $u = (1 - 12\,\nu^2)^{1/2}$. Thus, the roots are complex for $\nu^2 > \frac{1}{12}$, in which case $G(y)$ is monotonically increasing, and there is a unique solution to $G(y) = f_0^2$. Since $G(0) = 0 < f_0^2$, that solution satisfies $y > 0$. For $\nu^2 < \frac{1}{12}$, there are two local extrema at $y = y_\pm$. When $G_{\text{min}} = G(y_+) < f_0^2 < G(y_-) = G_{\text{max}}$, the cubic equation $G(y) = f_0^2$ has three real, positive roots. This is equivalent to the condition

$$-\tfrac{8}{27}\, u^3 + \tfrac{8}{9}\, u^2 < \tfrac{32}{27} - f_0^2 < \tfrac{8}{27}\, u^3 + \tfrac{8}{9}\, u^2 \quad . \tag{15.83}$$

We can say even more by exploring the behavior of eqs. (15.80) in the vicinity of the fixed points. Writing $A = A^* + \delta A$ and $\psi = \psi^* + \delta\psi$, we have

$$\frac{d}{dT_1} \begin{pmatrix} \delta A \\ \delta\psi \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\big(1 - \frac{3}{4} A^{*2}\big) & \nu A^* \\ -\nu/A^* & \frac{1}{2}\big(1 - \frac{1}{4} A^{*2}\big) \end{pmatrix} \begin{pmatrix} \delta A \\ \delta\psi \end{pmatrix} \quad . \tag{15.84}$$

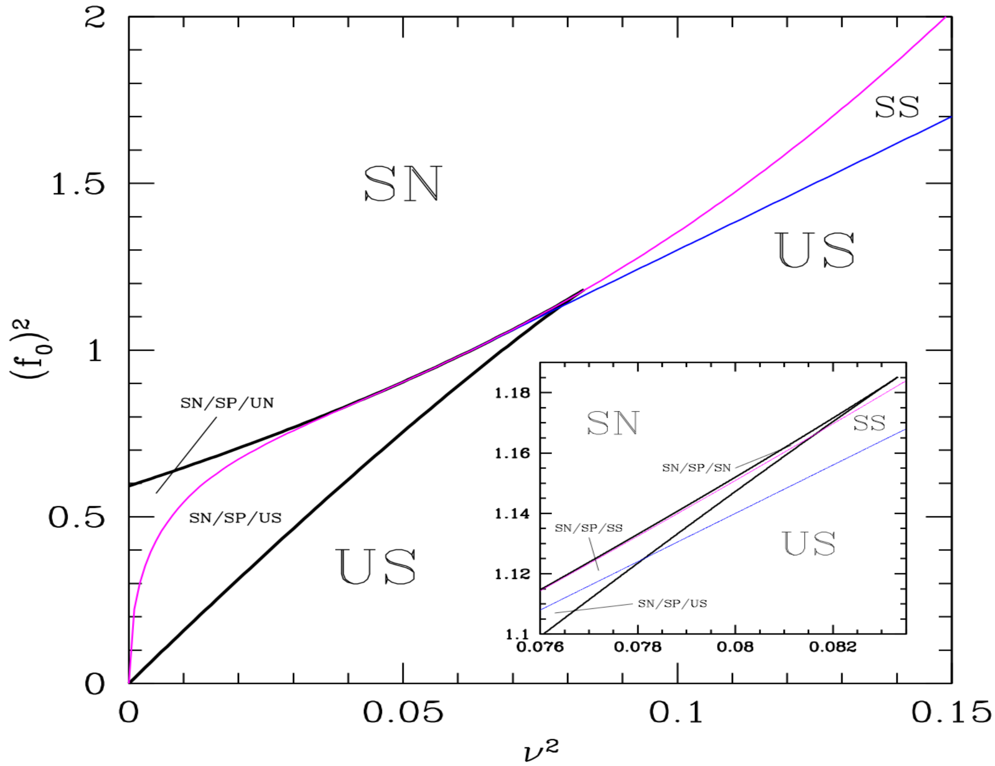


Figure 15.5: Phase diagram for the weakly forced van der Pol oscillator in the (ν^2, f_0^2) plane. Inset shows detail. Abbreviations for fixed point classifications are as in Fig. 15.4.

The eigenvalues of the linearized dynamics at the fixed point are given by $\lambda_\pm = \frac{1}{2}\big(T \pm \sqrt{T^2 - 4D}\big)$, where T and D are the trace and determinant of the linearized equation. Recall now the classification scheme for fixed points of two-dimensional phase flows. When $D < 0$, we have $\lambda_- < 0 < \lambda_+$ and the fixed point is a saddle. For $0 < 4D < T^2$, both eigenvalues have the same sign, so the fixed point is a node. For $4D > T^2$, the eigenvalues form a complex conjugate pair, and the fixed point is a spiral. A node/spiral fixed point is stable if $T < 0$ and unstable if $T > 0$. For our forced van der Pol oscillator, we have

$$\begin{aligned} T &= 1 - \tfrac{1}{2}A^{*2} \\ D &= \tfrac{1}{4}\big(1 - A^{*2} + \tfrac{3}{16}A^{*4}\big) + \nu^2 \quad . \end{aligned} \tag{15.85}$$

From these results we can obtain the plot of Fig. 15.4, where amplitude is shown *versus* detuning. We now ask: for what values of f_0^2 is there hysteretic behavior over a range $\nu \in \big[\nu_-, \nu_+\big]$? Suppose, following the curves of constant f_0^2 in Fig. 15.4, we start somewhere in the upper left corner of the diagram, in the region $D > 0$ and $f_0^2 < \frac{32}{27}$. Now ramp up ν^2 while keeping f_0^2 constant until we arrive on the upper branch of the $D = 0$ curve. An infinitesimal further increase in ν^2 will cause a discontinuous drop in $y = A^2$ to a value below the saddle point region. Clearly if $f_0^2 = 1$, we will wind up on a branch of this curve for which $A^2 < 2$, which is unstable, and so in order to end up on a stable branch, we must start with $f_0^2 > 1$. To find the minimum such value of f_0^2 for which this is possible, we first demand $G(y) = 0$ as well as $D = 0$. The second of these conditions is equivalent to $G'(y) = 0$.

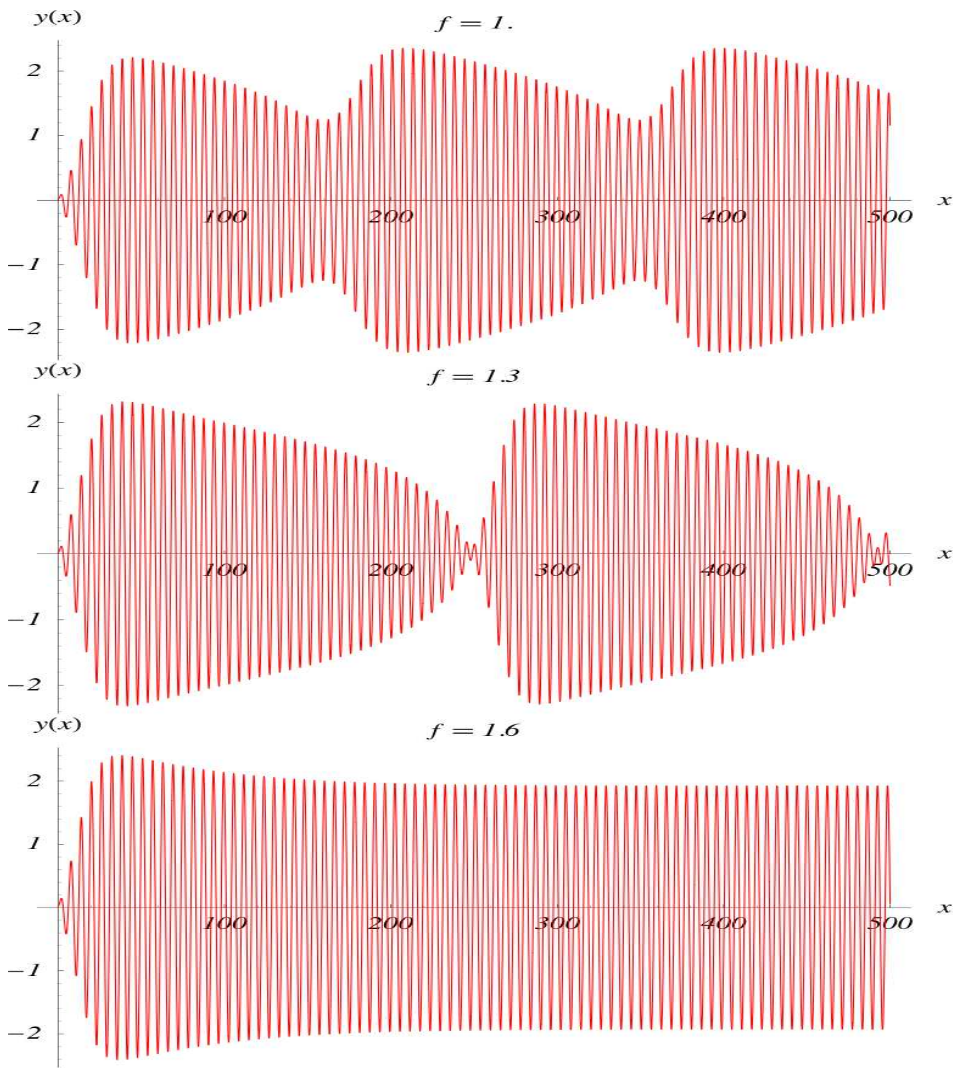


Figure 15.6: Forced van der Pol system with $\epsilon = 0.1$, $\nu = 0.4$ for three values of f_0. The limit entrained solution becomes unstable at $f_0 = 1.334$.

Eliminating y, we obtain the equation $\frac{8}{27}(2+u)^2(1-u) = f_0^2$, where $u = \sqrt{1-12\nu^2}$ as above. Next, we demand that $G(y) = 0$ at $y = 2$ (*i.e.* the blue line in Fig. 15.4) for the *same values* of f_0^2 and ν^2. Thus says $f_0^2 = \frac{1}{6}(7-4u^2)$. Eliminating f_0^2, we obtain the equation

$$\tfrac{8}{27}(2+u)^2\,(1-u) = \tfrac{1}{6}(7-4u^2) \quad , \tag{15.86}$$

which is equivalent to the factorized cubic $(4u-1)(2u+1)^2 = 0$. The root we seek is $u = \frac{1}{4}$, corresponding to $\nu^2 = \frac{15}{192}$ and $f_0^2 = \frac{9}{8}$. Thus, hysteretic behavior is possible only in the narrow regime $f_0^2 \in \left[\frac{9}{8}, \frac{32}{27}\right]$. The phase diagram in the (ν^2, f_0^2) plane is shown in Fig. 15.5. Hysteresis requires two among the three fixed points be stable, so the system can jump from one stable branch to another as ν is varied. These regions are so small they are only discernible in the inset.

Finally, we can make the following statement about the *global* dynamics (*i.e.* not simply in the vicinity of a fixed point). For large A, we have

$$\frac{dA}{dT_1} = -\tfrac{1}{8}A^3 + \ldots \qquad , \qquad \frac{d\psi}{dT_1} = -\nu + \ldots \quad . \tag{15.87}$$

This flow is inward, hence if the flow is not to a stable fixed point, it must be attracted to a limit cycle. The limit cycle necessarily involves several frequencies. This result – the generation of new frequencies by nonlinearities – is called *heterodyning.*

We can see heterodyning in action in the van der Pol system. In Fig. 15.5, the blue line which separates stable and unstable spiral solutions is given by $f_0^2 = 8\nu^2 + \frac{1}{2}$. For example, if we take $\nu = 0.40$ then the boundary lies at $f_0 = 1.334$. For $f_0 < 1.334$, we expect heterodyning, as the entrained solution is unstable. For $f > 1.334$ the solution is entrained and oscillates at a fixed frequency. This behavior is exhibited in Fig. 15.6.

15.4 Synchronization

Thus far we have assumed both the nonlinearity as well as the perturbation are weak. In many systems, we are confronted with a strong nonlinearity which we can perturb weakly. How does an attractive limit cycle in a strongly nonlinear system respond to weak periodic forcing? Here we shall follow the nice discussion in the book of Pikovsky *et al.*

Consider a forced dynamical system,

$$\dot{\boldsymbol{\varphi}} = \boldsymbol{V}(\boldsymbol{\varphi}) + \varepsilon\,\boldsymbol{f}(\boldsymbol{\varphi}, t) \quad . \tag{15.88}$$

When $\varepsilon = 0$, we assume that the system has at least one attractive limit cycle $\boldsymbol{\gamma}(t) = \boldsymbol{\gamma}(t + T_0)$. All points on the limit cycle are fixed under the T_0-advance map g_{T_0}, where $g_\tau \boldsymbol{\varphi}(t) = \boldsymbol{\varphi}(t + \tau)$. The idea is now to parameterize the points along the limit cycle by a phase angle ϕ which runs from 0 to 2π such that $\phi(t)$ increases by 2π with each orbit of the limit cycle, with ϕ increasing uniformly with time, so that $\dot{\phi} = \omega_0 = 2\pi/T_0$. Now consider the action of the T_0-advance map g_{T_0} on points in the vicinity of the limit cycle. Since each point $\boldsymbol{\gamma}(\phi)$ on the limit cycle is a fixed point, and since the limit cycle is presumed to be attractive, we can define the ϕ-*isochrone* as the set of points $\{\boldsymbol{\varphi}\}$ in phase space which flow to the fixed point $\boldsymbol{\gamma}(\phi)$ under repeated application of g_{T_0}. The isochrones are $(N-1)$-dimensional hypersurfaces.

Equivalently, consider a point $\boldsymbol{\varphi}_0 \in \Omega_\gamma$ lying within the basin of attraction Ω_γ of the limit cycle $\boldsymbol{\gamma}(t)$. We say that $\boldsymbol{\varphi}_0$ lies along the ϕ-isochrone if

$$\lim_{t\to\infty} \left| \boldsymbol{\varphi}(t) - \boldsymbol{\gamma}\Big(t + \frac{\phi}{2\pi}\,T_0\Big) \right| = 0 \quad , \tag{15.89}$$

where $\boldsymbol{\varphi}(0) = \boldsymbol{\varphi}_0$. For each $\boldsymbol{\varphi}_0 \in \Omega_\gamma$, there exists a unique corresponding value of $\phi(\boldsymbol{\varphi}_0) \in [0, 2\pi]$. This is called the *asymptotic* (or *latent*) *phase* of $\boldsymbol{\varphi}_0$.

To illustrate this, we analyze the example in Pikovsky *et al.* of the complex amplitude equation (CAE),

$$\frac{dA}{dt} = (1 + i\alpha)\,A - (1 + i\beta)\,|A|^2 A \quad , \tag{15.90}$$

where $A \in \mathbb{C}$ is a complex number. It is convenient to work in polar coordinates, writing $A = R\, e^{i\Theta}$, in which case the real and complex parts of the CAE become

$$\begin{aligned}\dot{R} &= (1 - R^2)\, R \\ \dot{\Theta} &= \alpha - \beta R^2 \quad .\end{aligned} \tag{15.91}$$

These equations can be integrated to yield the solution

$$\begin{aligned}R(t) &= \frac{R_0}{\sqrt{R_0^2 + (1 - R_0^2)\, e^{-2t}}} \\ \Theta(t) &= \Theta_0 + (\alpha - \beta)t - \tfrac{1}{2}\beta \log\big[R_0^2 + (1 - R_0^2)\, e^{-2t}\big] \\ &= \Theta_0 + (\alpha - \beta)\, t + \beta\, \log(R/R_0) \quad .\end{aligned} \tag{15.92}$$

As $t \to \infty$, we have $R(t) \to 1$ and $\dot{\Theta}(t) \to \omega_0$. Thus the limit cycle is the circle $R = 1$, and its frequency is $\omega_0 = \alpha - \beta$.

Since all points on each isochrone share the same phase, we can evaluate $\dot{\phi}$ along the limit cycle, and thus we have $\dot{\phi} = \omega_0$. The functional form of the isochrones is dictated by the rotational symmetry of the vector field, which requires $\phi(R, \Theta) = \Theta - f(R)$, where $f(R)$ is an as-yet undetermined function. Taking the derivative, we immediately find $f(R) = \beta \log R$, *i.e.*

$$\phi(R, \Theta) = \Theta - \beta \log R + c \quad , \tag{15.93}$$

where c is a constant. We can now check that

$$\dot{\phi} = \dot{\Theta} - \beta\, \frac{\dot{R}}{R} = \alpha - \beta = \omega_0 \quad . \tag{15.94}$$

Without loss of generality we may take $c = 0$. Thus the ϕ-isochrone is given by the curve $\Theta(R) = \phi + \beta\, \log R$, which is a logarithmic spiral. These isochrones are depicted in fig. 15.7.

At this point we have defined a phase function $\phi(\boldsymbol{\varphi})$ as the phase of the fixed point along the limit cycle to which $\boldsymbol{\varphi}$ flows under repeated application of the T_0-advance map g_{T_0}. Now let us examine the dynamics of ϕ for the weakly perturbed system of eqn. 15.88. We have

$$\begin{aligned}\frac{d\phi}{dt} &= \sum_{j=1}^{N} \frac{\partial \phi}{\partial \varphi_j}\, \frac{d\varphi_j}{dt} \\ &= \omega_0 + \varepsilon \sum_{j=1}^{N} \frac{\partial \phi}{\partial \varphi_j}\, f_j(\boldsymbol{\varphi}, t) \quad .\end{aligned} \tag{15.95}$$

We will assume that $\boldsymbol{\varphi}$ is close to the limit cycle, so that $\boldsymbol{\varphi} - \boldsymbol{\gamma}(\phi)$ is small. As an example, consider once more the complex amplitude equation (15.90), but now adding in a periodic forcing term.

$$\frac{dA}{dt} = (1 + i\alpha)\, A - (1 + i\beta)\, |A|^2 A + \varepsilon\, \cos\omega t \quad . \tag{15.96}$$

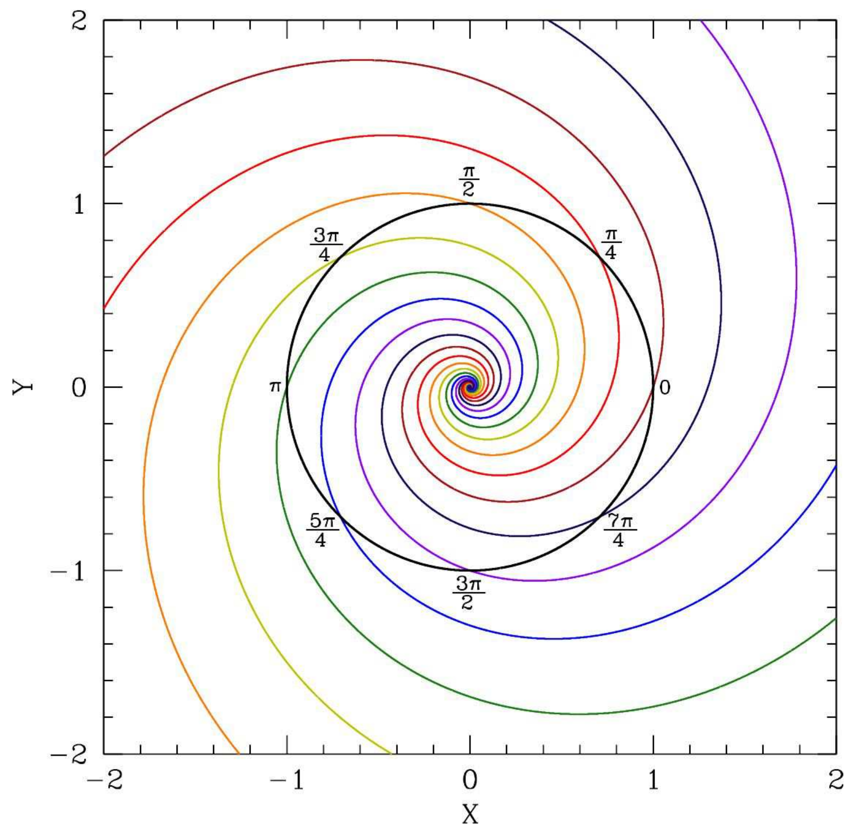


Figure 15.7: Isochrones of the complex amplitude equation $\dot{A} = (1 + i\alpha)A - (1 + i\beta)|A|^2 A$, where $A = X + iY$.

Writing $A = X + iY$, we have

$$\begin{aligned}\dot{X} &= X - \alpha Y - (X - \beta Y)(X^2 + Y^2) + \varepsilon \cos \omega t \\ \dot{Y} &= Y + \alpha X - (\beta X + Y)(X^2 + Y^2) \quad .\end{aligned} \tag{15.97}$$

In Cartesian coordinates, the isochrones for the $\varepsilon = 0$ system are

$$\phi = \tan^{-1}(Y/X) - \tfrac{1}{2}\beta \log(X^2 + Y^2) \quad , \tag{15.98}$$

hence

$$\begin{aligned}\frac{d\phi}{dt} &= \omega_0 + \varepsilon \frac{\partial\phi}{\partial X} \cos \omega t \\ &= \alpha - \beta - \varepsilon \left(\frac{\beta X + Y}{X^2 + Y^2}\right) \cos \omega t \\ &\approx \omega_0 - \varepsilon \, (\beta \cos \phi + \sin \phi) \, \cos \omega t \\ &= \omega_0 - \varepsilon \sqrt{1 + \beta^2} \, \cos(\phi - \phi_\beta) \cos \omega t \quad .\end{aligned} \tag{15.99}$$

where $\phi_\beta = \text{ctn}^{-1}\beta$. Note that in the third line above we have invoked $R \approx 1$, *i.e.* we assume that we are close to the limit cycle.

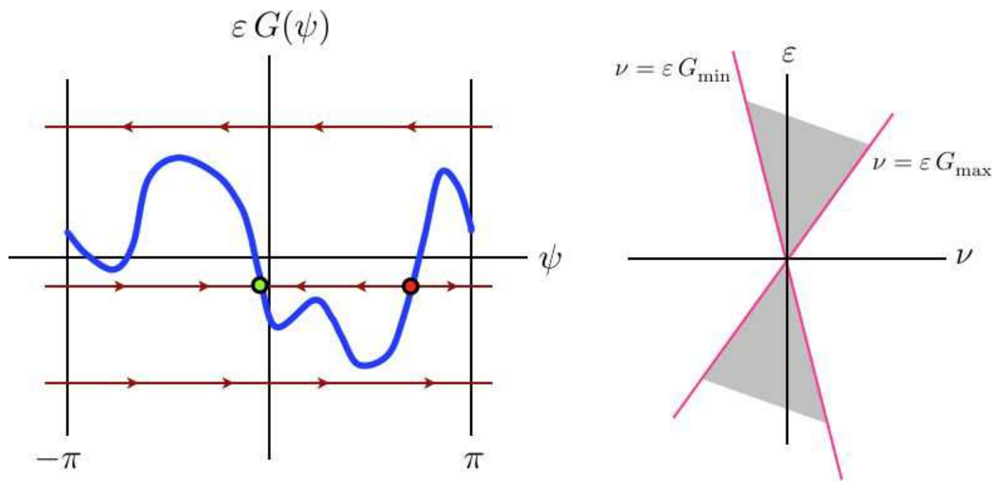


Figure 15.8: Left panel: graphical analysis of the equation $\dot{\psi} = -\nu + \varepsilon\, G(\psi)$. Right panel: Synchronization region (gray) as a function of detuning ν.

We now define the function

$$F(\phi, t) \equiv \sum_{j=1}^{N} \frac{\partial \phi}{\partial \varphi_j}\bigg|_{\boldsymbol{\gamma}(\phi)} f_j\,(\boldsymbol{\gamma}(\phi), t) \quad . \tag{15.100}$$

The phase dynamics for ϕ are now written as

$$\dot{\phi} = \omega_0 + \varepsilon\, F(\phi, t) \quad . \tag{15.101}$$

Now $F(\phi, t)$ is periodic in both its arguments, so we may write

$$F(\phi, t) = \sum_{k,l} F_{kl}\, e^{i(k\phi + l\omega t)} \quad . \tag{15.102}$$

For the unperturbed problem, we have $\dot{\phi} = \omega_0$, hence resonant terms in the above sum are those for which $k\omega_0 + l\omega \approx 0$. This occurs when $\omega \approx \frac{p}{q}\,\omega_0$, where p and q are relatively prime integers. In this case the resonance condition is satisfied for $k = jp$ and $l = -jq$ for all $j \in \mathbb{Z}$. We now separate the resonant from the nonresonant terms in the (k, l) sum, writing

$$\dot{\phi} = \omega_0 + \varepsilon \sum_j F_{jp,-jq}\, e^{ij(p\phi - q\omega t)} + \text{NRT} \quad , \tag{15.103}$$

where NRT denotes nonresonant terms, *i.e.* those for which $(k, l) \neq (jp, -jq)$ for some integer j. We now average over short time scales to eliminate the nonresonant terms, and focus on the dynamics of this averaged phase $\langle\phi\rangle$.

We define the angle $\psi \equiv p\langle\phi\rangle - q\omega t$, which obeys

$$\begin{aligned}\dot{\psi} &= p\,\langle\dot{\phi}\rangle - q\omega \\ &= (p\omega_0 - q\omega) + \varepsilon p \sum_j F_{jp,-jq}\, e^{ij\psi} = -\nu + \varepsilon\, G(\psi) \quad ,\end{aligned} \tag{15.104}$$

where $\nu \equiv q\omega - p\omega_0$ is the detuning and $G(\psi) = p\sum_j F_{jp,-jq}\, e^{ij\psi}$ is the sum over resonant terms. Note that the nonresonant terms have been eliminated by the aforementioned averaging procedure. This last equation is a simple $N = 1$ dynamical system on the circle – a system we have already studied. The dynamics are depicted in fig. 15.8. If the detuning ν falls within the range $[\varepsilon\, G_{\rm min}\, ,\, \varepsilon\, G_{\rm max}]$, then ψ flows to a fixed point, and the nonlinear oscillator is synchronized with the periodic external force, with $\langle\dot{\phi}\rangle \to \frac{q}{p}\,\omega$. If the detuning is too large and lies outside this region, then there is no synchronization. Rather, $\psi(t)$ increases on average linearly with time. In this case we have $\langle\phi(t)\rangle = \phi_0 + \frac{q}{p}\omega t + \frac{1}{p}\,\psi(t)$, where

$$dt = \frac{d\psi}{\varepsilon\, G(\psi) - \nu} \qquad \Longrightarrow \qquad T_\psi = \int\limits_{-\pi}^{\pi} \frac{d\psi}{\varepsilon\, G(\psi) - \nu} \quad . \tag{15.105}$$

Thus, $\psi(t) = \varOmega_\psi\, t + \varPsi(t)$, where $\varPsi(t) = \varPsi(t+T)$ is periodic with period $T_\psi = 2\pi/\varOmega_\psi$. This leads to heterodyning with a beat frequency $\varOmega_\psi(\nu,\varepsilon)$.

Why do we here find the general resonance condition $\omega = \frac{p}{q}\,\omega_0$, whereas for weakly forced, weakly nonlinear oscillators resonance could only occur for $\omega = \omega_0$? There are two reasons. The main reason is that in the latter case, the limit cycle is harmonic to zeroth order, with $x_0(t) = A\,\cos(t+\phi)$. There are only two frequencies, then, in the Fourier decomposition of the limit cycle: $\omega_0 = \pm 1$. In the strongly nonlinear case, the limit cycle is decomposed into what is in general a countably infinite set of frequencies which are all multiples of a fundamental ω_0. In addition, if the forcing $\boldsymbol{f}(\boldsymbol{\varphi}, t)$ is periodic in t, its Fourier decomposition in t will involve all integer multiples of some fundamental ω. Thus, the most general resonance condition is $k\omega_0 + l\omega = 0$.

Our analysis has been limited to the lowest order in ε, and we have averaged out the nonresonant terms. When one systematically accounts for both these features, there are two main effects. One is that the boundaries of the synchronous region are no longer straight lines as depicted in the right panel of fig. 15.8. The boundaries themselves can be curved. Moreover, even if there are no resonant terms in the (k,l) sum to lowest order, they can be generated by going to higher order in ε. In such a case, the width of the synchronization region $\Delta\nu$ will be proportional to a higher power of ε: $\varDelta\nu \propto \varepsilon^n$, where n is the order of ε where resonant forcing terms first appear in the analysis.

15.5 Relaxation Oscillations

We saw how to use multiple time scale analysis to identify the limit cycle of the van der Pol oscillator when ϵ is small. Consider now the opposite limit, where the coefficient of the damping term is very large. We generalize the van der Pol equation to

$$\ddot{x} + \mu\,\Phi(x)\,\dot{x} + x = 0 \quad , \tag{15.106}$$

and suppose $\mu \gg 1$. Define now the variable

$$y \equiv \frac{\dot{x}}{\mu} + \int\limits_0^x dx'\,\Phi(x') \equiv \frac{\dot{x}}{\mu} + F(x) \quad , \tag{15.107}$$

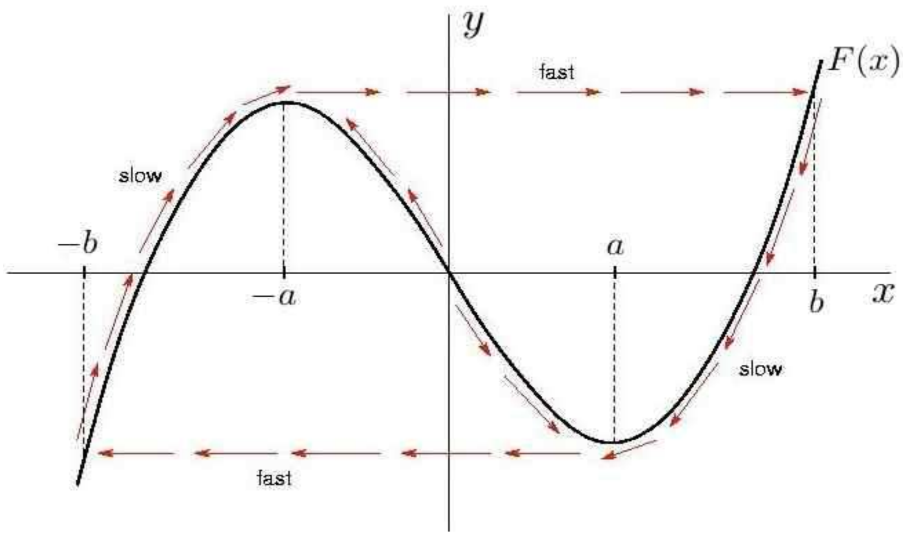


Figure 15.9: Relaxation oscillations in the Liènard plane (x, y). The system rapidly flows to a point on the curve $y = F(x)$, and then crawls slowly along this curve. The slow motion takes x from $-b$ to $-a$, after which the system executes a rapid jump to $x = +b$, then a slow retreat to $x = +a$, followed by a rapid drop to $x = -b$.

where $F'(x) = \Phi(x)$. (y is sometimes called the *Liènard variable*, and (x, y) the *Liènard plane*.) Then the original second order equation may be written as two coupled first order equations:

$$\begin{aligned}\dot{x} &= \mu\Big(y - F(x)\Big) \\ \dot{y} &= -\frac{x}{\mu} \quad .\end{aligned} \tag{15.108}$$

Since $\mu \gg 1$, the first of these equations is *fast* and the second one *slow*. The dynamics rapidly achieves $y \approx F(x)$, and then slowly evolves along the curve $y = F(x)$, until it is forced to make a large, fast excursion.

A concrete example is useful. Consider $F(x)$ of the form sketched in Fig. 15.9. This is what one finds for the van der Pol oscillator, where $\Phi(x) = x^2 - 1$ and $F(x) = \frac{1}{3}x^3 - x$. The limit cycle behavior $x_{\text{LC}}(t)$ is sketched in Fig. 15.10. We assume $\Phi(x) = \Phi(-x)$ for simplicity.

Assuming $\Phi(x) = \Phi(-x)$ is symmetric, $F(x)$ is antisymmetric. For the van der Pol oscillator and other similar cases, $F(x)$ resembles the sketch in fig. 15.9. There are two local extrema: a local maximum at $x = -a$ and a local minimum at $x = +a$. We define b such that $F(b) = F(-a)$, as shown in the figure; antisymmetry then entails $F(-b) = F(+a)$. Starting from an arbitrary initial condition, the y dynamics are slow, since $\dot{y} = -\mu^{-1}x$ (we assume $\mu \gg x(0)$). So y can be regarded as essentially constant for the fast dynamics in the second of eqn. 15.108, according to which $x(t)$ flows rapidly to the right if $y > F(x)$ and rapidly to the left if $y < F(x)$. This fast motion stops when $x(t)$ reaches a point where $y = F(x)$. At this point, the slow dynamics takes over. Assuming $y \approx F(x)$, we have

$$y \approx F(x) \qquad \Rightarrow \qquad \dot{y} = -\frac{x}{\mu} \approx F'(x)\,\dot{x} \quad , \tag{15.109}$$

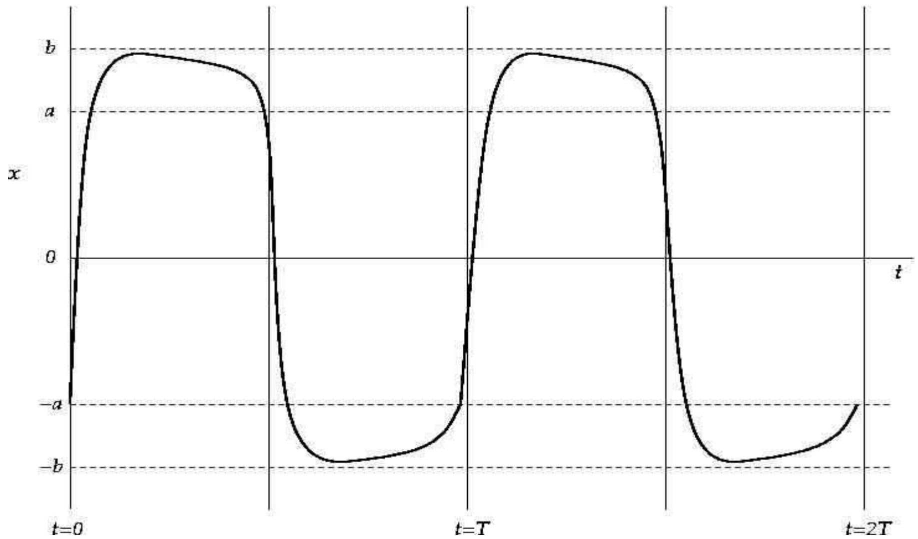


Figure 15.10: A sketch of the limit cycle for the relaxation oscillation studied in this section.

which says that

$$\dot{x} \approx -\frac{x}{\mu\, F'(x)} \qquad \text{if } y \approx F(x) \tag{15.110}$$

over the slow segments of the motion, which are the regions $x \in [-b, -a]$ and $x \in [a, b]$. The relaxation oscillation is then as follows. Starting at $x = -b$, $x(t)$ increases slowly according to eqn. 15.110. At $x = -a$, the motion can no longer follow the curve $y = F(x)$, since $\dot{y} = -\mu^{-1}x$ is still positive. The motion thus proceeds quickly to $x = +b$, with

$$\dot{x} \approx \mu\Big(F(b) - F(x)\Big) \qquad x \in \big[-a, +b\big] \quad . \tag{15.111}$$

After reaching $x = +b$, the motion once again is slow, and again follows eqn. 15.110, according to which $x(t)$ now decreases slowly until it reaches $x = +a$, at which point the motion is again fast, with

$$\dot{x} \approx \mu\Big(F(a) - F(x)\Big) \qquad x \in \big[-b, +a\big] \quad . \tag{15.112}$$

The cycle then repeats.

Thus, the limit cycle is given by the following segments:

$$x \in [-b, -a] \qquad (\dot{x} > 0) \;: \qquad \dot{x} \approx -\frac{x}{\mu\, F'(x)} \tag{15.113}$$

$$x \in [-a, b] \qquad (\dot{x} > 0) \;: \qquad \dot{x} \approx \mu\big[F(b) - F(x)\big] \tag{15.114}$$

$$x \in [a, b] \qquad (\dot{x} < 0) \;: \qquad \dot{x} \approx -\frac{x}{\mu\, F'(x)} \tag{15.115}$$

$$x \in [-b, a] \qquad (\dot{x} < 0) \;: \qquad \dot{x} \approx \mu\big[F(a) - F(x)\big] \quad . \tag{15.116}$$

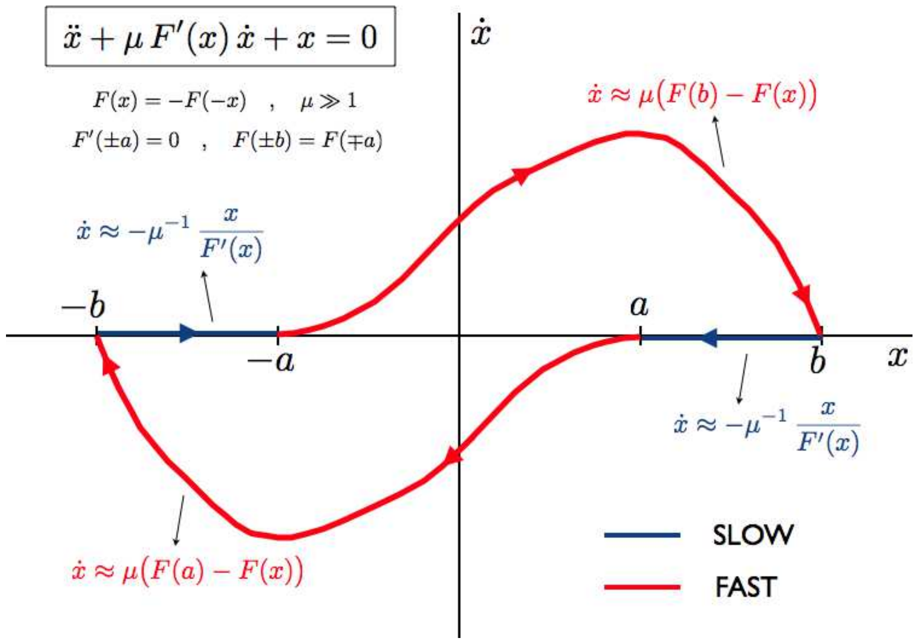


Figure 15.11: Limit cycle for large μ relaxation oscillations, shown in the phase plane $(x, \dot{x})$.

A sketch of the limit cycle is given in fig. 15.11, showing the slow and fast portions.

When $\mu \gg 1$ we can determine approximately the period of the limit cycle. Clearly the period is twice the time for either of the slow portions, hence

$$T \approx 2\mu \int\limits_a^b dx\, \frac{\Phi(x)}{x} \quad , \tag{15.117}$$

where $F'(\pm a) = \Phi(\pm a) = 0$ and $F(\pm b) = F(\mp a)$. For the van der Pol oscillator, with $\Phi(x) = x^2 - 1$, we have $a = 1$, $b = 2$, and $T \simeq (3 - 2\log 2)\,\mu$.

15.5.1 Example problem

Consider the equation

$$\ddot{x} + \mu\big(|x| - 1\big)\dot{x} + x = 0 \quad . \tag{15.118}$$

Sketch the trajectory in the Liènard plane, and find the approximate period of the limit cycle for $\mu \gg 1$.

Solution : We define

$$F'(x) = |x| - 1 \quad \Rightarrow \quad F(x) = \begin{cases} +\frac{1}{2}x^2 - x & \text{if } x > 0 \\ \\ -\frac{1}{2}x^2 - x & \text{if } x < 0 \quad . \end{cases} \tag{15.119}$$

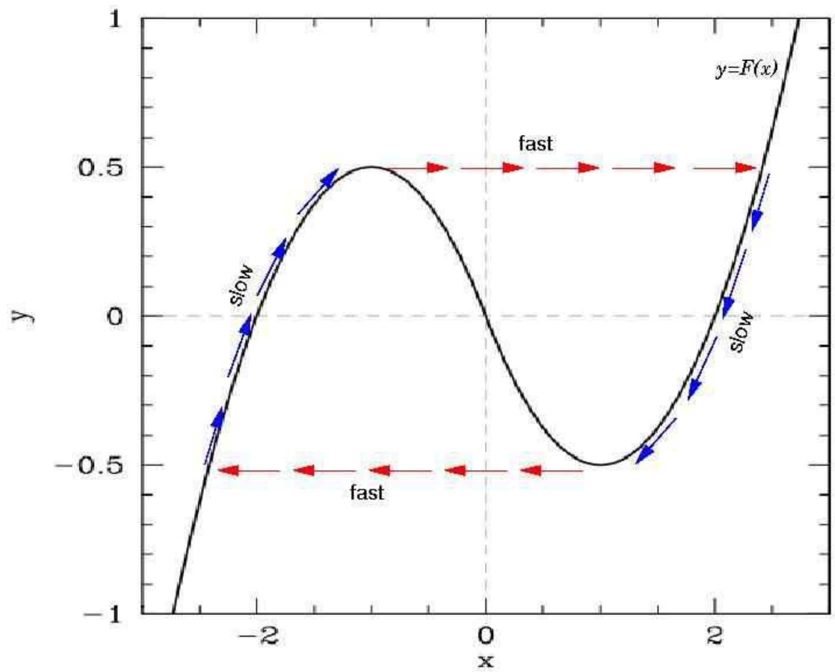


Figure 15.12: Relaxation oscillations for $\ddot{x} + \mu\big(|x| - 1\big)\,\dot{x} + x = 0$ plotted in the Liénard plane. The solid black curve is $y = F(x) = \frac{1}{2}x^2\,\mathrm{sgn}(x) - x$. The variable y is defined to be $y = \mu^{-1}\,\dot{x} + F(x)$. Along slow portions of the limit cycle, $y \simeq F(x)$.

We therefore have

$$\dot{x} = \mu\big\{y - F(x)\big\} \quad , \quad \dot{y} = -\frac{x}{\mu} \quad , \tag{15.120}$$

with $y \equiv \mu^{-1}\,\dot{x} + F(x)$.

Setting $F'(x) = 0$ we find $x = \pm a$, where $a = 1$ and $F(\pm a) = \mp\frac{1}{2}$. We also find $F(\pm b) = F(\mp a)$, where $b = 1 + \sqrt{2}$. Thus, the limit cycle is as follows: (i) fast motion from $x = -a$ to $x = +b$, (ii) slow relaxation from $x = +b$ to $x = +a$, (iii) fast motion from $x = +a$ to $x = -b$, and (iv) slow relaxation from $x = -b$ to $x = -a$. The period is approximately the time it takes for the slow portions of the cycle. Along these portions, we have $y \simeq F(x)$, and hence $\dot{y} \simeq F'(x)\,\dot{x}$. But $\dot{y} = -x/\mu$, so

$$F'(x)\,\dot{x} \simeq -\frac{x}{\mu} \quad \Rightarrow \quad dt = -\mu\,\frac{F'(x)}{x}\,dx \quad , \tag{15.121}$$

which we integrate to obtain

$$\begin{aligned} T &\simeq -2\mu\int_b^a \!dx\,\frac{F'(x)}{x} = 2\mu\int_1^{1+\sqrt{2}}\!\!dx\,\bigg(1 - \frac{1}{x}\bigg) \\ &= 2\mu\Big[\sqrt{2} - \log\big(1 + \sqrt{2}\big)\Big] \simeq 1.066\,\mu \quad . \end{aligned} \tag{15.122}$$

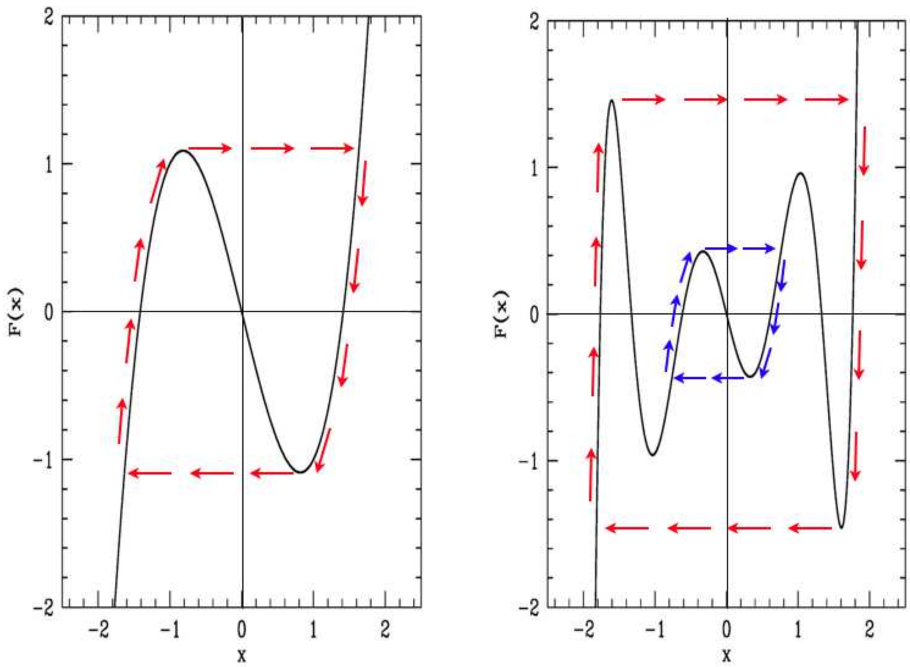


Figure 15.13: Liénard plots for systems with one (left) and two (right) relaxation oscillations.

15.5.2 Multiple limit cycles

For the equation

$$\ddot{x} + \mu\, F'(x)\, \dot{x} + x = 0 \quad , \tag{15.123}$$

it is illustrative to consider what sort of $F(x)$ would yield more than one limit cycle. Such an example is shown in fig. 15.13.

In polar coordinates, it is very easy to construct such examples. Consider, for example, the system

$$\begin{aligned} \dot{r} &= \sin(\pi r) + \epsilon \cos\theta \\ \dot{\theta} &= b\, r \quad , \end{aligned} \tag{15.124}$$

with $|\epsilon| < 1$. First consider the case $\epsilon = 0$. Clearly the radial flow is outward for $\sin(\pi r) > 0$ and inward for $\sin(\pi r) < 0$. Thus, we have stable limit cycles at $r = 2n + 1$ and unstable limit cycles at $r = 2n$, for all $n \in \mathbb{Z}$. With $0 < |\epsilon| < 1$, we have

$$\begin{aligned} r \in \big[2n + \tfrac{1}{\pi} \sin^{-1}\epsilon \,,\, 2n + 1 - \tfrac{1}{\pi} \sin^{-1}\epsilon\big] \qquad &\Rightarrow \qquad \dot{r} > 0 \\ r \in \big[2n + 1 + \tfrac{1}{\pi} \sin^{-1}\epsilon \,,\, 2n + 2 - \tfrac{1}{\pi} \sin^{-1}\epsilon\big] \qquad &\Rightarrow \qquad \dot{r} < 0 \end{aligned} \tag{15.125}$$

The Poincaré-Bendixson theorem then guarantees the existence of stable and unstable limit cycles. We can put bounds on the radial extent of these limit cycles.

$$\begin{aligned} r \in \big[2n + 1 - \tfrac{1}{\pi} \sin^{-1}\epsilon \,,\, 2n + 1 + \tfrac{1}{\pi} \sin^{-1}\epsilon\big] \qquad &\Rightarrow \qquad \text{stable limit cycle} \\ t \in \big[2n - \tfrac{1}{\pi} \sin^{-1}\epsilon \,,\, 2n + \tfrac{1}{\pi} \sin^{-1}\epsilon\big] \qquad &\Rightarrow \qquad \text{unstable limit cycle} \quad . \end{aligned} \tag{15.126}$$

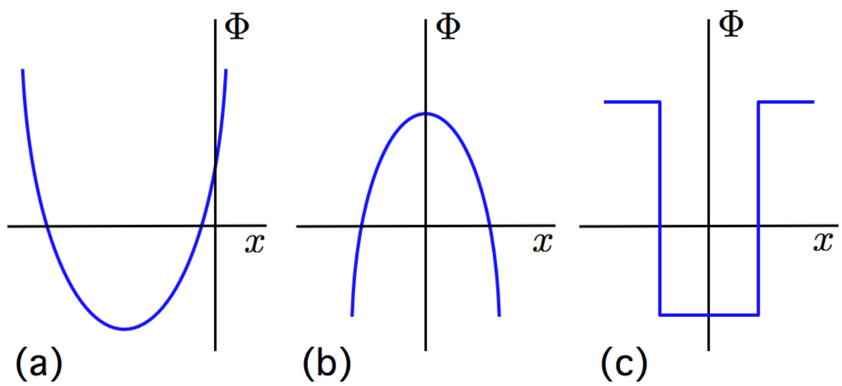


Figure 15.14: Three instances of $\Phi(x)$.

Note that an unstable limit cycle is a repeller, which is to say that it is stable (an attractor) if we run the dynamics backwards, sending $t \to -t$.

15.5.3 Example problem

Consider the nonlinear oscillator,

$$\ddot{x} + \mu\,\Phi(x)\,\dot{x} + x = 0 \quad , \tag{15.127}$$

with $\mu \gg 1$. For each case in fig. 15.14, sketch the flow in the Liènard plane, starting with a few different initial conditions. For which case(s) do relaxation oscillations occur?

Solution : Recall the general theory of relaxation oscillations. We define

$$y \equiv \frac{\dot{x}}{\mu} + \int\limits_0^x \!dx'\,\Phi(x') = \frac{\dot{x}}{\mu} + F(x) \quad , \tag{15.128}$$

in which case the second order ODE for the oscillator may be written as two coupled first order ODEs:

$$\dot{y} = -\frac{x}{\mu} \quad , \quad \dot{x} = \mu\Big(y - F(x)\Big) \quad . \tag{15.129}$$

Since $\mu \gg 1$, the first of these equations is *slow* and the second one *fast*. The dynamics rapidly achieves $y \approx F(x)$, and then slowly evolves along the curve $y = F(x)$, until it is forced to make a large, fast excursion.

To explore the dynamics in the Liènard plane, we plot $F(x)$ *versus* x, which means we must integrate $\Phi(x)$. This is done for each of the three cases in fig. 15.14.

Note that a fixed point corresponds to $x = 0$ and $\dot{x} = 0$. In the Liènard plane, this means $x = 0$ and $y = F(0)$. Linearizing by setting $x = \delta x$ and $y = F(0) + \delta y$, we have[1]

$$\frac{d}{dt}\begin{pmatrix}\delta x\\ \delta y\end{pmatrix} = \begin{pmatrix}\mu\,\delta y - \mu F'(0)\,\delta x\\ -\mu^{-1}\,\delta x\end{pmatrix} = \begin{pmatrix}-\mu F'(0) & \mu\\ -\mu^{-1} & 0\end{pmatrix}\begin{pmatrix}\delta x\\ \delta y\end{pmatrix} \quad . \tag{15.130}$$

[1] We could, of course, linearize about the fixed point in $(x, \dot{x})$ space and obtain the same results.

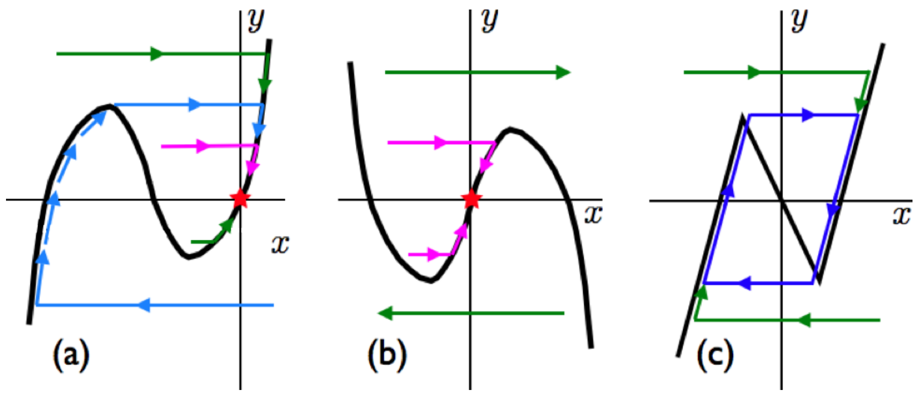


Figure 15.15: Phase flows in the Liénard plane for the three examples in fig. 15.14.

The linearized map has trace $T = -\mu\, F'(0)$ and determinant $D = 1$. Since $\mu \gg 1$ we have $0 < D < \frac{1}{4}T^2$, which means the fixed point is either a stable node, for $F'(0) > 0$, or an unstable node, for $F'(0) < 0$. In cases (a) and (b) the fixed point is a stable node, while in case (c) it is unstable. The flow in case (a) always collapses to the stable node. In case (b) the flow either is unbounded or else it collapses to the stable node. In case (c), all initial conditions eventually flow to a unique limit cycle exhibiting relaxation oscillations.

15.6 Appendix I: Multiple Time Scale Analysis to $\mathcal{O}(\epsilon^2)$

Problem : A particle of mass m moves in one dimension subject to the potential

$$U(x) = \tfrac{1}{2} m\, \omega_0^2\, x^2 + \tfrac{1}{3} \epsilon\, m\, \omega_0^2\, \frac{x^3}{a} \quad , \tag{15.131}$$

where ϵ is a dimensionless parameter.

(a) Find the equation of motion for x. Show that by rescaling x and t you can write this equation in dimensionless form as

$$\frac{d^2u}{ds^2} + u = -\epsilon u^2 \quad . \tag{15.132}$$

Solution : The equation of motion is

$$m\ddot{x} = -U'(x) = -m\omega_0^2 x - \epsilon m \omega_0^2\, \frac{x^2}{a} \quad . \tag{15.133}$$

We now define $s \equiv \omega_0 t$ and $u \equiv x/a$, yielding

$$\frac{d^2u}{ds^2} + u = -\epsilon u^2 \quad . \tag{15.134}$$

(b) You are now asked to perform an $\mathcal{O}(\epsilon^2)$ multiple time scale analysis of this problem, writing

$$T_0 = s \quad , \quad T_1 = \epsilon s \quad , \quad T_2 = \epsilon^2 s \quad , \tag{15.135}$$

and

$$u = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \ldots \quad . \tag{15.136}$$

This results in a hierarchy of coupled equations for the functions $\{u_n\}$. Derive the first three equations in the hierarchy.

Solution : We have

$$\frac{d}{ds} = \frac{\partial}{\partial T_0} + \epsilon \frac{\partial}{\partial T_1} + \epsilon^2 \frac{\partial}{\partial T_2} + \ldots \quad . \tag{15.137}$$

Therefore

$$\begin{aligned}\left(\frac{\partial}{\partial T_0} + \epsilon \frac{\partial}{\partial T_1} + \epsilon^2 \frac{\partial}{\partial T_2} + \ldots\right)^2 \Big(u_0 + \epsilon\, u_1 + \epsilon^2\, u_2 + \ldots\Big) + \Big(u_0 + \epsilon\, u_1 + \epsilon^2\, u_2 + \ldots\Big) \\ = -\epsilon \Big(u_0 + \epsilon\, u_1 + \epsilon^2\, u_2 + \ldots\Big)^2 \quad .\end{aligned} \tag{15.138}$$

Expanding and then collecting terms order by order in ϵ, we derive the hierarchy. The first three levels are

$$\begin{aligned}&\frac{\partial^2 u_0}{\partial T_0^2} + u_0 = 0 \\ &\frac{\partial^2 u_1}{\partial T_0^2} + u_1 = -2\, \frac{\partial^2 u_0}{\partial T_0\, \partial T_1} - u_0^2 \\ &\frac{\partial^2 u_2}{\partial T_0^2} + u_2 = -2\, \frac{\partial^2 u_0}{\partial T_0\, \partial T_2} - \frac{\partial^2 u_0}{\partial T_1^2} - 2\, \frac{\partial^2 u_1}{\partial T_0\, \partial T_1} - 2\, u_0\, u_1 \quad .\end{aligned} \tag{15.139}$$

(c) Show that there is no frequency shift to first order in ϵ.

Solution : At the lowest (first) level of the hierarchy, the solution is

$$u_0 = A(T_1, T_2)\, \cos\big(T_0 + \phi(T_1, T_2)\big) \quad . \tag{15.140}$$

At the second level, then,

$$\frac{\partial^2 u_1}{\partial T_0^2} + u_1 = 2\, \frac{\partial A}{\partial T_1}\, \sin(T_0 + \phi) + 2A\, \frac{\partial \phi}{\partial T_1}\, \cos(T_0 + \phi) - A^2 \cos^2(T_0 + \phi) \quad . \tag{15.141}$$

We eliminate the resonant forcing terms on the RHS by demanding

$$\frac{\partial A}{\partial T_1} = 0 \qquad \text{and} \qquad \frac{\partial \phi}{\partial T_1} = 0 \quad . \tag{15.142}$$

Thus, we must have $A = A(T_2)$ and $\phi = \phi(T_2)$. To $\mathcal{O}(\epsilon)$, then, ϕ is a constant, which means there is no frequency shift at this level of the hierarchy.

(d) Find $u_0(s)$ and $u_1(s)$.

Solution :The equation for u_1 is that of a non-resonantly forced harmonic oscillator. The solution is easily found to be

$$u_1 = -\tfrac{1}{2} A^2 + \tfrac{1}{6} A^2 \cos(2T_0 + 2\phi) \quad . \tag{15.143}$$

We now insert this into the RHS of the third equation in the hierarchy:

$$\begin{aligned}
\frac{\partial^2 u_2}{\partial T_0^2} + u_2 &= -2\frac{\partial^2 u_0}{\partial T_0\,\partial T_2} - 2\,u_0\,u_1 \qquad (15.144)\\
&= 2\,\frac{\partial A}{\partial T_2}\,\sin(T_0+\phi) + 2A\,\frac{\partial \phi}{\partial T_2}\,\cos(T_0+\phi) - 2A\cos(T_0+\phi)\,\Big\{-\tfrac{1}{2}A^2 + \tfrac{1}{6}A^2\cos(2T_0+2\phi)\Big\}\\
&= 2\,\frac{\partial A}{\partial T_2}\,\sin(T_0+\phi) + \Big(2A\,\frac{\partial \phi}{\partial T_2} + \tfrac{5}{6}A^3\Big)\cos(T_0+\phi) - \tfrac{1}{6}A^3\,\cos(3T_0+3\phi) \quad .
\end{aligned}$$

Setting the coefficients of the resonant terms on the RHS to zero yields

$$\begin{aligned}
\frac{\partial A}{\partial T_2} &= 0 \quad \Rightarrow \quad A = A_0\\
2A\,\frac{\partial \phi}{\partial T_2} + \tfrac{5}{6}A^3 &= 0 \quad \Rightarrow \quad \phi = -\tfrac{5}{12}\,A_0^2\,T_2 \quad .
\end{aligned} \qquad (15.145)$$

Therefore,

$$u(s) \;= \overbrace{A_0\,\cos\big(s - \tfrac{5}{12}\,\epsilon^2 A_0^2\,s\big)}^{u_0(s)} \;+\; \overbrace{\tfrac{1}{6}\,\epsilon\,A_0^2\,\cos\big(2s - \tfrac{5}{6}\,\epsilon^2 A_0^2\,s\big) - \tfrac{1}{2}\,\epsilon\,A_0^2}^{\epsilon\,u_1(s)} + \mathcal{O}\big(\epsilon^2\big) \qquad (15.146)$$

15.7 Appendix II: MSA and Poincaré-Lindstedt Methods

15.7.1 Problem using multiple time scale analysis

Consider the central force law $F(r) = -k\,r^{\beta^2-3}$.

(a) Show that a stable circular orbit exists at radius $r_0 = (\ell^2/\mu k)^{1/\beta^2}$.

Solution : For a circular orbit, the effective radial force must vanish:

$$F_{\text{eff}}(r) = \frac{\ell^2}{\mu r^3} + F(r) = \frac{\ell^2}{\mu r^3} - \frac{k}{r^{3-\beta^2}} = 0 \quad . \qquad (15.147)$$

Solving for $r = r_0$, we have $r_0 = (\ell^2/\mu k)^{1/\beta^2}$. The second derivative of $U_{\text{eff}}(r)$ at this point is

$$U''_{\text{eff}}(r_0) = -F'_{\text{eff}}(r_0) = \frac{3\ell^2}{\mu r_0^4} + (\beta^2 - 3)\,\frac{k}{r_0^{4-\beta^2}} = \frac{\beta^2\ell^2}{\mu r_0^4} \quad , \qquad (15.148)$$

which is manifestly positive. Thus, the circular orbit at $r = r_0$ is stable.

(b) Show that the geometric equation for the shape of the orbit may be written

$$\frac{d^2 s}{d\phi^2} + s = K(s) \qquad (15.149)$$

where $s = 1/r$, and

$$K(s) = s_0 \left(\frac{s}{s_0}\right)^{1-\beta^2} \quad , \tag{15.150}$$

with $s_0 = 1/r_0$.

Solution : We have previously derived (*e.g.* in the notes) the equation

$$\frac{d^2 s}{d\phi^2} + s = -\frac{\mu}{\ell^2 s^2}\, F(s^{-1}) \quad . \tag{15.151}$$

From the given $F(r)$, we then have

$$\frac{d^2 s}{d\phi^2} + s = \frac{\mu k}{\ell^2}\, s^{1-\beta^2} \equiv K(s) \quad , \tag{15.152}$$

where $s_0 \equiv (\mu k/\ell^2)^{1/\beta^2} = 1/r_0$, and where

$$K(s) = s_0 \left(\frac{s}{s_0}\right)^{1-\beta^2} \quad . \tag{15.153}$$

(c) Writing $s \equiv (1+u)\, s_0$, show that u satisfies

$$\frac{1}{\beta^2}\,\frac{d^2 u}{d\phi^2} + u = a_1\, u^2 + a_2\, u^3 + \ldots \quad . \tag{15.154}$$

Find a_1 and a_2.

Solution : Writing $s \equiv s_0\,(1+u)$, we have

$$\begin{aligned}\frac{d^2 u}{d\phi^2} + 1 + u &= (1+u)^{1-\beta^2}\\ &= 1 + (1-\beta^2)\, u + \tfrac{1}{2}(-\beta^2)(1-\beta^2)\, u^2\\ &\qquad + \tfrac{1}{6}(-1-\beta^2)(-\beta^2)(1-\beta^2)\, u^3 + \ldots \quad .\end{aligned} \tag{15.155}$$

Thus,

$$\frac{1}{\beta^2}\,\frac{d^2 u}{d\phi^2} + u = a_1\, u^2 + a_2\, u^3 + \ldots \quad , \tag{15.156}$$

where

$$a_1 = -\tfrac{1}{2}(1-\beta^2) \quad , \quad a_2 = \tfrac{1}{6}(1-\beta^4) \quad . \tag{15.157}$$

(d) Now let us associate a power of ε with each power of the deviation u and write

$$\frac{1}{\beta^2}\,\frac{d^2 u}{d\phi^2} + u = \varepsilon\, a_1\, u^2 + \varepsilon^2\, a_2\, u^3 + \ldots \quad , \tag{15.158}$$

Solve this equation using the method of multiple scale analysis (MSA). You will have to go to second order in the multiple scale expansion, writing

$$X \equiv \beta\phi \quad , \quad Y \equiv \varepsilon\,\beta\phi \quad , \quad Z \equiv \varepsilon^2\,\beta\phi \tag{15.159}$$

and hence

$$\frac{1}{\beta}\,\frac{d}{d\phi} = \frac{\partial}{\partial X} + \varepsilon\,\frac{\partial}{\partial Y} + \varepsilon^2\,\frac{\partial}{\partial Z} + \ldots \quad . \tag{15.160}$$

Further writing

$$u = u_0 + \varepsilon\,u_1 + \varepsilon^2\,u_2 + \ldots \quad , \tag{15.161}$$

derive the equations for the multiple scale analysis, up to second order in ε.

Solution : We now associate one power of ε with each additional power of u beyond order u^1. In this way, a uniform expansion in terms of ε will turn out to be an expansion in powers of the amplitude of the oscillations. We'll see how this works below. We then have

$$\frac{1}{\beta^2}\,\frac{d^2u}{d\phi^2} + u = a_1\,\varepsilon\,u^2 + a_2\,\varepsilon^2\,u^3 + \ldots \quad , \tag{15.162}$$

with $\varepsilon = 1$. We now perform a multiple scale analysis, writing

$$X \equiv \beta\phi \quad , \quad Y \equiv \varepsilon\,\beta\phi \quad , \quad Z \equiv \varepsilon^2\,\beta\phi \quad . \tag{15.163}$$

This entails

$$\frac{1}{\beta}\,\frac{d}{d\phi} = \frac{\partial}{\partial X} + \varepsilon\,\frac{\partial}{\partial Y} + \varepsilon^2\,\frac{\partial}{\partial Z} + \ldots \quad . \tag{15.164}$$

We also expand u in powers of ε, as

$$u = u_0 + \varepsilon\,u_1 + \varepsilon^2\,u_2 + \ldots \quad . \tag{15.165}$$

Thus, we obtain

$$\begin{aligned}
&\big(\partial_X + \varepsilon\,\partial_Y + \varepsilon^2\,\partial_Z + \ldots\big)^2\,(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \ldots) + (u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \ldots) \\
&\qquad = \varepsilon\,a_1\,(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \ldots)^2 + \varepsilon^2\,a_2\,(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \ldots)^3 + \ldots \quad .
\end{aligned} \tag{15.166}$$

We now extract a hierarchy of equations, order by order in powers of ε.

We find, out to order ε^2,

$$\begin{aligned}
\mathcal{O}(\varepsilon^0):&\quad \frac{\partial^2 u_0}{\partial X^2} + u_0 = 0 \\
\mathcal{O}(\varepsilon^1):&\quad \frac{\partial^2 u_1}{\partial X^2} + u_1 = -2\frac{\partial^2 u_0}{\partial Y\,\partial X} + a_1\,u_0^2 \\
\mathcal{O}(\varepsilon^2):&\quad \frac{\partial^2 u_2}{\partial X^2} + u_2 = -2\frac{\partial^2 u_0}{\partial Z\,\partial X} - \frac{\partial^2 u_0}{\partial Y^2} - 2\frac{\partial^2 u_1}{\partial Z\,\partial X} + 2a_1\,u_0\,u_1 + a_2\,u_0^3 \quad .
\end{aligned} \tag{15.167}$$

(e) Show that there is no shift of the angular period $\Delta\phi = 2\pi/\beta$ if one works only to leading order in ε.

Solution : The $\mathcal{O}(\varepsilon^0)$ equation in the hierarchy is solved by writing

$$u_0 = A\,\cos(X + \psi) \quad , \tag{15.168}$$

where

$$A = A(Y, Z) \quad , \quad \psi = \psi(Y, Z) \quad . \tag{15.169}$$

We define $\theta \equiv X + \psi(Y, Z)$, so we may write $u_0 = A\cos\theta$. At the next order, we obtain

$$\begin{aligned} \frac{\partial^2 u_1}{\partial\theta^2} + u_1 &= 2\frac{\partial A}{\partial Y}\,\sin\theta + 2A\,\frac{\partial\psi}{\partial Y}\,\cos\theta + a_1\,A^2\cos\theta \\ &= 2\frac{\partial A}{\partial Y}\,\sin\theta + 2A\,\frac{\partial\psi}{\partial Y}\,\cos\theta + \tfrac{1}{2}a_1 A^2 + \tfrac{1}{2}a_1 A^2\cos 2\theta \quad . \end{aligned} \tag{15.170}$$

In order that there be no resonantly forcing terms on the RHS of eqn. 16.36, we demand

$$\frac{\partial A}{\partial Y} = 0 \quad , \quad \frac{\partial\psi}{\partial Y} = 0 \quad \Rightarrow \quad A = A(Z) \quad , \quad \psi = \psi(Z) \quad . \tag{15.171}$$

The solution for u_1 is then

$$u_1(\theta) = \tfrac{1}{2}a_1 A^2 - \tfrac{1}{6}a_1 A^2\cos 2\theta \quad . \tag{15.172}$$

Were we to stop at this order, we could ignore $Z = \varepsilon^2\beta\phi$ entirely, since it is of order ε^2, and the solution would be

$$u(\phi) = A_0\cos(\beta\phi + \psi_0) + \tfrac{1}{2}\varepsilon a_1 A_0^2 - \tfrac{1}{6}\varepsilon a_1 A_0^2\cos(2\beta\phi + 2\psi_0) \quad . \tag{15.173}$$

The angular period is still $\Delta\phi = 2\pi/\beta$, and, starting from a small amplitude solution at order ε^0 we find that to order ε we must add a constant shift proportional to A_0^2, as well as a second harmonic term, also proportional to A_0^2.

(f) Carrying out the MSA to second order in ε, show that the shift of the angular period vanishes only if $\beta^2 = 1$ or $\beta^2 = 4$.

Solution : Carrying out the MSA to the next order, $\mathcal{O}(\varepsilon^2)$, we obtain

$$\begin{aligned} \frac{\partial^2 u_2}{\partial\theta^2} + u_2 &= 2\frac{\partial A}{\partial Z}\,\sin\theta + 2A\,\frac{\partial\psi}{\partial Z}\,\cos\theta + 2a_1 A\cos\theta\,\big(\tfrac{1}{2}a_1 A^2 - \tfrac{1}{6}a_1 A^2\cos 2\theta\big) + a_2 A^3\cos^3\!\theta \\ &= 2\frac{\partial A}{\partial Z}\,\sin\theta + 2A\,\frac{\partial\psi}{\partial Z}\,\cos\theta + \big(\tfrac{5}{6}a_1^2 + \tfrac{3}{4}a_2\big)\,A^3\cos\theta + \big(-\tfrac{1}{6}a_1^2 + \tfrac{1}{4}a_2\big)\,A^3\cos 3\theta \quad . \end{aligned} \tag{15.174}$$

Now in order to make the resonant forcing terms on the RHS vanish, we must choose

$$\frac{\partial A}{\partial Z} = 0 \tag{15.175}$$

as well as

$$\begin{aligned} \frac{\partial\psi}{\partial Z} &= -\big(\tfrac{5}{12}a_1^2 + \tfrac{3}{8}a_2\big)\,A^2 \\ &= -\tfrac{1}{24}(\beta^2 - 4)(\beta^2 - 1) \quad . \end{aligned} \tag{15.176}$$

The solutions to these equations are trivial:

$$A(Z) = A_0 \quad , \quad \psi(Z) = \psi_0 - \tfrac{1}{24}(\beta^2 - 1)(\beta^2 - 4) A_0^2 \, Z \quad . \tag{15.177}$$

With the resonant forcing terms eliminated, we may write

$$\frac{\partial^2 u_2}{\partial\theta^2} + u_2 = \big(-\tfrac{1}{6} a_1^2 + \tfrac{1}{4} a_2 \big) \, A^3 \cos 3\theta \quad , \tag{15.178}$$

with solution

$$\begin{aligned} u_2 &= \tfrac{1}{96}(2a_1^2 - 3a_2) \, A^3 \, \cos 3\theta \\ &= \tfrac{1}{96} \, \beta^2 \, (\beta^2 - 1) \, A_0^2 \, \cos\big(3X + 3\psi(Z)\big) \quad . \end{aligned} \tag{15.179}$$

The full solution to second order in this analysis is then

$$\begin{aligned} u(\phi) = A_0 \cos(\beta'\phi + \psi_0) + \tfrac{1}{2}\varepsilon a_1 A_0^2 - \tfrac{1}{6}\varepsilon a_1 A_0^2 \cos(2\beta'\phi + 2\psi_0) \\ + \tfrac{1}{96}\varepsilon^2 \, (2a_1^2 - 3a_2) \, A_0^3 \, \cos(3\beta'\phi + 3\psi_0) \quad . \end{aligned} \tag{15.180}$$

with

$$\beta' = \beta \cdot \Big\{ 1 - \tfrac{1}{24} \, \varepsilon^2 \, (\beta^2 - 1)(\beta^2 - 4) A_0^2 \Big\} \quad . \tag{15.181}$$

The angular period shifts:

$$\Delta\phi = \frac{2\pi}{\beta'} = \frac{2\pi}{\beta} \cdot \Big\{ 1 + \tfrac{1}{24} \, \varepsilon^2 \, (\beta^2 - 1)(\beta^2 - 4) A_0^2 \Big\} + \mathcal{O}(\varepsilon^3) \quad . \tag{15.182}$$

Note that there is no shift in the period, for any amplitude, if $\beta^2 = 1$ (*i.e.* Kepler potential) or $\beta^2 = 4$ (*i.e.* harmonic oscillator).

15.7.2 Solution using Poincaré-Lindstedt method

Recall that geometric equation for the shape of the (relative coordinate) orbit for the two body central force problem is

$$\begin{aligned} \frac{d^2 s}{d\phi^2} + s &= K(s) \\ K(s) &= s_0 \bigg(\frac{s}{s_0} \bigg)^{1-\beta^2} \end{aligned} \tag{15.183}$$

where $s = 1/r$, $s_0 = (l^2/\mu k)^{1/\beta^2}$ is the inverse radius of the stable circular orbit, and $f(r) = -kr^{\beta^2 - 3}$ is the central force. Expanding about the stable circular orbit, one has

$$\frac{d^2 y}{d\phi^2} + \beta^2 \, y = \tfrac{1}{2} K''(s_0) \, y^2 + \tfrac{1}{6} K'''(s_0) \, y^3 + \ldots \quad , \tag{15.184}$$

where $s = s_0(1+y)$, with

$$K'(s) = (1-\beta^2)\left(\frac{s_0}{s}\right)^{\beta^2}$$
$$K''(s) = -\beta^2\,(1-\beta^2)\left(\frac{s_0}{s}\right)^{1+\beta^2} \tag{15.185}$$
$$K'''(s) = \beta^2\,(1-\beta^2)\,(1+\beta^2)\left(\frac{s_0}{s}\right)^{2+\beta^2} \quad .$$

Thus,

$$\frac{d^2y}{d\phi^2} + \beta^2\,y = \epsilon\,a_1\,y^2 + \epsilon^2\,a_2\,y^3 \quad , \tag{15.186}$$

with $\epsilon = 1$ and

$$\begin{aligned} a_1 &= -\tfrac{1}{2}\,\beta^2\,(1-\beta^2) \\ a_2 &= +\tfrac{1}{6}\,\beta^2\,(1-\beta^2)\,(1+\beta^2) \quad . \end{aligned} \tag{15.187}$$

Note that we assign one factor of ϵ for each order of nonlinearity beyond order y^1. Note also that while y here corresponds to u in eqn. 15.156, the constants $a_{1,2}$ here are a factor of β^2 larger than those defined in eqn. 15.157.

We now apply the Poincaré-Lindstedt method, by defining $\theta = \Omega\,\phi$, with

$$\Omega^2 = \Omega_0^2 + \epsilon\,\Omega_1^2 + \epsilon^2\,\Omega_2^2 + \ldots \tag{15.188}$$

and

$$y(\theta) = y_0(\theta) + \epsilon\,y_1(\theta) + \epsilon^2\,y_2(\theta) + \ldots \quad . \tag{15.189}$$

We therefore have

$$\frac{d}{d\phi} = \Omega\,\frac{d}{d\theta} \tag{15.190}$$

and

$$\begin{aligned} \big(\Omega_0^2 + \epsilon\,\Omega_1^2 + \epsilon^2\,\Omega_2^2 + \ldots\big)\big(y_0'' + \epsilon\,y_1'' + \epsilon^2\,y_2'' + \ldots\big) + \beta^2\,\big(y_0 + \epsilon\,y_1 + \epsilon^2\,y_2 + \ldots\big) \\ = \epsilon\,a_1\,\big(y_0 + \epsilon\,y_1 + \epsilon^2\,y_2 + \ldots\big)^2 + \epsilon^2\,a_2\,\big(y_0 + \epsilon\,y_1 + \epsilon^2\,y_2 + \ldots\big)^3 \quad . \end{aligned} \tag{15.191}$$

We now extract equations at successive orders of ϵ. The first three in the hierarchy are

$$\begin{aligned} \Omega_0^2\,y_0'' + \beta^2\,y_0 &= 0 \\ \Omega_1^2\,y_0'' + \Omega_0^2\,y_1'' + \beta^2\,y_1 &= a_1 y_0^2 \\ \Omega_2^2\,y_0'' + \Omega_1^2\,y_1'' + \Omega_0^2\,y_2'' + \beta^2\,y_2 &= 2\,a_1\,y_0\,y_1 + a_2\,y_0^3 \quad , \end{aligned} \tag{15.192}$$

where prime denotes differentiation with respect to θ.

To order ϵ^0, the solution is $\Omega_0^2 = \beta^2$ and

$$y_0(\theta) = A\,\cos(\theta+\delta) \quad , \tag{15.193}$$

where A and δ are constants.

At order ϵ^1, we have

$$\begin{aligned}\beta^2\big(y_1'' + y_1\big) &= -\Omega_1^2\, y_0'' + a_1\, y_0^2 \\ &= \Omega_1^2\, A\,\cos(\theta+\delta) + a_1 A^2 \cos^2(\theta+\delta) \\ &= \Omega_1^2\, A\,\cos(\theta+\delta) + \tfrac{1}{2}\, a_1 A^2 + \tfrac{1}{2}\, a_1 A^2 \cos(2\theta+2\delta) \quad .\end{aligned} \tag{15.194}$$

The secular forcing terms on the RHS are eliminated by the choice $\Omega_1^2 = 0$. The solution is then

$$y_1(\theta) = \frac{a_1 A^2}{2\,\beta^2}\Big\{1 - \tfrac{1}{3}\cos(2\theta+2\delta)\Big\} \quad . \tag{15.195}$$

At order ϵ^2, then, we have

$$\begin{aligned}\beta^2\big(y_2'' + y_2\big) &= -\Omega_2^2\, y_0'' - \Omega_1^2\, y_1'' + 2\, a_1\, y_1\, y_1 + a_2\, y_0^3 \\ &= \Omega_2^2\, A\cos(\theta+\delta) + \frac{a_1^2 A^3}{\beta^2}\Big\{1 - \tfrac{1}{3}\cos(2\theta+2\delta)\Big\}\cos(\theta+\delta) + a_2 A^3 \cos^2(\theta+\delta) \\ &= \bigg\{\Omega_2^2 + \frac{5\, a_1^2 A^3}{6\beta^2} + \tfrac{3}{4}\, a_2 A^3\bigg\} A\cos(\theta+\delta) + \bigg\{-\frac{a_1^2 A^3}{6\beta^2} + \tfrac{1}{4}\, a_2 A^3\bigg\}\cos(3\theta+3\delta) \quad .\end{aligned} \tag{15.196}$$

The resonant forcing terms on the RHS are eliminated by the choice

$$\begin{aligned}\Omega_2^2 &= -\Big(\tfrac{5}{6}\,\beta^{-2}\, a_1^2 + \tfrac{3}{4}\, a_2\Big) A^3 \\ &= -\tfrac{1}{24}\,\beta^2\,(1-\beta^2)\,\Big[5\,(1-\beta^2) + 3\,(1+\beta^2)\Big] \\ &= -\tfrac{1}{12}\,\beta^2\,(1-\beta^2)\,(4-\beta^2) \quad .\end{aligned} \tag{15.197}$$

Thus, the frequency shift to this order vanishes whenever $\beta^2 = 0$, $\beta^2 = 1$, or $\beta^2 = 4$. Recall the force law is $F(r) = -C\, r^{\beta^2-3}$, so we see that there is no shift – hence no precession – for inverse cube, inverse square, or linear forces.

15.8 Appendix III: Modified van der Pol Oscillator

Consider the nonlinear oscillator

$$\ddot{x} + \epsilon\,(x^4 - 1)\,\dot{x} + x = 0 \quad . \tag{15.198}$$

Analyze this using the same approach we apply to the van der Pol oscillator.

(a) Sketch the vector field $\dot{\boldsymbol{\varphi}}$ for this problem. It may prove convenient to first identify the *nullclines*, which are the curves along which $\dot{x} = 0$ or $\dot{v} = 0$ (with $v = \dot{x}$). Argue that a limit cycle exists.

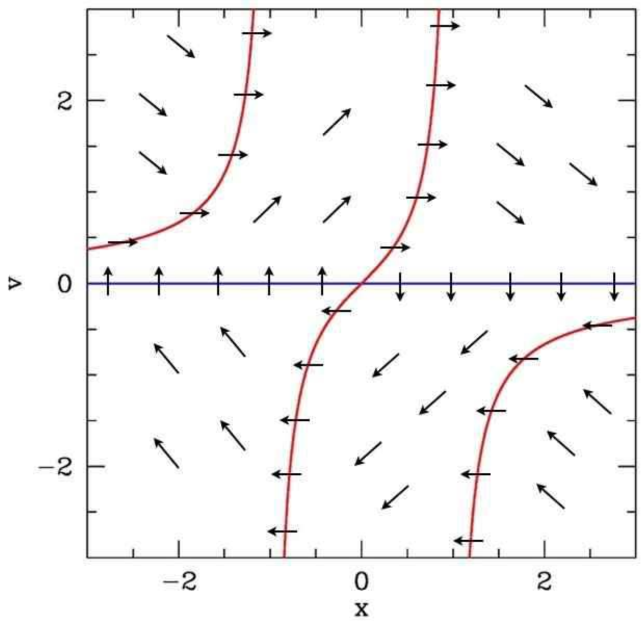


Figure 15.16: Phase flow and nullclines for the oscillator $\ddot{x} + \epsilon\,(x^4 - 1)\,\dot{x} + x = 0$. Red nullclines: $\dot{v} = 0$; blue nullcline: $\dot{x} = 0$.

Solution : There is a single fixed point, at the origin $(0, 0)$, for which the linearized dynamics obeys

$$\frac{d}{dt}\begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & \epsilon \end{pmatrix}\begin{pmatrix} x \\ v \end{pmatrix} + \mathcal{O}(x^4\, v) \quad . \tag{15.199}$$

One finds $T = \epsilon$ and $D = 1$ for the trace and determinant, respectively. The origin is an unstable spiral for $0 < \epsilon < 2$ and an unstable node for $\epsilon > 2$.

The nullclines are sketched in Fig. 15.16. One has

$$\dot{x} = 0 \;\leftrightarrow\; v = 0 \qquad , \qquad \dot{v} = 0 \;\leftrightarrow\; v = \frac{1}{\epsilon}\,\frac{x}{1 - x^4} \quad . \tag{15.200}$$

The flow at large distances from the origin winds once around the origin and spirals in. The flow close to the origin spirals out ($\epsilon < 2$) or flows radially out ($\epsilon > 2$). Ultimately the flow must collapse to a limit cycle, as can be seen in the accompanying figures.

(b) In the limit $0 < \varepsilon \ll 1$, use multiple time scale analysis to obtain a solution which reveals the approach to the limit cycle.

Solution : We seek to solve the equation

$$\ddot{x} + x = \epsilon\, h(x, \dot{x}) \quad , \tag{15.201}$$

with

$$h(x, \dot{x}) = (1 - x^4)\,\dot{x} \quad . \tag{15.202}$$

Employing the multiple time scale analysis to lowest nontrivial order, we write $T_0 \equiv t$, $T_1 \equiv \epsilon t$,

$$x = x_0 + \epsilon x_1 + \ldots \tag{15.203}$$

and identify terms order by order in ϵ. At $\mathcal{O}(\epsilon^0)$, this yields

$$\frac{\partial^2 x_0}{\partial T_0^2} + x_0 = 0 \qquad \Rightarrow \qquad x_0 = A\cos(T_0 + \phi) \quad , \tag{15.204}$$

where $A = A(T_1)$ and $\phi = \phi(T_1)$. At $\mathcal{O}(\epsilon^1)$, we have

$$\begin{aligned} \frac{\partial^2 x_1}{\partial T_0^2} + x_1 &= -2\,\frac{\partial^2 x_0}{\partial T_0\,\partial T_1} + h\bigg(x_0\,,\,\frac{\partial x_0}{\partial T_0}\bigg) \\ &= 2\,\frac{\partial A}{\partial T_1}\,\sin\theta + 2A\,\frac{\partial \phi}{\partial T_1}\,\cos\theta + h\big(A\cos\theta, -A\sin\theta\big) \end{aligned} \tag{15.205}$$

with $\theta = T_0 + \phi(T_1)$ as usual. We also have

$$\begin{aligned} h\big(A\cos\theta, -A\sin\theta\big) &= A^5 \sin\theta\cos\theta - A\sin\theta \\ &= \big(\tfrac{1}{8}A^5 - A\big)\,\sin\theta + \tfrac{3}{16}\,A^5\sin 3\theta + \tfrac{1}{16}\,A^5\sin 5\theta \quad . \end{aligned} \tag{15.206}$$

To eliminate the resonant terms in eqn. 15.205, we must choose

$$\frac{\partial A}{\partial T_1} = \tfrac{1}{2}A - \tfrac{1}{16}A^5 \qquad , \qquad \frac{\partial \phi}{\partial T_1} = 0 \quad . \tag{15.207}$$

The A equation is similar to the logistic equation. Clearly $A = 0$ is an unstable fixed point, and $A = 8^{1/4} \approx 1.681793$ is a stable fixed point. Thus, the amplitude of the oscillations will asymptotically approach $A^* = 8^{1/4}$. (Recall the asymptotic amplitude in the van der Pol case was $A^* = 2$.)

To integrate the A equation, substitute $y = \frac{1}{\sqrt{8}}A^2$, and obtain

$$dT_1 = \frac{dy}{y\,(1-y^2)} = \tfrac{1}{2}d\log\frac{y^2}{1-y^2} \quad \Rightarrow \quad y^2(T_1) = \frac{1}{1 + (y_0^{-2} - 1)\,\exp(-2T_1)} \quad . \tag{15.208}$$

We then have

$$A(T_1) = 8^{1/4}\sqrt{y(T_1)} = \left(\frac{8}{1 + (8A_0^{-4} - 1)\,\exp(-2T_1)}\right)^{1/4} \quad . \tag{15.209}$$

(c) In the limit $\epsilon \gg 1$, find the period of relaxation oscillations, using Liénard plane analysis. Sketch the orbit of the relaxation oscillation in the Liénard plane.

Solution : Our nonlinear oscillator may be written in the form

$$\ddot{x} + \epsilon\,\frac{dF(x)}{dt} + x = 0 \quad , \tag{15.210}$$

with

$$F(x) = \tfrac{1}{5}x^5 - x \quad . \tag{15.211}$$

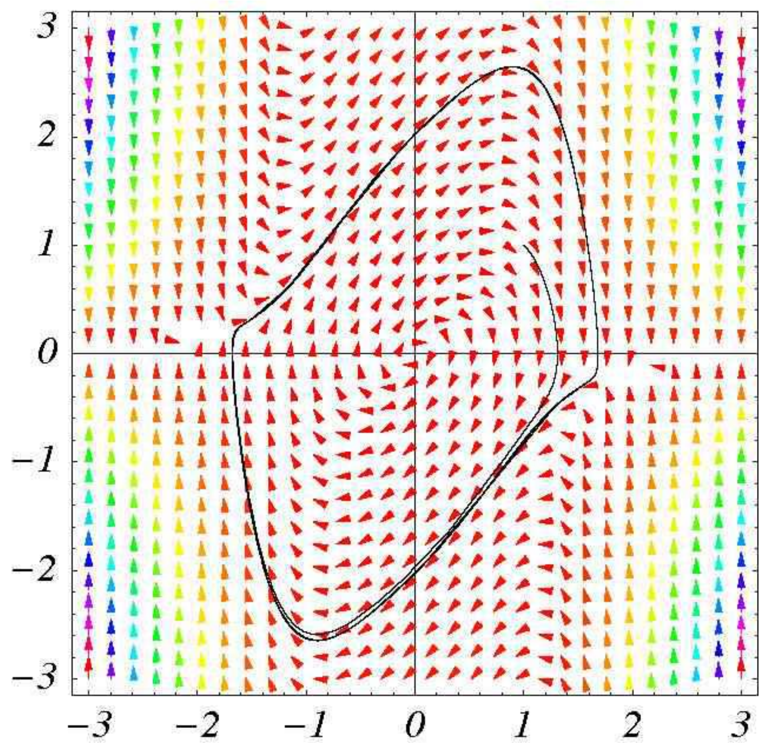


Figure 15.17: Vector field and phase curves for the oscillator $\ddot{x} + \epsilon\,(x^4 - 1)\,\dot{x} + x = 0$, with $\epsilon = 1$ and starting from $(x_0, v_0) = (1, 1)$.

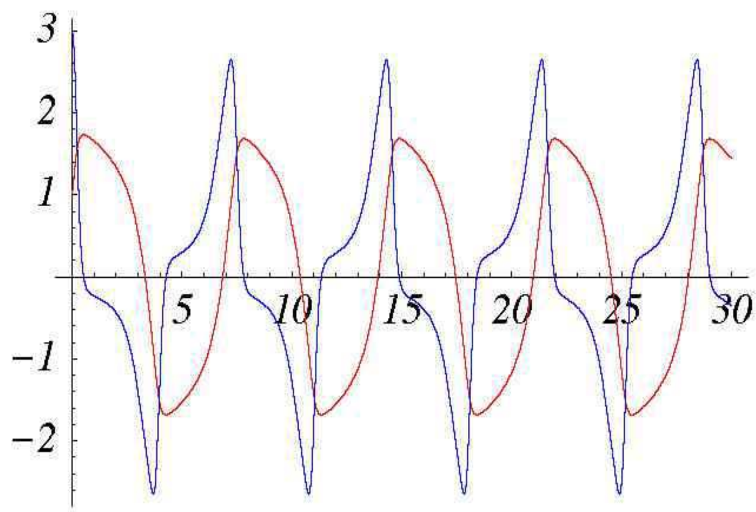


Figure 15.18: Solution to the oscillator equation $\ddot{x} + \epsilon\,(x^4 - 1)\,\dot{x} + x = 0$ with $\epsilon = 1$ and initial conditions $(x_0, v_0) = (1, 3)$. $x(t)$ is shown in red and $v(t)$ in blue. Note that $x(t)$ resembles a relaxation oscillation for this moderate value of ϵ.

Note $\dot{F} = (x^4 - 1)\,\dot{x}$. Now we define the Liénard variable

$$y \equiv \frac{\dot{x}}{\epsilon} + F(x) \quad , \tag{15.212}$$

and in terms of (x, y) we have

$$\dot{x} = \epsilon\Big[y - F(x)\Big] \qquad , \qquad \dot{y} = -\frac{x}{\epsilon} \quad . \tag{15.213}$$

As we have seen in the notes, for large ϵ the motion in the (x, y) plane is easily analyzed. $x(t)$ must move quickly over to the curve $y = F(x)$, at which point the motion slows down and slowly creeps along this curve until it can no longer do so, at which point another big fast jump occurs. The jumps take place between the local extrema of $F(x)$, which occur for $F'(a) = a^4 - 1 = 0$, *i.e.* at $a = \pm 1$, and points on the curve with the same values of $F(a)$. Thus, we solve $F(-1) = \frac{4}{5} = \frac{1}{5}b^5 - b$ and find the desired root at $b^* \approx 1.650629$. The period of the relaxation oscillations, for large ϵ, is

$$T \approx 2\epsilon \int\limits_a^b dx\, \frac{F'(x)}{x} = \epsilon \cdot \Big[\tfrac{1}{2}x^4 - 2\log x\Big]_a^b \approx 2.20935\,\epsilon \quad . \tag{15.214}$$

(d) Numerically integrate the equation (15.198) starting from several different initial conditions.

Solution : The accompanying `Mathematica` plots show $x(t)$ and $v(t)$ for this system for two representative values of ϵ.

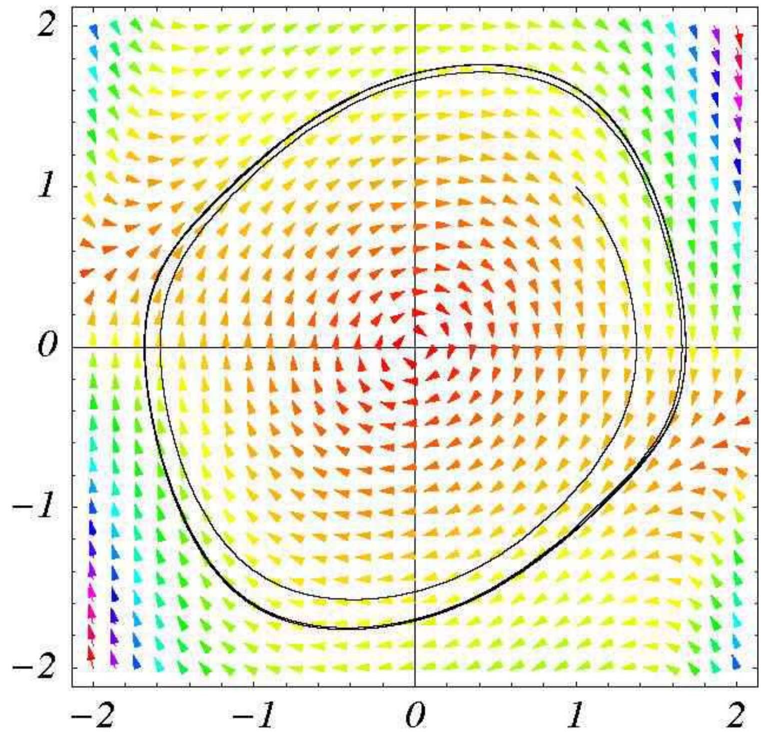


Figure 15.19: Vector field and phase curves for the oscillator $\ddot{x} + \epsilon\,(x^4 - 1)\,\dot{x} + x = 0$, with $\epsilon = 0.25$ and starting from $(x_0, v_0) = (1, 1)$. As $\epsilon \to 0$, the limit cycle is a circle of radius $A^* = 8^{1/4} \approx 1.682$.

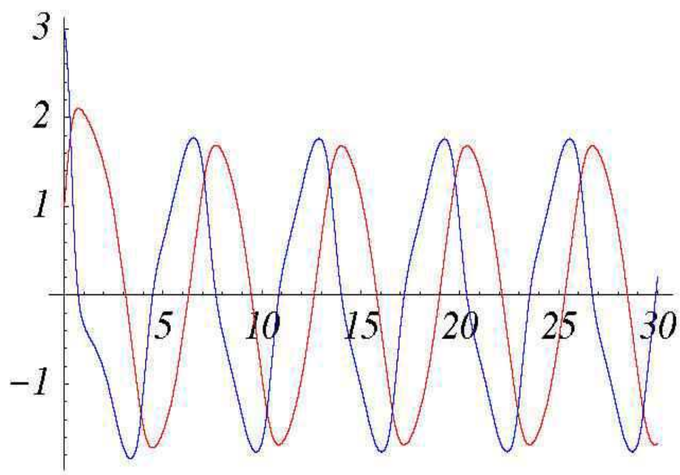


Figure 15.20: Solution to the oscillator equation $\ddot{x} + \epsilon\,(x^4 - 1)\,\dot{x} + x = 0$ with $\epsilon = 0.25$ and initial conditions $(x_0, v_0) = (1, 3)$. $x(t)$ is shown in red and $v(t)$ in blue. As $\epsilon \to 0$, the amplitude of the oscillations tends to $A^* = 8^{1/4} \approx 1.682$.

Chapter 16

Hamiltonian Mechanics

16.1 References

– R. Z. Sagdeev, D. A. Usikov, and G. M. Zaslavsky, *Nonlinear Physics* (Harwood, 1988)
A thorough treatment of nonlinear Hamiltonian particle and wave mechanics.

– E. Ott, *Chaos in Dynamical Systems* (Cambridge, 2002)
An excellent introductory text appropriate for graduate or advanced undergraduate students.

– W. Dittrich and M. Reuter, *Classical and Quantum Dynamics* (Springer, 2001)
More a handbook than a textbook, but reliably covers a large amount of useful material.

– G. M. Zaslavsky, *Hamiltonian Chaos & Fractional Dynamics* (Oxford, 2005)
An advanced text for graduate students and researchers.

– I. Percival and D. Richards, *Introduction to Dynamics* (Cambridge, 1994)
An excellent advanced undergraduate text.

– A. J. Lichenberg and M. A. Lieberman, *Regular and Stochastic Motion* (Springer, 1983)
An advanced graduate level text. Excellent range of topics, but quite technical and often lacking physical explanations.

16.2 The Hamiltonian

Recall that $L = L(q,\dot{q},t)$, and

$$p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma} \quad , \tag{16.1}$$

with $n = Nd$ for a system of N particles in d space dimensions. The Hamiltonian, $H(q,p,t)$ is obtained by a Legendre transformation,

$$H(q,p) = \sum_{\sigma=1}^{n} p_\sigma\, \dot{q}_\sigma - L(q,\dot{q},t) \quad . \tag{16.2}$$

Note that

$$\begin{aligned} dH &= \sum_{\sigma=1}^{n} \bigg(p_\sigma\, d\dot{q}_\sigma + \dot{q}_\sigma\, dp_\sigma - \frac{\partial L}{\partial q_\sigma}\, dq_\sigma - \frac{\partial L}{\partial \dot{q}_\sigma}\, d\dot{q}_\sigma \bigg) - \frac{\partial L}{\partial t}\, dt \\ &= \sum_{\sigma=1}^{n} \bigg(\dot{q}_\sigma\, dp_\sigma - \frac{\partial L}{\partial q_\sigma}\, dq_\sigma \bigg) - \frac{\partial L}{\partial t}\, dt \quad . \end{aligned} \tag{16.3}$$

Thus, we obtain Hamilton's equations of motion,

$$\frac{\partial H}{\partial p_\sigma} = \dot{q}_\sigma \qquad , \qquad \frac{\partial H}{\partial q_\sigma} = -\frac{\partial L}{\partial q_\sigma} = -\dot{p}_\sigma \tag{16.4}$$

and

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \quad . \tag{16.5}$$

Some remarks:

- As an example, consider a particle moving in three dimensions, described by spherical polar coordinates (r,θ,ϕ). Then

  $$L = \tfrac{1}{2} m \left(\dot{r}^2 + r^2\, \dot{\theta}^2 + r^2 \sin^2\!\theta\, \dot{\phi}^2 \right) - U(r,\theta,\phi) \quad . \tag{16.6}$$

  We have

  $$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \quad , \quad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\,\dot{\theta} \quad , \quad p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2\, \sin^2\!\theta\,\dot{\phi} \quad , \tag{16.7}$$

  and thus

  $$\begin{aligned} H &= p_r\,\dot{r} + p_\theta\,\dot{\theta} + p_\phi\,\dot{\phi} - L \\ &= \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\phi^2}{2mr^2 \sin^2\!\theta} + U(r,\theta,\phi) \quad . \end{aligned} \tag{16.8}$$

  Note that H is time-independent, hence $\frac{\partial H}{\partial t} = \frac{dH}{dt} = 0$, and therefore H is a constant of the motion.

- In order to obtain $H(q,p)$ we must invert the relation $p_\sigma = \frac{\partial L}{\partial \dot{q}_\sigma} = p_\sigma(q,\dot{q})$ to obtain $\dot{q}_\sigma(q,p)$. This is possible if the Hessian,

$$\frac{\partial p_\alpha}{\partial \dot{q}_\beta} = \frac{\partial^2 L}{\partial \dot{q}_\alpha \, \partial \dot{q}_\beta} \tag{16.9}$$

  is nonsingular. This is the content of the 'inverse function theorem' of multivariable calculus.

- Define the rank $2n$ vector, ξ, by its components,

$$\xi_i = \begin{cases} q_i & \text{if } 1 \le i \le n \\ p_{i-n} & \text{if } n < i \le 2n \end{cases} \quad . \tag{16.10}$$

  Then we may write Hamilton's equations compactly as

$$\dot{\xi}_i = \mathbb{J}_{ij} \, \frac{\partial H}{\partial \xi_j} \quad , \tag{16.11}$$

  where

$$\mathbb{J} = \begin{pmatrix} \mathbb{O}_{n\times n} & \mathbb{I}_{n\times n} \\ -\mathbb{I}_{n\times n} & \mathbb{O}_{n\times n} \end{pmatrix} \tag{16.12}$$

  is a rank $2n$ matrix. Note that $\mathbb{J}^{\rm t} = -\mathbb{J}$, *i.e.* $\mathbb{J}$ is antisymmetric, and that $\mathbb{J}^2 = -\mathbb{I}_{2n\times 2n}$. We shall utilize this 'symplectic structure' to Hamilton's equations shortly.

16.2.1 Modified Hamilton's principle

Let's vary the action now with respect to *both* $\{q_\sigma\}$ *and* $\{p_\sigma\}$, considering them as *independent variations*. We then have

$$\begin{aligned} 0 = \delta \int_{t_a}^{t_b} \! dt \, L &= \delta \int_{t_a}^{t_b} \! dt \, \big(p_\sigma \, \dot{q}_\sigma - H \big) \\ &= \int_{t_a}^{t_b} \! dt \, \bigg\{ p_\sigma \, \delta\dot{q}_\sigma + \dot{q}_\sigma \, \delta p_\sigma - \frac{\partial H}{\partial q_\sigma} \, \delta q_\sigma - \frac{\partial H}{\partial p_\sigma} \, \delta p_\sigma \bigg\} \\ &= \int_{t_a}^{t_b} \! dt \, \bigg\{ - \bigg(\dot{p}_\sigma + \frac{\partial H}{\partial q_\sigma} \bigg) \, \delta q_\sigma + \bigg(\dot{q}_\sigma - \frac{\partial H}{\partial p_\sigma} \bigg) \, \delta p_\sigma \bigg\} + \big(p_\sigma \, \delta q_\sigma \big) \Big|_{t_a}^{t_b} \quad . \end{aligned} \tag{16.13}$$

Assuming $\delta q_\sigma(t_a) = \delta q_\sigma(t_b) = 0$, and setting the coefficients of δq_σ and δp_σ to zero, we recover Hamilton's equations.

16.2.2 Phase flow is incompressible

A flow for which $\boldsymbol{\nabla}\cdot\boldsymbol{v}=0$ is *incompressible* – we shall see why in a moment. Let's check that the divergence of the phase space velocity does indeed vanish:

$$\begin{aligned}\boldsymbol{\nabla}\cdot\dot{\boldsymbol{\xi}} &= \sum_{\sigma=1}^{n}\left\{\frac{\partial \dot q_\sigma}{\partial q_\sigma}+\frac{\partial \dot p_\sigma}{\partial p_\sigma}\right\}\\ &= \sum_{i=1}^{2n}\frac{\partial \dot\xi_i}{\partial \xi_i}=\sum_{i,j}\mathbb{J}_{ij}\,\frac{\partial^2 H}{\partial\xi_i\,\partial\xi_j}=0 \quad .\end{aligned} \tag{16.14}$$

Now let $\rho(\boldsymbol{\xi},t)$ be a distribution on phase space. Continuity implies

$$\frac{\partial\rho}{\partial t}+\boldsymbol{\nabla}\cdot(\rho\,\dot{\boldsymbol{\xi}})=0 \quad . \tag{16.15}$$

Invoking $\boldsymbol{\nabla}\cdot\dot{\boldsymbol{\xi}}=0$, we have that

$$\frac{D\rho}{Dt}=\frac{\partial\rho}{\partial t}+\dot{\boldsymbol{\xi}}\cdot\boldsymbol{\nabla}\rho=0 \quad , \tag{16.16}$$

where $D\rho/Dt$ is sometimes called the *convective derivative* – it is the total derivative of the function $\rho\big(\boldsymbol{\xi}(t),t\big)$, evaluated at a point $\xi(t)$ in phase space which moves according to the dynamics. This says that the density in the "comoving frame" is locally constant.

16.2.3 Poincaré recurrence theorem

Let g_τ be the 'τ-advance mapping' which evolves points in phase space according to Hamilton's equations

$$\dot q_\sigma=+\frac{\partial H}{\partial p_\sigma} \qquad , \qquad \dot p_\sigma=-\frac{\partial H}{\partial q_\sigma} \tag{16.17}$$

for a time interval $\Delta t=\tau$. Consider a region Ω in phase space. Define $g_\tau^n\Omega$ to be the $n^{\rm th}$ image of Ω under the mapping g_τ. Clearly g_τ is invertible; the inverse is obtained by integrating the equations of motion backward in time. We denote the inverse of g_τ by g_τ^{-1}. By Liouville's theorem, g_τ is volume preserving when acting on regions in phase space, since the evolution of any given point is Hamiltonian. This follows from the continuity equation for the phase space density,

$$\frac{\partial\varrho}{\partial t}+\nabla\cdot(\boldsymbol{u}\varrho)=0 \tag{16.18}$$

where $\boldsymbol{u}=\{\dot{\boldsymbol{q}},\dot{\boldsymbol{p}}\}$ is the velocity vector in phase space, and Hamilton's equations, which say that the phase flow is incompressible, *i.e.* $\nabla\cdot\boldsymbol{u}=0$:

$$\begin{aligned}\nabla\cdot\boldsymbol{u} &= \sum_{\sigma=1}^{n}\left\{\frac{\partial \dot q_\sigma}{\partial q_\sigma}+\frac{\partial \dot p_\sigma}{\partial p_\sigma}\right\}\\ &= \sum_{\sigma=1}^{n}\left\{\frac{\partial}{\partial q_\sigma}\left(\frac{\partial H}{\partial p_\sigma}\right)+\frac{\partial}{\partial p_\sigma}\left(-\frac{\partial H}{\partial q_\sigma}\right)\right\}=0 \quad .\end{aligned} \tag{16.19}$$

Thus, we have that the convective derivative vanishes, *viz.*

$$\frac{D\varrho}{Dt} \equiv \frac{\partial\varrho}{\partial t} + \boldsymbol{u}\cdot\nabla\varrho = 0 \quad , \tag{16.20}$$

which guarantees that the density remains constant in a frame moving with the flow.

The proof of the recurrence theorem is simple. Assume that g_τ is invertible and volume-preserving, as is the case for Hamiltonian flow. Further assume that phase space volume is finite. Since the energy is preserved in the case of time-independent Hamiltonians, we simply ask that the volume of phase space *at fixed total energy* E be finite, *i.e.*

$$\int\! d\mu\, \delta\big(E - H(\boldsymbol{q},\boldsymbol{p})\big) < \infty \quad , \tag{16.21}$$

where $d\mu = \prod_i dq_i\, dp_i$ is the phase space uniform integration measure.

Theorem: In any finite neighborhood Ω of phase space there exists a point $\boldsymbol{\varphi}_0$ which will return to Ω after n applications of g_τ, where n is finite.

Proof: Assume the theorem fails; we will show this assumption results in a contradiction. Consider the set Υ formed from the union of all sets $g_\tau^m\,\Omega$ for all m:

$$\Upsilon = \bigcup_{m=0}^{\infty} g_\tau^m\,\Omega \tag{16.22}$$

We assume that the set $\{g_\tau^m\,\Omega \,|\, m \in \mathbb{Z}_{\geq 0}\}$ is disjoint. The volume of a union of disjoint sets is the sum of the individual volumes. Thus,

$$\mathsf{vol}(\Upsilon) = \sum_{m=0}^{\infty} \mathsf{vol}(g_\tau^m\,\Omega) = \mathsf{vol}(\Omega)\cdot\sum_{m=1}^{\infty} 1 = \infty \quad , \tag{16.23}$$

since $\mathsf{vol}(g_\tau^m\,\Omega) = \mathsf{vol}(\Omega)$ from volume preservation. But clearly Υ is a subset of the entire phase space, hence we have a contradiction, because by assumption phase space is of finite volume.

Thus, the assumption that the set $\{g_\tau^m\,\Omega \,|\, m \in \mathbb{Z} \quad, m \geq 0\}$ is disjoint fails. This means that there exists some pair of integers k and l, with $k \neq l$, such that $g_\tau^k\,\Omega \cap g_\tau^l\,\Omega \neq \emptyset$. Without loss of generality we may assume $k > l$. Apply the inverse g_τ^{-1} to this relation l times to get $g_\tau^{k-l}\,\Omega \cap \Omega \neq \emptyset$. Now choose any point $\boldsymbol{\varphi} \in g_\tau^n\,\Omega \cap \Omega$, where $n = k - l$, and define $\boldsymbol{\varphi}_0 = g_\tau^{-n}\boldsymbol{\varphi}$. Then by construction both $\boldsymbol{\varphi}_0$ and $g_\tau^n\,\boldsymbol{\varphi}_0$ lie within Ω and the theorem is proven.

Each of the two central assumptions – invertibility and volume preservation – is crucial. Without either of them, the proof fails. Consider, for example, a volume-preserving map which is not invertible. An example might be a mapping $f : \mathbb{R} \to \mathbb{R}$ which takes any real number to its fractional part. Thus, $f(\pi) = 0.14159265\ldots$. Let us restrict our attention to intervals of width less than unity. Clearly f is then volume preserving. The action of f on the interval $[2, 3)$ is to map it to the interval $[0, 1)$. But $[0, 1)$ remains fixed under the action of f, so no point within the interval $[2, 3)$ will ever return under repeated iterations of f. Thus, f does not exhibit Poincaré recurrence.

Consider next the case of the damped harmonic oscillator. In this case, phase space volumes contract. For a one-dimensional oscillator obeying $\ddot{x}+2\beta\dot{x}+\Omega_0^2 x = 0$ one has $\nabla{\cdot}\boldsymbol{u} = -2\beta < 0$ ($\beta > 0$ for damping). Thus the convective derivative is equal to $D_t\varrho = -(\nabla\cdot\boldsymbol{u})\varrho = +2\beta\varrho$ which says that the density increases exponentially in the comoving frame, as $\varrho(t) = e^{2\beta t}\,\varrho(0)$. Thus, phase space volumes collapse, and are not preserved by the dynamics. In this case, it is possible for the set Υ to be of finite volume, even if it is the union of an infinite number of sets $g_\tau^n\,\Omega$, because the volumes of these component sets themselves decrease exponentially, as $\mathsf{vol}(g_\tau^n\,\Omega) = e^{-2n\beta\tau}\,\mathsf{vol}(\Omega)$. A damped pendulum, released from rest at some small angle θ_0, will not return arbitrarily close to these initial conditions.

16.2.4 Poisson brackets

The time evolution of any function $F(\boldsymbol{q},\boldsymbol{p})$ over phase space is given by

$$\begin{aligned}\frac{d}{dt}\,F\big(\boldsymbol{q}(t),\boldsymbol{p}(t),t\big) &= \frac{\partial F}{\partial t} + \sum_{\sigma=1}^{n}\bigg\{\frac{\partial F}{\partial q_\sigma}\,\dot{q}_\sigma + \frac{\partial F}{\partial p_\sigma}\,\dot{p}_\sigma\bigg\} \\ &\equiv \frac{\partial F}{\partial t} + \big\{F,H\big\} \quad ,\end{aligned} \tag{16.24}$$

where the *Poisson bracket* $\{\cdot\,,\cdot\}$ is given by

$$\begin{aligned}\big\{A,B\big\} &\equiv \sum_{\sigma=1}^{n}\bigg(\frac{\partial A}{\partial q_\sigma}\,\frac{\partial B}{\partial p_\sigma} - \frac{\partial A}{\partial p_\sigma}\,\frac{\partial B}{\partial q_\sigma}\bigg) \\ &= \sum_{i,j=1}^{2n} \mathbb{J}_{ij}\,\frac{\partial A}{\partial \xi_i}\,\frac{\partial B}{\partial \xi_j} \quad .\end{aligned} \tag{16.25}$$

Properties of the Poisson bracket:

- Antisymmetry:
$$\big\{f,g\big\} = -\big\{g,f\big\} \quad . \tag{16.26}$$
- Bilinearity: if λ is a constant, and f, g, and h are functions on phase space, then
$$\big\{f+\lambda\,g,h\big\} = \big\{f,h\big\} + \lambda\big\{g,h\big\} \quad . \tag{16.27}$$
Linearity in the second argument follows from this and the antisymmetry condition.
- Associativity:
$$\big\{fg,h\big\} = f\big\{g,h\big\} + g\big\{f,h\big\} \quad . \tag{16.28}$$
- Jacobi identity:
$$\big\{f,\{g,h\}\big\} + \big\{g,\{h,f\}\big\} + \big\{h,\{f,g\}\big\} = 0 \quad . \tag{16.29}$$

Some other useful properties:

- If $\{A, H\} = 0$ and $\frac{\partial A}{\partial t} = 0$, then $\frac{dA}{dt} = 0$, *i.e.* $A(q, p)$ is a constant of the motion.
- If $\{A, H\} = 0$ and $\{B, H\} = 0$, then $\big\{\{A, B\}, H\big\} = 0$. If in addition A and B have no explicit time dependence, we conclude that $\{A, B\}$ is a constant of the motion.
- It is easily established that

$$\{q_\alpha, q_\beta\} = 0 \quad , \quad \{p_\alpha, p_\beta\} = 0 \quad , \quad \{q_\alpha, p_\beta\} = \delta_{\alpha\beta} \quad . \tag{16.30}$$

16.3 Canonical Transformations

16.3.1 Point transformations in Lagrangian mechanics

In Lagrangian mechanics, we are free to redefine our generalized coordinates, *viz.*

$$Q_\sigma = Q_\sigma(q_1, \ldots, q_n, t) \quad . \tag{16.31}$$

This is called a "point transformation." The transformation is invertible if

$$\det\left(\frac{\partial Q_\alpha}{\partial q_\beta}\right) \neq 0 \quad . \tag{16.32}$$

The transformed Lagrangian, $\tilde{L}$, written as a function of the new coordinates $\boldsymbol{Q}$ and velocities $\dot{\boldsymbol{Q}}$, is

$$\tilde{L}(\boldsymbol{Q}, \dot{\boldsymbol{Q}}, t) = L\big(\boldsymbol{q}(\boldsymbol{Q}, t), \dot{\boldsymbol{q}}(\boldsymbol{Q}, \dot{\boldsymbol{Q}}, t), t\big) + \frac{d}{dt} F\big(\boldsymbol{q}(\boldsymbol{Q}, t), t\big) \quad , \tag{16.33}$$

where $F(\boldsymbol{q}, t)$ is a function only of the coordinates $q_\sigma(\boldsymbol{Q}, t)$ and time[1]. Finally, Hamilton's principle,

$$\delta \int_{t_1}^{t_b} dt \, \tilde{L}(\boldsymbol{Q}, \dot{\boldsymbol{Q}}, t) = 0 \tag{16.34}$$

with $\delta Q_\sigma(t_a) = \delta Q_\sigma(t_b) = 0$, still holds, and the form of the Euler-Lagrange equations remains unchanged:

$$\frac{\partial \tilde{L}}{\partial Q_\sigma} - \frac{d}{dt}\left(\frac{\partial \tilde{L}}{\partial \dot{Q}_\sigma}\right) = 0 \quad . \tag{16.35}$$

The invariance of the equations of motion under a point transformation may be verified explicitly. We first evaluate

$$\frac{d}{dt}\left(\frac{\partial \tilde{L}}{\partial \dot{Q}_\sigma}\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha} \frac{\partial \dot{q}_\alpha}{\partial \dot{Q}_\sigma}\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha} \frac{\partial q_\alpha}{\partial Q_\sigma}\right) \quad , \tag{16.36}$$

[1]We must have that the relation $Q_\sigma = Q_\sigma(\boldsymbol{q}, t)$ is invertible.

where the relation $\partial\dot{q}_\alpha/\partial\dot{Q}_\sigma = \partial q_\alpha/\partial Q_\sigma$ follows from $\dot{q}_\alpha = \frac{\partial q_\alpha}{\partial Q_\sigma}\,\dot{Q}_\sigma + \frac{\partial q_\alpha}{\partial t}$. We know that adding a total time derivative of a function $\tilde{F}(\boldsymbol{Q},t) = F\big(\boldsymbol{q}(\boldsymbol{Q},t),t\big)$ to the Lagrangian does not alter the equations of motion. Hence we can set $F=0$ and compute

$$\begin{aligned}\frac{\partial\tilde{L}}{\partial Q_\sigma} &= \frac{\partial L}{\partial q_\alpha}\,\frac{\partial q_\alpha}{\partial Q_\sigma} + \frac{\partial L}{\partial \dot{q}_\alpha}\,\frac{\partial \dot{q}_\alpha}{\partial Q_\sigma} \\ &= \frac{\partial L}{\partial q_\alpha}\,\frac{\partial q_\alpha}{\partial Q_\sigma} + \frac{\partial L}{\partial \dot{q}_\alpha}\left(\frac{\partial^2 q_\alpha}{\partial Q_\sigma\,\partial Q_{\sigma'}}\,\dot{Q}_{\sigma'} + \frac{\partial^2 q_\alpha}{\partial Q_\sigma\,\partial t}\right) \\ &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right)\frac{\partial q_\alpha}{\partial Q_\sigma} + \frac{\partial L}{\partial \dot{q}_\alpha}\,\frac{d}{dt}\left(\frac{\partial q_\alpha}{\partial Q_\sigma}\right) \\ &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\,\frac{\partial q_\alpha}{\partial Q_\sigma}\right) = \frac{d}{dt}\left(\frac{\partial \tilde{L}}{\partial \dot{Q}_\sigma}\right)\quad ,\end{aligned} \tag{16.37}$$

where the last equality is what we obtained earlier in eqn. 16.36.

16.3.2 Canonical transformations in Hamiltonian mechanics

In Hamiltonian mechanics, we will deal with a much broader class of transformations – ones which mix all the q's and p's. The general form for a canonical transformation (CT) is

$$\begin{aligned} q_\sigma &= q_\sigma\big(Q_1,\ldots,Q_n\,;P_1,\ldots,P_n;t\big) \\ p_\sigma &= p_\sigma\big(Q_1,\ldots,Q_n\,;P_1,\ldots,P_n;t\big)\quad ,\end{aligned} \tag{16.38}$$

with $\sigma\in\{1,\ldots,n\}$. We may also write

$$\xi_i = \xi_i\big(\Xi_1,\ldots,\Xi_{2n};t\big)\quad , \tag{16.39}$$

with $i\in\{1,\ldots,2n\}$. The transformed Hamiltonian is $\tilde{H}(\boldsymbol{Q},\boldsymbol{P},t)$., where, as we shall see below, $\tilde{H}(\boldsymbol{Q},\boldsymbol{P},t) = H(\boldsymbol{q},\boldsymbol{p},t) + \frac{\partial}{\partial t}F(\boldsymbol{q},\boldsymbol{Q},t)$.

What sorts of transformations are allowed? Well, if Hamilton's equations are to remain invariant, then

$$\dot{Q}_\sigma = \frac{\partial\widetilde{H}}{\partial P_\sigma}\quad ,\quad \dot{P}_\sigma = -\frac{\partial\widetilde{H}}{\partial Q_\sigma}\quad , \tag{16.40}$$

which gives

$$\frac{\partial\dot{Q}_\sigma}{\partial Q_\sigma} + \frac{\partial\dot{P}_\sigma}{\partial P_\sigma} = 0 = \frac{\partial\dot{\Xi}_i}{\partial\Xi_i}\quad . \tag{16.41}$$

I.e. the flow remains incompressible in the new (Q,P) variables. We will also require that phase space volumes are preserved by the transformation, *i.e.*

$$\mathsf{det}\left(\frac{\partial\Xi_i}{\partial\xi_j}\right) = \left|\left|\frac{\partial(\boldsymbol{Q},\boldsymbol{P})}{\partial(\boldsymbol{q},\boldsymbol{p})}\right|\right| = 1\quad . \tag{16.42}$$

Additional conditions will be discussed below.

16.3.3 Hamiltonian evolution

Hamiltonian evolution itself defines a canonical transformation. Let $\xi_i = \xi_i(t)$ and let $\xi'_i = \xi_i(t+dt)$. Then from the dynamics $\dot{\xi}_i = \mathbb{J}_{ij}\, \partial H/\partial \xi_j$, we have

$$\xi_i(t+dt) = \xi_i(t) + \mathbb{J}_{ij}\, \frac{\partial H}{\partial \xi_j}\, dt + \mathcal{O}\big(dt^2\big) \quad . \tag{16.43}$$

Thus,

$$\begin{aligned}\frac{\partial \xi'_i}{\partial \xi_j} &= \frac{\partial}{\partial \xi_j}\bigg(\xi_i + \mathbb{J}_{ik}\, \frac{\partial H}{\partial \xi_k}\, dt + \mathcal{O}\big(dt^2\big)\bigg) \\ &= \delta_{ij} + \mathbb{J}_{ik}\, \frac{\partial^2 H}{\partial \xi_j\, \partial \xi_k}\, dt + \mathcal{O}\big(dt^2\big) \quad .\end{aligned} \tag{16.44}$$

Now, using the result $\det\big(1+\epsilon M\big) = 1 + \epsilon\ \mathsf{Tr}\ M + \mathcal{O}(\epsilon^2)$, we have

$$\bigg|\bigg|\frac{\partial \xi'_i}{\partial \xi_j}\bigg|\bigg| = 1 + \mathbb{J}_{jk}\, \frac{\partial^2 H}{\partial \xi_j\, \partial \xi_k}\, dt + \mathcal{O}\big(dt^2\big) = 1 + \mathcal{O}\big(dt^2\big) \quad . \tag{16.45}$$

16.3.4 Symplectic structure

We have that

$$\dot{\xi}_i = \mathbb{J}_{ij}\, \frac{\partial H}{\partial \xi_j} \quad . \tag{16.46}$$

Suppose we make a time-independent canonical transformation from $\{\xi_i\}$ to new phase space coordinates $\{\Xi_a\}$, where $\Xi_a = \Xi_a(\xi)$. We then have

$$\dot{\Xi}_a = \frac{\partial \Xi_a}{\partial \xi_j}\, \dot{\xi}_j = \frac{\partial \Xi_a}{\partial \xi_j}\, \mathbb{J}_{jk}\, \frac{\partial H}{\partial \xi_k} \quad . \tag{16.47}$$

But if the transformation is canonical, then the equations of motion are preserved, and we also have

$$\dot{\Xi}_a = \mathbb{J}_{ab}\, \frac{\partial \widetilde{H}}{\partial \Xi_b} = \mathbb{J}_{ab}\, \frac{\partial H}{\partial \xi_k}\, \frac{\partial \xi_k}{\partial \Xi_b} \quad . \tag{16.48}$$

Equating these two expressions, we have

$$M_{aj}\, \mathbb{J}_{jk}\, \frac{\partial H}{\partial \xi_k} = \mathbb{J}_{ab}\, M^{-1}_{kb}\, \frac{\partial H}{\partial \xi_k} \quad , \tag{16.49}$$

where $M_{aj} \equiv \partial \Xi_a / \partial \xi_j$ is the Jacobian of the transformation. Since the equality must hold for all ξ, we conclude

$$M\mathbb{J} = \mathbb{J}\big(M^{\rm t}\big)^{-1} \quad \Longrightarrow \quad M\mathbb{J}M^{\rm t} = \mathbb{J} \quad . \tag{16.50}$$

A matrix M satisfying $MM^{\rm t} = \mathbb{I}$ is an *orthogonal* matrix. A matrix M satisfying $M\mathbb{J}M^{\rm t} = \mathbb{J}$ is called *symplectic*. We write $M \in \mathsf{Sp}(2n)$, *i.e.* M is an element of the group of *symplectic matrices*[2] of rank $2n$.

[2]Note that the rank of a symplectic matrix is always even. Note also $M\mathbb{J}M^{\rm t} = \mathbb{J}$ implies $M^{\rm t}\mathbb{J}M = \mathbb{J}$.

The symplectic property of M guarantees that the Poisson brackets are preserved under a canonical transformation:

$$\begin{aligned} \big\{A,B\big\}_\xi &= \mathbb{J}_{ij}\,\frac{\partial A}{\partial \xi_i}\,\frac{\partial B}{\partial \xi_j} = \mathbb{J}_{ij}\,\frac{\partial A}{\partial \Xi_a}\,\frac{\partial \Xi_a}{\partial \xi_i}\,\frac{\partial B}{\partial \Xi_b}\,\frac{\partial \Xi_b}{\partial \xi_j} \\ &= \big(M_{ai}\,\mathbb{J}_{ij}\,M^{\rm t}_{jb}\big)\,\frac{\partial A}{\partial \Xi_a}\,\frac{\partial B}{\partial \Xi_b} = \mathbb{J}_{ab}\,\frac{\partial A}{\partial \Xi_a}\,\frac{\partial B}{\partial \Xi_b} = \big\{A,B\big\}_\Xi \quad . \end{aligned} \tag{16.51}$$

16.3.5 Generating functions for canonical transformations

For a transformation to be canonical, we require

$$\delta\int\limits_{t_a}^{t_b}\!dt\,\Big\{p_\sigma\,\dot q_\sigma - H(\boldsymbol{q},\boldsymbol{p},t)\Big\} = 0 = \delta\int\limits_{t_a}^{t_b}\!dt\,\Big\{P_\sigma\,\dot Q_\sigma - \widetilde H(\boldsymbol{Q},\boldsymbol{P},t)\Big\} \quad . \tag{16.52}$$

This is satisfied provided

$$\Big\{p_\sigma\,\dot q_\sigma - H(\boldsymbol{q},\boldsymbol{p},t)\Big\} = \lambda\,\bigg\{P_\sigma\,\dot Q_\sigma - \widetilde H(\boldsymbol{Q},\boldsymbol{P},t) + \frac{d}{dt}\,F(\boldsymbol{q},\boldsymbol{Q},t)\bigg\} \quad , \tag{16.53}$$

where λ is a constant. For canonical transformations[3], $\lambda=1$. Thus,

$$\widetilde H(\boldsymbol{Q},\boldsymbol{P},t) = H(\boldsymbol{q},\boldsymbol{p},t) + P_\sigma\,\dot Q_\sigma - p_\sigma\,\dot q_\sigma + \frac{\partial F}{\partial q_\sigma}\,\dot q_\sigma + \frac{\partial F}{\partial Q_\sigma}\,\dot Q_\sigma + \frac{\partial F}{\partial t} \quad . \tag{16.54}$$

Thus, we require

$$\frac{\partial F}{\partial q_\sigma} = p_\sigma \quad , \quad \frac{\partial F}{\partial Q_\sigma} = -P_\sigma \quad , \quad \frac{\partial F}{\partial p_\sigma} = 0 \quad , \quad \frac{\partial F}{\partial P_\sigma} = 0 \quad , \tag{16.55}$$

which says that $F = F(\boldsymbol{q},\boldsymbol{Q},t)$ is only a function of $(\boldsymbol{q},\boldsymbol{Q},t)$ and not a function of any of the momentum variables $\boldsymbol{p}$ and $\boldsymbol{P}$. The transformed Hamiltonian is then

$$\widetilde H(\boldsymbol{Q},\boldsymbol{P},t) = H(\boldsymbol{q},\boldsymbol{p},t) + \frac{\partial F(\boldsymbol{q},\boldsymbol{Q},t)}{\partial t} \quad . \tag{16.56}$$

There are four possibilities, corresponding to the freedom to make Legendre transformations with respect to the coordinate arguments of $F(q,Q,t)$:

$$F(\boldsymbol{q},\boldsymbol{Q},t) = \begin{cases} F_1(\boldsymbol{q},\boldsymbol{Q},t) & ; \quad p_\sigma = +\frac{\partial F_1}{\partial q_\sigma} \quad , \quad P_\sigma = -\frac{\partial F_1}{\partial Q_\sigma} \quad \text{(type I)} \\ \\ F_2(\boldsymbol{q},\boldsymbol{P},t) - P_\sigma\,Q_\sigma & ; \quad p_\sigma = +\frac{\partial F_2}{\partial q_\sigma} \quad , \quad Q_\sigma = +\frac{\partial F_2}{\partial P_\sigma} \quad \text{(type II)} \\ \\ F_3(\boldsymbol{p},\boldsymbol{Q},t) + p_\sigma\,q_\sigma & ; \quad q_\sigma = -\frac{\partial F_3}{\partial p_\sigma} \quad , \quad P_\sigma = -\frac{\partial F_3}{\partial Q_\sigma} \quad \text{(type III)} \\ \\ F_4(\boldsymbol{p},\boldsymbol{P},t) + p_\sigma\,q_\sigma - P_\sigma\,Q_\sigma & ; \quad q_\sigma = -\frac{\partial F_4}{\partial p_\sigma} \quad , \quad Q_\sigma = +\frac{\partial F_4}{\partial P_\sigma} \quad \text{(type IV)} \end{cases} \tag{16.57}$$

[3]Solutions of eqn. 16.53 with $\lambda\neq 1$ are known as *extended* canonical transformations. We can always rescale coordinates and/or momenta to achieve $\lambda=1$.

In each case ($\gamma = 1, 2, 3, 4$), we have

$$\widetilde{H}(\boldsymbol{Q}, \boldsymbol{P}, t) = H(\boldsymbol{q}, \boldsymbol{p}, t) + \frac{\partial F_\gamma}{\partial t} \quad . \tag{16.58}$$

Let's work out some examples:

- Consider the type-II transformation generated by

$$F_2(\boldsymbol{q}, \boldsymbol{P}) = A_\sigma(\boldsymbol{q})\, P_\sigma \quad , \tag{16.59}$$

where $A_\sigma(\boldsymbol{q})$ is an arbitrary function of the $\{q_\sigma\}$. We then have

$$Q_\sigma = \frac{\partial F_2}{\partial P_\sigma} = A_\sigma(\boldsymbol{q}) \qquad , \qquad p_\sigma = \frac{\partial F_2}{\partial q_\sigma} = \frac{\partial A_\alpha}{\partial q_\sigma}\, P_\alpha \quad . \tag{16.60}$$

Thus,

$$Q_\sigma = A_\sigma(\boldsymbol{q}) \qquad , \qquad P_\sigma = \frac{\partial q_\alpha}{\partial Q_\sigma}\, p_\alpha \quad . \tag{16.61}$$

This is a general point transformation of the kind discussed in eqn. 16.31. For a general linear point transformation, $Q_\alpha = M_{\alpha\beta}\, q_\beta$, we have $P_\alpha = p_\beta\, M^{-1}_{\beta\alpha}$, *i.e.* $\boldsymbol{Q} = M\boldsymbol{q}$, $\boldsymbol{P} = \boldsymbol{p}\, M^{-1}$. If $M_{\alpha\beta} = \delta_{\alpha\beta}$, this is the identity transformation. $F_2 = q_1 P_3 + q_3 P_1$ interchanges labels 1 and 3, *etc.*

- Consider the type-I transformation generated by

$$F_1(\boldsymbol{q}, \boldsymbol{Q}) = A_\sigma(\boldsymbol{q})\, Q_\sigma \quad . \tag{16.62}$$

We then have

$$\begin{aligned} p_\sigma &= \frac{\partial F_1}{\partial q_\sigma} = \frac{\partial A_\alpha}{\partial q_\sigma}\, Q_\alpha \\ P_\sigma &= -\frac{\partial F_1}{\partial Q_\sigma} = -A_\sigma(\boldsymbol{q}) \quad . \end{aligned} \tag{16.63}$$

Note that $A_\sigma(\boldsymbol{q}) = q_\sigma$ generates the transformation

$$\begin{pmatrix} \boldsymbol{q} \\ \boldsymbol{p} \end{pmatrix} \longrightarrow \begin{pmatrix} -\boldsymbol{P} \\ +\boldsymbol{Q} \end{pmatrix} \quad . \tag{16.64}$$

- A mixed transformation is also permitted. For example,

$$F(\boldsymbol{q}, \boldsymbol{Q}) = q_1\, Q_1 + (q_3 - Q_2)\, P_2 + (q_2 - Q_3)\, P_3 \tag{16.65}$$

is of type-I with respect to index $\sigma = 1$ and type-II with respect to indices $\sigma = 2, 3$. The transformation effected is

$$Q_1 = p_1 \quad , \quad Q_2 = q_3 \quad , \quad Q_3 = q_2 \quad , \quad P_1 = -q_1 \quad , \quad P_2 = p_3 \quad , \quad P_3 = p_2 \quad . \tag{16.66}$$

- Consider the $n = 1$ harmonic oscillator,

$$H(q,p) = \frac{p^2}{2m} + \tfrac{1}{2}kq^2 \quad . \tag{16.67}$$

If we could find a time-independent canonical transformation such that

$$p = \sqrt{2mf(P)}\,\cos Q \qquad , \qquad q = \sqrt{\frac{2\,f(P)}{k}}\,\sin Q \quad , \tag{16.68}$$

where $f(P)$ is some function of P, then we'd have $\widetilde{H}(Q,P) = f(P)$, which is cyclic in Q. To find this transformation, we take the ratio of p and q to obtain

$$p = \sqrt{mk}\,q\,\operatorname{ctn} Q \quad , \tag{16.69}$$

which suggests the type-I transformation

$$F_1(q,Q) = \tfrac{1}{2}\sqrt{mk}\,q^2 \operatorname{ctn} Q \quad . \tag{16.70}$$

This leads to

$$p = \frac{\partial F_1}{\partial q} = \sqrt{mk}\,q\,\operatorname{ctn} Q \quad , \quad P = -\frac{\partial F_1}{\partial Q} = \frac{\sqrt{mk}\,q^2}{2\,\sin^2\!Q} \quad . \tag{16.71}$$

Thus,

$$q = \frac{\sqrt{2P}}{\sqrt[4]{mk}}\,\sin Q \quad \Longrightarrow \quad f(P) = \sqrt{\frac{k}{m}}\,P = \omega P \quad , \tag{16.72}$$

where $\omega = \sqrt{k/m}$ is the oscillation frequency. We therefore have that $\widetilde{H}(Q,P) = \omega P$, whence $P = E/\omega$. The equations of motion are

$$\dot{P} = -\frac{\partial \widetilde{H}}{\partial Q} = 0 \qquad , \qquad \dot{Q} = \frac{\partial \widetilde{H}}{\partial P} = \omega \quad , \tag{16.73}$$

which yields

$$Q(t) = \omega t + \varphi_0 \qquad , \qquad q(t) = \sqrt{\frac{2E}{m\omega^2}}\,\sin\big(\omega t + \varphi_0\big) \quad . \tag{16.74}$$

16.4 Hamilton-Jacobi Theory

We've stressed the great freedom involved in making canonical transformations. Coordinates and momenta, for example, may be interchanged – the distinction between them is purely a matter of convention! We now ask: is there any specially preferred canonical transformation? In this regard, one obvious goal is to make the Hamiltonian $\widetilde{H}(\boldsymbol{Q},\boldsymbol{P},t)$ and the corresponding equations of motion as simple as possible.

Recall the general form of the canonical transformation:

$$\widetilde{H}(\boldsymbol{Q},\boldsymbol{P},t) = H(\boldsymbol{q},\boldsymbol{p},t) + \frac{\partial F(\boldsymbol{q},\boldsymbol{Q},t)}{\partial t} \quad , \tag{16.75}$$

with

$$\frac{\partial F}{\partial q_\sigma} = p_\sigma \quad , \quad \frac{\partial F}{\partial p_\sigma} = 0 \quad , \quad \frac{\partial F}{\partial Q_\sigma} = -P_\sigma \quad , \quad \frac{\partial F}{\partial P_\sigma} = 0 \quad . \tag{16.76}$$

We now ask that this transformation result in the simplest Hamiltonian possible, that is, $\widetilde{H}(\boldsymbol{Q}, \boldsymbol{P}, t) = 0$. This requires we find a function F such that

$$\frac{\partial F}{\partial t} = -H \quad , \quad \frac{\partial F}{\partial q_\sigma} = p_\sigma \quad . \tag{16.77}$$

The remaining functional dependence may be taken to be either on $\boldsymbol{Q}$ (type I) or on $\boldsymbol{P}$ (type II). As it turns out, the generating function F we seek is in fact the action, S, which is the integral of L with respect to time, expressed as a function of its endpoint values.

16.4.1 The action as a function of coordinates and time

We have seen how the action $S[\boldsymbol{\eta}(\tau)]$ is a *functional* of the path $\boldsymbol{\eta}(\tau)$ and a *function* of the endpoint values $\{\boldsymbol{q}_a, t_a\}$ and $\{\boldsymbol{q}_b, t_b\}$. Let us define the action *function* $S(\boldsymbol{q}, t)$ as

$$S(\boldsymbol{q}, t) = \int\limits_{t_a}^{t} d\tau \, L\big(\boldsymbol{\eta}, \dot{\boldsymbol{\eta}}, \tau\big) \quad , \tag{16.78}$$

where $\boldsymbol{\eta}(\tau)$ starts at $(\boldsymbol{q}_a, t_a)$ and ends at $(\boldsymbol{q}, t)$. We also require that $\boldsymbol{\eta}(\tau)$ satisfy the Euler-Lagrange equations,

$$\frac{\partial L}{\partial \eta_\sigma} - \frac{d}{d\tau}\bigg(\frac{\partial L}{\partial \dot{\eta}_\sigma}\bigg) = 0 \tag{16.79}$$

Let us now consider a new path, $\tilde{\boldsymbol{\eta}}(\tau)$, also starting at $(\boldsymbol{q}_a, t_a)$, but ending at $(\boldsymbol{q} + d\boldsymbol{q}, t + dt)$, and also satisfying the equations of motion. The differential of S is

$$\begin{aligned}
dS = S\big[\tilde{\boldsymbol{\eta}}(\tau)\big] - S\big[\boldsymbol{\eta}(\tau)\big] &= \int\limits_{t_a}^{t+dt} d\tau \, L(\tilde{\boldsymbol{\eta}}, \dot{\tilde{\boldsymbol{\eta}}}, \tau) - \int\limits_{t_a}^{t} d\tau \, L\big(\boldsymbol{\eta}, \dot{\boldsymbol{\eta}}, \tau\big) \\
&= \int\limits_{t_a}^{t} d\tau \left\{ \frac{\partial L}{\partial \eta_\sigma} \Big[\tilde{\eta}_\sigma(\tau) - \eta_\sigma(\tau)\Big] + \frac{\partial L}{\partial \dot{\eta}_\sigma} \Big[\dot{\tilde{\eta}}_\sigma(\tau) - \dot{\eta}_\sigma(\tau)\Big] \right\} + L\big(\tilde{\boldsymbol{\eta}}(t), \dot{\tilde{\boldsymbol{\eta}}}(t), t\big)\, dt \\
&= \int\limits_{t_a}^{t} d\tau \left\{ \frac{\partial L}{\partial \eta_\sigma} - \frac{d}{d\tau}\bigg(\frac{\partial L}{\partial \dot{\eta}_\sigma}\bigg) \right\} \Big[\tilde{\eta}_\sigma(\tau) - \eta_\sigma(\tau)\Big] + \frac{\partial L}{\partial \dot{\eta}_\sigma}\bigg|_t \Big[\tilde{\eta}_\sigma(t) - \eta_\sigma(t)\Big] + L\big(\tilde{\boldsymbol{\eta}}(t), \dot{\tilde{\boldsymbol{\eta}}}(t), t\big)\, dt \\
&= 0 + \pi_\sigma(t)\, \delta\eta_\sigma(t) + L\big(\boldsymbol{\eta}(t), \dot{\boldsymbol{\eta}}(t), t\big)\, dt + \mathcal{O}(\delta\boldsymbol{q}\, dt) \quad ,
\end{aligned} \tag{16.80}$$

where we have defined $\pi_\sigma = \partial L / \partial \dot{\eta}_\sigma$, and $\delta\eta_\sigma(\tau) \equiv \tilde{\eta}_\sigma(\tau) - \eta_\sigma(\tau)$.

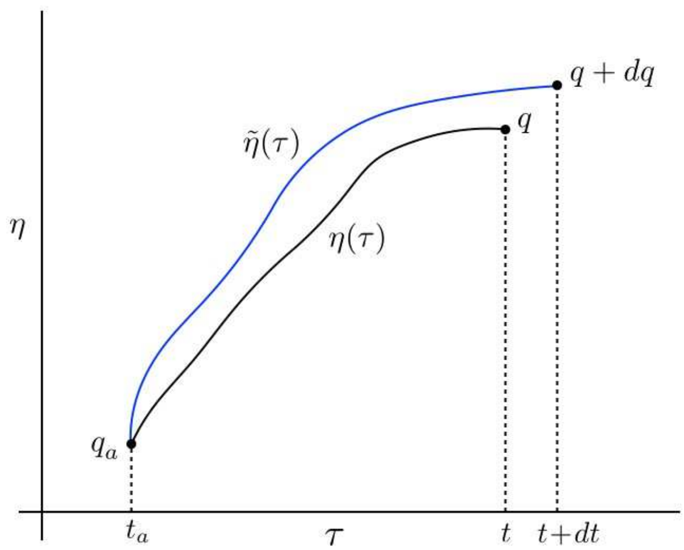


Figure 16.1: The paths $\boldsymbol{\eta}(\tau)$ and $\tilde{\boldsymbol{\eta}}(\tau)$.

Note that the differential dq_σ is given by

$$\begin{aligned} dq_\sigma &= \tilde{\eta}_\sigma(t+dt) - \eta_\sigma(t) \\ &= \tilde{\eta}_\sigma(t+dt) - \tilde{\eta}_\sigma(t) + \tilde{\eta}_\sigma(t) - \eta_\sigma(t) \\ &= \dot{\tilde{\eta}}_\sigma(t)\,dt + \delta\eta_\sigma(t) = \dot{q}_\sigma(t)\,dt + \delta\eta_\sigma(t) + \mathcal{O}(\delta\boldsymbol{q}\,dt) \quad . \end{aligned} \tag{16.81}$$

Thus, with $\pi_\sigma(t) \equiv p_\sigma$, we have

$$\begin{aligned} dS &= p_\sigma\,dq_\sigma + \big(L - p_\sigma\,\dot{q}_\sigma\big)\,dt \\ &= p_\sigma\,dq_\sigma - H\,dt \quad . \end{aligned} \tag{16.82}$$

We therefore obtain

$$\frac{\partial S}{\partial q_\sigma} = p_\sigma \quad , \quad \frac{\partial S}{\partial t} = -H \quad , \quad \frac{dS}{dt} = L \quad . \tag{16.83}$$

What about the lower limit at t_a? Clearly there are $n+1$ constants associated with this limit, and those are: $\big\{q_1(t_a), \ldots, q_n(t_a)\,;t_a\big\}$. Thus, we may write

$$S = S(q_1, \ldots, q_n; \Lambda_1, \ldots, \Lambda_n, t) + \Lambda_{n+1} \quad , \tag{16.84}$$

where our $n+1$ constants are $\{\Lambda_1, \ldots, \Lambda_{n+1}\}$. If we regard S as a mixed generator, which is type-I in some variables and type-II in others, then each Λ_σ for $1 \le \sigma \le n$ may be chosen to be either Q_σ or P_σ. We will define

$$\Gamma_\sigma = \frac{\partial S}{\partial \Lambda_\sigma} = \begin{cases} +Q_\sigma & \text{if } \Lambda_\sigma = P_\sigma \\ -P_\sigma & \text{if } \Lambda_\sigma = Q_\sigma \end{cases} \tag{16.85}$$

For each σ, the two possibilities $\Lambda_\sigma = Q_\sigma$ or $\Lambda_\sigma = P_\sigma$ are of course rendered equivalent by a canonical transformation $(Q_\sigma, P_\sigma) \to (P_\sigma, -Q_\sigma)$.

16.4.2 The Hamilton-Jacobi equation

Since the action $S(\boldsymbol{q},\boldsymbol{\Lambda},t)$ generates a canonical transformation for which $\tilde{H}(\boldsymbol{Q},\boldsymbol{P})=0$, this requirement may be written as

$$H\Big(q_1,\ldots,q_n,\frac{\partial S}{\partial q_1},\ldots,\frac{\partial S}{\partial q_n},t\Big)+\frac{\partial S}{\partial t}=0 \quad . \tag{16.86}$$

This is the *Hamilton-Jacobi equation* (HJE). It is a first order partial differential equation in $n+1$ variables, and in general is nonlinear (since kinetic energy is generally a quadratic function of momenta). Since $\widetilde{H}(\boldsymbol{Q},\boldsymbol{P},t)=0$, the equations of motion are trivial, and

$$Q_\sigma(t)=\text{const.} \qquad , \qquad P_\sigma(t)=\text{const.} \tag{16.87}$$

Once the HJE is solved, one must invert the relations $\Gamma_\sigma=\partial S(\boldsymbol{q},\boldsymbol{\Lambda},t)/\partial\Lambda_\sigma$ to obtain the $q_\sigma(\boldsymbol{Q},\boldsymbol{P},t)$. This is possible only if

$$\det\left(\frac{\partial^2 S}{\partial q_\alpha\,\partial\Lambda_\beta}\right)\neq 0 \quad , \tag{16.88}$$

which is known as the *Hessian condition*.

It is worth noting that the HJE may have several solutions. For example, consider the case of the free particle in one dimension, with $H(q,p)=p^2/2m$. The HJE is

$$\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2+\frac{\partial S}{\partial t}=0 \quad . \tag{16.89}$$

One solution of the HJE is

$$S(q,\Lambda,t)=\frac{m\,(q-\Lambda)^2}{2t} \quad . \tag{16.90}$$

For this we find

$$\Gamma=\frac{\partial S}{\partial\Lambda}=-\frac{m}{t}\,(q-\Lambda) \quad\Rightarrow\quad q(t)=\Lambda-\frac{\Gamma}{m}\,t \quad . \tag{16.91}$$

Here $\Lambda=q(0)$ is the initial value of q, and $\Gamma=-p$ is minus the momentum.

Another equally valid solution to the HJE is

$$S(q,\Lambda,t)=q\,\sqrt{2m\Lambda}-\Lambda\,t \quad . \tag{16.92}$$

This yields

$$\Gamma=\frac{\partial S}{\partial\Lambda}=q\,\sqrt{\frac{m}{2\Lambda}}-t \quad\Rightarrow\quad q(t)=\sqrt{\frac{2\Lambda}{m}}\,(t+\Gamma) \quad . \tag{16.93}$$

For this solution, $\Lambda=\frac{1}{2}mv^2$ is the energy and Γ may be related to the initial value $q(0)=\Gamma\sqrt{2\Lambda/m}$.

16.4.3 Time-independent Hamiltonians

When H has no explicit time dependence, we may reduce the order of the HJE by one, writing

$$S(\boldsymbol{q}, \boldsymbol{\Lambda}, t) = W(\boldsymbol{q}, \boldsymbol{\Lambda}) + T(\boldsymbol{\Lambda}, t) \quad . \tag{16.94}$$

The HJE becomes

$$H\left(\boldsymbol{q}\,, \frac{\partial W}{\partial \boldsymbol{q}}\right) = -\frac{\partial T}{\partial t} \quad . \tag{16.95}$$

Note that the LHS of the above equation is independent of t, and the RHS is independent of q. Therefore, each side must only depend on the constants Λ, which is to say that each side must be a constant, which, without loss of generality, we take to be Λ_1. Therefore

$$S(\boldsymbol{q}, \boldsymbol{\Lambda}, t) = W(\boldsymbol{q}, \boldsymbol{\Lambda}) - \Lambda_1 t \quad . \tag{16.96}$$

The function $W(\boldsymbol{q}, \boldsymbol{\Lambda})$ is called *Hamilton's characteristic function*. The HJE now takes the form

$$H\left(q_1, \ldots, q_n\,, \frac{\partial W}{\partial q_1}\,, \ldots\,, \frac{\partial W}{\partial q_n}\right) = \Lambda_1 \quad . \tag{16.97}$$

Note that adding an arbitrary constant C to S generates the same equation, and simply shifts the last constant $\Lambda_{n+1} \to \Lambda_{n+1} + C$. According to eqn. 16.96, this is equivalent to replacing t by $t - t_0$ with $t_0 = C/\Lambda_1$, *i.e.* it just redefines the zero of the time variable.

16.4.4 Example: one-dimensional motion

As an example of the method, consider the one-dimensional system,

$$H(q,p) = \frac{p^2}{2m} + U(q) \quad . \tag{16.98}$$

The HJE is

$$\frac{1}{2m}\left(\frac{\partial W}{\partial q}\right)^2 + U(q) = \Lambda \quad . \tag{16.99}$$

Clearly $\Lambda = E$ is the total energy. The HJE may be recast as

$$\frac{\partial W}{\partial q} = \pm\sqrt{2m\big[\Lambda - U(q)\big]} \quad , \tag{16.100}$$

with solution

$$W(q,\Lambda) = \pm\sqrt{2m}\int^q\! dq'\,\sqrt{\Lambda - U(q')} \quad , \tag{16.101}$$

with $S(q,\Lambda,t) = W(q,\Lambda) - \Lambda t$. We now have

$$p = \frac{\partial W}{\partial q} = \pm\sqrt{2m\big[\Lambda - U(q)\big]} \quad , \tag{16.102}$$

as well as

$$\Gamma = \frac{\partial S}{\partial \Lambda} = \frac{\partial W}{\partial \Lambda} - t = \pm\sqrt{\frac{m}{2}} \int^{q(t)} \frac{dq'}{\sqrt{\Lambda - U(q')}} - t \quad . \tag{16.103}$$

Thus, the motion $q(t)$ is given by quadrature:

$$\Gamma + t = \pm\sqrt{\frac{m}{2}} \int^{q(t)} \frac{dq'}{\sqrt{\Lambda - U(q')}} \quad , \tag{16.104}$$

where Λ and Γ are constants. The lower limit on the integral is arbitrary and merely shifts t by another constant. The characteristic function $W(q, \Lambda)$ is actually double-valued in q, corresponding to right-moving and left-moving parts of the motion.

16.4.5 Separation of variables

It is convenient to first work an example before discussing the general theory. Consider the following Hamiltonian, written in spherical polar coordinates:

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2\!\theta}\right) + \overbrace{A(r) + \frac{B(\theta)}{r^2} + \frac{C(\phi)}{r^2 \sin^2\!\theta}}^{\text{potential } U(r,\theta,\phi)} \quad . \tag{16.105}$$

We seek a characteristic function of the form $W(r, \theta, \phi) = W_r(r) + W_\theta(\theta) + W_\phi(\phi)$. The HJE is then

$$\begin{aligned} &\frac{1}{2m}\left(\frac{\partial W_r}{\partial r}\right)^2 + \frac{1}{2mr^2}\left(\frac{\partial W_\theta}{\partial \theta}\right)^2 + \frac{1}{2mr^2 \sin^2\!\theta}\left(\frac{\partial W_\phi}{\partial \phi}\right)^2 \\ &\qquad\qquad + A(r) + \frac{B(\theta)}{r^2} + \frac{C(\phi)}{r^2 \sin^2\!\theta} = \Lambda_1 = E \quad . \end{aligned} \tag{16.106}$$

Multiply through by $r^2 \sin^2\!\theta$ to obtain

$$\begin{aligned} \frac{1}{2m}\left(\frac{\partial W_\phi}{\partial \phi}\right)^2 + C(\phi) &= -\sin^2\!\theta \left\{\frac{1}{2m}\left(\frac{\partial W_\theta}{\partial \theta}\right)^2 + B(\theta)\right\} \\ &\qquad - r^2 \sin^2\!\theta \left\{\frac{1}{2m}\left(\frac{\partial W_r}{\partial r}\right)^2 + A(r) - \Lambda_1\right\} \quad . \end{aligned} \tag{16.107}$$

The LHS is independent of (r, θ), and the RHS is independent of ϕ. Therefore, we may set

$$\frac{1}{2m}\left(\frac{\partial W_\phi}{\partial \phi}\right)^2 + C(\phi) = \Lambda_2 \quad . \tag{16.108}$$

Proceeding, we replace the LHS in eqn. 16.107 with Λ_2, arriving at

$$\frac{1}{2m}\left(\frac{\partial W_\theta}{\partial \theta}\right)^2 + B(\theta) + \frac{\Lambda_2}{\sin^2\!\theta} = -r^2 \left\{\frac{1}{2m}\left(\frac{\partial W_r}{\partial r}\right)^2 + A(r) - \Lambda_1\right\} \quad . \tag{16.109}$$

The LHS of this equation is independent of r, and the RHS is independent of θ. Therefore,

$$\frac{1}{2m}\left(\frac{\partial W_\theta}{\partial\theta}\right)^2 + B(\theta) + \frac{\varLambda_2}{\sin^2\!\theta} = \varLambda_3 \quad . \tag{16.110}$$

We're left with

$$\frac{1}{2m}\left(\frac{\partial W_r}{\partial r}\right)^2 + A(r) + \frac{\varLambda_3}{r^2} = \varLambda_1 \quad . \tag{16.111}$$

The full solution is therefore

$$\begin{aligned} S(\boldsymbol{q},\boldsymbol{\varLambda},t) = \sqrt{2m}\int^{r}\! dr'\,\sqrt{\varLambda_1 - A(r') - \frac{\varLambda_3}{r'^2}} + \sqrt{2m}\int^{\theta}\! d\theta'\,\sqrt{\varLambda_3 - B(\theta') - \frac{\varLambda_2}{\sin^2\!\theta'}} \\ + \sqrt{2m}\int^{\phi}\! d\phi'\,\sqrt{\varLambda_2 - C(\phi')} - \varLambda_1 t \quad . \end{aligned} \tag{16.112}$$

We then have

$$\begin{aligned} \varGamma_1 &= \frac{\partial S}{\partial \varLambda_1} = \sqrt{\tfrac{m}{2}}\int^{r(t)}\!\frac{dr'}{\sqrt{\varLambda_1 - A(r') - \varLambda_3\, r'^{-2}}} - t \\ \varGamma_2 &= \frac{\partial S}{\partial \varLambda_2} = -\sqrt{\tfrac{m}{2}}\int^{\theta(t)}\!\frac{d\theta'}{\sin^2\!\theta'\,\sqrt{\varLambda_3 - B(\theta') - \varLambda_2\csc^2\!\theta'}} + \sqrt{\tfrac{m}{2}}\int^{\phi(t)}\!\frac{d\phi'}{\sqrt{\varLambda_2 - C(\phi')}} \\ \varGamma_3 &= \frac{\partial S}{\partial \varLambda_3} = -\sqrt{\tfrac{m}{2}}\int^{r(t)}\!\frac{dr'}{r'^2\,\sqrt{\varLambda_1 - A(r') - \varLambda_3\, r'^{-2}}} + \sqrt{\tfrac{m}{2}}\int^{\theta(t)}\!\frac{d\theta'}{\sqrt{\varLambda_3 - B(\theta') - \varLambda_2\csc^2\!\theta'}} \quad . \end{aligned} \tag{16.113}$$

The game plan here is as follows. The first of the above trio of equations is inverted to yield $r(t)$ in terms of t and constants. This solution is then invoked in the last equation (the upper limit on the first integral on the RHS) in order to obtain an implicit equation for $\theta(t)$, which is invoked in the second equation to yield an implicit equation for $\phi(t)$. The net result is the motion of the system in terms of time t and the six constants $(\varLambda_1, \varLambda_2, \varLambda_3, \varGamma_1, \varGamma_2, \varGamma_3)$. A seventh constant, associated with an overall shift of the zero of t, arises due to the arbitrary lower limits of the integrals.

In general, the separation of variables method begins with[4]

$$W(\boldsymbol{q},\boldsymbol{\varLambda}) = \sum_{\sigma=1}^{n} W_\sigma(q_\sigma\,,\boldsymbol{\varLambda}) \quad . \tag{16.114}$$

Each $W_\sigma(q_\sigma, \boldsymbol{\varLambda})$ may be regarded as a function of the single variable q_σ, and is obtained by satisfying an ODE of the form[5]

$$H_\sigma\!\left(q_\sigma, \frac{dW_\sigma}{dq_\sigma}\right) = \varLambda_\sigma \quad . \tag{16.115}$$

[4]Here we assume *complete separability*. A given system may only be *partially* separable.

[5]Note that $H_\sigma(q_\sigma, p_\sigma)$ may itself depend on several of the constants $\varLambda_\alpha$. For example, eqn. 16.111 is of the form $H_r\big(r, \partial_r W_r, \varLambda_3\big) = \varLambda_1$.

We then have

$$p_\sigma = \frac{\partial W_\sigma}{\partial q_\sigma} \quad , \quad \Gamma_\sigma = \frac{\partial W_\sigma}{\partial \Lambda_\sigma} + \delta_{\sigma,1}\, t \quad . \tag{16.116}$$

Note that while each W_σ depends on only a single q_σ, it may depend on several of the Λ_σ.

16.5 Action-angle vriables

16.5.1 Circular phase orbits: librations and rotations

In a completely integrable system, the Hamilton-Jacobi equation may be solved by separation of variables. Each momentum p_σ is a function of only its corresponding coordinate q_σ plus constants – no other coordinates enter:

$$p_\sigma = \frac{\partial W_\sigma}{\partial q_\sigma} = p_\sigma(q_\sigma, \boldsymbol{\Lambda}) \quad . \tag{16.117}$$

The motion satisfies $H_\sigma(q_\sigma, p_\sigma) = \Lambda_\sigma$. The level sets of H_σ are curves $\mathcal{C}_\sigma$. In general, these curves each depend on all of the constants $\boldsymbol{\Lambda}$, so we write $\mathcal{C}_\sigma = \mathcal{C}_\sigma(\boldsymbol{\Lambda})$. The curves $\mathcal{C}_\sigma$ are the *projections* of the full motion onto the (q_σ, p_σ) plane. In general we will assume the motion, and hence the curves $\mathcal{C}_\sigma$, is *bounded*. In this case, two types of projected motion are possible: librations and rotations. Librations are periodic oscillations about an equilibrium position. Rotations involve the advancement of an angular variable by 2π during a cycle. This is most conveniently illustrated in the case of the simple pendulum, for which

$$H(\phi, p_\phi) = \frac{p_\phi^2}{2I} + \tfrac{1}{2} I\omega^2 \left(1 - \cos\phi\right) \quad . \tag{16.118}$$

- When $E < I\,\omega^2$, the momentum p_ϕ vanishes at $\phi = \pm\cos^{-1}(2E/I\omega^2)$. The system executes librations between these extreme values of the angle ϕ.
- When $E > I\,\omega^2$, the kinetic energy is always positive, and the angle advances monotonically, executing rotations.

In a completely integrable system, each $\mathcal{C}_\sigma$ is either a libration or a rotation[6]. Both librations and rotations are closed curves. Thus, each $\mathcal{C}_\sigma$ is in general homotopic to (= "can be continuously distorted to yield") a circle, $\mathbb{S}^1$. For n freedoms, the motion is therefore confined to an n-torus, $\mathbb{T}^n$:

$$\mathbb{T}^n = \overbrace{\mathbb{S}^1 \times \mathbb{S}^1 \times \cdots \times \mathbb{S}^1}^{n \text{ times}} \quad . \tag{16.119}$$

These are called *invariant tori* (or *invariant manifolds*). There are many such tori, as there are many $\mathcal{C}_\sigma$ curves in each of the n two-dimensional submanifolds.

Invariant tori never intersect! This is ruled out by the uniqueness of the solution to the dynamical system, expressed as a set of coupled ordinary differential equations.

[6] $\mathcal{C}_\sigma$ may correspond to a separatrix, but this is a nongeneric state of affairs.

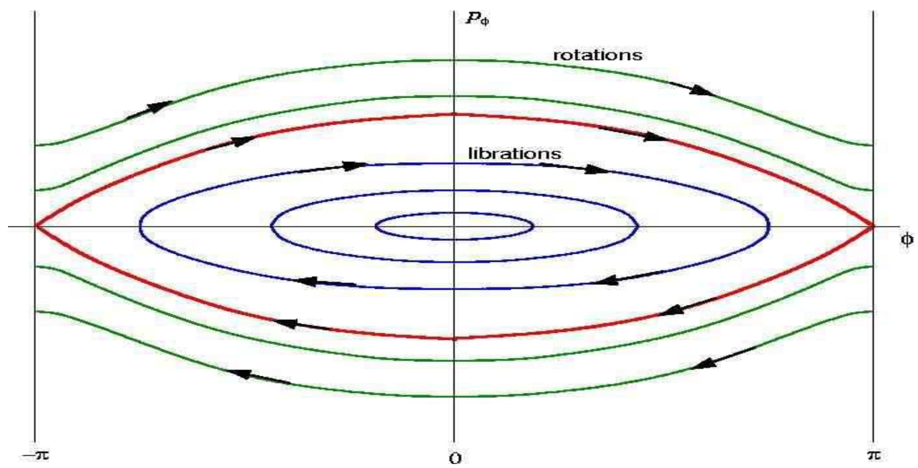


Figure 16.2: Phase curves for the simple pendulum, showing librations (in blue), rotations (in green), and the separatrix (in red). This phase flow is most correctly viewed as taking place on a cylinder, obtained from the above sketch by identifying the lines $\phi = \pi$ and $\phi = -\pi$.

Note also that phase space is of dimension $2n$, while the invariant tori are of dimension n. Phase space is 'covered' by the invariant tori, but it is in general difficult to conceive of how this happens. Perhaps the most accessible analogy is the $n = 1$ case, where the '1-tori' are just circles. Two-dimensional phase space is covered noninteracting circular orbits. (The orbits are *topologically* equivalent to circles, although *geometrically* they may be distorted.) It is challenging to think about the $n = 2$ case, where a four-dimensional phase space is filled by nonintersecting 2-tori.

16.5.2 Action-Angle Variables

For a completely integrable system, one can transform canonically from $(\boldsymbol{q}, \boldsymbol{p})$ to new coordinates $(\boldsymbol{\phi}, \boldsymbol{J})$ which specify a particular n-torus $\mathbb{T}^n$ as well as the location on the torus, which is specified by n angle variables. The $\{J_\sigma\}$ are 'momentum' variables which specify the torus itself; they are constants of the motion since the tori are invariant. They are called *action variables*. Since $\dot{J}_\sigma = 0$, we must have

$$\dot{J}_\sigma = -\frac{\partial H}{\partial \phi_\sigma} = 0 \quad \Longrightarrow \quad H = H(\boldsymbol{J}) \quad . \tag{16.120}$$

The $\{\phi_\sigma\}$ are the *angle variables*.

The coordinate ϕ_σ describes the projected motion along $\mathcal{C}_\sigma$, and is normalized by

$$\oint\limits_{\mathcal{C}_\sigma} d\phi_\sigma = 2\pi \qquad \text{(once around } \mathcal{C}_\sigma\text{)} \quad . \tag{16.121}$$

The dynamics of the angle variables are given by

$$\dot{\phi}_\sigma = \frac{\partial H}{\partial J_\sigma} \equiv \nu_\sigma(\boldsymbol{J}) \quad . \tag{16.122}$$

Thus, the motion is given by

$$\phi_\sigma(t) = \phi_\sigma(0) + \nu_\sigma(\boldsymbol{J})\, t \quad . \tag{16.123}$$

The $\{\nu_\sigma(\boldsymbol{J})\}$ are *frequencies* describing the rate at which the $\mathcal{C}_\sigma$ are traversed, and the period is $T_\sigma(\boldsymbol{J}) = 2\pi/\nu_\sigma(\boldsymbol{J})$.

16.5.3 Canonical transformation to action-angle variables

The $\{J_\sigma\}$ determine the $\{\mathcal{C}_\sigma\}$; each q_σ determines a point on $\mathcal{C}_\sigma$. This suggests a type-II transformation, with generator $F_2(\boldsymbol{q},\boldsymbol{J})$:

$$p_\sigma = \frac{\partial F_2}{\partial q_\sigma} \quad , \quad \phi_\sigma = \frac{\partial F_2}{\partial J_\sigma} \quad . \tag{16.124}$$

Note that[7]

$$2\pi = \oint\limits_{\mathcal{C}_\sigma} d\phi_\sigma = \oint\limits_{\mathcal{C}_\sigma} d\left(\frac{\partial F_2}{\partial J_\sigma}\right) = \oint\limits_{\mathcal{C}_\sigma} \frac{\partial^2 F_2}{\partial J_\sigma\, \partial q_\sigma}\, dq_\sigma = \frac{\partial}{\partial J_\sigma} \oint\limits_{\mathcal{C}_\sigma} p_\sigma\, dq_\sigma \quad , \tag{16.125}$$

which suggests the definition

$$J_\sigma = \frac{1}{2\pi} \oint\limits_{\mathcal{C}_\sigma} p_\sigma\, dq_\sigma \quad . \tag{16.126}$$

I.e. J_σ is $(2\pi)^{-1}$ times the area enclosed by $\mathcal{C}_\sigma$.

If, separating variables,

$$W(\boldsymbol{q},\boldsymbol{\Lambda}) = \sum_\sigma W_\sigma(q_\sigma,\boldsymbol{\Lambda}) \tag{16.127}$$

is Hamilton's characteristic function for the transformation $(\boldsymbol{q},\boldsymbol{p}) \to (\boldsymbol{Q},\boldsymbol{P})$, then

$$J_\sigma = \frac{1}{2\pi} \oint\limits_{\mathcal{C}_\sigma} \frac{\partial W_\sigma}{\partial q_\sigma}\, dq_\sigma = J_\sigma(\boldsymbol{\Lambda}) \tag{16.128}$$

is a function only of the $\{\Lambda_\alpha\}$ and not the $\{\Gamma_\alpha\}$. We then invert this relation to obtain $\boldsymbol{\Lambda}(\boldsymbol{J})$, to finally obtain

$$F_2(\boldsymbol{q},\boldsymbol{J}) = W\big(\boldsymbol{q},\boldsymbol{\Lambda}(\boldsymbol{J})\big) = \sum_\sigma W_\sigma\big(q_\sigma,\boldsymbol{\Lambda}(\boldsymbol{J})\big) \quad . \tag{16.129}$$

Thus, the recipe for canonically transforming to action-angle variable is as follows:

(1) Separate and solve the Hamilton-Jacobi equation for $W(\boldsymbol{q},\boldsymbol{\Lambda}) = \sum_\sigma W_\sigma(q_\sigma,\boldsymbol{\Lambda})$.

(2) Find the orbits $\mathcal{C}_\sigma(\boldsymbol{\Lambda})$, *i.e.* the level sets satisfying $H_\sigma(q_\sigma,p_\sigma) = \Lambda_\sigma$.

(3) Invert the relation $J_\sigma(\boldsymbol{\Lambda}) = \frac{1}{2\pi} \oint\limits_{\mathcal{C}_\sigma} \frac{\partial W_\sigma}{\partial q_\sigma}\, dq_\sigma$ to obtain $\boldsymbol{\Lambda}(\boldsymbol{J})$.

[7] In general, we should write $d\big(\frac{\partial F_2}{\partial J_\sigma}\big) = \frac{\partial^2 F_2}{\partial J_\sigma\, \partial q_\alpha}\, dq_\alpha$ with a sum over α. However, in eqn. 16.125 all coordinates and momenta other than q_σ and p_σ are held fixed. Thus, $\alpha = \sigma$ is the only term in the sum which contributes.

(4) $F_2(\boldsymbol{q},\boldsymbol{J}) = \sum_\sigma W_\sigma\big(q_\sigma, \boldsymbol{\Lambda}(\boldsymbol{J})\big)$ is the desired type-II generator[8].

16.5.4 Examples

Harmonic oscillator

The Hamiltonian is

$$H = \frac{p^2}{2m} + \tfrac{1}{2} m\omega_0^2\, q^2 \quad , \tag{16.130}$$

hence the Hamilton-Jacobi equation is

$$\bigg(\frac{dW}{dq}\bigg)^2 + m^2\omega_0^2\, q^2 = 2m\Lambda \quad . \tag{16.131}$$

Thus,

$$p = \frac{dW}{dq} = \pm\sqrt{2m\Lambda - m^2\omega_0^2 q^2} \quad . \tag{16.132}$$

We now define

$$q \equiv \sqrt{\frac{2\Lambda}{m\omega_0^2}}\, \sin\theta \qquad \Rightarrow \qquad p = \sqrt{2m\Lambda}\, \cos\theta \quad , \tag{16.133}$$

in which case

$$J = \frac{1}{2\pi}\oint p\, dq = \frac{1}{2\pi}\cdot\frac{2\Lambda}{\omega_0}\cdot\int\limits_0^{2\pi}\! d\theta\, \cos^2\!\theta = \frac{\Lambda}{\omega_0} \quad . \tag{16.134}$$

Solving the HJE, we write

$$\frac{dW}{d\theta} = \frac{\partial q}{\partial\theta}\cdot\frac{dW}{dq} = 2J\cos^2\!\theta \quad . \tag{16.135}$$

Integrating, we obtain

$$W = J\theta + \tfrac{1}{2} J\sin 2\theta \quad , \tag{16.136}$$

up to an irrelevant constant. We then have

$$\phi = \frac{\partial W}{\partial J}\bigg|_q = \theta + \tfrac{1}{2}\sin 2\theta + J\big(1+\cos 2\theta\big)\,\frac{\partial\theta}{\partial J}\bigg|_q \quad . \tag{16.137}$$

To find $(\partial\theta/\partial J)_q$, we differentiate $q = \sqrt{2J/m\omega_0}\,\sin\theta$:

$$dq = \frac{\sin\theta}{\sqrt{2m\omega_0 J}}\, dJ + \sqrt{\frac{2J}{m\omega_0}}\cos\theta\, d\theta \qquad \Rightarrow \qquad \frac{\partial\theta}{\partial J}\bigg|_q = -\frac{1}{2J}\,\tan\theta \quad . \tag{16.138}$$

Plugging this result into eqn. 16.137, we obtain $\phi = \theta$. Thus, the full transformation is

$$q = \sqrt{\frac{2J}{m\omega_0}}\sin\phi \quad , \quad p = \sqrt{2m\omega_0 J}\,\cos\phi \quad . \tag{16.139}$$

[8]Note that $F_2(\boldsymbol{q},\boldsymbol{J})$ is time-independent. *I.e.* we are not transforming to $\widetilde{H}=0$, but rather to $\widetilde{H}=\widetilde{H}(\boldsymbol{J})$.

The Hamiltonian is

$$H = \omega_0 \, J \quad , \tag{16.140}$$

hence $\dot{\phi} = \frac{\partial H}{\partial J} = \omega_0$ and $\dot{J} = -\frac{\partial H}{\partial \phi} = 0$, with solution $\phi(t) = \phi(0) + \omega_0 \, t$ and $J(t) = J(0)$.

Particle in a box

Consider a particle in an open box of dimensions $L_x \times L_y$ moving under the influence of gravity. The bottom of the box lies at $z = 0$. The Hamiltonian is

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m} + mgz \quad . \tag{16.141}$$

Step one is to solve the Hamilton-Jacobi equation via separation of variables. The Hamilton-Jacobi equation is written

$$\frac{1}{2m}\left(\frac{\partial W_x}{\partial x}\right)^2 + \frac{1}{2m}\left(\frac{\partial W_y}{\partial y}\right)^2 + \frac{1}{2m}\left(\frac{\partial W_z}{\partial z}\right)^2 + mgz = E \equiv \Lambda_z \quad . \tag{16.142}$$

We can solve for $W_{x,y}$ by inspection:

$$W_x(x) = \sqrt{2m\Lambda_x} \, x \quad , \quad W_y(y) = \sqrt{2m\Lambda_y} \, y \quad . \tag{16.143}$$

We then have[9]

$$\begin{aligned} W_z'(z) &= -\sqrt{2m\big(\Lambda_z - \Lambda_x - \Lambda_y - mgz\big)} \\ W_z(z) &= \frac{2\sqrt{2}}{3\sqrt{m}\, g}\big(\Lambda_z - \Lambda_x - \Lambda_y - mgz\big)^{3/2} \quad . \end{aligned} \tag{16.144}$$

Step two is to find the $\mathcal{C}_\sigma$. Clearly $p_{x,y} = \sqrt{2m\Lambda_{x,y}}$. For fixed p_x, the x motion proceeds from $x = 0$ to $x = L_x$ and back, with corresponding motion for y. For x, we have

$$p_z(z) = W_z'(z) = \sqrt{2m\big(\Lambda_z - \Lambda_x - \Lambda_y - mgz\big)} \quad , \tag{16.145}$$

and thus $\mathcal{C}_z$ is a truncated parabola, with $z_{\rm max} = (\Lambda_z - \Lambda_x - \Lambda_y)/mg$.

Step three is to compute $J(\Lambda)$ and invert to obtain $\Lambda(J)$. We have

$$\begin{aligned} J_x &= \frac{1}{2\pi}\oint\limits_{\mathcal{C}_x} p_x \, dx = \frac{1}{\pi}\int\limits_0^{L_x} dx \sqrt{2m\Lambda_x} = \frac{L_x}{\pi}\sqrt{2m\Lambda_x} \\ J_y &= \frac{1}{2\pi}\oint\limits_{\mathcal{C}_y} p_y \, dy = \frac{1}{\pi}\int\limits_0^{L_y} dy \sqrt{2m\Lambda_y} = \frac{L_y}{\pi}\sqrt{2m\Lambda_y} \end{aligned} \tag{16.146}$$

[9]Our choice of signs in taking the square roots for W_x', W_y', and W_z' is discussed below.

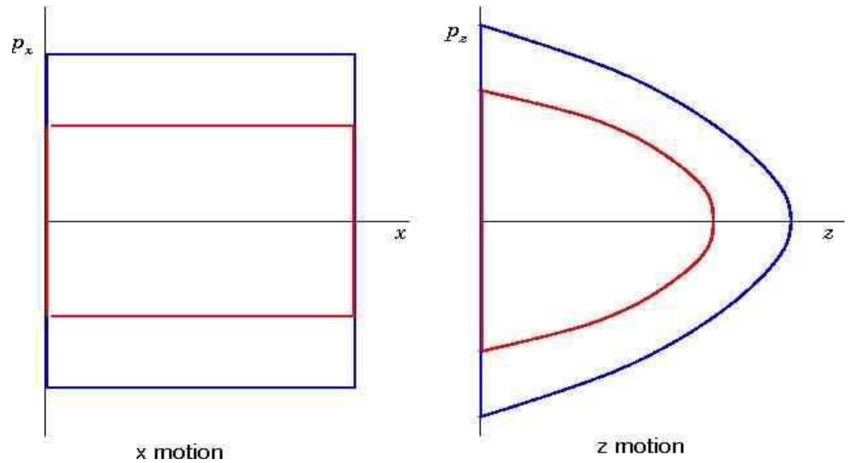


Figure 16.3: The librations $\mathcal{C}_z$ and $\mathcal{C}_x$. Not shown is $\mathcal{C}_y$, which is of the same shape as $\mathcal{C}_x$.

and

$$\begin{aligned} J_z &= \frac{1}{2\pi}\oint\limits_{\mathcal{C}_z} p_z\, dz = \frac{1}{\pi}\int\limits_0^{z_{\rm max}} dz\,\sqrt{2m\big(\Lambda_z - \Lambda_x - \Lambda_y - mgz\big)} \\ &= \frac{2\sqrt{2}}{3\pi\sqrt{m}\,g}\big(\Lambda_z - \Lambda_x - \Lambda_y\big)^{3/2} \quad . \end{aligned} \tag{16.147}$$

We now invert to obtain

$$\begin{aligned} \Lambda_x &= \frac{\pi^2}{2mL_x^2}\,J_x^2 \qquad , \qquad \Lambda_y = \frac{\pi^2}{2mL_y^2}\,J_y^2 \\ \Lambda_z &= \bigg(\frac{3\pi\sqrt{m}\,g}{2\sqrt{2}}\bigg)^{2/3} J_z^{2/3} + \frac{\pi^2}{2mL_x^2}\,J_x^2 + \frac{\pi^2}{2mL_y^2}\,J_y^2 \quad . \end{aligned} \tag{16.148}$$

$$F_2\big(x,y,z,J_x,J_y,J_z\big) = \frac{\pi x}{L_x}\,J_x + \frac{\pi y}{L_y}\,J_y + \pi\bigg(J_z^{2/3} - \frac{2m^{2/3}g^{1/3}z}{(3\pi)^{2/3}}\bigg)^{3/2} \quad . \tag{16.149}$$

We now find

$$\phi_x = \frac{\partial F_2}{\partial J_x} = \frac{\pi x}{L_x} \qquad , \qquad \phi_y = \frac{\partial F_2}{\partial J_y} = \frac{\pi y}{L_y} \tag{16.150}$$

and

$$\phi_z = \frac{\partial F_2}{\partial J_z} = \pi\sqrt{1 - \frac{2m^{2/3}g^{1/3}z}{(3\pi J_z)^{2/3}}} = \pi\sqrt{1 - \frac{z}{z_{\rm max}}} \quad , \tag{16.151}$$

where $z_{\rm max}(J_z) = (3\pi J_z/m)^{2/3}/2g^{1/3}$. The momenta are

$$p_x = \frac{\partial F_2}{\partial x} = \frac{\pi J_x}{L_x} \qquad , \qquad p_y = \frac{\partial F_2}{\partial y} = \frac{\pi J_y}{L_y} \tag{16.152}$$

and

$$p_z = \frac{\partial F_2}{\partial z} = -\sqrt{2m}\left(\left(\frac{3\pi\sqrt{m}\,g}{2\sqrt{2}}\right)^{2/3} J_z^{2/3} - mgz\right)^{1/2} \quad . \tag{16.153}$$

We note that the angle variables $\phi_{x,y,z}$ seem to be restricted to the range $[0,\pi]$, which seems to be at odds with eqn. 16.125. Similarly, the momenta $p_{x,y,z}$ all seem to be positive, whereas we know the momenta reverse sign when the particle bounces off a wall. The origin of the apparent discrepancy is that when we solved for the functions $W_{x,y,z}$, we had to take a square root in each case, and we chose a particular branch of the square root. So rather than $W_x(x) = \sqrt{2m\Lambda_x}\,x$, we should have taken

$$W_x(x) = \begin{cases} \sqrt{2m\Lambda_x}\,x & \text{if } p_x > 0 \\ \sqrt{2m\Lambda_x}\,(2L_x - x) & \text{if } p_x < 0 \end{cases} \quad . \tag{16.154}$$

The relation $J_x = (L_x/\pi)\sqrt{2m\Lambda_x}$ is unchanged, hence

$$W_x(x) = \begin{cases} (\pi x/L_x)\,J_x & \text{if } p_x > 0 \\ 2\pi J_x - (\pi x/L_x)\,J_x & \text{if } p_x < 0 \end{cases} \quad . \tag{16.155}$$

and

$$\phi_x = \begin{cases} \pi x/L_x & \text{if } p_x > 0 \\ \pi(2L_x - x)/L_x & \text{if } p_x < 0 \end{cases} \quad . \tag{16.156}$$

Now the angle variable ϕ_x advances by 2π during the cycle $\mathcal{C}_x$. Similar considerations apply to the y and z sectors.

16.6 Integrability and Motion on Invariant Tori

16.6.1 Librations and rotations

As discussed above, a completely integrable Hamiltonian system is solvable by separation of variables. The angle variables evolve as

$$\phi_\sigma(t) = \nu_\sigma(\boldsymbol{J})\,t + \phi_\sigma(0) \quad . \tag{16.157}$$

Thus, they wind around the invariant torus, specified by $\{J_\sigma\}$ at constant rates. In general, while each ϕ_σ executes periodic motion around a circle, the motion of the system as a whole is not periodic, since the frequencies $\nu_\sigma(\boldsymbol{J})$ are not, in general, commensurate. Periodic motion requires that there exists a time T such that $\nu_\sigma(\boldsymbol{J})\,T = 2\pi k_\sigma$ with $k_\sigma \in \mathbb{Z}$ for each $\sigma \in \{1,\ldots,n\}$ where each . This means the ratio of any two frequencies $\nu_\sigma/\nu_{\sigma'} = k_\sigma/k_{\sigma'} \in \mathbb{Q}$ must be a rational number. T is the smallest possible such period provided the set $\{k_1,\ldots,k_n\}$ has no common factors. On a given torus, there are several possible orbits, depending on initial conditions $\phi(0)$. However, since the frequencies are determined by the action variables, which specify each such invariant torus, on a given torus either all orbits are periodic, or none are.

In terms of the original coordinates $\boldsymbol{q}$, there are two possibilities:

$$q_\sigma(t) = \sum_{\ell_1=-\infty}^{\infty} \cdots \sum_{\ell_n=-\infty}^{\infty} A^{(\sigma)}_{\ell_1 \ell_2 \cdots \ell_n}\, e^{i\ell_1\phi_1(t)} \cdots e^{i\ell_n\phi_n(t)} \equiv \sum_{\boldsymbol{\ell}} A^\sigma_{\boldsymbol{\ell}}\, e^{i\boldsymbol{\ell}\cdot\boldsymbol{\phi}(t)} \qquad \text{(libration)} \tag{16.158}$$

or

$$q_\sigma(t) = \frac{q^\circ_\sigma\, \phi_\sigma(t)}{2\pi} + \sum_{\boldsymbol{\ell}} B^\sigma_{\boldsymbol{\ell}}\, e^{i\boldsymbol{\ell}\cdot\boldsymbol{\phi}(t)} \qquad \text{(rotation)} \quad . \tag{16.159}$$

For rotations, the variable $q_\sigma(t)$ increased by $\Delta q_\sigma = q^\circ_\sigma$.

I want to distinguish two important concepts. *Complete periodicity*, as we have defined, requires that there exists a time $T(\boldsymbol{J})$ such that $\nu_\sigma(\boldsymbol{J})\,T(\boldsymbol{J}) = 2\pi k_\sigma$, with $k_\sigma \in \mathbb{Z}$ for all $\sigma \in \{1,\ldots,n\}$. The period is then defined to be the smallest nonzero such value of $T(\boldsymbol{J})$. The second condition, *resonance* is weaker and only requires that there exists some $\boldsymbol{\ell} \in \mathbb{Z}^n$ such that $\boldsymbol{\ell}\cdot\boldsymbol{\nu}(\boldsymbol{J}) = 0$[10]. *Resonance is thus equivalent to periodicity on a lower-dimensional sub-torus* $\mathbb{T}^k$ *with* $k < n$. In other words, if the *projected dynamics* $\boldsymbol{\phi}(t)$ onto *any* 2-torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ spanned by coordinates $(\phi_\sigma, \phi_{\bar\sigma})$ is periodic, where $\sigma, \bar\sigma \in \{1,\ldots,n\}$, then the original n-torus is said to be resonant. Complete periodicity is thus a maximal state of resonance, where the motion projected onto *any* subtorus $\mathbb{T}^2$ is periodic.

16.6.2 Liouville-Arnol'd theorem

Another statement of complete integrability is the content of the *Liouville-Arnol'd theorem*, which says the following. Suppose that a time-independent Hamiltonian $H(\boldsymbol{q},\boldsymbol{p})$ has n *first integrals* $I_k(\boldsymbol{q},\boldsymbol{p})$ with $k \in \{1,\ldots,n\}$. This means that (see eqn. 16.24)

$$0 = \frac{d}{dt} I_k(\boldsymbol{q},\boldsymbol{p}) = \sum_{\sigma=1}^n \left(\frac{\partial I_k}{\partial q_\sigma}\, \dot q_\sigma + \frac{\partial I_k}{\partial p_\sigma}\, \dot p_\sigma \right) = \{ I_k \,,\, H \} \quad . \tag{16.160}$$

If the $\{I_k\}$ are *independent functions*, meaning that the phase space gradients $\{\boldsymbol{\nabla} I_k\}$ constitute a set of n linearly independent vectors at almost every point $(\boldsymbol{q},\boldsymbol{p}) \in \mathcal{M}$ in phase space, and the different first integrals *commute* with respect to the Poisson bracket, *i.e.* $\{I_k, I_l\} = 0$, then the set of Hamilton's equations of motion is completely solvable[11]. The theorem establishes that[12]

(i) The space $\mathcal{M}_{\boldsymbol{I}} = \big\{ (\boldsymbol{q},\boldsymbol{p}) \in \mathcal{M} \,\big|\, I_k(\boldsymbol{p},\boldsymbol{q}) = C_k \ \forall\, k \in \{1,\ldots,n\} \big\}$ is diffeomorphic to an n-torus $T^n \equiv \mathbb{S}^1 \times \mathbb{S}^1 \times \cdots \mathbb{S}^1$, on which one can introduce action-angle variables $(\boldsymbol{\phi},\boldsymbol{J})$ on patches, where $\boldsymbol{\phi}$ are coordinates on $\mathcal{M}_{\boldsymbol{I}}$ and $\boldsymbol{J}$ are the first integrals, *i.e.* $J_k(I_1,\ldots,I_n) = I_k$.

(ii) The equations of motion are $\dot I_k = 0$ and $\dot\phi_k = \omega_k(I_1,\ldots,I_n)$.

Note that the Liouville-Arnol'd theorem does *not* require that H that $\widetilde H(\boldsymbol{I}) = \sum_k \widetilde H^{(k)}(I_k)$, which would be a trivial state of affairs.

[10] Clearly if $\boldsymbol{\ell}\cdot\boldsymbol{\nu}(\boldsymbol{J}) = 0$, then replacing $\boldsymbol{\ell}$ by $p\boldsymbol{\ell}$ for any $p \in \mathbb{Z}$ also satisfies the resonance condition.

[11] Two first integrals I_k and I_l whose Poisson bracket $\{I_k, I_l\} = 0$ vanishes are said to be *in involution*.

[12] See chapter 1 of http://www.damtp.cam.ac.uk/user/md327/ISlecture_notes_2012.pdf for a proof.

16.7 Adiabatic Invariants

16.7.1 Slow perturbations

Adiabatic perturbations are slow, smooth, time-dependent perturbations to a dynamical system. A classic example: a pendulum with a slowly varying length $l(t)$. Suppose $\lambda(t)$ is the adiabatic parameter. We write $H = H\big(\boldsymbol{q}, \boldsymbol{p}\,; \lambda(t)\big)$. All explicit time-dependence to H comes through $\lambda(t)$. Typically, a dimensionless parameter ϵ may be associated with the perturbation:

$$\epsilon = \frac{1}{\omega_0} \left| \frac{d \log \lambda}{dt} \right| \quad , \tag{16.161}$$

where ω_0 is the natural frequency of the system when λ is constant. We require $\epsilon \ll 1$ for adiabaticity. In adiabatic processes, the action variables are conserved to a high degree of accuracy. These are the *adiabatic invariants*. For example, for the harmonic oscillator, the action is $J = E/\nu$. While E and ν may vary considerably during the adiabatic process, their ratio is very nearly fixed. As a consequence, assuming small oscillations,

$$E = \nu J = \tfrac{1}{2} mgl\, \theta_0^2 \quad \Rightarrow \quad \theta_0(l) \approx \frac{2J}{m\sqrt{g}\, l^{3/2}} \quad , \tag{16.162}$$

where $\theta_0(l)$ is the amplitude of the oscillation. Adiabatic invariance of J thus entails $\theta_0(\ell) \propto l^{-3/2}$.

Consider an $n = 1$ system, and suppose that for fixed λ the Hamiltonian is transformed to action-angle variables via the generator $S(q, J; \lambda)$. Now let $\lambda = \lambda(t)$. $S\big(q, J; \lambda(t)\big)$ is still a type-II generating function of a canonical transformation. The resulting transformed Hamiltonian is

$$\widetilde{H}(\phi, J, t) = H(J; \lambda) + \frac{\partial S}{\partial \lambda} \frac{d\lambda}{dt} \quad , \tag{16.163}$$

where

$$H(J; \lambda) = H\big(q(\phi, J; \lambda), p(\phi, J; \lambda); \lambda\big) \tag{16.164}$$

is a function only of J and the instantaneous value of λ. Hamilton's equations are now

$$\begin{aligned} \dot{\phi} &= +\frac{\partial \widetilde{H}}{\partial J} = \nu(J; \lambda) + \frac{\partial^2 S}{\partial \lambda\, \partial J} \frac{d\lambda}{dt} \\ \dot{J} &= -\frac{\partial \widetilde{H}}{\partial \phi} = -\frac{\partial^2 S}{\partial \lambda\, \partial \phi} \frac{d\lambda}{dt} \quad , \end{aligned} \tag{16.165}$$

where $\nu(J; \lambda) \equiv \partial H(J; \lambda)/\partial J$, and where $S(\phi, J; \lambda) = S\big(q(\phi, J; \lambda), J; \lambda\big)$ The second of eqns. 16.165 may then be Fourier decomposed as

$$\dot{J} = -i\dot{\lambda} \sum_{m=-\infty}^{\infty} m\, \frac{\partial S_m(J; \lambda)}{\partial \lambda}\, e^{im\phi} \quad , \tag{16.166}$$

hence

$$\Delta J = J(t=+\infty) - J(t=-\infty) = \sum_{m=-\infty}^{\infty} (-im) \int_{-\infty}^{\infty} dt\, \frac{\partial S_m(J;\lambda)}{\partial \lambda}\, \frac{d\lambda}{dt}\, e^{im\phi} \quad . \tag{16.167}$$

Since $\dot{\lambda}$ is small, we have $\phi(t) = \nu t + \beta$, to lowest order. We must therefore evaluate integrals such as

$$\mathcal{I}_m = \int_{-\infty}^{\infty} dt \left\{ \frac{\partial S_m(J;\lambda)}{\partial \lambda}\, \frac{d\lambda}{dt} \right\} e^{im\nu t} \quad . \tag{16.168}$$

The term in curly brackets is a smooth, slowly varying function of t. Call it $f(t)$. We presume $f(t)$ can be analytically continued off the real t axis, and that its closest singularities in the complex t plane lies at $\mathrm{Im}\, t = \pm\tau$, where $|\nu\tau| \gg 1$. In this case $\mathcal{I}_m$ behaves as $\exp(-|m|\nu\tau)$. Consider, for example, the Lorentzian,

$$f(t) = \frac{1}{\pi}\, \frac{\tau}{t^2+\tau^2} \quad \Rightarrow \quad \int_{-\infty}^{\infty} dt\, f(t)\, e^{im\nu t} = e^{-|m\nu\tau|} \quad , \tag{16.169}$$

which is exponentially small in the dimensionless product $|\nu\tau|$. Because of this, only $m = \pm 1$ need be considered. What this tells us is that the change ΔJ may be made arbitrarily small by a sufficiently slowly varying $\lambda(t)$.

16.7.2 Example: mechanical mirror

Consider a two-dimensional version of a mechanical mirror, depicted in fig. 16.4. A particle bounces between two curves, $y = \pm D(x)$, where $|D'(x)| \ll 1$. The bounce time given by $\tau_{\mathrm{b}\perp} = 2D/v_y$. We assume $\tau \ll L/v_x$, where $v_{x,y}$ are the components of the particle's velocity, and L is the total length of the system. There are, therefore, many bounces, which means the particle gets to sample the curvature in $D(x)$. The adiabatic invariant is the action,

$$J = \frac{1}{2\pi} \int_{-D}^{D} dy\, m\, v_y + \frac{1}{2\pi} \int_{D}^{-D} dy\, m\, (-v_y) = \frac{2}{\pi}\, m v_y\, D(x) \quad . \tag{16.170}$$

Thus,

$$E = \tfrac{1}{2} m \big(v_x^2 + v_y^2\big) = \tfrac{1}{2} m v_x^2 + \frac{\pi^2 J^2}{8 m D^2(x)} \quad , \tag{16.171}$$

or

$$v_x^2 = \frac{2E}{m} - \left(\frac{\pi J}{2 m D(x)}\right)^{\!2} \quad . \tag{16.172}$$

The particle is reflected in the throat of the device at horizontal coordinate x^*, where

$$D(x^*) = \frac{\pi J}{\sqrt{8mE}} \quad . \tag{16.173}$$

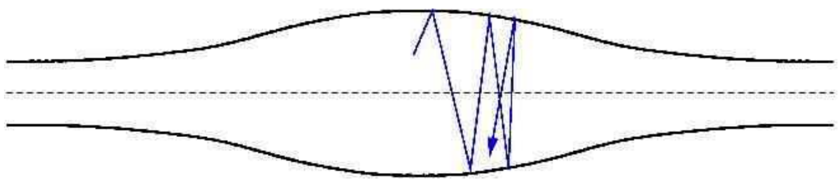


Figure 16.4: A mechanical mirror.

16.7.3 Example: magnetic mirror

Consider a particle of charge e moving in the presence of a uniform magnetic field $\boldsymbol{B} = B\hat{\boldsymbol{z}}$. Recall the basic physics: velocity in the parallel direction v_z is conserved, while in the plane perpendicular to $\boldsymbol{B}$ the particle executes circular 'cyclotron orbits', satisfying

$$\frac{mv_\perp^2}{\rho} = \frac{e}{c}\,v_\perp B \quad \Rightarrow \quad \rho = \frac{mcv_\perp}{eB} \quad , \tag{16.174}$$

where ρ is the radial coordinate in the plane perpendicular to $\boldsymbol{B}$. The period of the orbits is given by $T = 2\pi\rho v_\perp = 2\pi mc/eB$, hence their frequency is the cyclotron frequency $\omega_c = eB/mc$.

Now assume that the magnetic field is spatially dependent. Note that a spatially varying $\boldsymbol{B}$-field cannot be unidirectional:

$$\boldsymbol{\nabla}\cdot\boldsymbol{B} = \boldsymbol{\nabla}_\perp\cdot\boldsymbol{B}_\perp + \frac{\partial B_z}{\partial z} = 0 \quad . \tag{16.175}$$

The non-collinear nature of $\boldsymbol{B}$ results in the *drift* of the cyclotron orbits. Nevertheless, if the field $\boldsymbol{B}$ felt by the particle varies slowly on the time scale $T = 2\pi/\omega_c$, then the system possesses an adiabatic invariant:

$$\begin{aligned} J &= \frac{1}{2\pi}\oint_{\mathcal{C}} \boldsymbol{p}\cdot d\boldsymbol{\ell} = \frac{1}{2\pi}\oint_{\mathcal{C}} \big(m\boldsymbol{v} + \tfrac{e}{c}\boldsymbol{A}\big)\cdot d\boldsymbol{\ell} \\ &= \frac{m}{2\pi}\oint_{\mathcal{C}} \boldsymbol{v}\cdot d\boldsymbol{\ell} + \frac{e}{2\pi c}\oint_{\mathrm{int}(\mathcal{C})} \boldsymbol{B}\cdot\hat{\boldsymbol{n}}\,d\Sigma \quad . \end{aligned} \tag{16.176}$$

The last two terms are of opposite sign, and one has

$$\begin{aligned} J &= -\frac{m}{2\pi}\cdot\frac{\rho e B_z}{mc}\cdot 2\pi\rho + \frac{e}{2\pi c}\cdot B_z\cdot\pi\rho^2 \\ &= -\frac{eB_z\rho^2}{2c} = -\frac{e}{2\pi c}\cdot\Phi_B(\mathcal{C}) = -\frac{m^2v_\perp^2 c}{2eB_z} \quad , \end{aligned} \tag{16.177}$$

where $\Phi_B(\mathcal{C})$ is the magnetic flux enclosed by $\mathcal{C}$. The energy is $E = \frac{1}{2}mv_\perp^2 + \frac{1}{2}mv_z^2$, hence we have

$$v_z = \sqrt{\frac{2}{m}(E - MB)} \quad . \tag{16.178}$$

where

$$M \equiv -\frac{e}{mc}\,J = \frac{e^2}{2\pi mc^2}\,\Phi_B(\mathcal{C}) \tag{16.179}$$

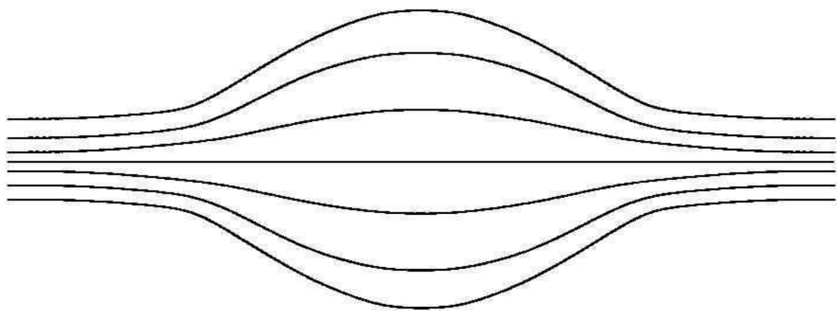


Figure 16.5: $\boldsymbol{B}$ field lines in a magnetic bottle.

is the *magnetic moment*. Note that v_z vanishes when $B = B_{\max} = E/M$. When this limit is reached, the particle turns around. This is a *magnetic mirror*. A pair of magnetic mirrors may be used to confine charged particles in a *magnetic bottle*, depicted in fig. 16.5.

Let $v_{\parallel,0}$, $\boldsymbol{v}_{\perp,0}$, and $B_{\parallel,0}$ be the longitudinal particle velocity, transverse particle velocity, and longitudinal component of the magnetic field, respectively, at the point of injection. Our two conservation laws, for J and E, guarantee $v_\parallel^2(z) + v_\perp^2(z) = v_{\parallel,0}^2 + v_{\perp,0}^2$ and

$$\frac{v_\perp^2(z)}{B_\parallel(z)} = \frac{v_{\perp,0}^2}{B_{\parallel,0}} \quad . \tag{16.180}$$

This leads to reflection at a longitudinal coordinate z^*, where

$$B_\parallel(z^*) = B_{\parallel,0}\left(1 + \frac{v_{\parallel,0}^2}{v_{\perp,0}^2}\right)^{1/2} \quad . \tag{16.181}$$

The physics is quite similar to that of the mechanical mirror.

16.7.4 Resonances

When $n > 1$, we have

$$\begin{aligned} j^\alpha &= -i\dot{\lambda} \sum_{\boldsymbol{m}\in\mathbb{Z}^n} m^\alpha \, \frac{\partial S_{\boldsymbol{m}}(\boldsymbol{J};\lambda)}{\partial\lambda} \, e^{i\boldsymbol{m}\cdot\boldsymbol{\phi}} \\ \Delta J^\alpha &= -i \sum_{\boldsymbol{m}\in\mathbb{Z}^n} m^\alpha \int\limits_{-\infty}^{\infty} \!dt \, \frac{\partial S_{\boldsymbol{m}}(\boldsymbol{J};\lambda)}{\partial\lambda} \, \frac{d\lambda}{dt} \, e^{i\boldsymbol{m}\cdot\boldsymbol{\nu}t} \, e^{i\boldsymbol{m}\cdot\boldsymbol{\beta}} \quad . \end{aligned} \tag{16.182}$$

Therefore, when $\boldsymbol{m}\cdot\boldsymbol{\nu}(J) = 0$ we have a resonance, and the integral grows linearly with its time limits, which is a violation of the adiabatic invariance of J^α.

16.8 Canonical Perturbation Theory

16.8.1 Canonical transformations and perturbation theory

Suppose we have a Hamiltonian

$$H(\boldsymbol{\xi},t) = H_0(\boldsymbol{\xi},t) + \epsilon H_1(\boldsymbol{\xi},t) \quad , \tag{16.183}$$

where ϵ is a small dimensionless parameter. Let's implement a type-II transformation, generated by $S(\boldsymbol{q},\boldsymbol{P},t)$:[13]

$$\widetilde{H}(\boldsymbol{Q},\boldsymbol{P},t) = H(\boldsymbol{q},\boldsymbol{p},t) + \frac{\partial}{\partial t}\, S(\boldsymbol{q},\boldsymbol{P},t) \quad . \tag{16.184}$$

Let's expand everything in powers of ϵ:

$$\begin{aligned} q_\sigma &= Q_\sigma + \epsilon\, q_{1,\sigma} + \epsilon^2\, q_{2,\sigma} + \ldots \\ p_\sigma &= P_\sigma + \epsilon\, p_{1,\sigma} + \epsilon^2\, p_{2,\sigma} + \ldots \\ \widetilde{H} &= \widetilde{H}_0 + \epsilon \widetilde{H}_1 + \epsilon^2 \widetilde{H}_2 + \ldots \\ S &= \underbrace{q_\sigma P_\sigma}_{\substack{\text{identity}\\ \text{transformation}}} + \epsilon S_1 + \epsilon^2 S_2 + \ldots \quad . \end{aligned} \tag{16.185}$$

Then

$$\begin{aligned} Q_\sigma = \frac{\partial S}{\partial P_\sigma} &= q_\sigma + \epsilon\, \frac{\partial S_1}{\partial P_\sigma} + \epsilon^2\, \frac{\partial S_2}{\partial P_\sigma} + \ldots \\ &= Q_\sigma + \bigg(q_{1,\sigma} + \frac{\partial S_1}{\partial P_\sigma}\bigg)\epsilon + \bigg(q_{2,\sigma} + \frac{\partial S_2}{\partial P_\sigma}\bigg)\epsilon^2 + \ldots \end{aligned} \tag{16.186}$$

and

$$\begin{aligned} p_\sigma = \frac{\partial S}{\partial q_\sigma} &= P_\sigma + \epsilon\, \frac{\partial S_1}{\partial q_\sigma} + \epsilon^2\, \frac{\partial S_2}{\partial q_\sigma} + \ldots \\ &= P_\sigma + \epsilon\, p_{1,\sigma} + \epsilon^2\, p_{2,\sigma} + \ldots \quad . \end{aligned} \tag{16.187}$$

We therefore conclude, order by order in ϵ,

$$q_{k,\sigma} = -\frac{\partial S_k}{\partial P_\sigma} \quad , \quad p_{k,\sigma} = +\frac{\partial S_k}{\partial q_\sigma} \quad . \tag{16.188}$$

Now let's expand the Hamiltonian:

$$\begin{aligned} \widetilde{H}(\boldsymbol{Q},\boldsymbol{P},t) &= H_0(\boldsymbol{q},\boldsymbol{p},t) + \epsilon H_1(q,p,t) + \frac{\partial S}{\partial t} \\ &= H_0(\boldsymbol{Q},\boldsymbol{P},t) + \frac{\partial H_0}{\partial Q_\sigma}\,(q_\sigma - Q_\sigma) + \frac{\partial H_0}{\partial P_\sigma}\,(p_\sigma - P_\sigma) + \epsilon H_1(\boldsymbol{Q},\boldsymbol{P},t) + \epsilon\, \frac{\partial}{\partial t}\, S_1(\boldsymbol{Q},\boldsymbol{P},t) + \mathcal{O}(\epsilon^2) \quad . \end{aligned} \tag{16.189}$$

[13] Here $S(\boldsymbol{q},\boldsymbol{P},t)$ is not meant to signify Hamilton's principal function.

Collecting terms, we have

$$\begin{aligned}\widetilde{H}(\boldsymbol{Q},\boldsymbol{P},t) &= H_0(\boldsymbol{Q},\boldsymbol{P},t) + \left(-\frac{\partial H_0}{\partial Q_\sigma}\frac{\partial S_1}{\partial P_\sigma} + \frac{\partial H_0}{\partial P_\sigma}\frac{\partial S_1}{\partial Q_\sigma} + \frac{\partial S_1}{\partial t} + H_1\right)\epsilon + \mathcal{O}(\epsilon^2) \\ &= H_0(\boldsymbol{Q},\boldsymbol{P},t) + \left(H_1 + \big\{S_1, H_0\big\} + \frac{\partial S_1}{\partial t}\right)\epsilon + \mathcal{O}(\epsilon^2) \quad .\end{aligned} \tag{16.190}$$

In the above expression, we evaluate $H_k(q,p,t)$ and $S_k(q,P,t)$ at $q = Q$ and $p = P$ and expand in the differences $q - Q$ and $p - P$. Thus, we have derived the relation

$$\widetilde{H}(\boldsymbol{Q},\boldsymbol{P},t) = \widetilde{H}_0(\boldsymbol{Q},\boldsymbol{P},t) + \epsilon\widetilde{H}_1(\boldsymbol{Q},\boldsymbol{P},t) + \ldots \tag{16.191}$$

with

$$\widetilde{H}_0(\boldsymbol{Q},\boldsymbol{P},t) = H_0(\boldsymbol{Q},\boldsymbol{P},t) \tag{16.192}$$

$$\widetilde{H}_1(\boldsymbol{Q},\boldsymbol{P},t) = H_1 + \big\{S_1, H_0\big\} + \frac{\partial S_1}{\partial t} \quad . \tag{16.193}$$

The problem, though, is this: we have one equation, eqn, 16.193, for the two unknowns $\widetilde{H}_1$ and S_1. Thus, the problem is underdetermined. Of course, we could choose $\widetilde{H}_1 = 0$, for example. But we might just as well demand that $\widetilde{H}_1$ satisfy some other desideratum, such as that $\widetilde{H}_0 + \epsilon\widetilde{H}_1$ be integrable.

Incidentally, this treatment is paralleled by one in quantum mechanics, where a unitary transformation may be implemented to eliminate a perturbation to lowest order in a small parameter. Consider the Schrödinger equation,

$$i\hbar\frac{\partial\psi}{\partial t} = (\mathcal{H}_0 + \epsilon\,\mathcal{H}_1)\,\psi \quad , \tag{16.194}$$

and define χ by $\psi \equiv e^{iS/\hbar}\,\chi$, with

$$S = \epsilon\, S_1 + \epsilon^2\, S_2 + \ldots \quad . \tag{16.195}$$

As before, the transformation $U \equiv \exp(iS/\hbar)$ collapses to the identity in the $\epsilon \to 0$ limit. Now let's write the Schrödinger equation for χ. Expanding in powers of ϵ, one finds

$$i\hbar\frac{\partial\chi}{\partial t} = \mathcal{H}_0\,\chi + \epsilon\left(\mathcal{H}_1 + \frac{1}{i\hbar}\big[S_1, \mathcal{H}_0\big] + \frac{\partial S_1}{\partial t}\right)\chi + \ldots \equiv \widetilde{\mathcal{H}}\,\chi \quad , \tag{16.196}$$

where $[A,B] = AB - BA$ is the commutator. Note the classical-quantum correspondence,

$$\{A,B\} \longleftrightarrow \frac{1}{i\hbar}\,[A,B] \quad . \tag{16.197}$$

Again, what should we choose for S_1? Usually the choice is made to make the $\mathcal{O}(\epsilon)$ term in $\widetilde{\mathcal{H}}$ vanish. But this is not the only possible simplifying choice.

16.8.2 Canonical perturbation theory for $n = 1$ systems

Here and henceforth we shall assume $H(\boldsymbol{q},\boldsymbol{p},t) = H(\boldsymbol{q},\boldsymbol{p})$ is time-independent, and we write the perturbed Hamiltonian as

$$H(\boldsymbol{q},\boldsymbol{p}) = H_0(\boldsymbol{q},\boldsymbol{p}) + \epsilon H_1(\boldsymbol{q},\boldsymbol{p}) \quad . \tag{16.198}$$

Let (ϕ_0, J_0) be the action-angle variables for H_0. Then

$$\widetilde{H}_0(\phi_0, J_0) = H_0\big(q(\phi_0,J_0),p(\phi_0,J_0)\big) = \widetilde{H}_0(J_0) \quad . \tag{16.199}$$

We define

$$\widetilde{H}_1(\phi_0, J_0) = H_1\big(q(\phi_0,J_0),p(\phi_0,J_0)\big) \quad . \tag{16.200}$$

We assume that $\widetilde{H} = \widetilde{H}_0 + \epsilon\widetilde{H}_1$ is integrable[14], so it, too, possesses action-angle variables, which we denote by (ϕ, J)[15]. Thus, there must be a canonical transformation taking $(\phi_0, J_0) \to (\phi, J)$, with

$$\widetilde{H}\big(\phi_0(\phi,J), J_0(\phi,J)\big) \equiv E(J) \quad . \tag{16.201}$$

We solve via a type-II canonical transformation:

$$S(\phi_0, J) = \phi_0 J + \epsilon\, S_1(\phi_0,J) + \epsilon^2\, S_2(\phi_0,J) + \ldots \quad , \tag{16.202}$$

where $\phi_0 J$ is the identity transformation. Then

$$\begin{aligned} J_0 &= \frac{\partial S}{\partial \phi_0} = J + \epsilon\,\frac{\partial S_1}{\partial \phi_0} + \epsilon^2\,\frac{\partial S_2}{\partial \phi_0} + \ldots \\ \phi &= \frac{\partial S}{\partial J} = \phi_0 + \epsilon\,\frac{\partial S_1}{\partial J} + \epsilon^2\,\frac{\partial S_2}{\partial J} + \ldots \quad , \end{aligned} \tag{16.203}$$

and

$$\begin{aligned} E(J) &= E_0(J) + \epsilon\, E_1(J) + \epsilon^2\, E_2(J) + \ldots \\ &= \widetilde{H}_0(\phi_0, J_0) + \epsilon \widetilde{H}_1(\phi_0, J_0) \quad . \end{aligned} \tag{16.204}$$

How is it that the second line terminates after order ϵ while the first line contains terms of order ϵ^k for all $k \geq 0$? The answer is that when we express (ϕ_0, J_0) in terms of (ϕ, J), the canonical transformation itself involve terms to all orders in ϵ, as we see from eqn. 16.203. In general, when a *nonlinear* system is perturbed, the response will include expressions to all orders in the perturbation.

We now expand $\widetilde{H}(\phi_0, J_0)$ in powers of $J_0 - J$, keeping in mind that $\widetilde{H}_0(\phi_0, J_0) = \widetilde{H}_0(J_0)$:

$$\begin{aligned} \widetilde{H}(\phi_0, J_0) &= \widetilde{H}_0(J_0) + \epsilon\,\widetilde{H}_1(\phi_0, J_0) \\ &= \widetilde{H}_0(J) + \frac{\partial \widetilde{H}_0}{\partial J}\,(J_0 - J) + \frac{1}{2}\,\frac{\partial^2 \widetilde{H}_0}{\partial J^2}\,(J_0-J)^2 + \epsilon\,\widetilde{H}_1(\phi_0, J) + \epsilon\,\frac{\partial \widetilde{H}_1}{\partial J}\bigg|_{\phi_0}(J_0 - J) + \ldots \quad . \end{aligned} \tag{16.205}$$

[14] This is always true, in fact, for $n = 1$.

[15] We assume the motion is bounded, so action-angle variables may be used.

Collecting terms,

$$\widetilde{H}(\phi_0, J_0) = \widetilde{H}_0(J) + \left(\widetilde{H}_1 + \frac{\partial \widetilde{H}_0}{\partial J}\,\frac{\partial S_1}{\partial \phi_0}\right)\epsilon + \left(\frac{\partial \widetilde{H}_0}{\partial J}\,\frac{\partial S_2}{\partial \phi_0} + \frac{1}{2}\,\frac{\partial^2 \widetilde{H}_0}{\partial J^2}\left(\frac{\partial S_1}{\partial \phi_0}\right)^2 + \frac{\partial \widetilde{H}_1}{\partial J}\,\frac{\partial S_1}{\partial \phi_0}\right)\epsilon^2 + \ldots \tag{16.206}$$

where all terms on the RHS are expressed as functions of ϕ_0 *and* J. Equating terms, then,

$$\begin{aligned} E_0(J) &= \widetilde{H}_0(J) \\ E_1(J) &= \widetilde{H}_1(\phi_0, J) + \frac{\partial \widetilde{H}_0}{\partial J}\,\frac{\partial S_1}{\partial \phi_0} \\ E_2(J) &= \frac{\partial \widetilde{H}_0}{\partial J}\,\frac{\partial S_2}{\partial \phi_0} + \frac{1}{2}\,\frac{\partial^2 \widetilde{H}_0}{\partial J^2}\left(\frac{\partial S_1}{\partial \phi_0}\right)^2 + \frac{\partial \widetilde{H}_1}{\partial J}\,\frac{\partial S_1}{\partial \phi_0} \quad . \end{aligned} \tag{16.207}$$

How, one might ask, can we be sure that the LHS of each equation in the above hierarchy depends only on J when each RHS seems to depend on ϕ_0 as well? The answer is that we use the freedom to choose each S_k to make this so. We demand each RHS be independent of ϕ_0, which means it must be equal to its average, $\langle\,\mathrm{RHS}(\phi_0)\,\rangle$, where

$$\big\langle f\big(\phi_0\big)\big\rangle = \int\limits_0^{2\pi} \frac{d\phi_0}{2\pi}\, f\big(\phi_0\big) \quad . \tag{16.208}$$

The average is performed *at fixed* J and *not* at fixed J_0. In this regard, we note that holding J constant and increasing ϕ_0 by 2π also returns us to the same starting point. Therefore, we are able to write

$$S_k(\phi_0, J) = \sum_{\ell=-\infty}^{\infty} S_{k,\ell}(J)\, e^{i\ell\phi_0} \tag{16.209}$$

for each $k > 0$, in which case

$$\left\langle \frac{\partial S_k}{\partial \phi_0} \right\rangle = \frac{1}{2\pi}\big[S_k(2\pi, J) - S_k(0, J)\big] = 0 \quad . \tag{16.210}$$

Let's see how this averaging works to the first two orders of the hierarchy. Since $\widetilde{H}_0(J)$ is independent of ϕ_0 and since $\partial S_1/\partial\phi_0$ is periodic, we have

$$E_1(J) = \big\langle \widetilde{H}_1(\phi_0, J)\big\rangle + \frac{\partial \widetilde{H}_0}{\partial J}\,\overbrace{\left\langle \frac{\partial S_1}{\partial \phi_0} \right\rangle}^{\text{this vanishes!}} \tag{16.211}$$

and hence S_1 must satisfy

$$\frac{\partial S_1}{\partial \phi_0} = \frac{\big\langle \widetilde{H}_1 \big\rangle - \widetilde{H}_1}{\nu_0(J)} \quad , \tag{16.212}$$

where $\nu_0(J) = \partial \widetilde{H}_0/\partial J$. Clearly the RHS of eqn. 16.212 has zero average, and must be a periodic function of ϕ_0. The solution is $S_1 = S_1(\phi_0, J) + f(J)$, where $f(J)$ is an arbitrary function of J. However,

$f(J)$ affects only the difference $\phi - \phi_0$, changing it by a constant value $f'(J)$. So there is no harm in taking $f(J) = 0$.

Next, let's go to second order in ϵ. We have

$$E_2(J) = \nu_0(J) \overbrace{\left\langle \frac{\partial S_2}{\partial \phi_0} \right\rangle}^{\text{this vanishes!}} + \frac{1}{2} \frac{\partial \nu_0}{\partial J} \left\langle \left(\frac{\partial S_1}{\partial \phi_0} \right)^2 \right\rangle + \left\langle \frac{\partial \widetilde{H}_1}{\partial J} \frac{\partial S_1}{\partial \phi_0} \right\rangle \quad . \tag{16.213}$$

The equation for S_2 is then

$$\begin{aligned} \frac{\partial S_2}{\partial \phi_0} = \frac{1}{\nu_0^2(J)} \Bigg\{ \left\langle \frac{\partial \widetilde{H}_1}{\partial J} \right\rangle \langle \widetilde{H}_1 \rangle - \left\langle \frac{\partial \widetilde{H}_1}{\partial J} \widetilde{H}_1 \right\rangle - \frac{\partial \widetilde{H}_1}{\partial J} \langle \widetilde{H}_1 \rangle + \frac{\partial \widetilde{H}_1}{\partial J} \widetilde{H}_1 \\ + \frac{1}{2} \frac{\partial \log \nu_0}{\partial J} \left(\langle \widetilde{H}_1^2 \rangle - 2 \langle \widetilde{H}_1 \rangle^2 + 2 \langle \widetilde{H}_1 \rangle \widetilde{H}_1 - \widetilde{H}_1^2 \right) \Bigg\} \quad . \end{aligned} \tag{16.214}$$

The expansion for the energy $E(J)$ is then

$$E(J) = \widetilde{H}_0(J) + \epsilon \langle \widetilde{H}_1 \rangle + \frac{\epsilon^2}{\nu_0(J)} \left\{ \left\langle \frac{\partial \widetilde{H}_1}{\partial J} \right\rangle \langle \widetilde{H}_1 \rangle - \left\langle \frac{\partial \widetilde{H}_1}{\partial J} \widetilde{H}_1 \right\rangle + \frac{1}{2} \frac{\partial \log \nu_0}{\partial J} \left(\langle \widetilde{H}_1^2 \rangle - \langle \widetilde{H}_1 \rangle^2 \right) \right\} + \mathcal{O}(\epsilon^3) \quad . \tag{16.215}$$

Note that we don't need S to find $E(J)$! The perturbed frequencies are $\nu(J) = \partial E / \partial J$. Sometimes the frequencies are all that is desired. However, we can of course obtain the full motion of the system via the succession of canonical transformations,

$$(\phi, J) \longrightarrow (\phi_0, J_0) \longrightarrow (q, p) \quad . \tag{16.216}$$

16.8.3 Example : quartic oscillator

Consider a harmonic oscillator with a quartic nonlinearity[16]. The Hamiltonian is

$$H(q,p) = \overbrace{\frac{p^2}{2m} + \tfrac{1}{2} m \nu_0^2 \, q^2}^{H_0} + \tfrac{1}{4} \epsilon \alpha q^4 \quad . \tag{16.217}$$

The action-angle variables for the harmonic oscillator Hamiltonian H_0 are

$$\phi_0 = \tan^{-1}\!\big(m \nu_0 q / p \big) \qquad , \qquad J_0 = \frac{p^2}{2m\nu_0} + \tfrac{1}{2} m \nu_0 q^2 \tag{16.218}$$

hence

$$q = \sqrt{\frac{2J_0}{m\nu_0}} \, \cos \phi_0 \qquad , \qquad p = \sqrt{2 J_0 m \nu_0} \, \sin \phi_0 \quad , \tag{16.219}$$

[16] In §15.11.5 below, we discuss the case of a cubic nonlinearity.

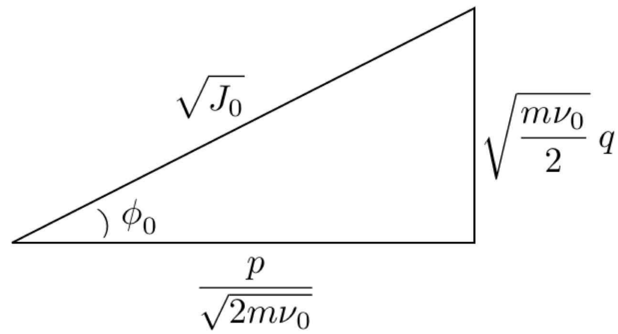


Figure 16.6: Action-angle variables for the harmonic oscillator.

as depicted in fig. 16.6. Note $H_0 = \nu_0\, J_0$. For the full Hamiltonian, we have

$$\begin{aligned}\widetilde{H}(\phi_0, J_0) &= \nu_0 J_0 + \tfrac{1}{4}\epsilon\alpha\bigg(\sqrt{\frac{2J_0}{m\nu_0}}\,\sin\phi_0\bigg)^4 \\ &= \nu_0 J_0 + \frac{\epsilon\alpha}{m^2\nu_0^2}\, J_0^2\,\sin^4\!\phi_0 \equiv H_0(\phi_0, J_0) + \epsilon\widetilde{H}_1(\phi_0, J_0) \quad .\end{aligned} \tag{16.220}$$

We may now evaluate

$$E_1(J) = \big\langle \widetilde{H}_1(\phi_0, J)\big\rangle = \frac{\alpha J^2}{m^2\nu_0^2}\int\limits_0^{2\pi}\frac{d\phi_0}{2\pi}\,\sin^4\!\phi_0 = \frac{3\alpha J^2}{8m^2\nu_0^2} \quad . \tag{16.221}$$

The frequency, to order ϵ, is

$$\nu(J) = \nu_0 + \frac{3\epsilon\alpha J}{4m^2\nu_0^2} \quad . \tag{16.222}$$

Now to lowest order in ϵ, we may replace J by $J_0 = \frac{1}{2}m\nu_0 A^2$, where A is the amplitude of the q motion:

$$\nu(A) = \nu_0 + \frac{3\epsilon\alpha A^2}{8m\nu_0} \quad . \tag{16.223}$$

This result agrees with that obtained via heavier lifting, using the Poincaré-Lindstedt method.

Next, let's evaluate the canonical transformation $(\phi_0, J_0) \to (\phi, J)$. We have

$$\begin{aligned}\nu_0\,\frac{\partial S_1}{\partial\phi_0} &= \frac{\alpha J^2}{m^2\nu_0^2}\Big(\tfrac{3}{8} - \sin^4\!\phi_0\Big) \qquad \Rightarrow \\ S(\phi_0, J) &= \phi_0\, J + \frac{\epsilon\alpha J^2}{8m^2\nu_0^3}\,\big(3 + 2\sin^2\!\phi_0\big)\sin\phi_0\cos\phi_0 + \mathcal{O}(\epsilon^2) \quad .\end{aligned} \tag{16.224}$$

Thus,

$$\begin{aligned}\phi &= \frac{\partial S}{\partial J} = \phi_0 + \frac{\epsilon\alpha J}{4m^2\nu_0^3}\,\big(3 + 2\sin^2\!\phi_0\big)\sin\phi_0\cos\phi_0 + \mathcal{O}(\epsilon^2) \\ J_0 &= \frac{\partial S}{\partial\phi_0} = J + \frac{\epsilon\alpha J^2}{8m^2\nu_0^3}\,\big(4\cos 2\phi_0 - \cos 4\phi_0\big) + \mathcal{O}(\epsilon^2) \quad .\end{aligned} \tag{16.225}$$

Again, to lowest order, we may replace J by J_0 in the above, whence

$$\begin{aligned} J &= J_0 - \frac{\epsilon\alpha J_0^2}{8m^2\nu_0^3}\big(4\cos 2\phi_0 - \cos 4\phi_0\big) + \mathcal{O}(\epsilon^2) \\ \phi &= \phi_0 + \frac{\epsilon\alpha J_0}{8m^2\nu_0^3}\,\big(3 + 2\sin^2\!\phi_0\big)\,\sin 2\phi_0 + \mathcal{O}(\epsilon^2) \quad . \end{aligned} \tag{16.226}$$

Writing $q = (2J_0/m\nu_0)^{1/2}\sin\phi_0$ and $p = (2m\nu_0 J_0)^{1/2}\cos\phi_0$, one can substitute the above relations, replacing (ϕ_0, J_0) with (ϕ, J) in the $\mathcal{O}(\epsilon)$ terms on the RHS of each equation, to obtain (q,p) in terms of (ϕ, J), valid to $\mathcal{O}(\epsilon)$.

16.8.4 $n > 1$ systems: degeneracies and resonances

Generalizing the procedure we derived for $n = 1$, we obtain

$$\begin{aligned} J_0^\alpha &= \frac{\partial S}{\partial \phi_0^\alpha} = J^\alpha + \epsilon\,\frac{\partial S_1}{\partial \phi_0^\alpha} + \epsilon^2\,\frac{\partial S_2}{\partial \phi_0^\alpha} + \ldots \\ \phi^\alpha &= \frac{\partial S}{\partial J^\alpha} = \phi_0^\alpha + \epsilon\,\frac{\partial S_1}{\partial J^\alpha} + \epsilon^2\,\frac{\partial S_2}{\partial J^\alpha} + \ldots \end{aligned} \tag{16.227}$$

and

$$\begin{aligned} E_0(\boldsymbol{J}) &= \widetilde{H}_0(\boldsymbol{J}) \\ E_1(\boldsymbol{J}) &= \widetilde{H}_1 + \nu_0^\alpha\,\frac{\partial S_1}{\partial \phi_0^\alpha} \\ E_2(\boldsymbol{J}) &= \nu_0^\alpha\,\frac{\partial S_2}{\partial \phi_0^\alpha} + \frac{1}{2}\,\frac{\partial \nu_0^\alpha}{\partial J^\beta}\,\frac{\partial S_1}{\partial \phi_0^\alpha}\,\frac{\partial S_1}{\partial \phi_0^\beta} + \frac{\partial \widetilde{H}_1}{\partial J^\alpha}\,\frac{\partial S_1}{\partial \phi_0^\alpha} \quad , \end{aligned} \tag{16.228}$$

where $\nu_0^\alpha(\boldsymbol{J}) = \partial \widetilde{H}_0(\boldsymbol{J})/\partial J^\alpha$. We now implement the averaging procedure, with

$$\big\langle f(\phi_0^1,\ldots,\phi_0^n, J^1,\ldots,J^n)\big\rangle = \int\limits_0^{2\pi}\!\frac{d\phi_0^1}{2\pi}\,\cdots \int\limits_0^{2\pi}\!\frac{d\phi_0^n}{2\pi}\; f\big(\phi_0^1,\ldots,\phi_0^n, J^1,\ldots,J^n\big) \quad . \tag{16.229}$$

The equation for S_1 is

$$\nu_0^\alpha\,\frac{\partial S_1}{\partial \phi_0^\alpha} = \big\langle \widetilde{H}_1(\boldsymbol{\phi}_0, \boldsymbol{J})\big\rangle - \widetilde{H}_1(\boldsymbol{\phi}_0, \boldsymbol{J}) \equiv -{\sum_{\boldsymbol{\ell}}}' \hat{V}_{\boldsymbol{\ell}}(\boldsymbol{J})\,e^{i\boldsymbol{\ell}\cdot\boldsymbol{\phi}_0} \quad , \tag{16.230}$$

where $\boldsymbol{\ell} = \{\ell^1, \ell^2, \ldots, \ell^n\}$, with each ℓ^σ an integer, and with $\boldsymbol{\ell} \neq 0$. The solution is

$$S_1(\boldsymbol{\phi}_0, \boldsymbol{J}) = i\,{\sum_{l}}' \frac{\hat{V}_{\boldsymbol{\ell}}(\boldsymbol{J})}{\boldsymbol{\ell}\cdot\boldsymbol{\nu}_0(\boldsymbol{J})}\,e^{i\boldsymbol{\ell}\cdot\boldsymbol{\phi}_0} \quad . \tag{16.231}$$

where $\boldsymbol{\ell}\cdot\boldsymbol{\nu}_0 = \sum_{\alpha=1}^n l^\alpha \nu_0^\alpha$. When two or more of the frequencies $\nu_0^\alpha(\boldsymbol{J})$ are *commensurate*, there exists a set of integers $\boldsymbol{\ell}$ such that the denominator of $D(l)$ vanishes. But even when the frequencies are not rationally related, one can approximate the ratios $\nu_0^\alpha/\nu_0^{\alpha'}$ by rational numbers, and for large enough $|\boldsymbol{\ell}|$ the denominator can become arbitrarily small.

16.8.5 Nonlinear oscillator with two degrees of freedom

As an example of how to implement canonical perturbation theory for $n > 1$, consider the nonlinear oscillator system,

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + \tfrac{1}{2}m\omega_1^2\, q_1^2 + \tfrac{1}{2}m\omega_2^2\, q_2^2 + \tfrac{1}{4}\epsilon\, b\,\omega_1^2\,\omega_2^2\, q_1^2\, q_2^2 \quad . \tag{16.232}$$

Writing $H = H_0 + \epsilon H_1$, we have, in terms of the action-angle variables $(\phi_0^{(1,2)}, J_0^{(1,2)})$,

$$\widetilde{H}_0(\boldsymbol{J}_0) = \omega_1 J_0^{(1)} + \omega_2 J_0^{(2)} \tag{16.233}$$

with $q_k = (2J_0^k/m\omega_k)^{1/2}\sin\phi_0^k$ and $p_k = (2m\omega_k J_0^k)^{1/2}\cos\phi_0^k$ with $k \in \{1,2\}$. We then have

$$\widetilde{H}_1(\boldsymbol{\phi}_0, \boldsymbol{J}) = b\,\omega_1\,\omega_2\, J^{(1)} J^{(2)}\ \sin^2\!\phi_0^{(1)}\ \sin^2\!\phi_0^{(2)} \quad . \tag{16.234}$$

We therefore have $E(\boldsymbol{J}) = E_0(\boldsymbol{J}) + \epsilon E_1(\boldsymbol{J})$ with $E_0(\boldsymbol{J}) = H_0(\boldsymbol{J}) = \omega_1 J^{(1)} + \omega_2 J^{(2)}$ and

$$E_1(\boldsymbol{J}) = \big\langle \widetilde{H}_1(\boldsymbol{\phi}_0, \boldsymbol{J})\big\rangle = \tfrac{1}{4} b\,\omega_1\omega_2\, J^{(1)} J^{(2)} \quad . \tag{16.235}$$

Next, we work out the generator $S_1(\boldsymbol{\phi}_0, \boldsymbol{J})$ from eqn. 16.230:

$$\begin{aligned}\big\langle \widetilde{H}_1(\boldsymbol{\phi}_0, \boldsymbol{J})\big\rangle - \widetilde{H}_1(\boldsymbol{\phi}_0, \boldsymbol{J}) &= b\,\omega_1\omega_2\, J^{(1)} J^{(2)} \left\{\tfrac{1}{4} - \sin^2\!\phi_0^{(1)}\ \sin^2\!\phi_0^{(2)}\right\} \\ &= b\,\omega_1\omega_2\, J^{(1)} J^{(2)} \Big\{ -\tfrac{1}{2}\cos\!\big(2\phi_0^{(1)} + 2\phi_0^{(2)}\big) - \tfrac{1}{2}\cos\!\big(2\phi_0^{(1)} - 2\phi_0^{(2)}\big) \\ &\qquad\qquad + \cos 2\phi_0^{(1)} + \cos 2\phi_0^{(2)} \Big\} \quad ,\end{aligned} \tag{16.236}$$

and therefore, from eqn. 16.231,

$$S_1(\boldsymbol{\phi}_0, \boldsymbol{J}) = \tfrac{1}{4} b\,\omega_1\omega_2\, J^{(1)} J^{(2)} \left\{ -\frac{\sin(2\phi_0^{(1)} + 2\phi_0^{(2)})}{\omega_1 + \omega_2} - \frac{\sin(2\phi_0^{(1)} - 2\phi_0^{(2)})}{\omega_1 - \omega_2} + \frac{2\sin 2\phi_0^{(1)}}{\omega_1} + \frac{2\sin 2\phi_0^{(2)}}{\omega_2} \right\} \quad . \tag{16.237}$$

We see that there is a vanishing denominator if $\omega_1 = \omega_2$.

16.8.6 Periodic time-dependent perturbations

Periodic time-dependent perturbations present a similar problem. Consider the system

$$H(\boldsymbol{\phi}, \boldsymbol{J}, t) = H_0(\boldsymbol{J}) + \epsilon\, V(\boldsymbol{\phi}, \boldsymbol{J}, t) \quad , \tag{16.238}$$

where $V(t+T) = V(t)$. This means we may write

$$\begin{aligned} V(\boldsymbol{\phi}, \boldsymbol{J}, t) &= \sum_k \hat{V}_k(\boldsymbol{\phi}, \boldsymbol{J})\, e^{-ik\Omega t} \\ &= \sum_{\boldsymbol{k}} \sum_{\boldsymbol{\ell}} \hat{V}_{k,\boldsymbol{\ell}}(\boldsymbol{J})\, e^{i\boldsymbol{\ell}\cdot\boldsymbol{\phi}}\, e^{-ik\Omega t} \quad .\end{aligned} \tag{16.239}$$

by Fourier transforming from both time and angle variables; here $\Omega = 2\pi/T$. Note that $V(\boldsymbol{\phi}, \boldsymbol{J}, t)$ is real if $V^*_{k,\boldsymbol{\ell}} = \hat{V}_{-\boldsymbol{k},-l}$. The equations of motion are

$$\begin{aligned} \dot{J}^\alpha &= -\frac{\partial H}{\partial \phi^\alpha} = -i\epsilon \sum_{k,\boldsymbol{\ell}} l^\alpha \, \hat{V}_{k,\boldsymbol{\ell}}(\boldsymbol{J}) \, e^{i\boldsymbol{\ell}\cdot\boldsymbol{\phi}} \, e^{-ik\Omega t} \\ \dot{\phi}^\alpha &= +\frac{\partial H}{\partial J^\alpha} = \nu_0^\alpha(\boldsymbol{J}) + \epsilon \sum_{k,\boldsymbol{\ell}} \frac{\partial \hat{V}_{k,\boldsymbol{\ell}}(\boldsymbol{J})}{\partial J^\alpha} \, e^{i\boldsymbol{\ell}\cdot\boldsymbol{\phi}} \, e^{-ik\Omega t} \quad . \end{aligned} \tag{16.240}$$

We now expand in ϵ:

$$\begin{aligned} \phi^\alpha &= \phi_0^\alpha + \epsilon \, \phi_1^\alpha + \epsilon^2 \, \phi_2^\alpha + \ldots \\ J^\alpha &= J_0^\alpha + \epsilon \, J_1^\alpha + \epsilon^2 \, J_2^\alpha + \ldots \quad . \end{aligned} \tag{16.241}$$

To order ϵ^0, we have $J^\alpha = J_0^\alpha$ and $\phi_0^\alpha = \nu_0^\alpha \, t + \beta_0^\alpha$. To order ϵ^1,

$$\dot{J}_1^\alpha = -i \sum_{k,l} l^\alpha \, \hat{V}_{k,\boldsymbol{\ell}}(\boldsymbol{J}_0) \, e^{i(\boldsymbol{\ell}\cdot\boldsymbol{\nu}_0 - k\Omega)\, t} \, e^{i\boldsymbol{\ell}\cdot\boldsymbol{\beta}_0} \tag{16.242}$$

and

$$\dot{\phi}_1^\alpha = \frac{\partial \nu_0^\alpha}{\partial J^\beta} \, J_1^\beta + \sum_{k,\boldsymbol{\ell}} \frac{\partial \hat{V}_{k,\boldsymbol{\ell}}(\boldsymbol{J})}{\partial J^\alpha} \, e^{i(\boldsymbol{\ell}\cdot\boldsymbol{\nu}_0 - k\Omega)\, t} \, e^{i\boldsymbol{\ell}\cdot\boldsymbol{\beta}_0} \quad , \tag{16.243}$$

where derivatives are evaluated at $\boldsymbol{J} = \boldsymbol{J}_0$. The solution is:

$$\begin{aligned} J_1^\alpha &= \sum_{k,\boldsymbol{\ell}} \frac{l^\alpha \, \hat{V}_{k,\boldsymbol{\ell}}(\boldsymbol{J}_0)}{k\Omega - \boldsymbol{\ell} \cdot \boldsymbol{\nu}_0} \, e^{i(\boldsymbol{\ell}\cdot\boldsymbol{\nu}_0 - k\Omega)\, t} \, e^{i\boldsymbol{\ell}\cdot\boldsymbol{\beta}_0} \\ \phi_1^\alpha &= \left\{ \frac{\partial \nu_0^\alpha}{\partial J_0^\beta} \, \frac{l^\beta \, \hat{V}_{k,\boldsymbol{\ell}}(\boldsymbol{J}_0)}{(k\Omega - \boldsymbol{\ell}\cdot\boldsymbol{\nu}_0)^2} + \frac{\partial \hat{V}_{k,\boldsymbol{\ell}}(\boldsymbol{J}_0)}{\partial J_0^\alpha} \, \frac{1}{k\Omega - \boldsymbol{\ell}\cdot\boldsymbol{\nu}_0} \right\} e^{i(\boldsymbol{\ell}\cdot\boldsymbol{\nu}_0 - k\Omega)\, t} \, e^{i\boldsymbol{\ell}\cdot\boldsymbol{\beta}_0} \quad . \end{aligned} \tag{16.244}$$

When the resonance condition $k\Omega = \boldsymbol{\ell}\cdot\boldsymbol{\nu}_0(\boldsymbol{J}_0)$ is satisfied, the denominators vanish, and the perturbation theory breaks down.

16.8.7 Particle-wave Interaction

Consider a particle of charge e moving in the presence of a constant magnetic field $\boldsymbol{B} = B\hat{\boldsymbol{z}}$ and a space- and time-varying electric field $\boldsymbol{E}(\boldsymbol{x}, t)$, described by the Hamiltonian

$$H = \frac{1}{2m}\big(\boldsymbol{p} - \tfrac{e}{c}\boldsymbol{A}\big)^2 + \epsilon \, e\hat{V}_0 \, \cos(k_\perp x + k_z z - \omega t) \quad , \tag{16.245}$$

where ϵ is a dimensionless expansion parameter. This is an $n = 3$ system with canonical pairs (x, p_x), (y, p_y), and (z, p_z).

Working in the gauge $\boldsymbol{A} = Bx\hat{\boldsymbol{y}}$, we transform the first two pairs (x, y, p_x, p_y) to convenient variables (Q, P, ϕ, J), explicitly discussed in §16.11.2 below), such that

$$H = \omega_{\rm c} J + \frac{p_z^2}{2m} + \epsilon\, e\hat{V}_0\, \cos\!\bigg(k_z z + \frac{k_\perp P}{m\omega_{\rm c}} + k_\perp \sqrt{\frac{2J}{m\omega_{\rm c}}}\, \sin\phi - \omega t\bigg) \quad . \tag{16.246}$$

Here,

$$x = \frac{P}{m\omega_{\rm c}} + \sqrt{\frac{2J}{m\omega_{\rm c}}}\, \sin\phi \qquad , \qquad y = Q + \sqrt{\frac{2J}{m\omega_{\rm c}}}\, \cos\phi \quad , \tag{16.247}$$

with $\omega_{\rm c} = eB/mc$, the cyclotron frequency. Here, (Q, P) describe the *guiding center* degrees of freedom, and (ϕ, J) the *cyclotron* degrees of freedom.

We now make a mixed canonical transformation, generated by

$$F = \phi \tilde{J} + \Big(k_z z + \frac{k_\perp P}{m\omega_{\rm c}} - \omega t\Big)\tilde{K} - P\tilde{Q} \quad , \tag{16.248}$$

where the new sets of conjugate variables are $\big\{(\tilde{\phi}, \tilde{J})\,,\, (\tilde{Q}, \tilde{P})\,,\, (\tilde{\psi}, \tilde{K})\big\}$. We then have

$$\tilde{\phi} = \frac{\partial F}{\partial \tilde{J}} = \phi \qquad\qquad J = \frac{\partial F}{\partial \phi} = \tilde{J} \tag{16.249}$$

$$Q = -\frac{\partial F}{\partial P} = -\frac{k_\perp \tilde{K}}{m\omega_{\rm c}} + \tilde{Q} \qquad\qquad \tilde{P} = -\frac{\partial F}{\partial \tilde{Q}} = P \tag{16.250}$$

$$\tilde{\psi} = \frac{\partial F}{\partial \tilde{K}} = k_z z + \frac{k_\perp P}{m\omega_{\rm c}} - \omega t \qquad\qquad p_z = \frac{\partial F}{\partial z} = k_z \tilde{K} \quad . \tag{16.251}$$

The transformed Hamiltonian is

$$\begin{aligned} H' &= H + \frac{\partial F}{\partial t} \\ &= \omega_{\rm c} \tilde{J} + \frac{k_z^2}{2m}\, \tilde{K}^2 - \omega \tilde{K} + \epsilon\, e\hat{V}_0\, \cos\bigg(\tilde{\psi} + k_\perp \sqrt{\frac{2\tilde{J}}{m\omega_{\rm c}}}\, \sin\tilde{\phi}\bigg) \quad . \end{aligned} \tag{16.252}$$

Note the guiding center pair $(\tilde{Q}, \tilde{P})$ doesn't appear in the transformed Hamiltonian H'.

We now drop the tildes and the prime on H and write $H = H_0 + \epsilon\, H_1$, with

$$\begin{aligned} H_0 &= \omega_{\rm c} J + \frac{k_z^2}{2m}\, K^2 - \omega K \\ H_1 &= e\hat{V}_0\, \cos\bigg(\psi + k_\perp \sqrt{\frac{2J}{m\omega_{\rm c}}}\, \sin\phi\bigg) \quad . \end{aligned} \tag{16.253}$$

When $\epsilon = 0$, the frequencies associated with the ϕ and ψ motion are

$$\omega_\phi^0 = \frac{\partial H_0}{\partial J} = \omega_{\rm c} \qquad , \qquad \omega_\psi^0 = \frac{\partial H_0}{\partial K} = \frac{k_z^2 K}{m} - \omega = k_z v_z - \omega \quad , \tag{16.254}$$

where $v_z = p_z/m$ is the z-component of the particle's velocity.

We are now in position to implement the time-independent canonical perturbation theory approach. We invoke a generator

$$S(\phi,\mathcal{J},\psi,\mathcal{K}) = \phi\mathcal{J} + \psi\mathcal{K} + \epsilon\, S_1(\phi,\mathcal{J},\psi,\mathcal{K}) + \epsilon^2 S_2(\phi,\mathcal{J},\psi,\mathcal{K}) + \ldots \tag{16.255}$$

to transform from (ϕ, J, ψ, K) to $(\varPhi, \mathcal{J}, \varPsi, \mathcal{K})$. We must now solve eqn. 16.230:

$$\omega^0_\phi\, \frac{\partial S_1}{\partial\phi} + \omega^0_\psi\, \frac{\partial S_1}{\partial\psi} = \langle\, H_1 \,\rangle - H_1 \quad . \tag{16.256}$$

That is,

$$\begin{aligned}\omega_{\rm c}\, \frac{\partial S_1}{\partial\phi} + \left(\frac{k_z^2\,\mathcal{K}}{m} - \omega\right)\frac{\partial S_1}{\partial\psi} &= -eA_0 \cos\!\left(\psi + k_\perp\sqrt{\frac{2\mathcal{J}}{m\omega_{\rm c}}}\,\sin\phi\right)\\ &= -eA_0 \sum_{n=-\infty}^\infty J_n\!\left(k_\perp\sqrt{\frac{2\mathcal{J}}{m\omega_{\rm c}}}\right)\,\cos(\psi + n\phi) \quad ,\end{aligned}$$

where we have used the result

$$e^{iz\sin\theta} = \sum_{n=-\infty}^\infty J_n(z)\, e^{in\theta} \quad . \tag{16.257}$$

The solution for S_1 is then

$$S_1(\phi,\mathcal{J},\psi,\mathcal{K}) = \sum_n \frac{e\hat{V}_0}{\omega - n\omega_{\rm c} - k_z^2\,\mathcal{K}/m}\, J_n\!\left(k_\perp\sqrt{\frac{2\mathcal{J}}{m\omega_{\rm c}}}\right)\,\sin(\psi + n\phi) \quad . \tag{16.258}$$

We then have new action variables $\mathcal{J}$ and $\mathcal{K}$, where

$$\begin{aligned} J &= \mathcal{J} + \epsilon\,\frac{\partial S_1}{\partial\phi} + \mathcal{O}(\epsilon^2)\\ K &= \mathcal{K} + \epsilon\,\frac{\partial S_1}{\partial\psi} + \mathcal{O}(\epsilon^2) \quad .\end{aligned} \tag{16.259}$$

Defining the dimensionless variable

$$\lambda \equiv k_\perp\sqrt{\frac{2J}{m\omega_{\rm c}}} \quad , \tag{16.260}$$

we obtain the result[17]

$$\left(\frac{m\omega_{\rm c}^2}{2e\hat{V}_0 k_\perp^2}\right)\varLambda^2 = \left(\frac{m\omega_{\rm c}^2}{2e\hat{V}_0 k_\perp^2}\right)\lambda^2 - \epsilon\sum_n \frac{nJ_n(\varLambda)\cos(\psi + n\phi)}{\omega/\omega_{\rm c} - n - k_z^2\,\mathcal{K}/m\omega_{\rm c}} + \mathcal{O}(\epsilon^2) \quad , \tag{16.261}$$

where $\varLambda \equiv k_\perp (2\mathcal{J}/m\omega_{\rm c})^{1/2}$.

We see that resonances occur whenever

$$\frac{\omega}{\omega_{\rm c}} - \frac{k_z^2\,\mathcal{K}}{m\omega_{\rm c}} = n \quad , \tag{16.262}$$

[17] Note that the argument of J_n in eqn. 16.261 is λ and not $\varLambda$. This arises because we are computing the new action $\mathcal{J}$ in terms of the old variables (ϕ, J) and (ψ, K).

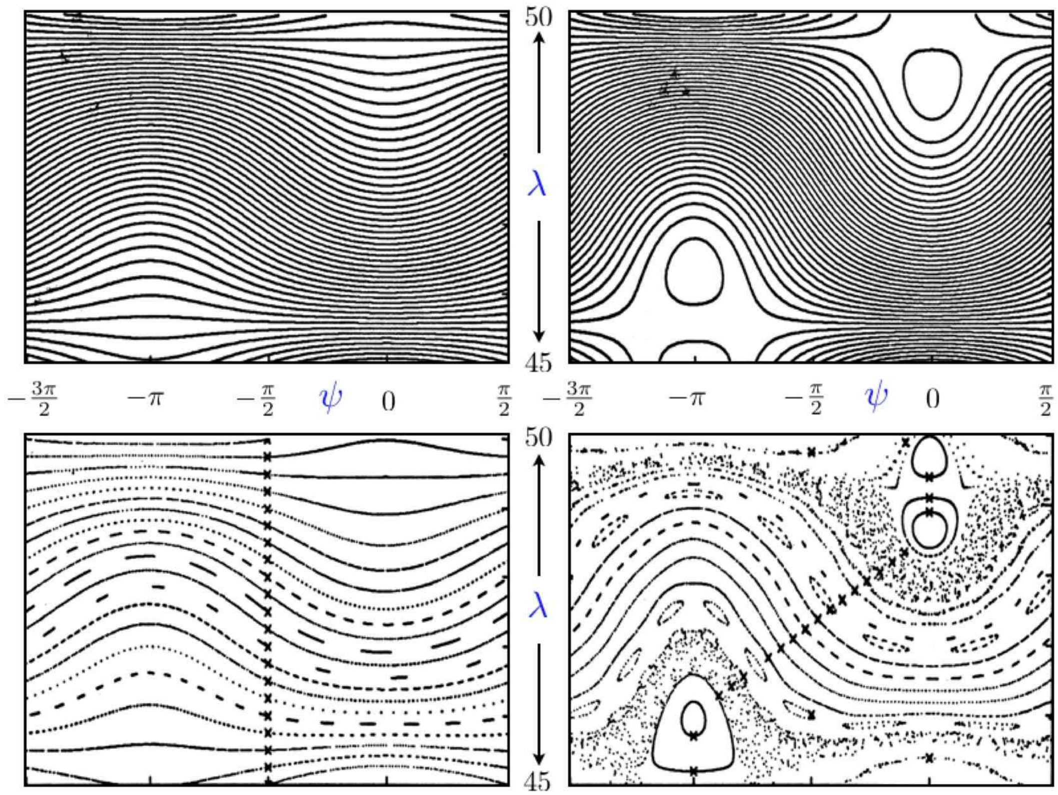


Figure 16.7: Plot of Λ *versus* ψ for $\phi = 0$ (Poincaré section) for $\omega = 30.11\,\omega_c$ Top panels are nonresonant invariant curves calculated to first order. Bottom panels are exact numerical dynamics, with x symbols marking the initial conditions. Left panels: weak amplitude (no trapping). Right panels: stronger amplitude (shows trapping). From Lichtenberg and Lieberman (1983).

for any integer n. Let us consider the case $k_z = 0$, in which the resonance condition is $\omega = n\omega_c$. We then have

$$\frac{\Lambda^2}{2\alpha} = \frac{\lambda^2}{2\alpha} - \epsilon \sum_n \frac{n\, J_n(\Lambda)\cos(\psi + n\phi)}{\omega/\omega_c - n} \quad , \tag{16.263}$$

where

$$\alpha = \frac{E_0}{B} \cdot \frac{ck_\perp}{\omega_c} \tag{16.264}$$

is a dimensionless measure of the strength of the perturbation, with $E_0 \equiv k_\perp \hat{V}_0$. In fig. 16.7 we plot the level sets for the RHS of the above equation $\lambda(\psi)$ for $\phi = 0$, for two different values of the dimensionless amplitude α, for $\omega/\omega_c = 30.11$ (*i.e.* off resonance). Thus, when the amplitude is small, the level sets are far from a primary resonance, and the analytical and numerical results are very similar (left panels). When the amplitude is larger, resonances may occur which are not found in the lowest order perturbation treatment. However, as is apparent from the plots, the gross features of the phase diagram are reproduced by perturbation theory. What is missing is the existence of 'chaotic islands' which initially emerge in the vicinity of the trapping regions.

16.9 Removal of Resonances in Perturbation Theory

We follow the treatment in chapter 3 of Lichtenberg and Lieberman.

16.9.1 The case of $n = \frac{3}{2}$ degrees of freedom

Consider the time-dependent Hamiltonian,

$$H(\phi, J, t) = H_0(J) + \epsilon\, V(\phi, J, t) \quad , \tag{16.265}$$

where $V(\phi, J, t) = V(\phi + 2\pi, J, t) = V(\phi, J, t + T)$ is periodic in time as well as in the angle variable ϕ. We may express the perturbation as a double Fourier sum,

$$V(\phi, J, t) = \sum_{k,\ell} \hat{V}_{k,\ell}(J)\, e^{ik\phi}\, e^{-i\ell\Omega t} \quad , \tag{16.266}$$

where $\Omega = 2\pi/T$. Hamilton's equations of motion are

$$\begin{aligned}
\dot{J} &= -\frac{\partial H}{\partial \phi} = -i\epsilon \sum_{k,\ell} k\, \hat{V}_{k,l}(J)\, e^{ik\phi}\, e^{-i\ell\Omega t} \\
\dot{\phi} &= +\frac{\partial H}{\partial J} = \omega_0(J) + \epsilon \sum_{k,\ell} \frac{\partial \hat{V}_{k,l}(J)}{\partial J}\, e^{ik\phi}\, e^{-i\ell\Omega t} \quad ,
\end{aligned} \tag{16.267}$$

where $\omega_0(J) \equiv \partial H_0/\partial J$.[18] The resonance condition is obtained by inserting the zeroth order solution $\phi(t) = \omega_0(J)\, t + \beta$ into the perturbation terms. When $k\, \omega_0(J) = l\Omega$, the perturbation results in a secular forcing, leading to a linear time increase and a failure of the solution at sufficiently large values of t.

To resolve this crisis, we focus on a particular resonance, where $(k, \ell) = \pm(k_0, \ell_0)$. The resonance condition $k_0\, \omega(J) = \ell_0 \Omega$ fixes the value of J. There may be several solutions, and we focus on a particular one, which we write as $J = J_0$. There is still an infinite set of possible (k, l) values, because if (k_0, ℓ_0) yields a solution for $J = J_0$, so does $(k, \ell) = (pk_0, p\ell_0)$ for $p \in \mathbb{Z}$. However, the amplitude of the Fourier components $\hat{V}_{pk_0, p\ell_0}$ is, in general, a rapidly decreasing function of $|p|$, provided $V(J, \phi, t)$ is smooth in ϕ and t. Furthermore, $p = 0$ always yields a solution. Therefore, we will assume k_0 and ℓ_0 are relatively prime and take $p = 0$ and $p = \pm 1$. This simplifies the system in eqn. 16.267 to

$$\begin{aligned}
\dot{J} &= 2\epsilon\, k_0 \hat{V}_1(J)\, \sin(k_0\phi - \ell_0 \Omega t + \delta) \\
\dot{\phi} &= \omega_0(J) + \epsilon\, \frac{\partial \hat{V}_0(J)}{\partial J} + 2\epsilon\, \frac{\partial \hat{V}_1(J)}{\partial J}\, \cos(k_0\phi - \ell_0\Omega t + \delta) \quad ,
\end{aligned} \tag{16.268}$$

where $\hat{V}_{0,0}(J) \equiv \hat{V}_0(J)$ and $\hat{V}_{k_0,\ell_0}(J) = V^*_{-k_0,-\ell_0}(J) \equiv \hat{V}_1(J)\, e^{i\delta}$, where $\hat{V}_0(J)$ and $\hat{V}_1(J)$ are both real. We then expand, writing

$$J = J_0 + \Delta J \quad , \quad \psi = k_0\phi - \ell_0\Omega t + \delta + \pi \quad , \tag{16.269}$$

[18] In this section we write $\partial H_0/\partial J = \omega_0(J)$ rather than $\nu_0(J)$ in order to obviate any confusion between the frequency ν_0 and the potential $\hat{V}_1$ and its various Fourier components.

resulting in the system

$$\begin{aligned}\frac{d\Delta J}{dt} &= -2\epsilon\, k_0 \hat{V}_1(J_0)\sin\psi \\ \frac{d\psi}{dt} &= k_0\,\omega_0'(J_0)\,\Delta J + \epsilon\, k_0 \hat{V}_0'(J_0) - 2\epsilon\, k_0 \hat{V}_1'(J_0)\cos\psi \quad ,\end{aligned} \tag{16.270}$$

which follow from the Hamiltonian

$$K(\Delta J,\psi) = \tfrac{1}{2}k_0\,\omega_0'(J_0)\,(\Delta J)^2 + \epsilon k_0 \hat{V}_0'(J_0)\,\Delta J - 2\epsilon k_0 \hat{V}_1(J_0+\Delta J)\cos\psi \quad , \tag{16.271}$$

with $d\psi/dt = \partial K/\partial(\Delta J)$ and $d(\Delta J)/dt = -\partial K/\partial\psi$. Concerning the last term, we can drop the ΔJ term in the argument of $\hat{V}_1$, leaving $\hat{V}_1(J_0)$, because it will yield a term of second order in smallness in the equation of motion for ψ. The remaining term in K linear in ΔJ can then be removed by a shift of $\Delta J \to \Delta J - \epsilon\,\hat{V}_0'(J_0)/\omega_0'(J_0)$. This is tantamount to shifting the value of J_0, which we could have done at the outset by absorbing the term $\epsilon \hat{V}_0(J)$ into $H_0(J)$, and defining $\omega(J) \equiv \omega_0(J) + \epsilon\,\partial\hat{V}_0/\partial J$. We are left with a simple pendulum, with

$$\ddot{\psi} + \gamma^2 \sin\psi = 0 \tag{16.272}$$

with $\gamma = \sqrt{2\epsilon\, k_0^2\,\omega'(J_0)\,\hat{V}_1(J_0)}$. In Fig. 16.8, we plot the level sets of the function

$$\check{K}(\Delta J,\psi) \equiv \tfrac{1}{2}k_0\,\omega_0'(J_0)\,(\Delta J)^2 + \epsilon k_0 \hat{V}_0'(J_0)\,\Delta J - 2\epsilon k_0 \hat{V}_1(J_0)\cos\psi \tag{16.273}$$

in the *rotating* (q,p) plane, *i.e.* in the $(\check{q},\check{p})$ plane, where

$$\check{q} \propto (J_0+\Delta J)^{1/2}\cos(k_0\,\check{\phi}) \qquad , \qquad \check{p} \propto (J_0+\Delta J)^{1/2}\sin(k_0\,\check{\phi}) \quad , \tag{16.274}$$

where $\check{\phi} = \phi - l_0\,\Omega t/k_0$.

What do we conclude? The original 1-torus (*i.e.* circle) with $J = J_0$ and $\phi(t) = \omega_0(J_0)\,t + \beta$ is destroyed. It and its neighboring tori are replaced, in the c ase $k_0 = 1$, by the separatrix in the left panel of fig. 16.8 and the neighboring librational and rotational phase curves. The structure for $k_0 = 6$ is shown in the right panel. The amplitude of the separatrix is $(\Delta J)_{\rm max} = \sqrt{8\epsilon\hat{V}_1(J_0)/\omega_0'(J_0)}$. In order for the approximations leading to this structure to be justified, we need $(\Delta J)_{\rm max} \ll J_0$ and $\Delta\omega \ll \omega_0$, where $\Delta\omega = \gamma$. These conditions may be written as

$$\epsilon \ll \alpha \ll \frac{1}{\epsilon} \quad , \tag{16.275}$$

where $\alpha = d\log\omega_0/d\log J\big|_{J_0} = J_0\,|\omega_0'|/\omega_0$.

16.9.2 $n = 2$ systems

Consider now the time-independent Hamiltonian $H = H_0(\boldsymbol{J}) + \epsilon H_1(\boldsymbol{\phi},\boldsymbol{J})$ with $n = 2$ degrees of freedom, *i.e.* $\boldsymbol{J} = (J_1, J_2)$ and $\boldsymbol{\phi} = (\phi_1,\phi_2)$. We Fourier expand

$$H_1(\boldsymbol{\phi},\boldsymbol{J}) = \sum_{\boldsymbol{\ell}} \hat{V}_{\boldsymbol{\ell}}(\boldsymbol{J})\, e^{i\boldsymbol{\ell}\cdot\boldsymbol{\phi}} \quad , \tag{16.276}$$

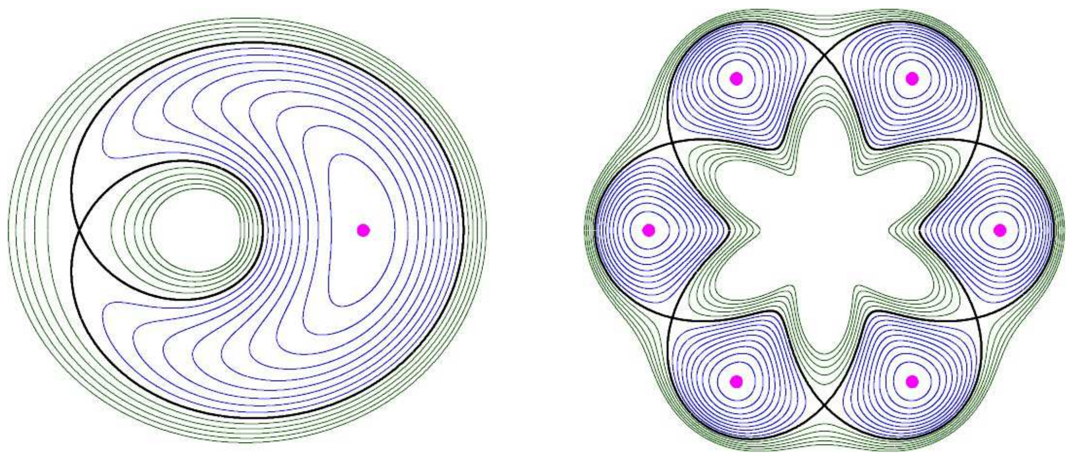


Figure 16.8: Librations, separatrices, and rotations for $k_0 = 1$ (left) and $k_0 = 6$ (right), plotted in the (q,p) phase plane. Elliptical fixed points are shown in magenta. Hyperbolic fixed points are located at the self-intersection of the separatrices (black curves).

with $\boldsymbol{\ell} = (\ell_1, \ell_2)$ and $\hat{V}_{-\boldsymbol{\ell}}(\boldsymbol{J}) = V^*_{\boldsymbol{\ell}}(\boldsymbol{J})$ since $\hat{V}_{\boldsymbol{\ell}}(\boldsymbol{J})$ are the Fourier components of a real function. A resonance exists between the frequencies $\omega_{1,2} = \partial H_0/\partial J_{1,2}$ if there exist nonzero integers r and s such that $r\omega_1 = s\omega_2$. We eliminate the resonance in two steps. First, we employ a canonical transformation $(\boldsymbol{\phi}, \boldsymbol{J}) \to (\boldsymbol{\varphi}, \boldsymbol{\mathcal{J}})$, generated by

$$F_2(\boldsymbol{\phi}, \boldsymbol{\mathcal{J}}) = (r\phi_1 - s\phi_2)\mathcal{J}_1 + \phi_2 \mathcal{J}_2 \quad . \tag{16.277}$$

We then have

$$J_1 = \frac{\partial F_2}{\partial \phi_1} = r\mathcal{J}_1 \qquad \varphi_1 = \frac{\partial F_2}{\partial \mathcal{J}_1} = r\phi_1 - s\phi_2 \tag{16.278}$$

$$J_2 = \frac{\partial F_2}{\partial \phi_2} = \mathcal{J}_2 - s\mathcal{J}_1 \qquad \varphi_2 = \frac{\partial F_2}{\partial \mathcal{J}_2} = \phi_2 \quad . \tag{16.279}$$

This transforms us to a rotating frame in which $\dot{\varphi}_1 = r\dot{\phi}_1 - s\dot{\phi}_2$ is slowly varying, while $\dot{\varphi}_2 = \dot{\phi}_2 \approx \omega_2$. Note that we could have chosen $F_2 = \phi_1 \mathcal{J}_1 + (r\phi_1 - s\phi_2)\mathcal{J}_2$, in which case we'd have obtained $\varphi_1 = \phi_1$ with an unperturbed natural frequency of ω_1 and $\varphi_2 = r\phi_1 - s\phi_2$ slowly varying, *i.e.* with an unperturbed natural frequency of zero. Which transformation are we to choose? The answer is that we want to end up averaging over the *slower* of $\omega_{1,2}$, so the generator in eqn. 16.277 is appropriate if $\omega_1 > \omega_2$. The reason has to do with what happens when there are higher order resonances to be removed – a state of affairs we shall discuss in the following section.

At this stage, our transformed Hamiltonian is

$$\begin{aligned}\widetilde{H}(\boldsymbol{\varphi}, \boldsymbol{\mathcal{J}}) &= H_0\big(\boldsymbol{J}(\boldsymbol{\mathcal{J}})\big) + \epsilon H_1\big(\boldsymbol{\phi}(\boldsymbol{\varphi}), \boldsymbol{J}(\boldsymbol{\mathcal{J}})\big) \\ &\equiv \widetilde{H}_0(\boldsymbol{\mathcal{J}}) + \epsilon \sum_{\boldsymbol{\ell}} \hat{\widetilde{V}}_{\boldsymbol{\ell}}(\boldsymbol{\mathcal{J}}) \, \exp\left[\frac{i\ell_1}{r}\,\varphi_1 + i\left(\frac{s\,\ell_1}{r} + \ell_2\right)\varphi_2\right] \quad ,\end{aligned} \tag{16.280}$$

where $\widetilde{H}(\boldsymbol{\mathcal{J}}) \equiv H_0\big(\boldsymbol{J}(\boldsymbol{\mathcal{J}})\big)$ and $\hat{\tilde{V}}_{\boldsymbol{\ell}}(\boldsymbol{\mathcal{J}}) \equiv \hat{\tilde{V}}_{\boldsymbol{\ell}}\big(\boldsymbol{J}(\boldsymbol{\mathcal{J}})\big)$. Note that $\phi_1 = \varphi_1/r + s\,\varphi_2/r$. We now average over the angle φ_2, which requires $s\ell_1 + r\ell_2 = 0$. Thus, $\ell_1 = pr$ and $\ell_2 = -ps$ for some $p \in \mathbb{Z}$, and

$$\langle H_1 \rangle = \sum_p \hat{\tilde{V}}_{pr,-ps}(\boldsymbol{\mathcal{J}})\, e^{-ip\varphi_1} \quad . \tag{16.281}$$

The averaging is valid close to the resonance, where $|\dot{\varphi}_2| \gg |\dot{\varphi}_1|$. We are now left with the Hamiltonian

$$\mathcal{H}(\varphi_1, \boldsymbol{\mathcal{J}}) = \widetilde{H}_0(\boldsymbol{\mathcal{J}}) + \epsilon \sum_p \hat{\tilde{V}}_{pr,-ps}(\boldsymbol{\mathcal{J}})\, e^{-ip\varphi_1} \quad . \tag{16.282}$$

Here, $\mathcal{J}_2$ is to be regarded as a parameter which itself has no dynamics: $\dot{\mathcal{J}}_2 = 0$. Note $\mathcal{J}_2 = (s/r)J_1 + J_2$ is the new invariant.

At this point, φ_2 has been averaged out, $\mathcal{J}_2$ is a constant, and only the $(\varphi_1, \mathcal{J}_1)$ variables are dynamical. A stationary point for these dynamics, satisfying $\partial\mathcal{H}/\partial\mathcal{J}_1 = \partial\mathcal{H}/\partial\varphi_1 = 0$ corresponds to a periodic solution to the original perturbed Hamiltonian, since we are now in a rotating frame. Since the Fourier amplitudes $\hat{\tilde{V}}_{-pr,ps}(\boldsymbol{\mathcal{J}})$ generally decrease rapidly with increasing $|p|$, we make the approximation of restricting to $p = 0$ and $p = \pm 1$. Thus,

$$\mathcal{H}(\varphi_1, \boldsymbol{\mathcal{J}}) \approx \widetilde{H}_0(\boldsymbol{\mathcal{J}}) + \epsilon\, \hat{\tilde{V}}_{0,0}(\boldsymbol{\mathcal{J}}) + 2\epsilon\, \hat{\tilde{V}}_{r,-s}(\boldsymbol{\mathcal{J}}) \cos\varphi_1 \quad , \tag{16.283}$$

where we have absorbed any phase in the Fourier amplitude $\hat{\tilde{V}}_{r,-s}(\boldsymbol{\mathcal{J}})$ into a shift of φ_1, and subsequently take $\hat{\tilde{V}}_{r,-s}(\boldsymbol{\mathcal{J}})$ to be real. The fixed points $\big(\varphi_1^{(0)}, \mathcal{J}_1^{(0)}\big)$ of the $(\varphi_1, \mathcal{J}_1)$ dynamics satisfy

$$\begin{aligned} 0 &= \frac{\partial \widetilde{H}_0}{\partial \mathcal{J}_1} + \epsilon\, \frac{\partial \hat{\tilde{V}}_{0,0}}{\partial \mathcal{J}_1} + 2\epsilon\, \frac{\partial \hat{\tilde{V}}_{r,-s}}{\partial \mathcal{J}_1}\, \cos\varphi_1 \\ 0 &= 2\epsilon\, \hat{\tilde{V}}_{r,-s}\, \sin\varphi_1 \quad . \end{aligned} \tag{16.284}$$

Thus, $\varphi_1 = 0$ or π at the fixed points. Note that

$$\frac{\partial \widetilde{H}_0}{\partial \mathcal{J}_1} = \frac{\partial H_0}{\partial J_1}\,\frac{\partial J_1}{\partial \mathcal{J}_1} + \frac{\partial H_0}{\partial J_2}\,\frac{\partial J_2}{\partial \mathcal{J}_1} = r\omega_1 - s\omega_2 = 0 \quad , \tag{16.285}$$

and therefore fixed points occur for solutions $\mathcal{J}_1^{(0)}$ to

$$\frac{\partial \hat{\tilde{V}}_{0,0}}{\partial \mathcal{J}_1} \pm 2\, \frac{\partial \hat{\tilde{V}}_{r,-s}}{\partial \mathcal{J}_1} = 0 \quad , \tag{16.286}$$

where the upper sign corresponds to $\varphi_1^{(0)} = 0$ and the lower sign to $\varphi_1^{(0)} = \pi$. We now consider two cases.

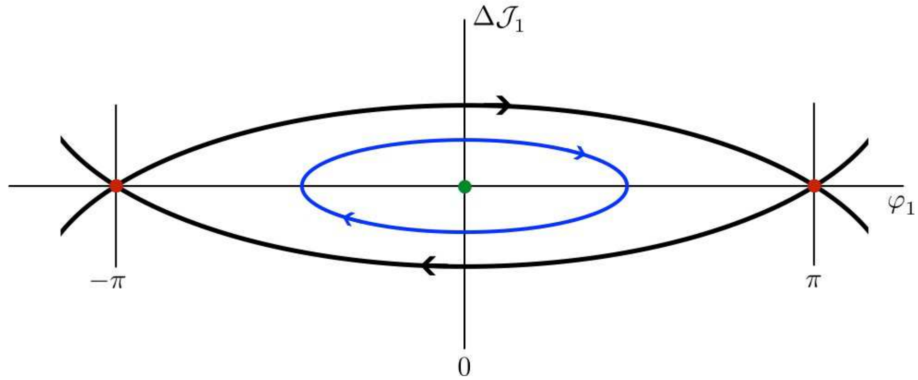


Figure 16.9: Motion in the vicinity of a resonance, showing elliptical fixed point in green, hyperbolic fixed point in red, and separatrix in black.

(i) accidental degeneracy

In the case of accidental degeneracy, the resonance condition $r\omega_1 = s\omega_2$ is satisfied only for particular values of (J_1, J_2), *i.e.* on a set $J_2 = J_2(J_1)$. This corresponds to the case where $H_0(J_1, J_2)$ is a generic function of its two arguments. According to eqn. 16.283, excursions of $\mathcal{J}_1$ relative to its value $\mathcal{J}_1^{(0)}$ at the fixed points are on the order of $\epsilon\hat{\tilde{V}}_{r,-s}$, while excursions of φ_1 are $\mathcal{O}(1)$. We may then expand

$$\widetilde{H}_0(\mathcal{J}_1, \mathcal{J}_2) = \widetilde{H}_0\big(\mathcal{J}_1^{(0)}, \mathcal{J}_2\big) + \frac{\partial \widetilde{H}_0}{\partial \mathcal{J}_1}\,\Delta\mathcal{J}_1 + \frac{1}{2}\frac{\partial^2 \widetilde{H}_0}{\partial \mathcal{J}_1^2}\,(\Delta\mathcal{J}_1)^2 + \ldots \quad , \tag{16.287}$$

where the derivatives are evaluated at $\mathcal{J}_1 = \mathcal{J}_1^{(0)}$. Thus, we arrive at what is often called the *standard Hamiltonian*,

$$\mathcal{H}(\varphi_1, \Delta\mathcal{J}_1) = \tfrac{1}{2}G\,(\Delta\mathcal{J}_1)^2 - F\cos\varphi_1 \quad , \tag{16.288}$$

with

$$G(\mathcal{J}_2) = \frac{\partial^2 \widetilde{H}_0}{\partial \mathcal{J}_1^2}\bigg|_{\mathcal{J}_1^{(0)}} \qquad , \qquad F(\mathcal{J}_2) = -2\epsilon\,\hat{\tilde{V}}_{r,-s}\big(\mathcal{J}_1^{(0)}, \mathcal{J}_2\big) \quad . \tag{16.289}$$

Thus, the motion in the vicinity of every resonance is like that of a pendulum, meaning libration, separatrix, and rotation in the phase plane. F is the amplitude of the first Fourier mode of the perturbation (*i.e.* $|p| = 1$), and G the 'nonlinearity parameter'. For $FG > 0$ the elliptic fixed point (EFP) is at $\varphi_1 = 0$ and the hyperbolic fixed point (HFP) at $\varphi_1 = \pi$. For $FG < 0$, the locations are switched. The frequency of libration about the EFP is given by $\nu_1 = \sqrt{FG} = \mathcal{O}\Big(\sqrt{\epsilon\hat{\tilde{V}}_{r,-s}}\Big)$. The frequency decreases to zero as the separatrix is approached. The maximum excursion along the separatrix is $(\Delta\mathcal{J}_1)_{\rm max} = 2\sqrt{F/G}$ which is also $\mathcal{O}\Big(\sqrt{\epsilon\hat{\tilde{V}}_{r,-s}}\Big)$. The ratio of semiminor to semimajor axis lengths for motion in the vicinity of the EFP is

$$\frac{(\Delta\mathcal{J}_1)_{\rm max}}{(\Delta\varphi_1)_{\rm max}} = \sqrt{\frac{F}{G}} = \mathcal{O}\big(\epsilon^{1/2}\big) \quad . \tag{16.290}$$

(ii) intrinsic degeneracy

In this case, $H_0(J_1, J_2)$ is a function of only the combination $sJ_1 + rJ_2 = r\mathcal{J}_2$, so

$$\mathcal{H}(\varphi_1, \boldsymbol{\mathcal{J}}) = \widetilde{H}_0(\mathcal{J}_2) + \epsilon\,\hat{\tilde{V}}_{0,0}(\boldsymbol{\mathcal{J}}) + 2\epsilon\,\hat{\tilde{V}}_{r,-s}(\boldsymbol{\mathcal{J}})\cos\varphi_1 \quad . \tag{16.291}$$

In this case excursions of $\mathcal{J}_1$ and φ_1 are both $\mathcal{O}\big(\epsilon\,\hat{\tilde{V}}_{\bullet,\bullet}\big)$, and we are not licensed to expand in $\Delta\mathcal{J}_1$. However, in the vicinity of an EFP, we may expand, both in $\Delta\mathcal{J}_1$ and $\Delta\varphi_1$, resulting in

$$\mathcal{H} = \tfrac{1}{2}G\,(\Delta\mathcal{J}_1)^2 + \tfrac{1}{2}F\,(\Delta\varphi_1)^2 \quad , \tag{16.292}$$

where

$$G(\mathcal{J}_2) = \left[\frac{\partial^2\widetilde{H}_0}{\partial\mathcal{J}_1^2} + \epsilon\,\frac{\partial^2\hat{\tilde{V}}_{0,0}}{\partial\mathcal{J}_1^2} + 2\epsilon\,\frac{\partial^2\hat{\tilde{V}}_{r,-s}}{\partial\mathcal{J}_1^2}\right]_{(\mathcal{J}_1^{(0)},\,\mathcal{J}_2)} \quad , \quad F(\mathcal{J}_2) = -2\epsilon\,\hat{\tilde{V}}_{r,-s}\big(\mathcal{J}_1^{(0)}, \mathcal{J}_2\big) \quad . \tag{16.293}$$

For the case of intrinsic degeneracy, the first term in brackets on the RHS of the equation for $G(\mathcal{J}_2)$ vanishes, since $\widetilde{H}_0$ is a function only of $\mathcal{J}_2$. Hence F and G are both $\mathcal{O}\big(\epsilon\,\hat{\tilde{V}}_{\bullet,\bullet}\big)$, hence $\nu_1 = \sqrt{FG} = \mathcal{O}(\epsilon)$ and the ratio of semiminor to semimajor axis lengths of the motion is

$$\frac{(\Delta\mathcal{J}_1)_{\rm max}}{(\Delta\varphi_1)_{\rm max}} = \sqrt{\frac{F}{G}} = \mathcal{O}(1) \quad . \tag{16.294}$$

16.9.3 Secondary resonances

By averaging over the φ_2 motion and expanding about the EFP, we obtained the Hamiltonian in Eqns. 16.292 and 16.293. In so doing, we dropped all terms on the RHS of eqn. 16.280 with $s\ell_1 + r\ell_2 \neq 0$. We now restore those terms, and continue to expand about the EFP. The first step is to transform the harmonic oscillator Hamiltonian in eqn. 16.292 to action-angle variables; this was already done in §16.8.3. The canonical transformation from $(\Delta\varphi_1, \Delta\mathcal{J}_1)$ to (χ_1, I_1) is given by

$$\Delta\mathcal{J}_1 = (2RI_1)^{1/2}\cos\chi_1 \quad , \quad \Delta\varphi_1 = (2R^{-1}I_1)^{1/2}\sin\chi_1 \quad , \tag{16.295}$$

with $R = (F/G)^{1/2}$. We will also define $I_2 \equiv \mathcal{J}_2$ and $\chi_2 \equiv \varphi_2$. Then we may write

$$\mathcal{H}(\varphi_1, \boldsymbol{\mathcal{J}}) \longrightarrow \widetilde{\mathcal{H}}_0(\boldsymbol{I}) = \widetilde{H}_0\big(\mathcal{J}_1^{(0)}, I_2\big) + \nu_1(I_2)\,I_1 - \tfrac{1}{16}\,G(I_2)\,I_1^2 + \ldots \quad , \tag{16.296}$$

where the last term on the RHS before the ellipses is from nonlinear terms in $\Delta\varphi_1$. The missing terms we seek are

$$\widetilde{H}_1' = \sum_{\boldsymbol{\ell}} \hat{\tilde{V}}_{\boldsymbol{\ell}}\big(\mathcal{J}_1^{(0)}, I_2\big)\,\exp\big[ir^{-1}\ell_1(2R^{-1}I_1)^{1/2}\sin\chi_1\big]\,\exp\big[i(r^{-1}s\,\ell_1 + \ell_2)\,\chi_2\big] \quad . \tag{16.297}$$

Note that we set $\mathcal{J}_1 = \mathcal{J}_1^{(0)}$ in the argument of $\hat{\tilde{V}}_{\boldsymbol{\ell}}(\boldsymbol{\mathcal{J}})$, because $\Delta\mathcal{J}_1$ is of order $\epsilon^{1/2}$. Next we invoke the Bessel function identity,

$$e^{iu\sin\chi} = \sum_{-\infty}^{\infty} J_n(u)\,e^{in\chi} \quad , \tag{16.298}$$

so we write

$$\widetilde{H}_1' \longrightarrow \mathcal{H}_1(\boldsymbol{\chi},\boldsymbol{I}) = \sum_{\boldsymbol{\ell}} \sum_n W_{\boldsymbol{\ell},n}(\boldsymbol{I})\, e^{in\chi_1}\, e^{i(r^{-1}s\,\ell_1+\ell_2)\,\chi_2} \quad , \tag{16.299}$$

where

$$W_{\boldsymbol{\ell},n}(\boldsymbol{I}) = \hat{\tilde{V}}_{\boldsymbol{\ell}}\big(\mathcal{J}_1^{(0)}, I_2\big)\, J_n\!\left(\frac{\ell_1}{r}\sqrt{\frac{2I_1}{R}}\right) \quad . \tag{16.300}$$

We now write

$$\mathcal{H}(\boldsymbol{\chi},\boldsymbol{I}) = \mathcal{H}_0(\boldsymbol{I}) + \tilde{\epsilon}\,\mathcal{H}_1(\boldsymbol{\chi},\boldsymbol{I}) \quad . \tag{16.301}$$

Here, while $\tilde{\epsilon} = \epsilon$ it is convenient to use a new symbol since ϵ itself appears within $\mathcal{H}_0$.

We now see that a secondary resonance will occur if $r'\nu_1 = s'\nu_2$, with $\nu_j(\boldsymbol{I}) = \partial\mathcal{H}_0/\partial I_j$ and $r', s' \in \mathbb{Z}$. But note that $\nu_1 = \mathcal{O}(\epsilon^{1/2})$ while $\nu_2 = \mathcal{O}(1)$ in the case of an accidental primary resonance. As before, we may eliminate this new resonance by transforming to a moving frame in which the resonance shifts to zero frequency to zeroth order and then averaging over the remaining motion. That is, we canonically transform $(\boldsymbol{\chi},\boldsymbol{I}) \to (\boldsymbol{\psi},\boldsymbol{\mathcal{I}})$ via a type-II generator $F_2' = (r'\chi_1 - s'\chi_2)\,\mathcal{I}_1 + \chi_2\,\mathcal{I}_2$, yielding

$$I_1 = \frac{\partial F_2'}{\partial \chi_1} = r'\mathcal{I}_1 \qquad\qquad \psi_1 = \frac{\partial F_2'}{\partial \mathcal{I}_1} = r'\chi_1 - s'\chi_2 \tag{16.302}$$

$$I_2 = \frac{\partial F_2'}{\partial \chi_2} = \mathcal{I}_2 - s'\mathcal{I}_1 \qquad\qquad \psi_2 = \frac{\partial F_2'}{\partial \mathcal{I}_2} = \chi_2 \quad . \tag{16.303}$$

The phase angle in eqn. 16.299 is then

$$n\chi_1 + \left(\frac{s}{r}\,\ell_1 + \ell_2\right)\chi_2 = \frac{n}{r'}\,\psi_1 + \left(\frac{ns'}{r'} + \frac{s}{r}\,\ell_1 + \ell_2\right)\psi_2 \quad . \tag{16.304}$$

Averaging over $\psi_2(t)$ then requires $nrs' + sr'\ell_1 + rr'\ell_2 = 0$, which is satisfied when

$$n = jr' \quad , \quad \ell_1 = kr \quad , \quad \ell_2 = -js' - ks \tag{16.305}$$

for some $j, k \in \mathbb{Z}$. The result of the averaging is

$$\langle\mathcal{H}\rangle_{\psi_2} = \mathcal{H}_0\big(\boldsymbol{I}(\boldsymbol{\mathcal{I}})\big) + \tilde{\epsilon}\sum_j \Gamma_{jr',-js'}(\boldsymbol{\mathcal{I}})\, e^{-ij\psi_1} \tag{16.306}$$

where

$$\Gamma_{jr',-js'}(\boldsymbol{\mathcal{I}}) = W_{kr,-js'-ks,jr'}\big(\boldsymbol{I}(\boldsymbol{\mathcal{I}})\big) = \hat{\tilde{V}}_{kr,-js'-ks}\big(\mathcal{J}_1^{(0)}, I_2\big)\, J_{jr'}\!\left(k\sqrt{\frac{2I_1(\boldsymbol{\mathcal{I}})}{R}}\right) \quad . \tag{16.307}$$

Since $\langle\mathcal{H}\rangle_{\psi_2}$ is independent of ψ_2, the corresponding action $\mathcal{I}_2 = (s'/r')\,I_1 + I_2$ is the adiabatic invariant for the new oscillation.

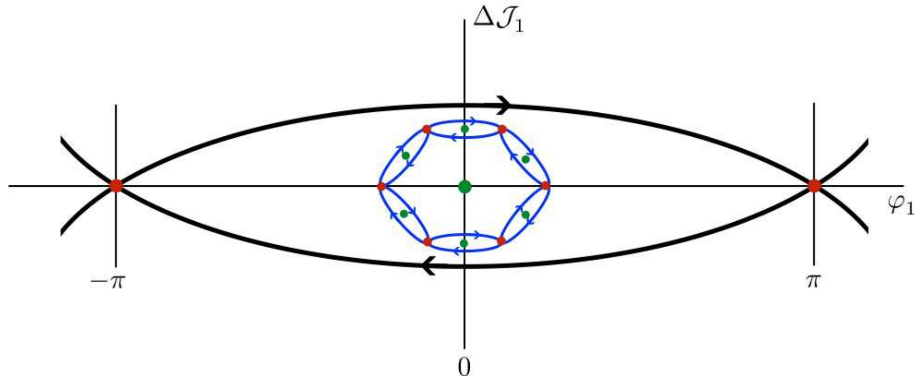


Figure 16.10: Motion in the vicinity of a secondary resonance with $r' = 6$ and $s' = 1$. Elliptical fixed points are in green, hyperbolic fixed points in red, and separatrices in black and blue.

Strength of island resonances

To assess the strength of the secondary resonances, we consider $r = s = j = k = s' = 1$, in which case $r' = \nu_2/\nu_1 = \mathcal{O}(\epsilon^{-1/2})$ is parametrically large. The resulting structure in the phase plane is depicted in fig. 16.10 for $r' = 6$. The amplitude of the $\mathcal{M}_1$ oscillations is proportional to

$$J_r\big((2I_1(\boldsymbol{\mathcal{I}})/2R)^{1/2}\big) \sim \frac{\big(2I_1(\boldsymbol{\mathcal{I}})/2R\big)^{r'/2}}{r'!} = \mathcal{O}\bigg(\frac{1}{(\epsilon^{-1/2})!}\bigg) \quad . \tag{16.308}$$

The frequency of the island oscillations is of the same order of magnitude. Successive higher order resonances result in an increasingly tiny island chain amplitude.

16.10 Whither Integrability?

We are left with the following question: what happens when we perturb an integrable Hamiltonian, $H(\boldsymbol{\phi}, \boldsymbol{J}) = H_0(\boldsymbol{J}) + \epsilon\, H_1(\boldsymbol{\phi}, \boldsymbol{J})$? Two extreme conjectures, and their refutations:

(i) $H(\boldsymbol{\phi}, \boldsymbol{J})$ is always integrable, even though we may not always be able to obtain the corresponding action-angle variables. Tori are deformed but not destroyed. If this were the case, there would be n conserved quantities, *i.e.* the first integrals of motion I_j. This would violate the fundamental tenets of equilibrium statistical physics, as the canonical Gibbs distribution $\varrho = \exp(-\beta H)/Z$ would be replaced with the *pseudo-Gibbs* distribution, $\varrho = \exp(-\lambda_j I_j)/Z$, where $\{\lambda_j\}$ are a set of Lagrange multipliers[19].

(ii) Integrability is destroyed for any $\epsilon > 0$, in which case $E = H(\boldsymbol{\phi}, \boldsymbol{J})$ is the only conserved quantity[20]. If this were the case, the solar system would be unstable, and we wouldn't be here to study Hamiltonian mechanics.

[19] The corresponding microcanonical distribution would be $\prod_{j=1}^{n} \delta\big(I_j - \langle I_j \rangle\big)$, as opposed to $\delta(H - E)$.

[20] Without loss of generality, we may assume $\epsilon \geq 0$.

So the truth lies somewhere in between, and is the focus of the celebrated KAM theorem[21]. We have already encountered the problem of resonances, which arise for tori which satisfy $\boldsymbol{\ell}\cdot\boldsymbol{\omega}_0(\boldsymbol{J})=0$ for some integers $\boldsymbol{\ell}=\{\ell_1,\ldots,\ell_n\}$. Such tori, which are dense in the phase space $\mathcal{M}$ yet still of Lebesgue measure zero, are destroyed by arbitrarily small perturbations, as we have seen. This observation dates back to Poincaré. For a given torus with an $(n-1)$-dimensional family of periodic orbits, $J_n=J_n(J_1,\ldots,J_{n-1})$, it is generally the case that only a finite number of periodic orbits survive the perturbation. Since, in a nondegenerate system, the set of resonant tori is dense, it seems like the situation is hopeless and that arbitrarily small ϵ will induce ergodicity on each energy surface. Until the early 1950s, it was generally believed that this was the case, and the stability of the solar system was regarded as a deep mystery.

Enter Andrey Nikolaevich Kolmogorov, who in 1954 turned conventional wisdom on its head, showing that, in fact, the *majority* of tori survive. Specifically, Kolmogorov proved that *strongly nonresonant* tori survive small perturbations. A strongly nonresonant torus is one for which there exist constants $\alpha>0$ and $\tau>0$ such that $\big|\boldsymbol{\ell}\cdot\boldsymbol{\omega}_0(\boldsymbol{J})\big|\geq\alpha\,|\boldsymbol{\ell}|^{-\tau}$, where $|\boldsymbol{\ell}|\equiv|\ell_1|+\ldots+|\ell_n|$. From a measure theoretic point of view, almost all tori are strongly nonresonant for any $\tau>n-1$, but in order to survive the perturbation, it is necessary that $\epsilon\ll\alpha^2$. For these tori, perturbation theory converges, although not quite in the naïve form we have derived, *i.e.* from the generator $S(\boldsymbol{\phi},\boldsymbol{\mathcal{J}})=S_0+\epsilon S_1+\epsilon^2 S_2+\ldots$, but rather using the 'superconvergent' method pioneered by Kolmogorov.

Since the arithmetic of the strongly nonresonant tori is a bit unusual, let's first convince ourselves that such tori actually exist[22]. Let Δ^τ_α denote the set of all $\boldsymbol{\omega}\in\mathbb{R}^n$ satisfying, for fixed α and τ, the infinitely many conditions $\boldsymbol{\ell}\cdot\boldsymbol{\omega}\geq\alpha|\boldsymbol{\ell}|^{-\tau}$, for all nonzero $\boldsymbol{\ell}\in\mathbb{Z}^n$. Clearly Δ^τ_α is the complement of the open and dense set $R^\tau_\alpha=\bigcup_{0\neq\boldsymbol{\ell}\in\mathbb{Z}^n}R^\tau_{\alpha,\boldsymbol{\ell}}$, where

$$R^\tau_{\alpha,\boldsymbol{\ell}}=\Big\{\boldsymbol{\omega}\in\mathbb{R}^n\ :\ |\boldsymbol{\ell}\cdot\boldsymbol{\omega}|<\alpha\,|\boldsymbol{\ell}|^{-\tau}\Big\}\quad. \tag{16.309}$$

For any bounded region $\Omega\in\mathbb{R}^n$, we can estimate the Lebesgue measure of the set $R^\tau_\alpha\cap\Omega$ from the calculation

$$\mu\big(R^\tau_\alpha\cap\Omega\big)\leq\sum_{\boldsymbol{\ell}\neq 0}\mu\big(R^\tau_{\alpha,\boldsymbol{\ell}}\cap\Omega\big)=\mathcal{O}(\alpha)\quad, \tag{16.310}$$

The sum converges provided $\tau>n-1$ since $\mu\big(R^\tau_{\alpha,\boldsymbol{\ell}}\cap\Omega\big)=\mathcal{O}\big(\alpha/|\boldsymbol{\ell}|^{\tau+1}\big)$. Taking the intersection over *all* $\alpha>0$, we conclude $R^\tau=\bigcap_{\alpha>0}R^\tau_\alpha$ is a set of measure zero, and therefore its complement, $\Delta^\tau=\bigcup_{\alpha>0}\Delta^\tau_\alpha$, is a set of full measure in $\mathbb{R}^n$. This means that almost every $\boldsymbol{\omega}\in\mathbb{R}^n$ belongs to the set Δ^τ, which is the set of all $\boldsymbol{\omega}$ satisfying the *Diophantine condition* $|\boldsymbol{\ell}\cdot\boldsymbol{\omega}|\geq\alpha\,|\boldsymbol{\ell}|^{-\tau}$ for *some* value of α, again provided $\tau>n-1$.

We say that a torus *survives* the perturbation if for $\epsilon>0$ there exists a deformed torus in phase space homotopic to that for $\epsilon=0$, and for which the frequencies satisfy $\boldsymbol{\omega}_\epsilon=f(\epsilon)\,\boldsymbol{\omega}_0$, with $\lim_{\epsilon\to 0}f(\epsilon)=1$. Note this says $\omega_j/\omega_k=\omega_{0,j}/\omega_{0,k}$. Only tori with frequencies in Δ^τ_α with $\alpha\gg\sqrt{\epsilon}$ survive. The KAM theorem says that the measure of the space of surviving tori approaches unity as $\epsilon\to 0$.

[21]KAM = Kolmogorov-Arnol'd-Moser, who developed the theory in a series of papers during the 1950s and 1960s. For a "friendly introduction to the content, history, and significance" of KAM, I highly recommend H. Scott Dumas, *The KAM Story* (World Scientific, 2014).

[22]See J. Pöschel, *A Lesson on the Classical KAM Theorem*, *Proc. Symp. Pure Math.* **69**, 707 (2001), in §1.d.

16.11 Appendix: Examples

16.11.1 Hamilton-Jacobi theory for point charge plus electric field

Consider a potential of the form

$$U(r) = \frac{k}{r} - Fz \quad , \tag{16.311}$$

which corresponds to a charge in the presence of an external point charge plus an external electric field. This problem is amenable to separation in parabolic coordinates, (ξ, η, φ):

$$x = \sqrt{\xi\eta}\,\cos\varphi \quad , \quad y = \sqrt{\xi\eta}\,\sin\varphi \quad , \quad z = \tfrac{1}{2}(\xi - \eta) \quad . \tag{16.312}$$

Note that

$$\begin{aligned} \rho &\equiv \sqrt{x^2 + y^2} = \sqrt{\xi\eta} \\ r &= \sqrt{\rho^2 + z^2} = \tfrac{1}{2}(\xi + \eta) \quad . \end{aligned} \tag{16.313}$$

The kinetic energy is

$$\begin{aligned} T &= \tfrac{1}{2}m\big(\dot\rho^2 + \rho^2\,\dot\varphi^2 + \dot z^2\big) \\ &= \tfrac{1}{8}m\,(\xi+\eta)\left(\frac{\dot\xi^2}{\xi} + \frac{\dot\eta^2}{\eta}\right) + \tfrac{1}{2}m\,\xi\eta\,\dot\varphi^2 \quad , \end{aligned} \tag{16.314}$$

and hence the Lagrangian is

$$L = \tfrac{1}{8}m\,(\xi+\eta)\left(\frac{\dot\xi^2}{\xi} + \frac{\dot\eta^2}{\eta}\right) + \tfrac{1}{2}m\,\xi\eta\,\dot\varphi^2 - \frac{2k}{\xi+\eta} + \tfrac{1}{2}F\,(\xi-\eta) \quad . \tag{16.315}$$

Thus, the conjugate momenta are

$$\begin{aligned} p_\xi &= \frac{\partial L}{\partial\dot\xi} = \tfrac{1}{4}m\,(\xi+\eta)\,\frac{\dot\xi}{\xi} \\ p_\eta &= \frac{\partial L}{\partial\dot\eta} = \tfrac{1}{4}m\,(\xi+\eta)\,\frac{\dot\eta}{\eta} \\ p_\varphi &= \frac{\partial L}{\partial\dot\varphi} = m\,\xi\eta\,\dot\varphi \quad , \end{aligned} \tag{16.316}$$

and the Hamiltonian is

$$\begin{aligned} H &= p_\xi\,\dot\xi + p_\eta\,\dot\eta + p_\varphi\,\dot\varphi \\ &= \frac{2}{m}\left(\frac{\xi\,p_\xi^2 + \eta\,p_\eta^2}{\xi+\eta}\right) + \frac{p_\varphi^2}{2m\xi\eta} + \frac{2k}{\xi+\eta} - \tfrac{1}{2}F\,(\xi-\eta) \quad . \end{aligned} \tag{16.317}$$

Notice that $\partial H/\partial t = 0$, which means $dH/dt = 0$, *i.e.* $H = E \equiv \Lambda_1$ is a constant of the motion. Also, φ is cyclic in H, so its conjugate momentum p_φ is a constant of the motion.

We write

$$\begin{aligned} S(q,\Lambda) &= W(q,\Lambda) - Et \\ &= W_\xi(\xi,\Lambda) + W_\eta(\eta,\Lambda) + W_\varphi(\varphi,\Lambda) - Et \quad . \end{aligned} \tag{16.318}$$

with $E = \Lambda_1$. Clearly we may take

$$W_\varphi(\varphi,\Lambda) = P_\varphi\,\varphi \quad , \tag{16.319}$$

where $P_\varphi = \Lambda_2$. Multiplying the Hamilton-Jacobi equation by $\frac{1}{2}m\,(\xi+\eta)$ then gives

$$\xi\left(\frac{dW_\xi}{d\xi}\right)^2 + \frac{P_\varphi^2}{4\xi} + mk - \tfrac{1}{4}F\xi^2 - \tfrac{1}{2}mE\xi = -\eta\left(\frac{dW_\eta}{d\eta}\right)^2 - \frac{P_\varphi^2}{4\eta} - \tfrac{1}{4}F\eta^2 + \tfrac{1}{2}mE\eta \equiv \Upsilon \quad , \tag{16.320}$$

where $\Upsilon = \Lambda_3$ is the third constant: $\Lambda = (E, P_\varphi, \Upsilon)$. Thus,

$$\begin{aligned} S(\overbrace{\xi,\eta,\varphi}^{q};\underbrace{E,P_\varphi,\Upsilon}_{\Lambda}) &= \int^\xi \! d\xi'\sqrt{\tfrac{1}{2}mE + \frac{\Upsilon - mk}{\xi'} + \tfrac{1}{4}mF\xi' - \frac{P_\varphi^2}{4\xi'^2}} \\ &\qquad + \int^\eta \! d\eta'\sqrt{\tfrac{1}{2}mE - \frac{\Upsilon}{\eta'} - \tfrac{1}{4}mF\eta' - \frac{P_\varphi^2}{4\eta'^2}} + P_\varphi\,\varphi - Et \quad . \end{aligned} \tag{16.321}$$

16.11.2 Hamilton-Jacobi theory for charged particle in a magnetic field

The Hamiltonian is

$$H = \frac{1}{2m}\Big(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}\Big)^2 \quad . \tag{16.322}$$

We choose the gauge $\boldsymbol{A} = Bx\hat{\boldsymbol{y}}$, and we write

$$S(x,y,P_1,P_2) = W_x(x,P_1,P_2) + W_y(y,P_1,P_2) - P_1\,t \quad . \tag{16.323}$$

Note that here we will consider S to be a function of $\{q_\sigma\}$ and $\{P_\sigma\}$.

The Hamilton-Jacobi equation is then

$$\left(\frac{\partial W_x}{\partial x}\right)^2 + \left(\frac{\partial W_y}{\partial y} - \frac{eBx}{c}\right)^2 = 2mP_1 \quad . \tag{16.324}$$

We solve by writing

$$W_y = P_2\,y \qquad \Rightarrow \qquad \left(\frac{dW_x}{dx}\right)^2 + \left(P_2 - \frac{eBx}{c}\right)^2 = 2mP_1 \quad . \tag{16.325}$$

This equation suggests the substitution

$$x = \frac{cP_2}{eB} + \frac{c}{eB}\sqrt{2mP_1}\,\sin\theta \quad . \tag{16.326}$$

in which case

$$\frac{\partial x}{\partial \theta} = \frac{c}{eB}\sqrt{2mP_1}\,\cos\theta \tag{16.327}$$

and

$$\frac{\partial W_x}{\partial x} = \frac{\partial W_x}{\partial \theta}\cdot\frac{\partial \theta}{\partial x} = \frac{eB}{c\sqrt{2mP_1}}\,\frac{1}{\cos\theta}\,\frac{\partial W_x}{\partial \theta} \quad . \tag{16.328}$$

Substitution into eqn. 16.325, we have $\partial W_x/\partial\theta = (2mcP_1/eB)\cos^2\!\theta$ which integrates to

$$W_x = \frac{mcP_1}{eB}\,\theta + \frac{mcP_1}{2eB}\,\sin(2\theta) \quad . \tag{16.329}$$

We then have

$$p_x = \frac{\partial W_x}{\partial x} = \frac{\partial W_x}{\partial \theta}\bigg/\frac{\partial x}{\partial \theta} = \sqrt{2mP_1}\,\cos\theta \tag{16.330}$$

and $p_y = \partial W_y/\partial y = P_2$. The type-II generator we seek is then

$$S(q,P,t) = \frac{mcP_1}{eB}\,\theta + \frac{mcP_1}{2eB}\,\sin(2\theta) + P_2\,y - P_1\,t \quad , \tag{16.331}$$

where

$$\theta = \frac{eB}{c\sqrt{2mP_1}}\,\sin^{-1}\!\bigg(x - \frac{cP_2}{eB}\bigg) \quad . \tag{16.332}$$

Note that, from eqn. 16.326, we may write

$$dx = \frac{c}{eB}\,dP_2 + \frac{mc}{eB}\,\frac{1}{\sqrt{2mP_1}}\,\sin\theta\,dP_1 + \frac{c}{eB}\sqrt{2mP_1}\,\cos\theta\,d\theta \quad , \tag{16.333}$$

from which we derive

$$\frac{\partial\theta}{\partial P_1} = -\frac{\tan\theta}{2P_1} \quad , \qquad \frac{\partial\theta}{\partial P_2} = -\frac{1}{\sqrt{2mP_1}\,\cos\theta} \quad . \tag{16.334}$$

These results are useful in the calculation of Q_1 and Q_2 :

$$\begin{aligned} Q_1 = \frac{\partial S}{\partial P_1} &= \frac{mc}{eB}\,\theta + \frac{mcP_1}{eB}\,\frac{\partial\theta}{\partial P_1} + \frac{mc}{2eB}\,\sin(2\theta) + \frac{mcP_1}{eB}\,\cos(2\theta)\,\frac{\partial\theta}{\partial P_1} - t \\ &= \frac{\theta}{\omega_c} - t \end{aligned} \tag{16.335}$$

where $\omega_c = eB/mc$ is the 'cyclotron frequency', and

$$\begin{aligned} Q_2 = \frac{\partial S}{\partial P_2} &= y + \frac{mcP_1}{eB}\big[1 + \cos(2\theta)\big]\,\frac{\partial\theta}{\partial P_2} \\ &= y - \frac{c}{eB}\sqrt{2mP_1}\,\cos\theta \quad . \end{aligned} \tag{16.336}$$

Now since $\tilde{H}(P,Q)=0$, we have that $\dot{Q}_\sigma=0$, which means that each Q_σ is a constant. We therefore have the following solution:

$$\begin{aligned} x(t) &= x_0 + A\,\sin(\omega_{\rm c} t+\delta) \\ y(t) &= y_0 + A\,\cos(\omega_{\rm c} t+\delta) \quad , \end{aligned} \tag{16.337}$$

and

$$x_0 = \frac{cP_2}{eB} \quad , \quad y_0 = Q_2 \quad , \quad \delta \equiv \omega_{\rm c}\, Q_1 \quad , \quad A = \frac{c}{eB}\sqrt{2mP_1} \quad . \tag{16.338}$$

16.11.3 Action-angle variables for the Kepler problem

This is discussed in detail in standard texts, such as Goldstein. The potential is $V(r)=-k/r$, and the problem is separable. We write[23]

$$W(r,\theta,\varphi) = W_r(r) + W_\theta(\theta) + W_\varphi(\varphi) \quad , \tag{16.339}$$

hence

$$\frac{1}{2m}\bigg(\frac{\partial W_r}{\partial r}\bigg)^2 + \frac{1}{2mr^2}\bigg(\frac{\partial W_\theta}{\partial \theta}\bigg)^2 + \frac{1}{2mr^2\sin^2\!\theta}\bigg(\frac{\partial W_\varphi}{\partial \varphi}\bigg)^2 + V(r) = E \equiv \varLambda_r \quad . \tag{16.340}$$

Separating, we have

$$\frac{1}{2m}\bigg(\frac{dW_\varphi}{d\varphi}\bigg)^2 = \varLambda_\varphi \quad \Rightarrow \quad J_\varphi = \oint\limits_{\mathcal{C}_\varphi} d\varphi\,\frac{dW_\varphi}{d\varphi} = 2\pi\sqrt{2m\varLambda_\varphi} \quad . \tag{16.341}$$

Next we deal with the θ coordinate. We have

$$\frac{1}{2m}\bigg(\frac{dW_\theta}{d\theta}\bigg)^2 = \varLambda_\theta - \frac{\varLambda_\varphi}{\sin^2\!\theta} \quad , \tag{16.342}$$

and therefore

$$\begin{aligned} J_\theta &= 4\sqrt{2m\varLambda_\theta}\int\limits_{\theta_0}^{\pi/2}\!d\theta\,\sqrt{1-\big(\varLambda_\varphi/\varLambda_\theta\big)\csc^2\!\theta} \\ &= 2\pi\sqrt{2m}\,\Big(\sqrt{\varLambda_\theta}-\sqrt{\varLambda_\varphi}\Big) \quad , \end{aligned} \tag{16.343}$$

where $\theta_0=\sin^{-1}(\varLambda_\varphi/\varLambda_\theta)$. Finally, we have for the radial coordinate

$$\frac{1}{2m}\bigg(\frac{dW_r}{dr}\bigg)^2 = E + \frac{k}{r} - \frac{\varLambda_\theta}{r^2} \quad , \tag{16.344}$$

[23] We denote the azimuthal angle by φ to distinguish it from the AA variable ϕ.

and so[24]

$$\begin{aligned} J_r &= \oint\limits_{\mathcal{C}_r} dr\,\sqrt{2m\bigg(E + \frac{k}{r} - \frac{\Lambda_\theta}{r^2}\bigg)} \\ &= -(J_\theta + J_\varphi) + \pi k \sqrt{\frac{2m}{|E|}} \quad , \end{aligned} \tag{16.345}$$

where we've assumed $E < 0$, *i.e.* bound motion.

Thus, we find

$$H = E = -\frac{2\pi^2 m k^2}{\big(J_r + J_\theta + J_\varphi\big)^2} \quad . \tag{16.346}$$

Note that the frequencies are completely degenerate:

$$\nu \equiv \nu_{r,\theta,\varphi} = \frac{\partial H}{\partial J_{r,\theta,\varphi}} = \frac{4\pi^2 m k^2}{\big(J_r + J_\theta + J_\varphi\big)^3} = \bigg(\frac{\pi^2 m k^2}{2\,|E|^3}\bigg)^{1/2} \quad . \tag{16.347}$$

This threefold degeneracy may be removed by a transformation to new AA variables,

$$\Big\{(\phi_r, J_r),\ (\phi_\theta, J_\theta),\ (\phi_\varphi, J_\varphi)\Big\} \longrightarrow \Big\{(\chi_1, \mathcal{J}_1),\ (\chi_2, \mathcal{J}_2),\ (\chi_3, \mathcal{J}_3)\Big\} \quad , \tag{16.348}$$

using the type-II generator

$$F_2(\phi_r, \phi_\theta, \phi_\varphi; \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3) = (\phi_\varphi - \phi_\theta)\,\mathcal{J}_1 + (\phi_\theta - \phi_r)\,\mathcal{J}_2 + \phi_r\,\mathcal{J}_3 \quad , \tag{16.349}$$

which results in

$$\chi_1 = \frac{\partial F_2}{\partial \mathcal{J}_1} = \phi_\varphi - \phi_\theta \qquad J_r = \frac{\partial F_2}{\partial \phi_r} = \mathcal{J}_3 - \mathcal{J}_2 \tag{16.350}$$

$$\chi_2 = \frac{\partial F_2}{\partial \mathcal{J}_2} = \phi_\theta - \phi_r \qquad J_\theta = \frac{\partial F_2}{\partial \phi_\theta} = \mathcal{J}_2 - \mathcal{J}_1 \tag{16.351}$$

$$\chi_3 = \frac{\partial F_2}{\partial \mathcal{J}_3} = \phi_r \qquad J_\varphi = \frac{\partial F_2}{\partial \phi_\varphi} = \mathcal{J}_1 \quad . \tag{16.352}$$

The new Hamiltonian is

$$H(\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3) = -\frac{2\pi^2 m k^2}{\mathcal{J}_3^2} \quad , \tag{16.353}$$

whence $\nu_1 = \nu_2 = 0$ and $\nu_3 = \nu$.

[24]The details of performing the integral around $\mathcal{C}_r$ are discussed in *e.g.* Goldstein.

16.11.4 Action-angle variables for charged particle in a magnetic field

For the case of the charged particle in a magnetic field, studied above in section 16.11.2, we found

$$x = \frac{cP_2}{eB} + \frac{c}{eB}\sqrt{2mP_1}\,\sin\theta \tag{16.354}$$

with $p_x = \sqrt{2mP_1}\,\cos\theta$ and $p_y = P_2$. The action variable J is then

$$J = \oint p_x\,dx = \frac{2mcP_1}{eB}\int\limits_0^{2\pi} d\theta\,\cos^2\!\theta = \frac{mcP_1}{eB} \quad . \tag{16.355}$$

We then have

$$W = J\theta + \tfrac{1}{2}J\sin(2\theta) + Py \quad , \tag{16.356}$$

where $P \equiv P_2$. Thus,

$$\begin{aligned}\phi = \frac{\partial W}{\partial J} &= \theta + \tfrac{1}{2}\sin(2\theta) + J\big[1 + \cos(2\theta)\big]\,\frac{\partial\theta}{\partial J} \\ &= \theta + \tfrac{1}{2}\sin(2\theta) + 2J\cos^2\!\theta\cdot\left(-\frac{\tan\theta}{2J}\right) = \theta \quad .\end{aligned} \tag{16.357}$$

The other canonical pair is (Q, P), where

$$Q = \frac{\partial W}{\partial P} = y - \sqrt{\frac{2cJ}{eB}}\,\cos\phi \quad . \tag{16.358}$$

Therefore, we have

$$x = \frac{cP}{eB} + \sqrt{\frac{2cJ}{eB}}\,\sin\phi \qquad , \qquad y = Q + \sqrt{\frac{2cJ}{eB}}\,\cos\phi \tag{16.359}$$

and

$$p_x = \sqrt{\frac{2eBJ}{c}}\,\cos\phi \qquad , \qquad p_y = P \quad . \tag{16.360}$$

The Hamiltonian is

$$\begin{aligned}H &= \frac{P_x^2}{2m} + \frac{1}{2m}\left(p_y - \frac{eBx}{c}\right)^2 \\ &= \frac{eBJ}{mc}\,\cos^2\!\phi + \frac{eBJ}{mc}\,\sin^2\!\phi = \omega_c\,J \quad ,\end{aligned} \tag{16.361}$$

where $\omega_c = eB/mc$. The equations of motion are

$$\dot\phi = \frac{\partial H}{\partial J} = \omega_c \qquad , \qquad \dot J = -\frac{\partial H}{\partial\phi} = 0 \tag{16.362}$$

and

$$\dot Q = \frac{\partial H}{\partial P} = 0 \qquad , \qquad \dot P = -\frac{\partial H}{\partial Q} = 0 \quad . \tag{16.363}$$

Thus, Q, P, and J are constants, and $\phi(t) = \phi_0 + \omega_c\,t$.

16.11.5 Canonical perturbation theory for the cubic oscillator

Consider the Hamiltonian

$$H = \frac{p^2}{2m} + \tfrac{1}{2} m\,\omega_0^2\, q^2 + \tfrac{1}{3} \epsilon\, m\,\omega_0^2\, \frac{q^3}{a} \quad , \tag{16.364}$$

where ϵ is a small dimensionless parameter.

(a) Show that the oscillation frequency satisfies $\nu(J) = \omega_0 + \mathcal{O}(\epsilon^2)$. That is, show that the first order (in ϵ) frequency shift vanishes.

Solution: It is good to recall the basic formulae

$$q = \sqrt{\frac{2J_0}{m\omega_0}}\, \sin\phi_0 \qquad , \qquad p = \sqrt{2m\,\omega_0\, J_0}\, \cos\phi_0 \tag{16.365}$$

as well as the results

$$\begin{aligned} J_0 &= \frac{\partial S}{\partial \phi_0} = J + \epsilon\, \frac{\partial S_1}{\partial \phi_0} + \epsilon^2\, \frac{\partial S_2}{\partial \phi_0} + \ldots \\ \phi &= \frac{\partial S}{\partial J} = \phi_0 + \epsilon\, \frac{\partial S_1}{\partial J} + \epsilon^2\, \frac{\partial S_2}{\partial J} + \ldots \quad , \end{aligned} \tag{16.366}$$

and

$$\begin{aligned} E_0(J) &= \widetilde{H}_0(J) \\ E_1(J) &= \widetilde{H}_1(\phi_0, J) + \frac{\partial \widetilde{H}_0}{\partial J}\, \frac{\partial S_1}{\partial \phi_0} \\ E_2(J) &= \frac{\partial \widetilde{H}_0}{\partial J}\, \frac{\partial S_2}{\partial \phi_0} + \frac{1}{2}\, \frac{\partial^2 \widetilde{H}_0}{\partial J^2} \left(\frac{\partial S_1}{\partial \phi_0}\right)^2 + \frac{\partial \widetilde{H}_1}{\partial J}\, \frac{\partial S_1}{\partial \phi_0} \quad . \end{aligned} \tag{16.367}$$

Expressed in action-angle variables,

$$\begin{aligned} \widetilde{H}_0(\phi_0, J) &= \omega_0\, J \\ \widetilde{H}_1(\phi_0, J) &= \frac{2}{3} \sqrt{\frac{2\omega_0}{ma^2}}\, J^{3/2}\, \sin^3\!\phi_0 \quad . \end{aligned} \tag{16.368}$$

Thus, $\nu_0 = \frac{\partial \widetilde{H}_0}{\partial J} = \omega_0$.

Averaging the equation for $E_1(J)$ yields

$$E_1(J) = \big\langle \widetilde{H}_1(\phi_0, J) \big\rangle = \frac{2}{3} \sqrt{\frac{2\omega_0}{ma^2}}\, J^{3/2}\, \big\langle \sin^3\!\phi_0 \big\rangle = 0 \quad . \tag{16.369}$$

(b) Compute the frequency shift $\nu(J)$ to second order in ϵ.

Solution : From the equation for E_1, we also obtain

$$\frac{\partial S_1}{\partial \phi_0} = \frac{1}{\nu_0}\Big(\langle \widetilde{H}_1 \rangle - \widetilde{H}_1\Big) \quad . \tag{16.370}$$

Inserting this into the equation for $E_2(J)$ and averaging then yields

$$E_2(J) = \frac{1}{\nu_0}\bigg\langle \frac{\partial \widetilde{H}_1}{\partial J}\Big(\langle \widetilde{H}_1 \rangle - \widetilde{H}_1\Big)\bigg\rangle = -\frac{1}{\nu_0}\bigg\langle \widetilde{H}_1\,\frac{\partial \widetilde{H}_1}{\partial J}\bigg\rangle = -\frac{4\nu_0 J^2}{3ma^2}\,\langle \sin^6\!\phi_0 \rangle \tag{16.371}$$

In computing the average of $\sin^6\!\phi_0$, it is good to recall the binomial theorem, or the Fibonacci tree. The sixth order coefficents are easily found to be $\{1, 6, 15, 20, 15, 6, 1\}$, whence

$$\begin{aligned} \sin^6\!\phi_0 &= \frac{1}{(2i)^6}\big(e^{i\phi_0} - e^{-i\phi_0}\big)^6 \\ &= \tfrac{1}{64}\big(-2\sin 6\phi_0 + 12\sin 4\phi_0 - 30\sin 2\phi_0 + 20\big) \quad . \end{aligned} \tag{16.372}$$

Thus $\langle \sin^6\!\phi_0 \rangle = \frac{5}{16}$, whence

$$E(J) = \omega_0\, J - \tfrac{5}{12}\epsilon^2\,\frac{J^2}{ma^2} \tag{16.373}$$

and

$$\nu(J) = \frac{\partial E}{\partial J} = \omega_0 - \tfrac{5}{6}\epsilon^2\,\frac{J}{ma^2} \quad . \tag{16.374}$$

(c) Find $q(t)$ to order ϵ. Your result should be finite for all times.

Solution : From the equation for $E_1(J)$, we have

$$\frac{\partial S_1}{\partial \phi_0} = -\frac{2}{3}\sqrt{\frac{2J^3}{m\omega_0 a^2}}\,\sin^3\!\phi_0 \quad . \tag{16.375}$$

Integrating, we obtain

$$\begin{aligned} S_1(\phi_0, J) &= \frac{2}{3}\sqrt{\frac{2J^3}{m\omega_0 a^2}}\,\big(\cos\phi_0 - \tfrac{1}{3}\cos^3\!\phi_0\big) \\ &= \frac{J^{3/2}}{\sqrt{2m\omega_0 a^2}}\,\big(\cos\phi_0 - \tfrac{1}{9}\cos 3\phi_0\big) \quad . \end{aligned} \tag{16.376}$$

Thus, with

$$S(\phi_0, J) = \phi_0\, J + \epsilon\, S_1(\phi_0, J) + \ldots \quad , \tag{16.377}$$

we have

$$\begin{aligned} \phi &= \frac{\partial S}{\partial J} = \phi_0 + \frac{3}{2}\,\frac{\epsilon\, J^{1/2}}{\sqrt{2m\omega_0 a^2}}\,\big(\cos\phi_0 - \tfrac{1}{9}\cos 3\phi_0\big) \\ J_0 &= \frac{\partial S}{\partial \phi_0} = J - \frac{\epsilon\, J^{3/2}}{\sqrt{2m\omega_0 a^2}}\,\big(\sin\phi_0 - \tfrac{1}{3}\sin 3\phi_0\big) \quad . \end{aligned} \tag{16.378}$$

Inverting, we may write ϕ_0 and J_0 in terms of ϕ and J:

$$\begin{aligned}\phi_0 &= \phi + \frac{3}{2}\,\frac{\epsilon\, J^{1/2}}{\sqrt{2m\omega_0 a^2}}\,\big(\tfrac{1}{9}\cos 3\phi - \cos\phi\big) \\ J_0 &= J + \frac{\epsilon\, J^{3/2}}{\sqrt{2m\omega_0 a^2}}\,\big(\tfrac{1}{3}\sin 3\phi - \sin\phi\big) \quad .\end{aligned} \tag{16.379}$$

Thus,

$$\begin{aligned} q(t) &= \sqrt{\frac{2J_0}{m\omega_0}}\,\sin\phi_0 \\ &= \sqrt{\frac{2J}{m\omega_0}}\,\sin\phi\cdot\Big(1+\frac{\delta J}{2J}+\ldots\Big)\Big(\sin\phi + \delta\phi\,\cos\phi+\ldots\Big) \\ &= \sqrt{\frac{2J}{m\omega_0}}\,\sin\phi - \frac{\epsilon\, J}{m\omega_0 a}\,\big(1+\tfrac{1}{3}\cos 2\phi\big) + \mathcal{O}\big(\epsilon^2\big) \quad , \end{aligned} \tag{16.380}$$

with

$$\phi(t) = \phi(0) + \nu(J)\,t \quad . \tag{16.381}$$

Chapter 17

Maps, Strange Attractors, and Chaos

17.1 Motion on Resonant Tori

Consider an integrable Hamiltonian with two degrees of freedom. The energy $E(J_1, J_2)$ is then a function of the two action coordinates, so at fixed energy we may regard $J_2(J_1; E)$ as being determined by J_1. The motion is then given by $\phi_j(t) = \omega_j(J_1; E)\, t + \beta_j$, and, at fixed E, is confined to a two-torus $\mathbb{T}^2$ specified by the action J_1, as depicted in fig. 17.1. For a system with N freedoms (*i.e.* a phase space of dimension $2n$), integrable motion is confined to an n-torus $\mathbb{T}^N = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$. The frequencies $\omega_j = \partial E/\partial J_j = \omega_j(J_1, \ldots, J_{N-1}; E)$ are specified, at fixed E, by $N-1$ action variables. If, on a given torus, the frequency ratios $\omega_j/\omega_{j'}$ are rational numbers for all j and j', then the motion is periodic, and all frequencies are said to be in resonance. In this case we may write $\omega_j = k_j\, \omega_0$ for some set $\{k_1, \ldots, k_N\} \in \mathbb{Z}^N$, and some quantity ω_0 which has dimensions of frequency. One can Fourier decompose the original coordinates $q_\sigma(\boldsymbol{\phi}, \boldsymbol{J})$ as

$$q_\sigma(\boldsymbol{\phi}, \boldsymbol{J}) = \sum_{\boldsymbol{m}} \hat{q}_{\sigma,\boldsymbol{m}}(\boldsymbol{J})\, e^{i\boldsymbol{m}\cdot\boldsymbol{\phi}} \quad , \tag{17.1}$$

and similarly for $p_\sigma(\boldsymbol{\phi}, \boldsymbol{J})$, where $\boldsymbol{m} \in \mathbb{Z}^N$. Invoking the solution $\phi_\sigma(t) = k_\sigma \omega_0 t + \beta_\sigma$, one sees that the motion is periodic in time with period $T = 2\pi/\omega_0$. That all the frequencies are in resonance further means that for some of the $\boldsymbol{m}$ vectors, one has $\boldsymbol{m} \cdot \boldsymbol{k} = 0$.

17.1.1 The twist map

Consider the motion $\boldsymbol{\phi}(t) = \boldsymbol{\omega}(\boldsymbol{J})\, t + \boldsymbol{\beta}$ along a resonant torus, and let us plot consecutive intersections of the trajectory with the (J_1, ϕ_1) plane, *i.e.* the subset of phase space where both E (or J_2) and ϕ_2 are fixed. Such a plot is called a *surface of section*. Successive intersections of this surface occur at time interval $\Delta t = 2\pi/\omega_2$, during which the angle ϕ_1 changes by $\Delta\phi_1 = \omega_1 \Delta t = 2\pi\alpha$, where $\alpha = \omega_1/\omega_2$. Focusing only on the surface of section, we write $\phi \equiv \phi_1$ and $J \equiv J_1$. The relation between (ϕ, J) values

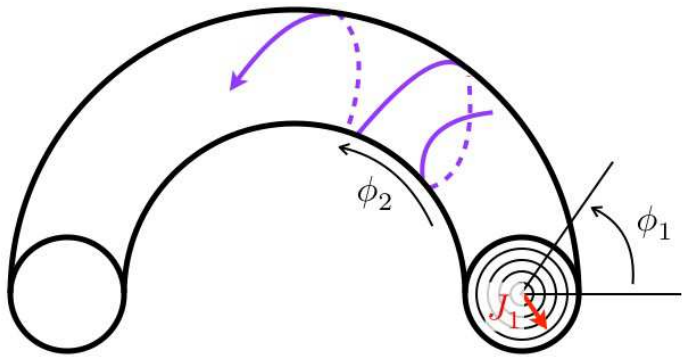


Figure 17.1: Motion of an $n = 2$ system on an invariant torus specified by action J_1, with $E(J_1, J_2)$ fixed.

at successive crossings of this surface is

$$\begin{aligned} \phi_{n+1} &= \phi_n + 2\pi\alpha(J_{n+1}) \\ J_{n+1} &= J_n \quad . \end{aligned} \tag{17.2}$$

Formally, we may write the map as $\boldsymbol{\varphi}_{n+1} = \hat{T}\boldsymbol{\varphi}_n$, where $\boldsymbol{\varphi}_n = (\phi_n, J_n)$ and $\hat{T}$ is the map. Note that the action variable is unchanged during the motion[1], and hence is fixed under the map. We are left with a mapping of the circle onto itself, called the *twist map*. Since the map faithfully represents Hamiltonian evolution, it must be canonical, meaning

$$\big\{\phi_{n+1}, J_{n+1}\big\}_{(\phi_n,J_n)} = \text{det}\, \frac{\partial(\phi_{n+1}, J_{n+1})}{\partial(\phi_n, J_n)} = \frac{\partial\phi_{n+1}}{\partial\phi_n}\,\frac{\partial J_{n+1}}{\partial J_n} - \frac{\partial\phi_{n+1}}{\partial J_n}\,\frac{\partial J_{n+1}}{\partial\phi_n} = 1 \quad , \tag{17.3}$$

which is indeed satisfied. If $\alpha(J) \in \mathbb{Q}$ is rational, certain iterations of the map leave the circle fixed. Specifically, let $\alpha = r/s$. Then $\hat{T}^s$ acts as the identity, leaving the entire circle (and indeed the entire (ϕ, J) plane) fixed.

For systems with N degrees of freedom, with (ϕ_N, J_N) or (ϕ_N, E) fixing the surface of section, one defines $\boldsymbol{\varphi} = (\phi_1, \ldots, \boldsymbol{\phi}_{N-1})$ and $\boldsymbol{J} = (J_1, \ldots, J_{N-1})$, and with $\boldsymbol{\alpha} = (\omega_1/\omega_N, \ldots, \omega_{N-1}/\omega_N)$ one has

$$\begin{aligned} \boldsymbol{\varphi}_{n+1} &= \boldsymbol{\phi}_n + 2\pi\boldsymbol{\alpha}(\boldsymbol{J}_{n+1}) \\ \boldsymbol{J}_{n+1} &= \boldsymbol{J}_n \quad . \end{aligned} \tag{17.4}$$

One can check that this map is also area-preserving (canonical).

17.1.2 The perturbed twist map

Now consider a perturbed Hamiltonian $H(\boldsymbol{\phi}, \boldsymbol{J}) = H_0(\boldsymbol{J}) + \epsilon H_1(\boldsymbol{\phi}, \boldsymbol{J})$, again for $N = 2$. Once more we consider the surface of section defined by the (ϕ_1, J_1) plane. We expect a perturbed twist map $\hat{T}_\epsilon$ of the form

$$\begin{aligned} \phi_{n+1} &= \phi_n + 2\pi\alpha(J_{n+1}) + \epsilon\, f(\phi_n, J_{n+1}) \\ J_{n+1} &= J_n + \epsilon\, g(\phi_n, J_{n+1}) \quad , \end{aligned} \tag{17.5}$$

[1]It is for this reason that we may write $\alpha(J_{n+1})$ in the first equation, rather than $\alpha(J_n)$. The reason will soon be apparent.

for some functions f and g. Is the perturbed twist map canonical? We could investigate this by computing the Poisson bracket $\{\phi_{n+1}, J_{n+1}\}$, but here we take another approach, which is to exhibit explicitly a type-II generator $F_2(\phi_n, J_{n+1})$ which effects the canonical transformation $(\phi_n, J_n) \to (\phi_{n+1}, J_{n+1})$. Consider the generator

$$F_2(\phi_n, J_{n+1}) = \phi_n J_{n+1} + 2\pi A(J_{n+1}) + \epsilon B(\phi_n, J_{n+1}) \quad . \tag{17.6}$$

The CT generated is

$$\begin{aligned} \phi_{n+1} &= \frac{\partial F_2}{\partial J_{n+1}} = \phi_n + 2\pi \frac{\partial A}{\partial J_{n+1}} + \epsilon \frac{\partial B}{\partial J_{n+1}} \\ J_n &= \frac{\partial F_2}{\partial \phi_n} = J_{n+1} + \epsilon \frac{\partial B}{\partial \phi_n} \quad . \end{aligned} \tag{17.7}$$

We therefore identify $\alpha(J_{n+1}) = A'(J_{n+1})$ as well as

$$f(J_{n+1}, \phi_n) = \frac{\partial B}{\partial J_{n+1}} \qquad , \qquad g(J_{n+1}, \phi_n) = -\frac{\partial B}{\partial \phi_n} \quad . \tag{17.8}$$

This, in turn, requires

$$\frac{\partial f}{\partial \phi_n} + \frac{\partial g}{\partial J_{n+1}} = 0 \quad , \tag{17.9}$$

which is a necessary and sufficient condition in order that the map $\hat{T}_\epsilon$ be canonical.

In the case $g = g(\phi_n)$, the above condition requires $f = f(J_{n+1})$, and we may absorb $f(J)$ into the definition of $\alpha(J)$. We then have the map

$$\begin{aligned} \phi_{n+1} &= \phi_n + 2\pi\alpha(J_{n+1}) \\ J_{n+1} &= J_n + \epsilon\, g(\phi_n) \quad . \end{aligned} \tag{17.10}$$

For $\alpha(J) = J$ and $g(\phi) = -\sin\phi$, we obtain the *standard map*, about which we shall have more to say below.

17.2 From Time-Dependent Hamiltonian Systems to Maps

17.2.1 Parametric oscillator

Consider the equation

$$\ddot{x} + \omega_0^2(t)\, x = 0 \quad , \tag{17.11}$$

where the oscillation frequency is a function of time. Equivalently,

$$\frac{d}{dt}\begin{pmatrix} x \\ \dot{x} \end{pmatrix} = \overbrace{\begin{pmatrix} 0 & 1 \\ -\omega_0^2(t) & 0 \end{pmatrix}}^{M(t)} \overbrace{\begin{pmatrix} x \\ \dot{x} \end{pmatrix}}^{\varphi(t)} \quad . \tag{17.12}$$

The formal solution is the path-ordered exponential,

$$\boldsymbol{\varphi}(t) = \mathcal{P}\exp\left\{\int\limits_0^t dt'\, M(t')\right\}\boldsymbol{\varphi}(0) \quad . \tag{17.13}$$

Let's consider an example in which

$$\omega(t) = \begin{cases} (1+\epsilon)\,\omega_0 & \text{if } 2n\tau \le t < (2n+1)\tau \\ \\ (1-\epsilon)\,\omega_0 & \text{if } (2n+1)\tau \le t < (2n+2)\tau \quad . \end{cases} \tag{17.14}$$

Define $\boldsymbol{\varphi}_n \equiv \boldsymbol{\varphi}(2n\tau)$. Then

$$\boldsymbol{\varphi}_{n+1} = \exp(M_-\tau)\, \exp(M_+\tau)\,\boldsymbol{\varphi}_n \equiv \mathcal{U}\,\boldsymbol{\varphi}_n \quad , \tag{17.15}$$

where

$$M_\pm = \begin{pmatrix} 0 & 1 \\ -\omega_\pm^2 & 0 \end{pmatrix} \quad , \tag{17.16}$$

with $\omega_\pm \equiv (1\pm\epsilon)\,\omega_0$. Note that $M_\pm^2 = -\omega_\pm^2\,\mathbb{I}$ is a multiple of the identity. Evaluating the Taylor series for the exponential, one finds

$$\mathcal{U}_\pm \equiv \exp(M_\pm t) = \begin{pmatrix} \cos\omega_\pm\tau & \omega_\pm^{-1}\sin\omega_\pm\tau \\ -\omega_\pm\,\sin\omega_\pm\tau & \cos\omega_\pm\tau \end{pmatrix} \quad , \tag{17.17}$$

from which we derive the evolution matrix

$$\mathcal{U} \equiv \mathcal{U}_-\mathcal{U}_+ = \begin{pmatrix} \cos\omega_-\tau & \omega_-^{-1}\sin\omega_-\tau \\ -\omega_-\,\sin\omega_-\tau & \cos\omega_-\tau \end{pmatrix}\begin{pmatrix} \cos\omega_+\tau & \omega_+^{-1}\sin\omega_+\tau \\ -\omega_+\,\sin\omega_+\tau & \cos\omega_+\tau \end{pmatrix} \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with

$$\begin{aligned} a &= \cos\omega_-\tau\,\cos\omega_+\tau - \frac{\omega_+}{\omega_-}\,\sin\omega_-\tau\,\sin\omega_+\tau \\ b &= \frac{1}{\omega_+}\,\cos\omega_-\tau\sin\omega_+\tau + \frac{1}{\omega_-}\,\sin\omega_-\tau\cos\omega_+\tau \\ c &= -\omega_+\,\cos\omega_-\tau\sin\omega_+\tau - \omega_-\,\sin\omega_-\tau\cos\omega_+\tau \\ d &= \cos\omega_-\tau\,\cos\omega_+\tau - \frac{\omega_-}{\omega_+}\,\sin\omega_-\tau\,\sin\omega_+\tau \quad . \end{aligned} \tag{17.18}$$

Note that $\mathcal{U}_\pm$ are each symplectic, hence $\det\exp(M_\pm\tau) = 1$, and therefore $\mathcal{U}$ is also symplectic with $\det\mathcal{U} = 1$. Also note that

$$P(\lambda) = \det\big(\mathcal{U} - \lambda\cdot\mathbb{I}\big) = \lambda^2 - T\lambda + \varDelta \quad , \tag{17.19}$$

where $T = a + d = \mathrm{Tr}\,\mathcal{U}$ and $\varDelta = ad - bc = \det\,\mathcal{U}$. The eigenvalues of $\mathcal{U}$ are

$$\lambda_\pm = \tfrac{1}{2}T \pm \tfrac{1}{2}\sqrt{T^2 - 4\varDelta} \quad . \tag{17.20}$$

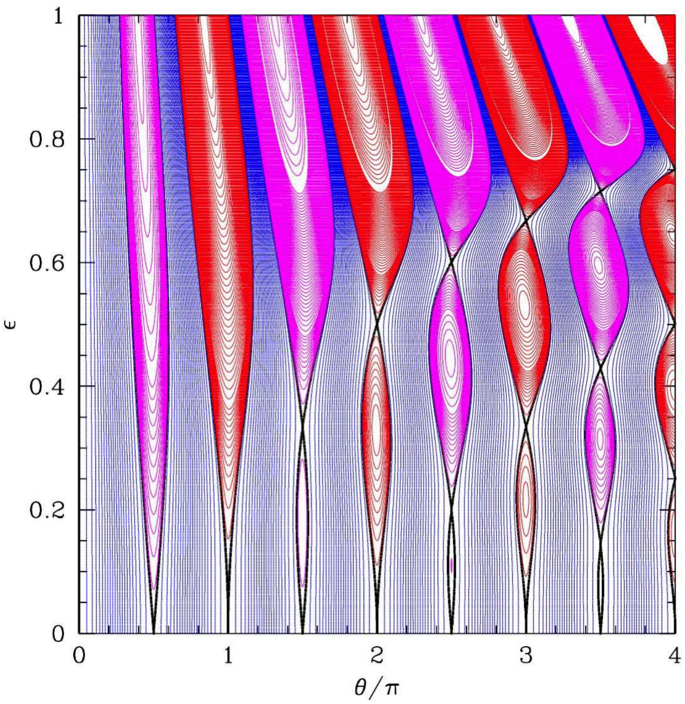


Figure 17.2: Phase diagram for the parametric oscillator in the (θ, ϵ) plane. Thick black lines correspond to $T = \pm 2$. Blue regions: $|T| < 2$. Red regions: $T > 2$. Magenta regions: $T < -2$.

In our case, $\Delta = 1$. There are two cases to consider:

$$\begin{aligned} |T| < 2 &: \lambda_+ = \lambda_-^* = e^{i\delta} \quad , \quad \delta = \cos^{-1} \tfrac{1}{2} T \\ |T| > 2 &: \lambda_+ = \lambda_-^{-1} = \pm e^{\mu} \quad , \quad \mu = \cosh^{-1} \tfrac{1}{2} |T| \quad . \end{aligned} \tag{17.21}$$

When $|T| < 2$, φ remains bounded; when $|T| > 2$, $|\varphi|$ increases exponentially with time. Note that phase space volumes are preserved by the dynamics.

To investigate more fully, let $\theta \equiv \omega_0 \tau$. The period of the frequency oscillations is $\Delta t = 2\tau$, *i.e.* $\omega_{\text{pump}} = \pi/\tau$ is the frequency at which the system is 'pumped', so

$$\frac{\theta}{\pi} = \frac{\omega_0}{\omega_{\text{pump}}} = \frac{T_{\text{pump}}}{T_0} \quad , \tag{17.22}$$

where $T_0 = 2\pi/\omega_0$ is the unperturbed natural frequency and $T_{\text{pump}} = \Delta t = 2\tau$. One finds $T = \text{Tr}\,\mathcal{U}$ is given by

$$T = \frac{2\cos(2\theta) - 2\epsilon^2 \cos(2\epsilon\theta)}{1 - \epsilon^2} \quad . \tag{17.23}$$

We are interested in the boundaries in the (θ, ϵ) plane where $|T| = 2$. Setting $T = +2$, we write $\theta = n\pi + \delta$, which means $T_{\text{pump}} \approx n T_0$. Expanding for small δ and ϵ, we obtain the relation

$$\delta^2 = n^2 \pi^2 \epsilon^4 \quad \Rightarrow \quad \epsilon = \pm \left| \frac{\delta}{n\pi} \right|^{1/2} \quad . \tag{17.24}$$

Setting $T = -2$, we write $\theta = (n + \frac{1}{2})\pi + \delta$, *i.e.* $T_{\rm pump} \approx (n + \frac{1}{2})\, T_0$. This gives

$$\delta^2 = \epsilon^2 \quad \Rightarrow \quad \epsilon = \pm\delta \quad . \tag{17.25}$$

The full phase diagram in the (θ, ϵ) plane is shown in fig. 17.2. A physical example is pumping a swing. By extending your legs periodically, you effectively change the length $\ell(t)$ of the pendulum, resulting in a time-dependent $\omega_0(t) = \sqrt{g/\ell(t)}$.

17.2.2 Kicked dynamics

A related model is described by the *kicked dynamics* of the Hamiltonian

$$H(t) = T(p) + V(q)\, K(t) \quad , \tag{17.26}$$

where

$$K(t) = \tau \sum_{n=-\infty}^{\infty} \delta(t - n\tau) \tag{17.27}$$

is the kicking function. The potential thus winks on and off with period τ. Note that

$$\lim_{\tau\to 0} K(t) = 1 \quad . \tag{17.28}$$

In the $\tau \to 0$ limit, the system is continuously kicked, and is equivalent to motion in a time-independent external potential $V(q)$.

The equations of motion are

$$\dot q = T'(p) \qquad , \qquad \dot p = -V'(q)\, K(t) \quad . \tag{17.29}$$

Integrating these equations, we obtain the map

$$\begin{aligned} q_{n+1} &= q_n + \tau\, T'(p_n) \\ p_{n+1} &= p_n - \tau\, V'(q_{n+1}) \quad , \end{aligned} \tag{17.30}$$

where $q_n = q(t = n\tau^+)$ and $p_n = p(t = n\tau^+)$. Note that the determinant of Jacobean of the map is unity:

$$\textsf{det}\, \frac{\partial(q_{n+1}\, ,\, p_{n+1})}{\partial(q_n\, ,\, p_n)} = \textsf{det} \begin{pmatrix} 1 & \tau\, T''(p_n) \\ -\tau\, V''(q_{n+1}) & 1 - \tau^2\, T''(p_n)\, V''(q_{n+1}) \end{pmatrix} = 1 \quad . \tag{17.31}$$

This means that the map preserves phase space volumes.

Consider, for example, the Hamiltonian $H(t) = \frac{L^2}{2I} - V\cos(\phi)\, K(t)$, where L is the angular momentum conjugate to ϕ. This results in the map

$$\begin{aligned} \phi_{n+1} &= \phi_n + 2\pi\epsilon\, J_n \\ J_{n+1} &= J_n - \epsilon \sin\phi_{n+1} \quad , \end{aligned} \tag{17.32}$$

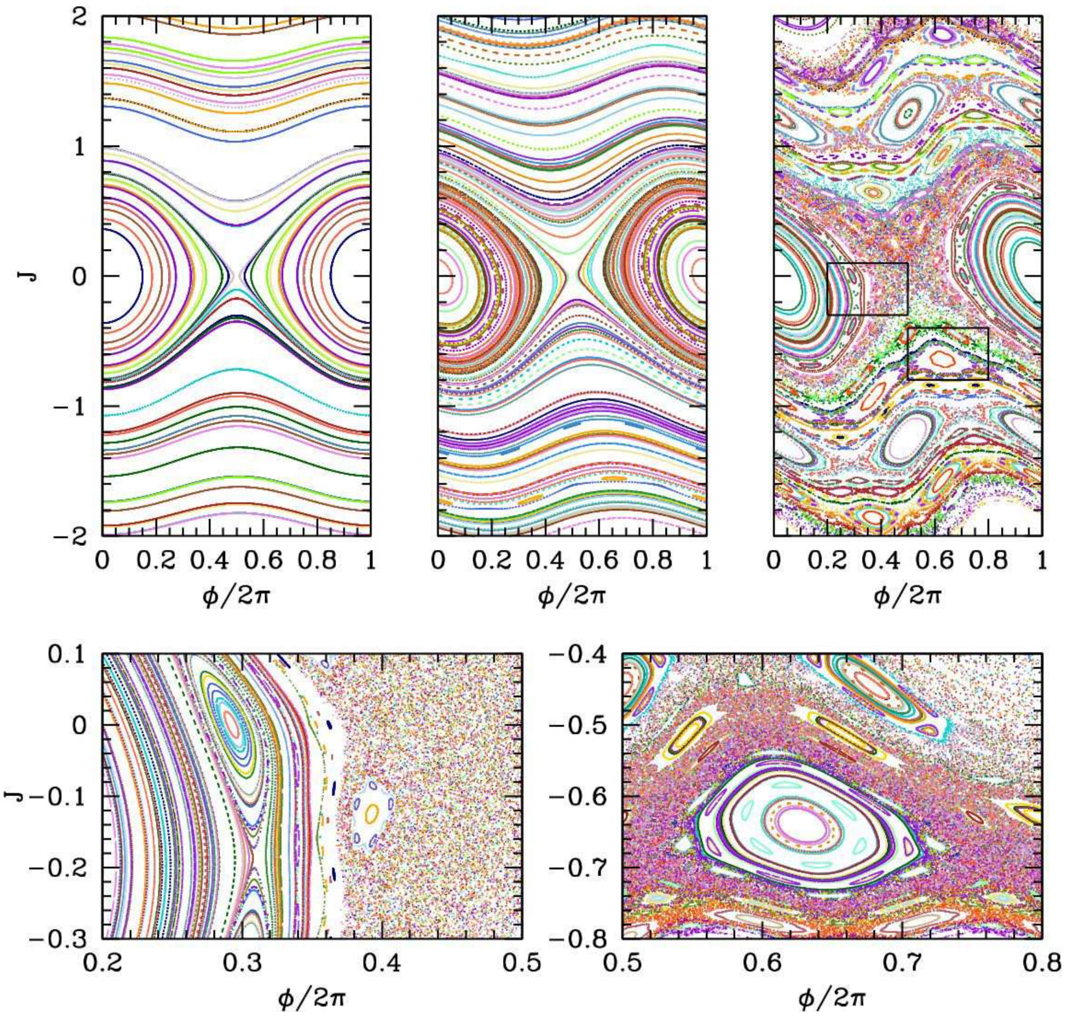


Figure 17.3: Top: the standard map, as defined in the text. Three values of the ϵ parameter are shown: $\epsilon = 0.01$ (left), $\epsilon = 0.2$ (center), and $\epsilon = 0.4$ (right). Bottom: details of the $\epsilon = 0.4$ map.

where $J_n = L_n/\sqrt{2\pi IV}$ and $\epsilon = \tau\sqrt{V/2\pi I}$. This is the standard map[2], which we encountered earlier, albeit in a slightly different form. In the limit $\epsilon \to 0$, we may define $\dot{\phi} = (\phi_{n+1} - \phi_n)/\epsilon$ and $\dot{J} = (J_{n+1} - J_n)/\epsilon$, and we recover the continuous time dynamics $\dot{\phi} = 2\pi J$ and $\dot{J} = -\sin\phi$. These dynamics preserve the energy function $E = \pi J^2 - \cos\phi$. There is a separatrix at $E = 1$, given by $J(\phi) = \pm\frac{2}{\pi}\big|\cos(\phi/2)\big|$. We see from fig. 17.3 that this separatrix is the first structure to be replaced by a chaotic fuzz as ϵ increases from zero to a small finite value.

Another well-studied system is the *kicked Harper model*, for which

$$H(t) = -V_1 \cos\left(\frac{2\pi p}{P}\right) - V_2 \cos\left(\frac{2\pi q}{Q}\right) K(t) \quad . \tag{17.33}$$

[2]The standard map us usually written in the form $x_{n+1} = x_n + \mathcal{J}_n$ and $\mathcal{J}_{n+1} = \mathcal{J}_n - k\sin(2\pi x_{n+1})$. We can recover our version by rescaling $\phi_n = 2\pi x_n$, $\mathcal{J}_n \equiv \sqrt{k}\, J_n$ and defining $\epsilon \equiv \sqrt{k}$.

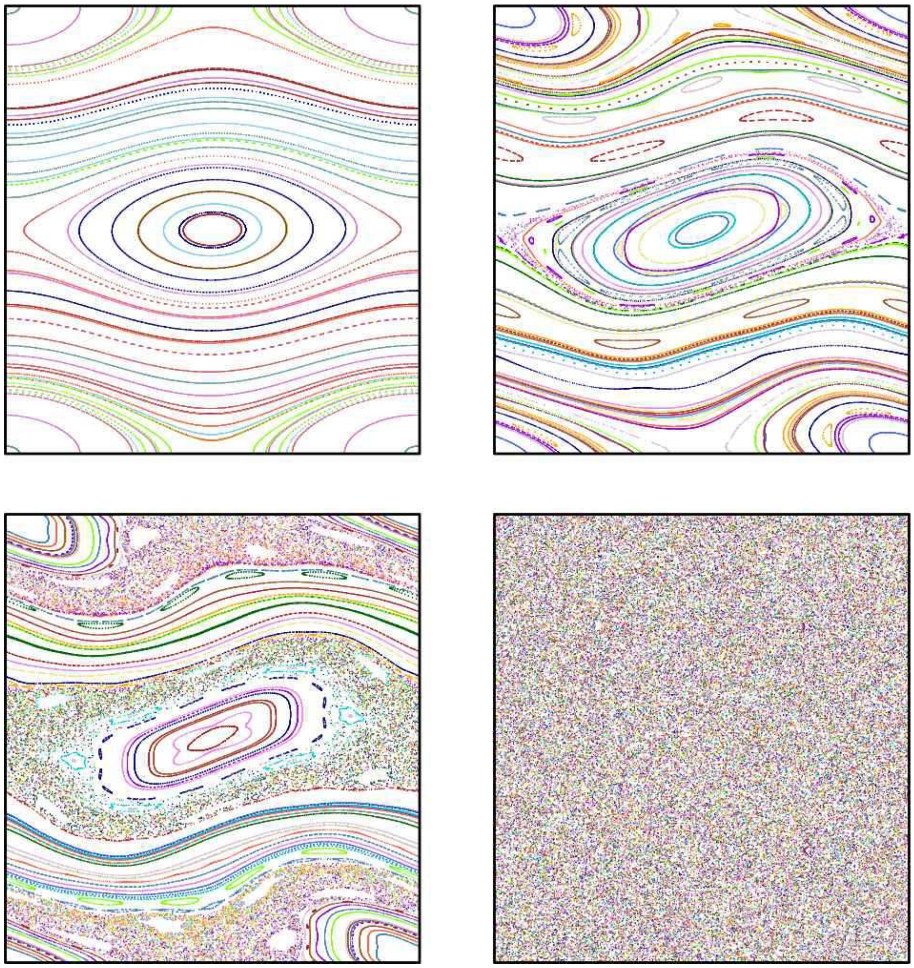


Figure 17.4: The kicked Harper map, with $\alpha = 2$, and with $\epsilon = 0.01$ (upper left), $\epsilon = 0.125$ (upper right), $\epsilon = 0.2$ (lower left) and $\epsilon = 5.0$ (lower right). The phase space here is the unit torus, $\mathbb{T}^2 = [0,1] \times [0,1]$.

With $x = q/Q$ and $y = p/P$, Hamilton's equations generate the map

$$\begin{aligned} x_{n+1} &= x_n + \epsilon\,\alpha\,\sin(2\pi y_n) \\ y_{n+1} &= y_n - \frac{\epsilon}{\alpha}\,\sin(2\pi x_{n+1}) \quad , \end{aligned} \tag{17.34}$$

where $\epsilon = 2\pi\tau\sqrt{V_1 V_2}/PQ$ and $\alpha = \sqrt{V_1/V_2}$ are dimensionless parameters. In this case, the conserved energy is

$$E = -\alpha^{-1}\cos(2\pi x) - \alpha\cos(2\pi y) \quad . \tag{17.35}$$

There are then two separatrices, at $E = \pm(\alpha - \alpha^{-1})$, with equations $\alpha\cos(\pi y) = \pm\sin(\pi x)$ and $\alpha\sin(\pi y) = \pm\cos(\pi x)$. Again, as is apparent from fig. 17.4, the separatrix is the first structure to be destroyed at finite ϵ. This also occurs for the standard map – there is a transition to *global stochasticity* at a critical value of ϵ.

Note that the kicking function may be written as

$$K(t) = \tau \sum_{n=-\infty}^{\infty} \delta(t - n\tau) = \sum_{m=-\infty}^{\infty} \cos\left(\frac{2\pi m t}{\tau}\right) \quad , \tag{17.36}$$

a particularly handy result known as the *Poisson summation formula*. This, a kicked Hamiltonian may be written as

$$H(J, \phi, t) = H_0(J) + V(\phi) \sum_{m=-\infty}^{\infty} \cos\left(\frac{2\pi m t}{\tau}\right) \quad . \tag{17.37}$$

The $m = 0$ term generates the continuous time dynamics $\dot{\phi} = \omega_0(J)$, $\dot{J} = -V'(\phi)$. For the standard map, these are the dynamics of a simple pendulum. The $m \neq 0$ terms are responsible for resonances and the formation of so-called 'stochastic layers'.

17.3 Local Stability and Lyapunov Exponents

17.3.1 The fate of nearly separated initial conditions under iteration

Consider a map $\hat{T}$ acting on a phase space of dimension $2N$ (*i.e.* N position degrees of freedom). We ask what is the fate of two nearby initial conditions, $\boldsymbol{\xi}_0$ and $\boldsymbol{\xi}_0 + d\boldsymbol{\xi}$, under the iterated map. Under the first iteration, we have $\boldsymbol{\xi}_0 \to \boldsymbol{\xi}_1 = \hat{T}\boldsymbol{\xi}_0$ and

$$\boldsymbol{\xi}_0 + d\boldsymbol{\xi} \longrightarrow \boldsymbol{\xi}_1 + M(\boldsymbol{\xi}_0)\, d\boldsymbol{\xi} \quad , \tag{17.38}$$

where $M(\boldsymbol{\xi})$ is a matrix given by the *linearization of* $\hat{T}$ *at* $\boldsymbol{\xi}$, *viz.*

$$M_{ij}(\boldsymbol{\xi}) = \frac{\partial(\hat{T}\boldsymbol{\xi})_i}{\partial \xi_j} \quad . \tag{17.39}$$

Let's iterate again. Clearly $\boldsymbol{\xi}_1 \to \boldsymbol{\xi}_2 = \hat{T}^2\boldsymbol{\xi}_0$ and

$$\boldsymbol{\xi}_1 + M(\boldsymbol{\xi}_0)\, d\boldsymbol{\xi} \longrightarrow \boldsymbol{\xi}_2 + M(\boldsymbol{\xi}_1) M(\boldsymbol{\xi}_0)\, d\boldsymbol{\xi} \quad . \tag{17.40}$$

After n iterations, we clearly have $\hat{T}^n\boldsymbol{\xi}_0 = \boldsymbol{\xi}_n$ and

$$\hat{T}^n(\boldsymbol{\xi}_0 + d\boldsymbol{\xi}) = \boldsymbol{\xi}_n + M(\boldsymbol{\xi}_{n-1}) \cdots M(\boldsymbol{\xi}_0)\, d\boldsymbol{\xi} \quad , \tag{17.41}$$

and we define $R^{(n)}(\boldsymbol{\xi}) = M\big(\hat{T}^{(n-1)}\boldsymbol{\xi}\big) \cdots M(\hat{T}\boldsymbol{\xi}) M(\boldsymbol{\xi})$, a linear operator – *i.e.* a matrix – whose matrix elements are given by $R^{(n)}_{ij}(\boldsymbol{\xi}) = \partial(\hat{T}^n\boldsymbol{\xi})_i / \partial \xi_j$.

Since the map $\hat{T}$ is presumed to be canonical, at each stage $M(\boldsymbol{\xi}_j) \in \mathsf{Sp}(2N)$, and since the product of symplectic matrices is a symplectic matrix, $R^{(n)}(\boldsymbol{\xi}) \in \mathsf{Sp}(2N)$. It is easy to see that for any real symplectic matrix R, the eigenvalues come in unimodular conjugate pairs $\{e^{i\delta}, e^{-i\delta}\}$, in real pairs $\{\lambda, \lambda^{-1}\}$ with $\lambda \in \mathbb{R}$, or in quartets $\{\lambda, \lambda^{-1}, \lambda^*, \lambda^{*-1}\}$ with $\lambda \in \mathbb{C}$, where λ^* is the complex conjugate of λ. This follows from analysis of the characteristic polynomial $P(\lambda) = \text{det}(\lambda - R)$ given the symplectic condition[3] $R^{\text{t}}\,\mathbb{J}\,R = \mathbb{J}$. Let $\{\lambda_j^{(n)}(\boldsymbol{\xi})\}$ be the eigenvalues of $R^{(n)}(\boldsymbol{\xi})$, with $j \in \{1, \ldots, 2N\}$. One defines the *Lyapunov exponents*,

$$\nu_j(\boldsymbol{\xi}) = \lim_{n\to\infty} \frac{1}{n} \log\left|\lambda_j^{(n)}(\boldsymbol{\xi})\right| \quad . \tag{17.42}$$

These may be ordered such that $\nu_1 \le \nu_2 \le \cdots \le \nu_{2N}$. Positive Lyapunov exponents correspond to an exponential stretching (as a function of the iteration number n), while negative ones correspond to an exponential squeezing.

As an example, consider the Arnol'd cat map, which is an automorphism of the torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$, given by[4]

$$\begin{aligned} q_{n+1} &= (K+1)\,q_n + p_n \\ p_{n+1} &= K q_n + p_n \quad , \end{aligned} \tag{17.43}$$

where $K \in \mathbb{Z}$, and where both q_n and p_n are defined modulo unity, so $(q_n, p_n) \in [0,1]\times[0,1]$. Note that K must be an integer in order for the map to be smooth on the torus, *i.e.* it is left unchanged by displacing either coordinate by an integer distance. The map is already linear, hence we can read off

$$M = \frac{\partial(q_{n+1}, p_{n+1})}{\partial(q_n, p_n)} = \begin{pmatrix} K+1 & 1 \\ K & 1 \end{pmatrix} \quad , \tag{17.44}$$

which is independent of (q_n, p_n). The inverse map also has integer coefficients:

$$M^{-1} = \begin{pmatrix} 1 & -1 \\ -K & K+1 \end{pmatrix} \quad . \tag{17.45}$$

Since $\text{det}\, M = 1$, the cat map is canonical, *i.e.* it preserves phase space volumes. The eigenvalues of M are the roots of the characteristic polynomial $P(\lambda) = \lambda^2 - (K+2)\lambda - K$, and are given by

$$\lambda_\pm = 1 + \tfrac{1}{2}K \pm \sqrt{K + \tfrac{1}{4}K^2} \quad . \tag{17.46}$$

Thus, for $K \in \{-4, -3, -2, -1, 0\}$, the eigenvalues come in pairs $e^{\pm i\delta_K}$, with $\delta_{-4} = \pi$, $\delta_{-3} = \frac{2}{3}\pi$, $\delta_{-2} = \frac{1}{2}\pi$, $\delta_{-1} = \frac{1}{3}\pi$, and $\delta_0 = 0$. For $K < -4$ or $K > 0$, the eigenvalues are (λ, λ^{-1}) with $|\lambda| > 1$ and $0 < |\lambda|^{-1} < 1$, corresponding, respectively, to stretching and squeezing. The Lyapunov exponents are $\nu_\pm = \log|\lambda_\pm|$.

17.3.2 Kolmogorov-Sinai entropy

Let $\mathit{\Gamma} < \infty$ be our phase space (at constant energy, for a Hamiltonian system), and $\{\mathit{\Delta}_j\}$ a partition of disjoint sets whose union is $\mathit{\Gamma}$. The simplest arrangement to think of is for each $\mathit{\Delta}_j$ to correspond to a little hypercube; stacking up all the hypercube builds the entire phase space. Now apply the inverse

[3] One has $P(\lambda) = \text{det}(\lambda - R) = \text{det}(\lambda - R^{\text{t}}) = \text{det}(\lambda + \mathbb{J}R^{-1}\mathbb{J}) = \text{det}(\lambda^{-1} - R)\cdot\lambda^{2N} / \,\text{det}\, R$ and therefore if λ is a root of the characteristic polynomial, then so is λ^{-1}. Since $R = R^*$, one also has $P(\lambda^*) = \big[P(\lambda)\big]^*$, hence if λ is a root, then so is λ^*. From $\text{Pf}(R^{\text{t}}\,\mathbb{J}\,R) = \text{det}(R)\,\text{Pf}(\mathbb{J})$, where Pf is the Pfaffian, one has $\text{det}\, R = 1$.

[4] The map in eqn. 17.43 is a generalized version of Arnol'd's original cat map, which had $K = 1$.

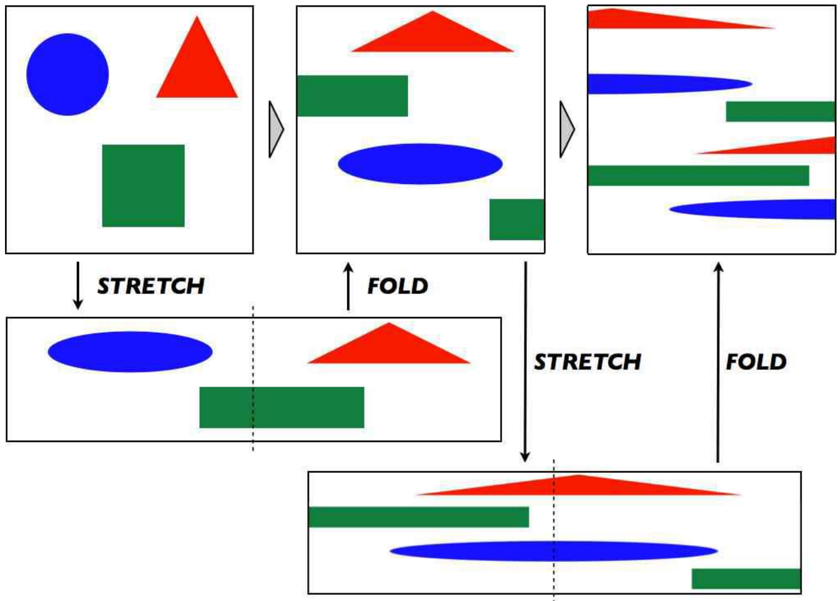


Figure 17.5: The baker's transformation involves stretching/squeezing and 'folding' (cutting and restacking).

map $\hat{T}^{-1}$ to each $\varDelta_j$, and form the intersections $\varDelta_{jk} \equiv \varDelta_j \cap \hat{T}^{-1}\varDelta_k$. If $\sum_j \mu(\varDelta_j) = \mu(\varGamma) \equiv 1$, then $\sum_{j,k} \mu(\varDelta_{jk}) = 1$. Iterating further, we obtain $\varDelta_{jkl} \equiv \varDelta_j \cap \hat{T}^{-1}\varDelta_k \cap \hat{T}^{-2}\varDelta_l$, *etc.*

The entropy of a distribution $\{p_a\}$ is defined to be $S = -\sum_a p_a \log p_a$. Accordingly we define

$$S_L(\boldsymbol{\varDelta}) = -\sum_{j_1} \cdots \sum_{j_L} \mu(\varDelta_{j_1 \cdots j_L}) \log \mu(\varDelta_{j_1 \cdots j_L}) \quad . \tag{17.47}$$

This is a function of both the iteration number L as well as the initial set $\boldsymbol{\varDelta} = \{\varDelta_1, \ldots, \varDelta_r\}$, where r is the number of subregions in our original partition. We then define the Kolmogorov-Sinai entropy to be

$$h_{\rm KS} \equiv \sup_{\boldsymbol{\varDelta}} \lim_{L\to\infty} \frac{1}{L}\, S_L(\boldsymbol{\varDelta}) \quad . \tag{17.48}$$

Here sup stands for *supremum*, meaning we maximize over all partitions $\boldsymbol{\varDelta}$.

Consider, for example, the *baker's transformation* (see fig. 17.5), which stretches, cuts, stacks, and compresses the torus according to

$$(q', p') = \hat{T}(q, p) = \begin{cases} (2q\,,\, \frac{1}{2}p) & \text{if } 0 \le q < \frac{1}{2} \\ (2q-1, \frac{1}{2}p + \frac{1}{2}) & \text{if } \frac{1}{2} \le q < 1 \end{cases} \tag{17.49}$$

It is not difficult to convince oneself that the KS entropy for the baker's transformation is $h_{\rm KS} = \log 2$. On the other hand, for a simple translation map which takes $(q, p) \to (q', p') = (q + \alpha\,,\, p + \beta)$, it is easy to see that $h_{\rm KS} = 0$. The KS entropy is related to the Lyapunov exponents through the formula

$$h_{\rm KS} = \sum_j \nu_j \, \Theta(\nu_j) \quad . \tag{17.50}$$

The RHS is the sum over all the positive Lyapunov exponents $\nu_j > 0$. Actually, this formula presumes that the ν_j do not vary in phase space, but in general this is not the case. The more general result is known as *Pesin's entropy formula*,

$$h_{\rm KS} = \int\limits_{\Gamma} d\mu(\boldsymbol{\xi}) \sum_j \nu_j(\boldsymbol{\xi})\,\Theta\big(\nu_j(\boldsymbol{\xi})\big) \quad . \tag{17.51}$$

17.4 The Poincaré-Birkhoff Theorem

Let's return to our discussion of the perturbed twist map,

$$\begin{pmatrix} \phi_{n+1} \\ J_{n+1} \end{pmatrix} = \hat{T}_\epsilon \begin{pmatrix} \phi_n \\ J_n \end{pmatrix} = \begin{pmatrix} \phi_n + 2\pi\alpha(J_{n+1}) + \epsilon f(\phi_n, J_{n+1}) \\ J_n + \epsilon g(\phi_n, J_{n+1}) \end{pmatrix} \quad , \tag{17.52}$$

with $\frac{\partial f}{\partial \phi_n} + \frac{\partial g}{\partial J_{n+1}} = 0$ in order that the map be canonical. For $\epsilon = 0$, the map $\hat{T}_0$ leaves J invariant, and takes circles to circles. If $\alpha(J) \neq \mathbb{Q}$, the images of the iterated map become dense on the circle.

Consider now a circle with fixed J for which $\alpha(J) = r/s$ is rational (may assume r and s are relatively prime), and without loss of generality let us presume $\alpha'(J) > 0$ so that on circles $J_\pm = J \pm \Delta J$ we have $\alpha(J_+) > r/s$ and $\alpha(J_-) < r/s$. Under $\hat{T}_0^s$, all the points on the circle $\mathcal{C} = \mathcal{C}(J)$ are fixed, whereas those on $\mathcal{C}_+$ rotate slightly counterclockwise, and those on $\mathcal{C}_-$ slightly clockwise (see left panel of fig. 17.6), where $\mathcal{C}_\pm = \mathcal{C}(J_\pm)$. Now consider the action of the iterated perturbed map $\hat{T}_\epsilon^s$. Acting on $\mathcal{C}_+$, the action is still a net counterclockwise shift (assuming $\epsilon \ll \frac{\Delta J}{J}$), with some small $\mathcal{O}(\epsilon)$ radial component. Similarly,

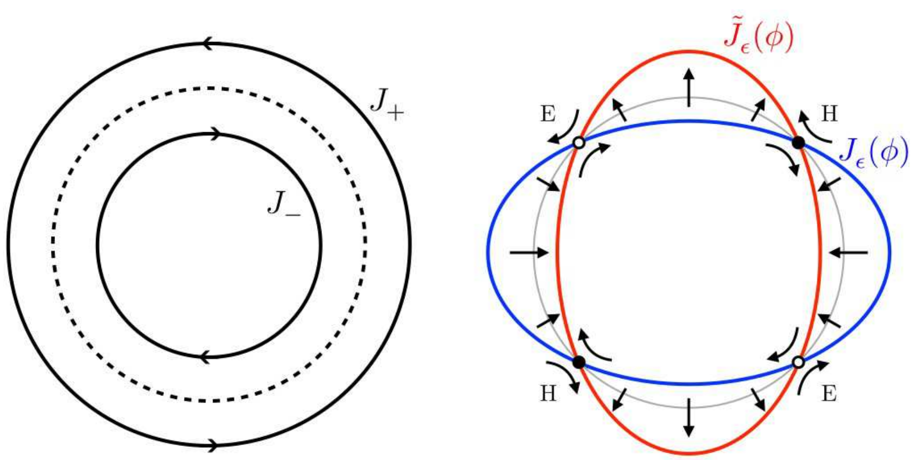


Figure 17.6: Left: The action of the iterated map $\hat{T}_0^s$ leaves the circle with $\alpha(J) = r/s$ invariant (dotted curve), but rotates slightly counterclockwise and clockwise on the circles $J = J_+$ and $J = J_-$, respectively. Right: The blue curve $J_\epsilon(\phi)$ is the locus of points where $\hat{T}_\epsilon^s$ acts purely radially and preserves ϕ, resulting in the red curve $\tilde{J}_\epsilon(\phi)$. Since $\hat{T}_\epsilon^s$ is volume-preserving, these curves must intersect in an alternating sequence of elliptic and hyperbolic fixed points.

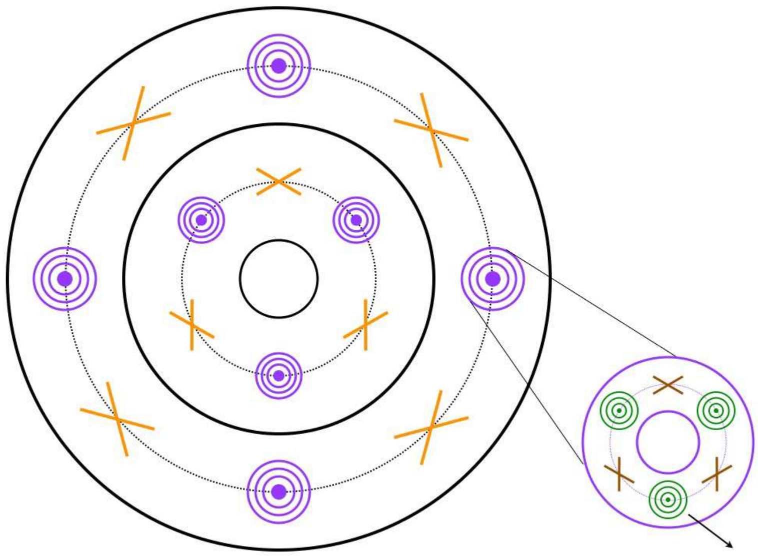


Figure 17.7: Self-similar structures in the iterated twist map.

acting on $\mathcal{C}_-$, the action is a net clockwise shift plus $\mathcal{O}(\epsilon)$ radial component. By the intermediate value theorem, for a given fixed ϕ, as one proceeds radially outward from $\mathcal{C}_-$ to $\mathcal{C}_+$, there must be a point $J = J_\epsilon(\phi)$ where the angular shift vanishes. This defines an entire curve $J_\epsilon(\phi)$ along which the action of $\hat{T}^s_\epsilon$ is purely radial. Now consider the curve $\tilde{J}_\epsilon(\phi) = \hat{T}^s_\epsilon J_\epsilon(\phi)$, *i.e.* the action of $\hat{T}^s_\epsilon$ on the curve $J_\epsilon(\phi)$. We know that each point along $J_\epsilon(\phi)$ is displaced radially, but we also know that $\hat{T}^s_\epsilon$ is volume-preserving. *Therefore, the curves $J_\epsilon(\phi)$ and $\tilde{J}_\epsilon(\phi)$ must intersect at an even number of points*[5]. These intersections are *fixed points* of the map $\hat{T}^s_\epsilon$. The situation is depicted in the right panel of fig. 17.6. From the figure, it is clear that the set $J_\epsilon(\phi) \cap \tilde{J}_\epsilon(\phi)$ consists of alternating elliptic and hyperbolic fixed points (EFPs and HFPs).

What we have just described is the content of the *Poincaré-Birkhoff theorem*: A small perturbation of a resonant torus with $\alpha(J) = r/s$ results in an equal number elliptic and hyperbolic fixed points for the iterated map $\hat{T}^s_\epsilon$. Since T_ϵ has period s acting on these fixed points, the number of EFPs and HFPs must be equal and also a multiple of s. In the vicinity of the EFPs, this structure repeats, as depicted above in fig. 17.7.

Stable/unstable manifolds and homoclinic/heteroclinic intersections

Now consider the HFPs. Emanating from a given HFP $\boldsymbol{\xi}^*$ are *stable and unstable manifolds*, $\Sigma^{\rm S/U}(\boldsymbol{\xi}^*)$, defined by:

$$\begin{aligned}\boldsymbol{\xi} \in \Sigma^{\rm S}(\boldsymbol{\xi}^*) &\Rightarrow \lim_{n\to\infty} \hat{T}^{+ns}_\epsilon \boldsymbol{\xi} = \boldsymbol{\xi}^* \quad \text{(to } \boldsymbol{\xi}^*\text{)} \\ \boldsymbol{\xi} \in \Sigma^{\rm U}(\boldsymbol{\xi}^*) &\Rightarrow \lim_{n\to\infty} \hat{T}^{-ns}_\epsilon \boldsymbol{\xi} = \boldsymbol{\xi}^* \quad \text{(from } \boldsymbol{\xi}^*\text{)} \quad .\end{aligned} \tag{17.53}$$

Note that $\Sigma^{\rm U}(\boldsymbol{\xi}^*_i)$ can never intersect $\Sigma^{\rm U}(\boldsymbol{\xi}^*_j)$ for any i and j, nor can $\Sigma^{\rm S}(\boldsymbol{\xi}^*_i)$ ever intersect $\Sigma^{\rm S}(\boldsymbol{\xi}^*_j)$. However, $\Sigma^{\rm U}(\boldsymbol{\xi}^*_i)$ *can* intersect $\Sigma^{\rm S}(\boldsymbol{\xi}^*_j)$. If $i = j$, such an intersection is called a *homoclinic point*, while for

[5] Tangency is nongeneric and broken by a small change in ϵ.

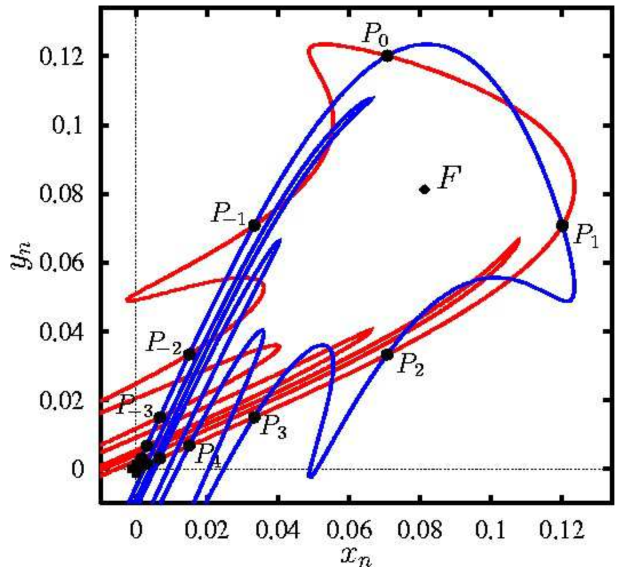


Figure 17.8: Homoclinic tangle for the map $x_{n+1} = y_n$ and $y_{n+1} = (a + by_n^2)\, y_n - x_n$ for parameters $a = 2.693$, $b = -104.888$. Stable (blue) and unstable (red) manifolds emanating from the HFP at $(0, 0)$ are shown.

$i \neq j$ the intersection is called a *heteroclinic point*. Because $\hat{T}^s_\epsilon$ is continuous and invertible, its action at a homoclinic/heteroclinic point will produce a *new* homoclinic/heteroclinic point, *ad infinitum*! For homoclinic intersections, the resulting structure is known as a *homoclinic tangle*, an example of which is shown in fig. 17.8.

17.5 One-Dimensional Maps

Consider now an even simpler case of a purely one-dimensional map,

$$x_{n+1} = f(x_n) \quad . \tag{17.54}$$

A fixed point of the map satisfies $x = f(x)$. Writing the solution as x^* and expanding about the fixed point, we write $x = x^* + u$ and obtain

$$u_{n+1} = f'(x^*)\, u_n + \mathcal{O}(u^2) \quad . \tag{17.55}$$

Thus, the fixed point is stable if $\left|f'(x^*)\right| < 1$, since successive iterates of u then get smaller and smaller. The fixed point is unstable if $\left|f'(x^*)\right| > 1$.

Perhaps the most important and most studied of the one-dimensional maps is the logistic map, where $f(x) = rx(1 - x)$, defined on the interval $x \in [0, 1]$. This has a fixed point at $x^* = 1 - r^{-1}$ if $r > 1$. We then have $f'(x^*) = 2 - r$, so the fixed point is stable if $r \in (1, 3)$. What happens for $r > 3$? We can explore the behavior of the iterated map by drawing a *cobweb diagram*, shown in fig. 17.9. We sketch, on the same graph, the curves $y = x$ (in blue) and $y = f(x)$ (in black). Starting with a point x on the line

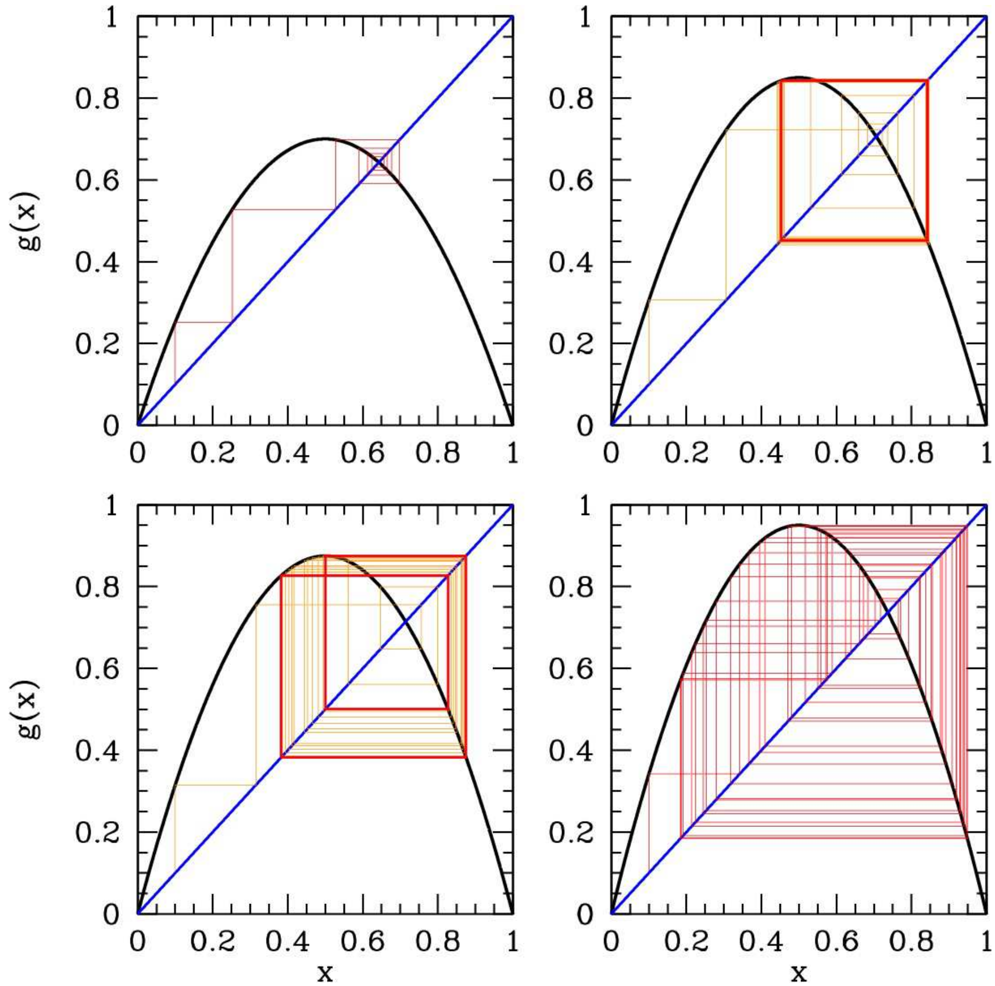


Figure 17.9: Cobweb diagram showing iterations of the logistic map $f(x) = rx(1-x)$ for $r = 2.8$ (upper left), $r = 3.4$ (upper right), $r = 3.5$ (lower left), and $r = 3.8$ (lower right). Note the single stable fixed point for $r = 2.8$, the stable two-cycle for $r = 3.4$, the stable four-cycle for $r = 3.5$, and the chaotic behavior for $r = 3.8$.

$y = x$, we move vertically until we reach the curve $y = f(x)$. To iterate, we then move horizontally to the line $y = x$ and repeat the process. We see that for $r = 3.4$ the fixed point x^* is unstable, but there is a stable two-cycle, defined by the equations

$$\begin{aligned} x_2 &= rx_1(1-x_1) \\ x_1 &= rx_2(1-x_2) \quad . \end{aligned} \tag{17.56}$$

The second iterate of $f(x)$ is then

$$f^{(2)}(x) = f\big(f(x)\big) = r^2 x(1-x)\big(1 - rx + rx^2\big) \quad . \tag{17.57}$$

Setting $x = f^{(2)}(x)$, we obtain a cubic equation. Since $x - x^*$ must be a factor, we can divide out by this

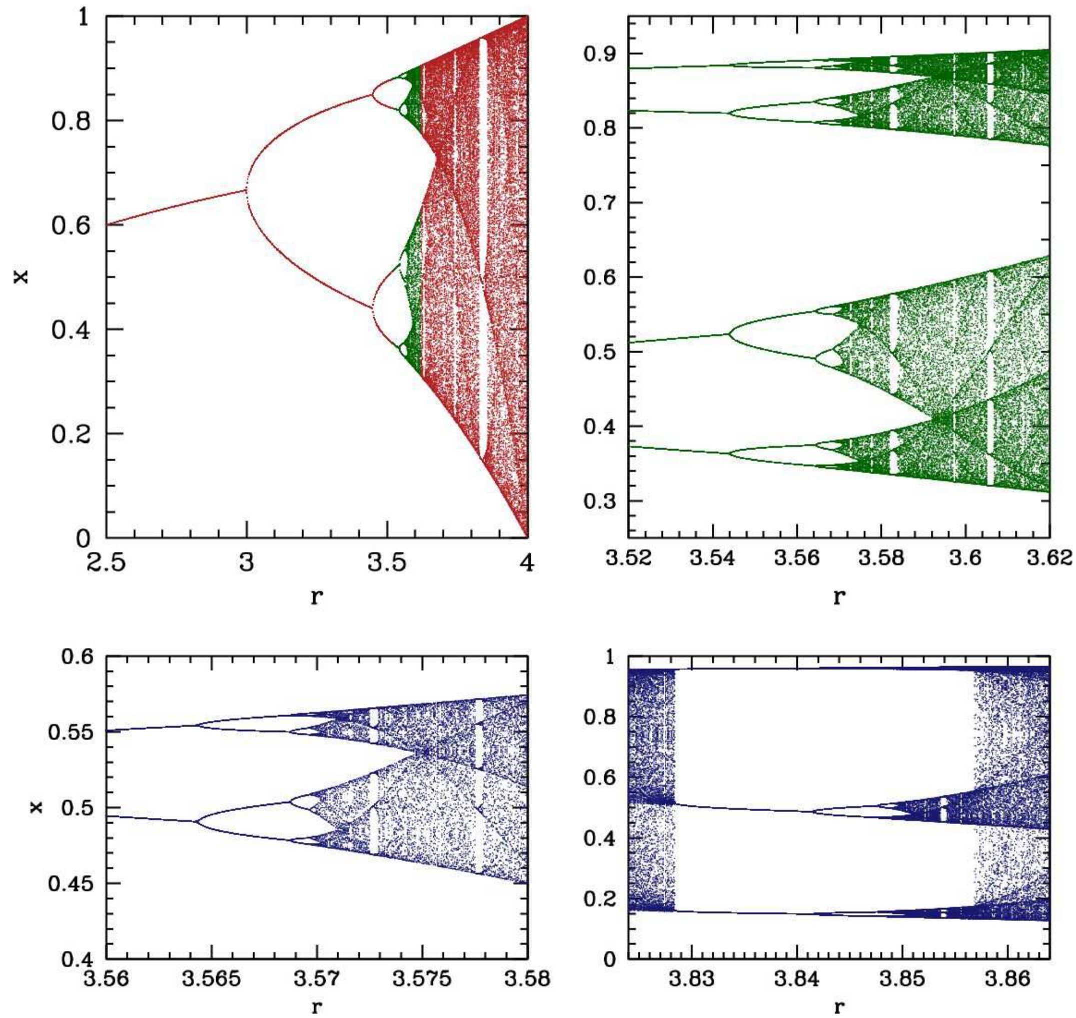


Figure 17.10: Iterates of the logistic map $f(x) = rx(1-x)$.

monomial and obtain a quadratic equation for x_1 and x_2. We find

$$x_{1,2} = \frac{1 + r \pm \sqrt{(r+1)(r-3)}}{2r} \quad . \tag{17.58}$$

How stable is this 2-cycle? We find

$$\frac{d}{dx} f^{(2)}(x) = r^2(1-2x_1)(1-2x_2) = -r^2 + 2r + 4 \quad . \tag{17.59}$$

The condition that the 2-cycle be stable is then

$$-1 < r^2 - 2r - 4 < 1 \qquad \Longrightarrow \qquad r \in \big[3\,,\,1+\sqrt{6}\big] \quad . \tag{17.60}$$

At $r = 1 + \sqrt{6} = 3.4494897\ldots$ there is a bifurcation to a 4-cycle, as can be seen in fig. 17.10.

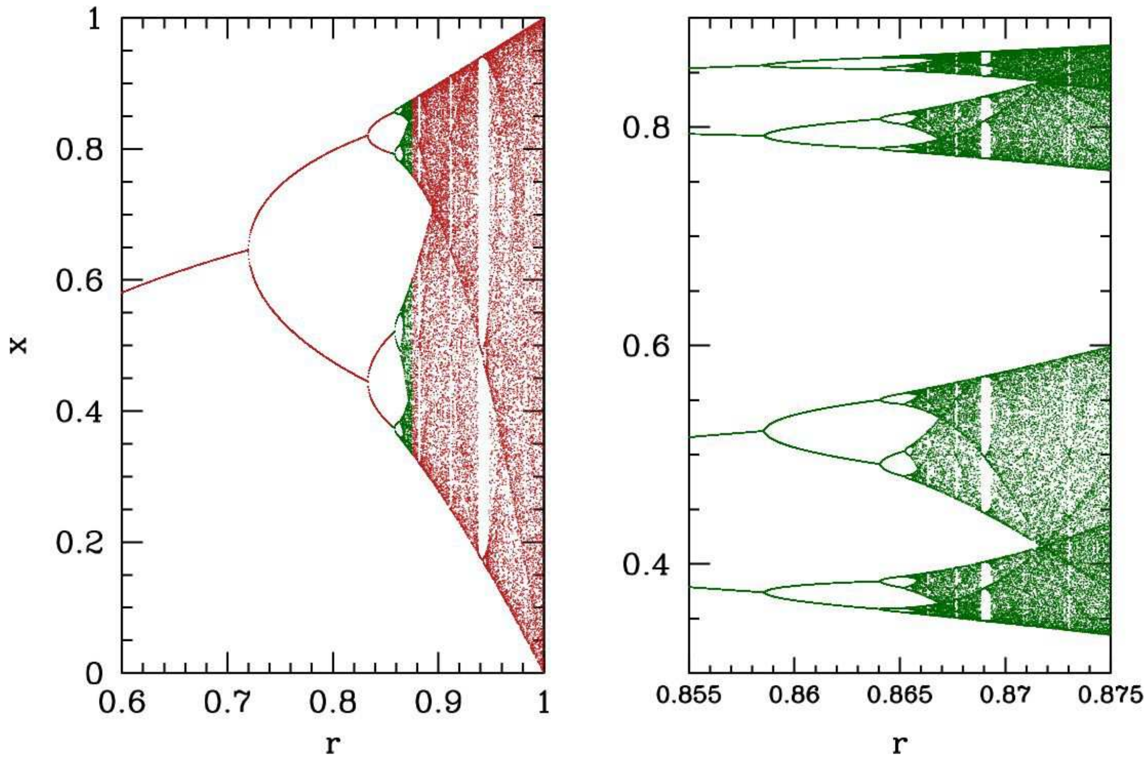


Figure 17.11: Iterates of the sine map $f(x) = r\sin(\pi x)$.

17.5.1 Chaos in the logistic map

What happens in the logistic map for $r > 1 + \sqrt{6}$? At this point, the 2-cycle becomes unstable and a stable 4-cycle develops. However, this soon goes unstable and is replaced by a stable 8-cycle, as the right hand panel of fig. 17.10 shows. The first eight values of r where bifurcations occur are given by

$$\begin{aligned} &r_1 = 3 \quad , \; r_2 = 1 + \sqrt{6} = 3.4494897 \quad , \; r_3 = 3.544096 \quad , \; r_4 = 3.564407 \quad , \\ &r_5 = 3.568759 \quad , \; r_6 = 3.569692 \quad , \; r_7 = 3.569891 \quad , \; r_8 = 3.569934 \quad , \; \ldots \end{aligned} \tag{17.61}$$

Feigenbaum noticed that these numbers seemed to be converging exponentially. With the *Ansatz*

$$r_\infty - r_k \sim \frac{c}{\delta^k} \quad , \tag{17.62}$$

one finds

$$\delta = \lim_{k\to\infty} \frac{r_k - r_{k-1}}{r_{k+1} - r_k} \quad , \tag{17.63}$$

and taking the limit $k \to \infty$ from the above data one finds

$$\delta = 4.669202 \quad , \quad c = 2.637 \quad , \quad r_\infty = 3.5699456 \quad . \tag{17.64}$$

There's a very nifty way of thinking about the chaos in the logistic map at the special value $r = 4$. If we define $x_n \equiv \sin^2\theta_n$, then we find

$$\theta_{n+1} = 2\theta_n \quad . \tag{17.65}$$

Now let us write

$$\theta_0 = \pi \sum_{k=1}^{\infty} \frac{b_k}{2^k} \quad , \tag{17.66}$$

where each b_k is either 0 or 1. In other words, the $\{b_k\}$ are the digits in the binary decimal expansion of θ_0/π. Now $\theta_n = 2^n \theta_0$, hence

$$\theta_n = \pi \sum_{k=1}^{\infty} \frac{b_{n+k}}{2^k} \quad . \tag{17.67}$$

We now see that the logistic map has the effect of *shifting* to the left the binary digits of θ_n/π to yield θ_{n+1}/π. With each such shift, leftmost digit falls off the edge of the world, as it were, since it results in an overall contribution to θ_{n+1} which is zero modulo π. This very emphatically demonstrates the sensitive dependence on initial conditions which is the hallmark of chaotic behavior, for eventually two very close initial conditions, differing by $\Delta\theta \sim 2^{-m}$, will, after m iterations of the logistic map, come to differ by $\mathcal{O}(1)$.

17.5.2 Lyapunov exponents

The *Lyapunov exponent* $\lambda(x)$ of the iterated map $f(x)$ at point x is defined to be

$$\lambda(x) = \lim_{n\to\infty} \frac{1}{n} \log \left| \frac{df^{(n)}(x)}{dx} \right| = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \log \left| f'(x_j) \right| \quad , \tag{17.68}$$

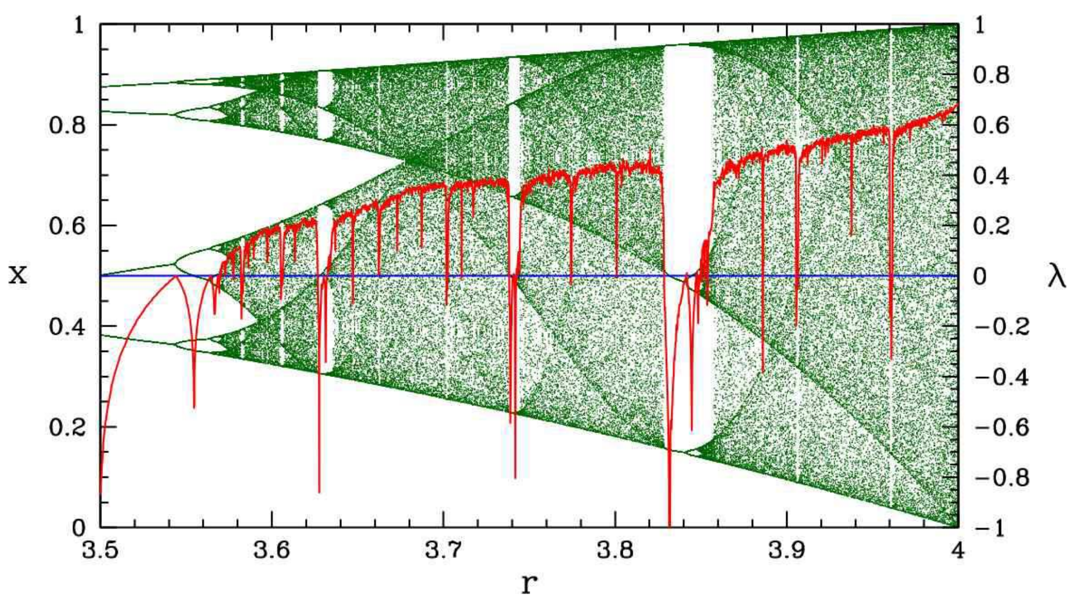


Figure 17.12: Lyapunov exponent (in red) for the logistic map.

where $x_{j+1} \equiv f(x_j)$. The significance of the Lyapunov exponent is the following. If $\mathsf{Re}\big(\lambda(x)\big) > 0$ then two initial conditions near x will exponentially separate under the iterated map. For the *tent map*,

$$f(x) = \begin{cases} 2rx & \text{if } x < \frac{1}{2} \\ 2r(1-x) & \text{if } x \geq \frac{1}{2} \end{cases} \quad , \tag{17.69}$$

one easily finds $\lambda(x) = \log(2r)$ independent of x. Thus, if $r > \frac{1}{2}$ the Lyapunov exponent is positive, meaning that every neighboring pair of initial conditions will eventually separate exponentially under repeated application of the map. The Lyapunov exponent for the logistic map is depicted in fig. 17.12.

17.5.3 Intermittency

Successive period doubling is one route to chaos, as we've just seen. Another route is *intermittency*. Intermittency works like this. At a particular value of our control parameter r, the map exhibits a stable periodic cycle, such as the stable 3-cycle at $r = 3.829$, as shown in the bottom panel of fig. 17.13. If we then vary the control parameter slightly in a certain direction, the periodic behavior persists for a finite number of iterations, followed by a *burst*, which is an interruption of the regular periodicity, followed again by periodic behavior, *ad infinitum*. There are three types of intermittent behavior, depending on whether the Lyapunov exponent λ goes through $\mathsf{Re}(\lambda) = 0$ while $\mathsf{Im}(\lambda) = 0$ (type-I intermittency), or with $\mathsf{Im}(\lambda) = \pi$ (type-III intermittency), or, as is possible for two-dimensional maps, with $\mathsf{Im}(\lambda) = \eta$, a general real number.

17.6 Attractors

An *attractor* of a dynamical system $\dot{\boldsymbol{\varphi}} = \boldsymbol{V}(\boldsymbol{\varphi})$ is the set of $\boldsymbol{\varphi}$ values that the system evolves to after a sufficiently long time. For $N = 1$ the only possible attractors are stable fixed points. For $N = 2$, we have stable nodes and spirals, but also stable limit cycles. For $N > 2$ the situation is qualitatively different, and a fundamentally new type of set, the *strange attractor,* emerges.

A strange attractor is basically a bounded set on which nearby orbits diverge exponentially (*i.e.* there exists at least one positive Lyapunov exponent). To envision such a set, consider a flat rectangle, like a piece of chewing gum. Now fold the rectangle over, stretch it, and squash it so that it maintains its original volume. Keep doing this. Two points which started out nearby to each other will eventually, after a sufficiently large number of folds and stretches, grow far apart. Formally, a strange attractor is a *fractal*, and may have *noninteger Hausdorff dimension.* (We won't discuss fractals and Hausdorff dimension here.)

17.7 The Lorenz Model

The canonical example of an $N = 3$ strange attractor is found in the Lorenz model. E. N. Lorenz, in a seminal paper from the early 1960's, reduced the essential physics of the coupled *partial* differential

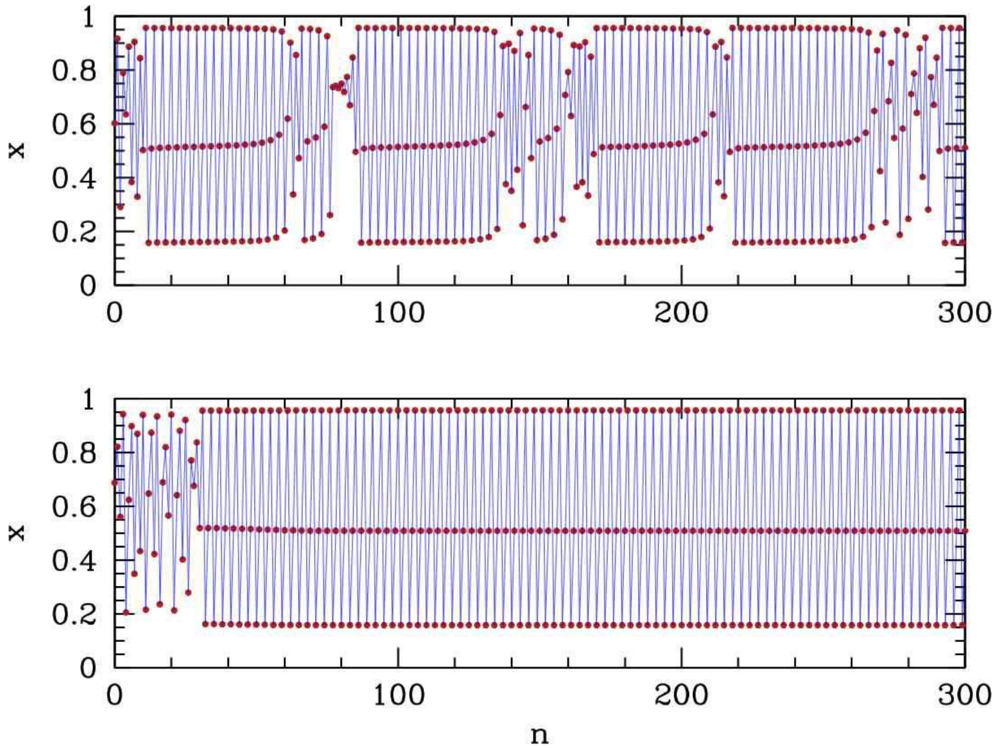


Figure 17.13: Intermittency in the logistic map in the vicinity of the 3-cycle Top panel: $r = 3.828$, showing intermittent behavior. Bottom panel: $r = 3.829$, showing a stable 3-cycle.

equations describing Rayleigh-Benard convection (a fluid slab of finite thickness, heated from below – in Lorenz's case a model of the atmosphere warmed by the ocean) to a set of twelve coupled nonlinear *ordinary* differential equations. Lorenz's intuition was that his weather model should exhibit recognizable patterns over time. What he found instead was that in some cases, changing his initial conditions by a part in a thousand rapidly led to totally different behavior. This *sensitive dependence on initial conditions* is a hallmark of chaotic systems.

The essential physics (or mathematics?) of Lorenz's $N = 12$ system is elicited by the reduced $N = 3$ system,

$$\begin{aligned} \dot{X} &= -\sigma X + \sigma Y \\ \dot{Y} &= rX - Y - XZ \\ \dot{Z} &= XY - bZ \quad , \end{aligned} \tag{17.70}$$

where σ, r, and b are all real and positive. Here t is the familiar time variable (appropriately scaled), and (X, Y, Z) represent linear combinations of physical fields, such as global wind current and poleward temperature gradient. These equations possess a symmetry under $(X, Y, Z) \to (-X, -Y, Z)$, but what is most important is the presence of nonlinearities in the second and third equations.

The Lorenz system is *dissipative* because phase space volumes contract:

$$\boldsymbol{\nabla} \cdot \boldsymbol{V} = \frac{\partial \dot{X}}{\partial X} + \frac{\partial \dot{Y}}{\partial Y} + \frac{\partial \dot{Z}}{\partial Z} = -(\sigma + b + 1) \quad . \tag{17.71}$$

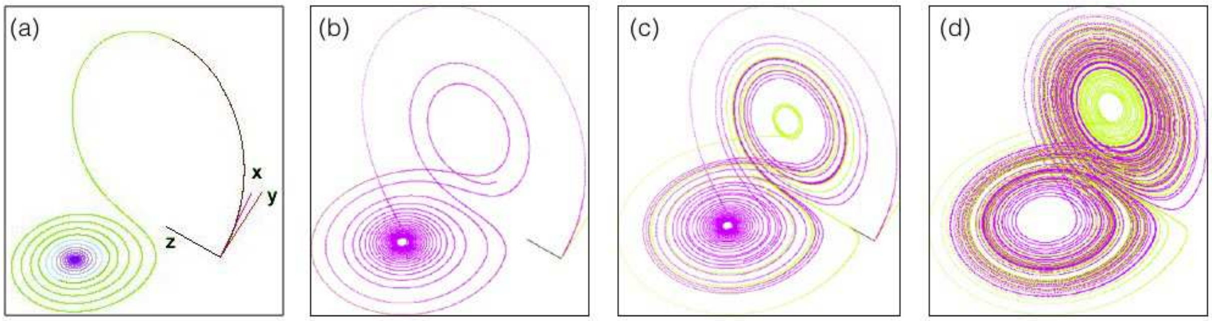


Figure 17.14: (a) Evolution of the Lorenz equations for $\sigma = 10$, $b = \frac{8}{3}$, and $r = 15$, with initial conditions $(X, Y, Z) = (0, 1, 0)$, projected onto the (X, Z) plane. The attractor is a stable spiral. (b) - (d) Chaotic regime ($r = 28$) evolution showing sensitive dependence on initial conditions. The magenta and green curves differ in their initial X coordinate by 10^{-5}. (Source: Wikipedia)

Thus, volumes contract under the flow. Another property is the following. Let

$$F(X, Y, Z) = \tfrac{1}{2}X^2 + \tfrac{1}{2}Y^2 + \tfrac{1}{2}(Z - r - \sigma)^2 \quad . \tag{17.72}$$

Then

$$\begin{aligned} \dot{F} &= X\dot{X} + Y\dot{Y} + (Z - r - \sigma)\dot{Z} \\ &= -\sigma X^2 - Y^2 - b\big(Z - \tfrac{1}{2}r - \tfrac{1}{2}\sigma\big)^2 + \tfrac{1}{4}b(r + \sigma)^2 \quad . \end{aligned} \tag{17.73}$$

Thus, $\dot{F} < 0$ outside an ellipsoid, which means that all solutions must remain bounded in phase space for all times.

17.7.1 Fixed point analysis

Setting $\dot{X} = \dot{Y} = \dot{Z} = 0$, we find three solutions. One solution which is always present is $X^* = Y^* = Z^* = 0$. If we linearize about this solution, we obtain

$$\frac{d}{dt}\begin{pmatrix} \delta X \\ \delta Y \\ \delta Z \end{pmatrix} = \begin{pmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{pmatrix} \begin{pmatrix} \delta X \\ \delta Y \\ \delta Z \end{pmatrix} \quad . \tag{17.74}$$

The eigenvalues of the linearized dynamics are found to be

$$\begin{aligned} \lambda_{1,2} &= -\tfrac{1}{2}(1 + \sigma) \pm \tfrac{1}{2}\sqrt{(1 + \sigma)^2 + 4\sigma(r - 1)} \\ \lambda_3 &= -b \quad , \end{aligned} \tag{17.75}$$

and thus if $0 < r < 1$ all three eigenvalues are negative, and the fixed point is a stable node. If, however, $r > 1$, then $\lambda_2 > 0$ and the fixed point is attractive in two directions but repulsive in a third, corresponding to a three-dimensional version of a saddle point.

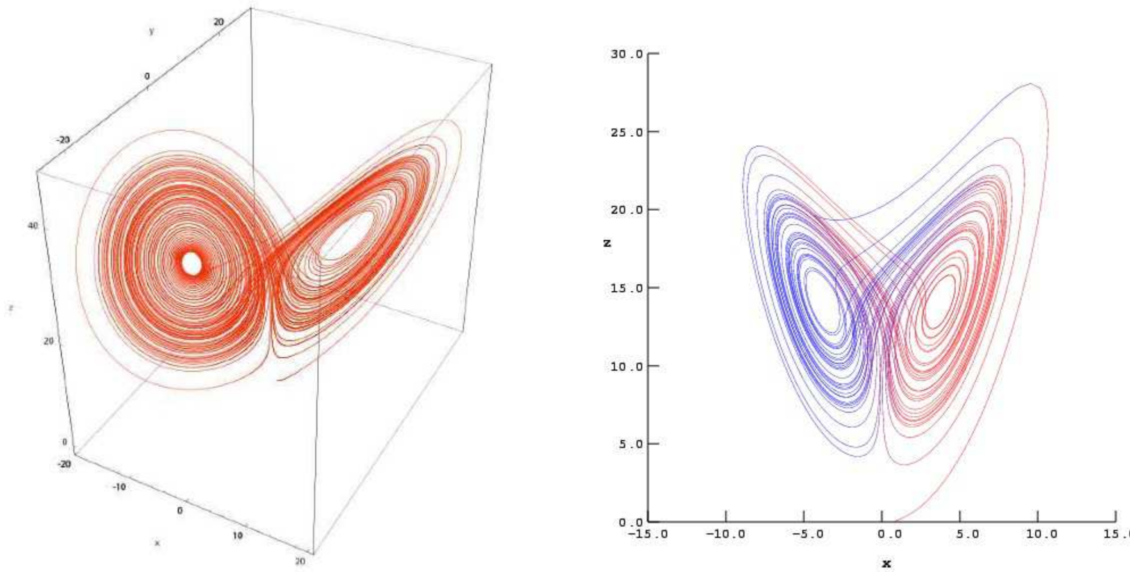


Figure 17.15: Left: Evolution of the Lorenz equations for $\sigma = 10$, $b = \frac{8}{3}$, and $r = 28$, with initial conditions $(X_0, Y_0, Z_0) = (0, 1, 0)$, showing the 'strange attractor'. Right: The Lorenz attractor, projected onto the (X, Z) plane. (Source: Wikipedia)

For $r > 1$, a new pair of solutions emerges, with

$$X^* = Y^* = \pm\sqrt{b(r-1)} \quad , \quad Z^* = r - 1 \quad . \tag{17.76}$$

Linearizing about either one of these fixed points, we find

$$\frac{d}{dt}\begin{pmatrix} \delta X \\ \delta Y \\ \delta Z \end{pmatrix} = \begin{pmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & -X^* \\ X^* & X^* & -b \end{pmatrix}\begin{pmatrix} \delta X \\ \delta Y \\ \delta Z \end{pmatrix} \quad . \tag{17.77}$$

The characteristic polynomial of the linearized map is

$$P(\lambda) = \lambda^3 + (b + \sigma + 1)\,\lambda^2 + b(\sigma + r)\,\lambda + 2b(r-1) \quad . \tag{17.78}$$

Since b, σ, and r are all positive, $P'(\lambda) > 0$ for all $\lambda \geq 0$. Since $P(0) = 2b(r-1) > 0$, we may conclude that there is always at least one eigenvalue λ_1 which is real and negative. The remaining two eigenvalues are either both real and negative, or else they occur as a complex conjugate pair: $\lambda_{2,3} = \alpha \pm i\beta$. The fixed point is stable provided $\alpha < 0$. The stability boundary lies at $\alpha = 0$. Thus, we set

$$P(i\beta) = \Big[2b(r-1) - (b + \sigma + 1)\beta^2\Big] + i\Big[b(\sigma + r) - \beta^2\Big]\beta = 0 \quad , \tag{17.79}$$

which results in two equations. Solving these two equations for $r(\sigma, b)$, we find

$$r_c = \frac{\sigma(\sigma + b + 3)}{\sigma - b - 1} \quad . \tag{17.80}$$

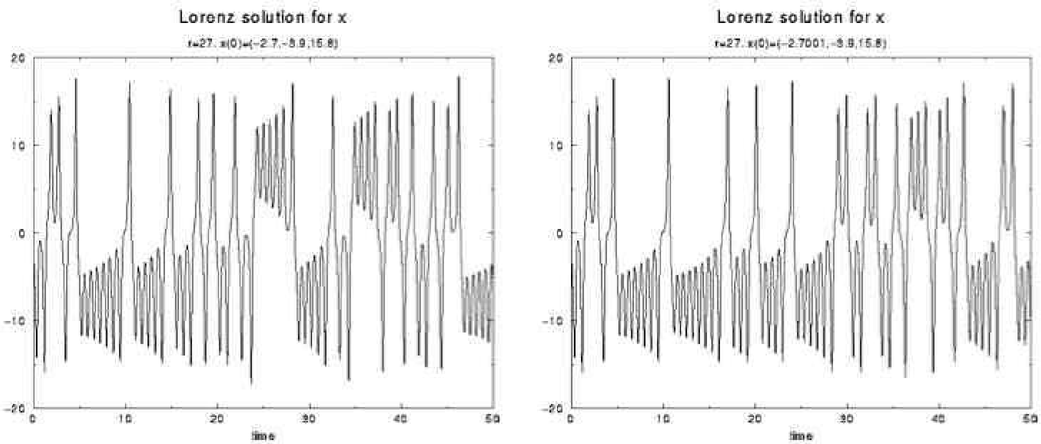


Figure 17.16: $X(t)$ for the Lorenz equations with $\sigma = 10$, $b = \frac{8}{3}$, $r = 28$, and initial conditions $(X_0, Y_0, Z_0) = (-2.7, -3.9, 15.8)$, and initial conditions $(X_0, Y_0, Z_0) = (-2.7001, -3.9, 15.8)$.

The fixed point is stable for $r \in [1, r_c]$. These fixed points correspond to steady convection. The approach to this fixed point is shown in fig. 17.14.

The Lorenz system has commonly been studied with $\sigma = 10$ and $b = \frac{8}{3}$. This means that the volume collapse is very rapid, since $\boldsymbol{\nabla} \cdot \boldsymbol{V} = -\frac{41}{3} \approx -13.67$, leading to a volume contraction of $e^{-41/3} \simeq 1.16 \times 10^{-6}$ per unit time. For these parameters, one also has $r_c = \frac{470}{19} \approx 24.74$. The capture by the strange attractor is shown in fig. 17.15.

In addition to the new pair of fixed points, a strange attractor appears for $r > r_s \simeq 24.06$. In the narrow interval $r \in [24.06, 24.74]$ there are then *three* stable attractors, two of which correspond to steady convection and the third to chaos. Over this interval, there is also hysteresis. *I.e.* starting with a convective state for $r < 24.06$, the system remains in the convective state until $r = 24.74$, when the convective fixed point becomes unstable. The system is then driven to the strange attractor, corresponding to chaotic dynamics. Reversing the direction of r, the system remains chaotic until $r = 24.06$, when the strange attractor loses its own stability.

17.7.2 Poincaré section

One method used by Lorenz in analyzing his system was to plot its *Poincaré section*. This entails placing one constraint on the coordinates (X, Y, Z) to define a two-dimensional surface Σ, and then considering the intersection of this surface Σ with a given phase curve for the Lorenz system. Lorenz chose to set $\dot{Z} = 0$, which yields the surface $Z = b^{-1}XY$. Note that since $\dot{Z} = 0$, $Z(t)$ takes its maximum and minimum values on this surface; see the left panel of fig. 17.17. By plotting the values of the maxima Z_N as the integral curve successively passed through this surface, Lorenz obtained results such as those shown in the right panel of fig. 17.17, which has the form of a one-dimensional map and may be analyzed as such. Thus, chaos in the Lorenz attractor can be related to chaos in a particular one-dimensional map, known as the *return map* for the Lorenz system.

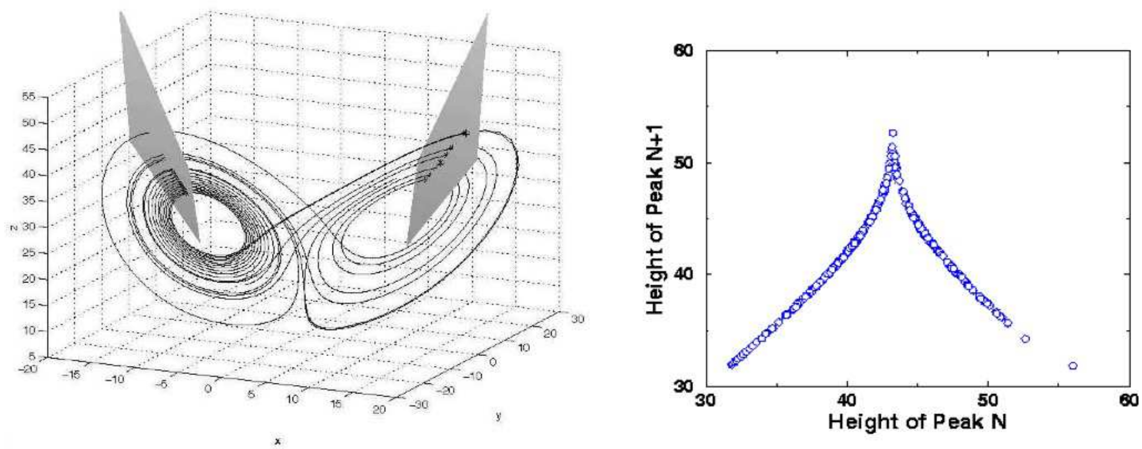


Figure 17.17: Left: Lorenz attractor for $b = \frac{8}{3}$, $\sigma = 10$, and $r = 28$. Maxima of Z are depicted by stars. Right: Relation between successive maxima Z_N along the strange attractor.

17.7.3 Rössler system

The strange attractor is one of the hallmarks of the Lorenz system. Another simple dynamical system which possesses a strange attractor is the Rössler system. This is also described by $N = 3$ coupled ordinary differential equations, *viz.*

$$\begin{aligned} \dot{X} &= -Y - Z \\ \dot{Y} &= Z + aY \\ \dot{Z} &= b + Z(X - c) \quad , \end{aligned} \tag{17.81}$$

typically studied as a function of c for $a = b = \frac{1}{5}$. In fig. 17.19, we present results from work by Crutchfield *et al.* (1980). The transition from simple limit cycle to strange attractor proceeds via a sequence of period-doubling bifurcations, as shown in the figure. A convenient diagnostic for examining this period-doubling route to chaos is the *power spectral density*, or PSD, defined for a function $F(t)$ as

$$\Phi_F(\omega) = \left| \int\limits_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, F(t)\, e^{-i\omega t} \right|^2 = \big|\hat{F}(\omega)\big|^2 \quad . \tag{17.82}$$

As one sees in fig. 17.19, as c is increased past each critical value, the PSD exhibits a series of frequency halvings (*i.e.* period doublings). All harmonics of the lowest frequency peak are present. In the chaotic region, where $c > c_\infty \approx 4.20$, the PSD also includes a noisy broadband background.

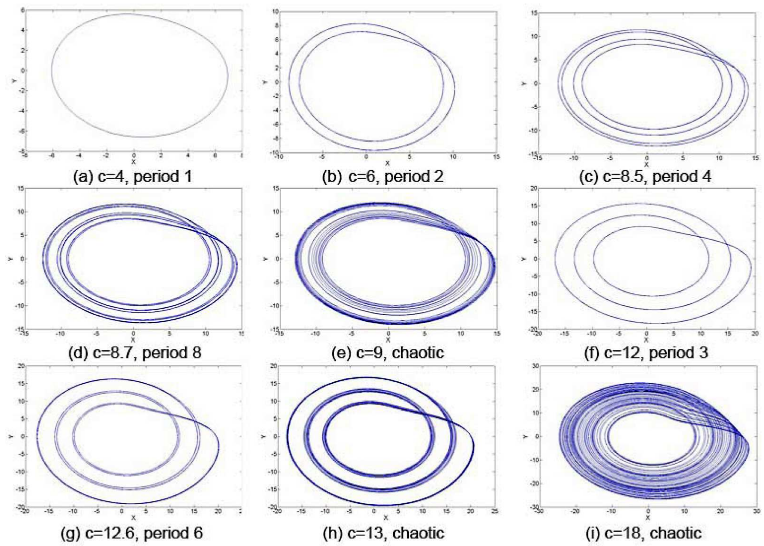


Figure 17.18: Period doubling bifurcations of the Rössler attractor, projected onto the (x, y) plane, for nine values of c, with $a = b = \frac{1}{10}$.

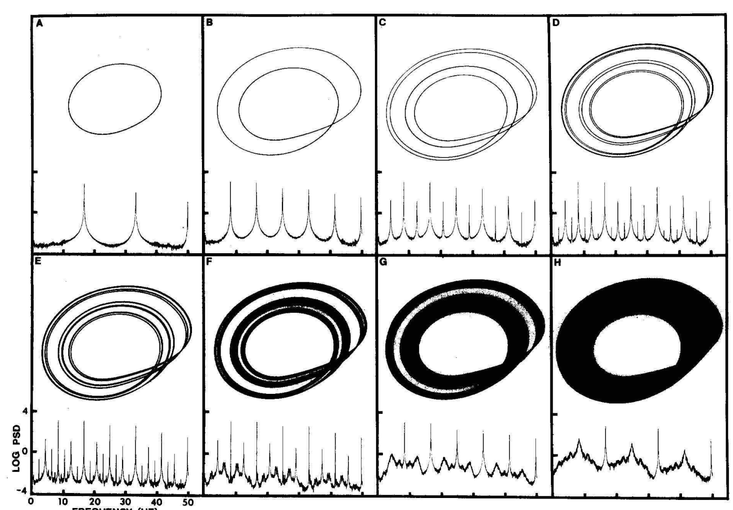


Figure 17.19: Period doubling bifurcations of the Rössler attractor with $a = b = \frac{1}{5}$, projected onto the (X,Y) plane, for eight values of c, and corresponding power spectral density for Z(t). (a) $c = 2.6$; (b) $c = 3.5$; (c) $c = 4.1$; (d) $c = 4.18$; (e) $c = 4.21$; (f) $c = 4.23$; (g) $c = 4.30$; (h) $c = 4.60$.