Contents

List of Figures

List of Tables

Chapter 1

Introduction to Dynamics

1.1 What is Dynamics?

Loosely speaking, dynamics is the study of mathematical models of "what happens next," which is to say how systems evolve in time. There are four main elements to dynamics:

  (i) The **initial conditions**, or "how things are now."

 (ii) The **equations of motion**, which encode how a given system evolves. There are two broad classes to speak of: *difference equations* which describe evolution in discrete time steps, and *differential equations* which describe continuous time evolution.

(iii) There may be a *random* component to the evolution, in which case the equations of motion are said to be **stochastic**.

(iv) The *solution* of the equations of motion, given the initial conditions, tells us the **motion** of the system, *i.e.* "how things will be in the future." For stochastic systems, we cannot compute the motion itself, but rather only statistical properties thereof, such as the average position(s) at some future time.

Our main concern will be in applying these mathematical models to physical mechanical systems: balls and springs, celestial bodies, spinning tops, *etc.*, which are the purview of *classical mechanics*. In classical mechanics, the equations of motion describe continuous time dynamics of each system's various *degrees of freedom* in the form of coupled second order ordinary differential equations[1], which are nothing more than Newton's second law $F = ma$. In one space dimension, for example, we have $m\,d^2x/dt^2 = F(x)$. Such systems are special, and constitute a restricted class of the general family of continuous time dynamical systems. For example, if the forces are derivable from a potential energy function, then there is

---

[1]The degrees of freedom are the positional coordinates for point particles and the orientational coordinates for rigid bodies. In the case of nonrigid continuous systems, like strings, membranes, and elastic media, the equations of motion are partial differential equations involving both space and time. Continuum mechanics is discussed below in chapter 10.

a conserved quantity, which is the total energy[2]. We will derive the equations of motion, *i.e.* Newton's laws, using a powerful variational principle known as the *principle of extremal action,* which lies at the foundation of Lagrange's approach to mechanics. A related and even more powerful approach, due to Hamilton, is the subject of graduate level mechanics courses.

Lets start by considering some examples.

1.1.1   Simple difference equation

Consider the difference equation

$$x_{n+1} = x_n + \alpha \quad , \tag{1.1}$$

where $x_n \in \mathbb{R}$ is the position of a point object at discrete time step n, and $\alpha \in \mathbb{R}$ is a real number. The initial conditions are specified by x_0, which is the position at discrete time step $n = 0$.

Clearly the position advances by α with each step, and thus the motion of the system is given by

$$x_n = x_0 + n\alpha \quad . \tag{1.2}$$

1.1.2   Another difference equation: Fibonacci numbers

Next, consider the difference equation

$$x_{n+1} = x_n + x_{n-1} \quad . \tag{1.3}$$

The initial conditions are now specified by two values, x_0 and x_1. Given these, we can compute $x_2 = x_1 + x_0$, $x_3 = x_2 + x_1 = 2x_1 + x_0$, *etc.* Can we obtain a general expression for x_n? Yes we can! Let's try a solution of the form $x_n = A\lambda^n$ where A and λ are as yet undetermined. We stick this into eqn. 1.3 and obtain the relation

$$\lambda^2 - \lambda - 1 = 0 \quad , \tag{1.4}$$

which has two solutions,

$$\lambda_\pm = \tfrac{1}{2}(1 \pm \sqrt{5}) = \{1 + \phi, \, -\phi\} \quad , \tag{1.5}$$

where $\phi = \tfrac{1}{2}(\sqrt{5} - 1) = 0.618034\ldots$ is the golden mean. Thus we write

$$x_n = A_+ \, \lambda_+^n + A_- \, \lambda_-^n \quad . \tag{1.6}$$

Imposing the initial conditions by setting $n = 0$ and $n = 1$ then yields the relations

$$\begin{pmatrix} 1 & 1 \\ \lambda_+ & \lambda_- \end{pmatrix} \begin{pmatrix} A_+ \\ A_- \end{pmatrix} = \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \quad , \tag{1.7}$$

and thus

$$\begin{pmatrix} A_+ \\ A_- \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \lambda_+ & \lambda_- \end{pmatrix}^{-1} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \frac{1}{\lambda_+ - \lambda_-} \begin{pmatrix} x_1 - \lambda_- x_0 \\ \lambda_+ x_0 - x_1 \end{pmatrix} \quad . \tag{1.8}$$

---

[2]More precisely, the conserved quantity is the *Hamiltonian H*, which may differ from the total energy E, as we shall discuss in §4.13.2 below.

The full motion of the system is then given by

$$x_n = \frac{1}{1 + 2\phi}\left\{(1 + \phi)^n\left[\phi x_0 + x_1\right] + (-\phi)^n\left[(1 + \phi)x_0 + x_1\right]\right\} \tag{1.9}$$

For the initial conditions $x_0 = 0$ and $x_1 = 1$, we obtain

$$x_n = \frac{1}{1 + 2\phi}\left\{(1 + \phi)^n - (-\phi)^n\right\} = F_n \quad, \tag{1.10}$$

i.e. the nth Fibonacci number: $F_n = \{1, 1, 2, 3, 5, 8, 13, \ldots\}$ starting from $n = 1$. Did you know that such a closed form expression for all the Fibonacci numbers can be derived?

1.1.3 Stochastic difference equation: diffusion

Now consider the stochastic difference equation

$$x_{n+1} = x_n + \sigma_n \quad, \tag{1.11}$$

where the $\{\sigma_n\}$ are independent, identically distributed ('IID' in statistics parlance) random numbers whose distribution is given by

$$\mathsf{Prob}[\sigma_n = \varepsilon] = p\,\delta_{\varepsilon,+1} + q\,\delta_{\varepsilon,-1} = \begin{cases} p & \text{if } \varepsilon = +1 \\ q & \text{if } \varepsilon = -1 \end{cases}, \tag{1.12}$$

with $p \in [0, 1]$. Since there are only two possibilities for each σ_n, the sum of their probabilities must be unity, *i.e.* $p + q = 1$, which fixes $q = 1 - p$. This system corresponds to a one-dimensional random walk, where the probability of a step to the right, *i.e.* $x_{n+1} = x_n + 1$, is p, and the probability of a step to the left, *i.e.* $x_{n+1} = x_n - 1$, is q. The initial conditions are given by the value of x_0. Clearly we have

$$x_n = x_0 + \sigma_1 + \sigma_2 + \ldots + \sigma_n = x_0 + \sum_{j=1}^{n}\sigma_j \quad. \tag{1.13}$$

We can now compute averages with respect to the random distribution:

$$\begin{aligned} \langle x_n \rangle &= x_0 + \sum_{j=1}^{n}\langle \sigma_j \rangle \\ \langle x_n^2 \rangle &= x_0^2 + 2x_0\sum_{j=1}^{n}\langle \sigma_j \rangle + \sum_{j=1}^{n}\sum_{k=1}^{n}\langle \sigma_j \sigma_k \rangle \quad. \end{aligned} \tag{1.14}$$

We will need

$$\begin{aligned} \langle \sigma_j \rangle &= \sum_{\varepsilon=\pm 1}\varepsilon\,\mathsf{Prob}[\sigma_j = \varepsilon] \\ &= p - q = 2p - 1 \end{aligned} \tag{1.15}$$

and

$$\langle \sigma_j \sigma_k \rangle = \begin{cases} 1 & \text{if } j = k \\ (2p-1)^2 & \text{if } j \neq k \end{cases} , \tag{1.16}$$

because $\sigma_j^2 = 1 \Rightarrow \langle \sigma_j^2 \rangle = 1$ and $\langle \sigma_j \sigma_k \rangle = \langle \sigma_j \rangle \langle \sigma_k \rangle = (2p-1)^2$ for $j \neq k$ (IID). Thus,

$$\begin{aligned} \langle x_n \rangle &= x_0 + (2p-1)n \\ \langle x_n \rangle^2 &= x_0^2 + 2x_0(2p-1)n + (2p-1)^2 n^2 \\ \langle x_n^2 \rangle &= x_0^2 + 2x_0(2p-1)n + (2p-1)^2 n(n-1) + n \end{aligned} \tag{1.17}$$

and

$$\begin{aligned} \langle (\Delta x_n)^2 \rangle &\equiv \langle \left[x_n - \langle x_n \rangle \right]^2 \rangle \\ &= \langle x_n^2 \rangle - \langle x_n \rangle^2 = 4p(1-p)n \quad . \end{aligned} \tag{1.18}$$

When $p \neq q$ the random walk is *biased*, and there is an unequal probability of stepping to the right and to the left. Thus $\langle x_n \rangle = x_0 + (2p-1)n$ on average changes by $\langle \sigma_j \rangle = p - q = 2p - 1$ during each time step. If we start at x_0 and then execute our random walk, after n time steps we know that we will end up at some point x_n between $x_0 - n$ (all steps to the left) and $x_0 + n$ (all steps to the right). Where we end up will likely change each time we rerun the experiment, but if we average over a great many experiments, we will obtain $\langle x_n \rangle = x_0 + (2p-1)n$. But while the average of the difference Δx_n between x_n and its mean $\langle x_n \rangle$ vanishes, the average of its square $\langle (\Delta x_n)^2 \rangle$ grows linearly with n. The *root mean square* variation then grows as $n^{1/2}$, *viz.*

$$\Delta x_n^{\text{RMS}} = \sqrt{\langle (\Delta x_n)^2 \rangle} = 2\sqrt{pqn} \quad . \tag{1.19}$$

This is an example of *diffusion*.

1.1.4   Nonlinear discrete dynamics: the logistic map

Consider the simple case of a general one-dimensional *map*,

$$x_{n+1} = g(x_n) \quad , \tag{1.20}$$

where $g(x) : \mathbb{R} \to \mathbb{R}$ is a real function of a real number. A *fixed point* of this map satisfies $g(x) = x$. Some maps have no fixed points, such as $g(x) = x + 1$. For $g(x) = x$, *every* point is a fixed point. This last example is highly nongeneric; generically the set of fixed points - if there are any fixed points at all - is discrete.

Let's focus in on what happens when x is close to some fixed point x^* and write $x_n = x^* + u_n$ with $|u_n| \ll 1$. Then

$$u_{n+1} = g(x^* + u_n) - x^* = g'(x^*)\, u_n + \tfrac{1}{2}\, g''(x^*)\, u_n^2 + \dots \quad . \tag{1.21}$$

Here we have used Taylor's theorem to expand $g(x^* + u_n)$ in powers of the small quantity u_n. If we drop all the terms in the Taylor series which are beyond linear in u_n, we obtain the equation $u_{n+1} = \kappa\, u_n$, where $\kappa = g'(x^*)$. The solution is $u_n = \kappa^n u_0$ and we conclude
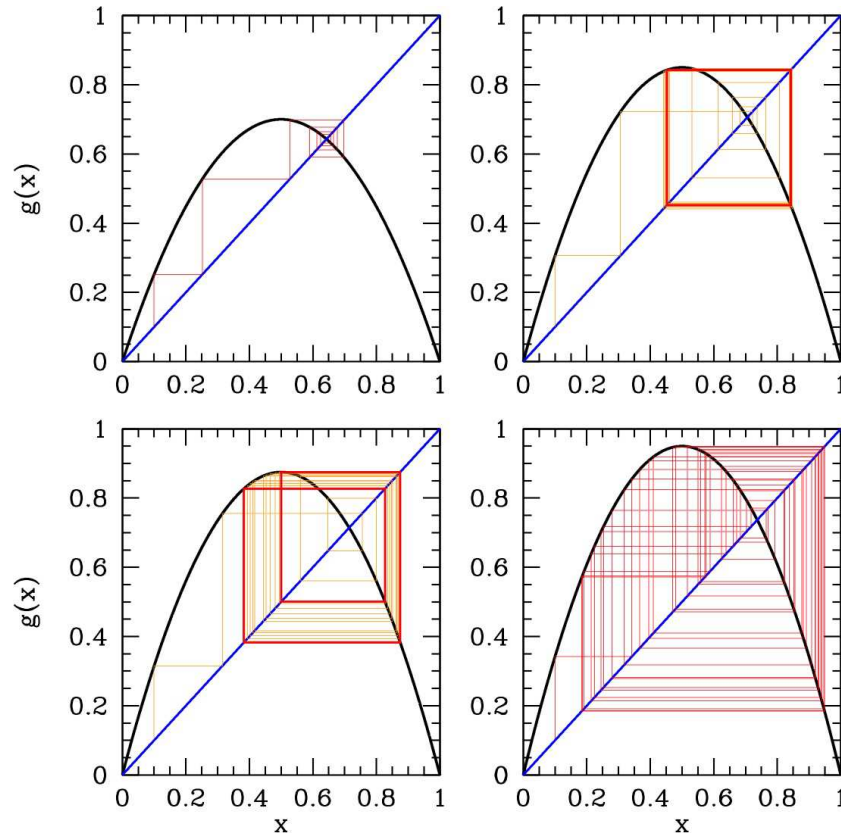
Figure 1.1: Cobweb diagram showing iterations of the logistic map $g(x) = rx(1 - x)$ for $r = 2.8$ (upper left), $r = 3.4$ (upper right), $r = 3.5$ (lower left), and $r = 3.8$ (lower right). Note the single stable fixed point for $r = 2.8$, the stable two-cycle for $r = 3.4$, the stable four-cycle for $r = 3.5$, and the chaotic behavior for $r = 3.8$.

- If $|g'(x^*)| < 1$ then $|u_{n+1}| < |u_n|$ and the magnitude of u_n decreases exponentially with n:

$$u_n = (\pm 1)^n e^{-\alpha n} u_0 \quad , \tag{1.22}$$

where $\alpha = -\log|g(x^*)| > 0$ and we take the $+$ sign if $g(x^*) > 0$ and the $-$ sign if $g(x^*) < 0$. The approximation to neglect higher order terms in the Taylor series expansion of $g(x^* + u_n)$ gets better and better as n increases. A fixed point x^* with $|g'(x^*)| < 1$ is called a *stable fixed point* (SFP).

- If $|g(x^*)| > 1$, then $|u_{n+1}| > |u_n|$ and the magnitude of u_n increases exponentially with n. Successive iterations of the map move us further and further away from x^*. However, at some point the higher order terms which we've neglected in the Taylor expansion of $g(x^* + u_n)$ become non-negligible, and the behavior is no longer exponential. A fixed point x^* for which $|g(x^*)| > 1$ is called an *unstable fixed point* (UFP).

Perhaps the most important and most studied of the one-dimensional maps is the logistic map, where $g(x) = rx(1 - x)$, defined on the interval $x \in [0, 1]$, with $r \in [0, 4]$. There is a fixed point at $x = 0$ which is stable for $r < 1$ and unstable for $r > 1$. When $r > 1$, a new fixed point is present, at $x^* = 1 - r^{-1}$ if
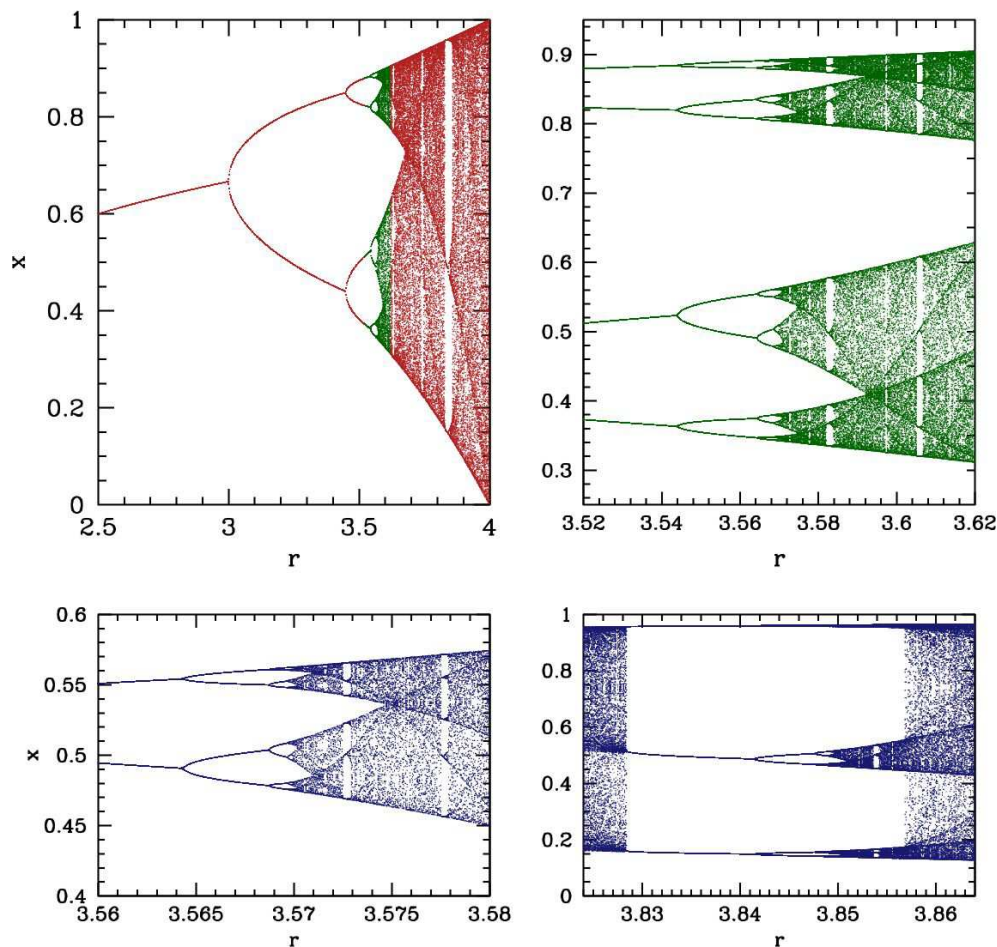
Figure 1.2: Iterates of the logistic map $g(x) = rx(1 - x)$.

$r > 1$. We then have $g'(x^*) = 2 - r$, so the fixed point is stable if $r \in (1, 3)$. What happens for $r > 3$? We can explore the behavior of the iterated map by drawing a *cobweb diagram*, shown in fig. 1.1. We sketch, on the same graph, the curves $y = x$ (in blue) and $y = g(x)$ (in black). Starting with a point x on the line $y = x$, we move vertically until we reach the curve $y = g(x)$. To iterate, we then move horizontally to the line $y = x$ and repeat the process. We see that for $r = 3.4$ the fixed point x^* is unstable, but there is a stable two-cycle, defined by the equations

$$
\begin{aligned}
x_2 &= rx_1(1 - x_1) \\
x_1 &= rx_2(1 - x_2) \, .
\end{aligned}
\tag{1.23}
$$

The second iterate of $g(x)$ is then

$$
g^{(2)}(x) = g\big(g(x)\big) = r^2 x (1 - x)\big(1 - rx + rx^2\big) \, .
\tag{1.24}
$$

Setting $x = g^{(2)}(x)$, we obtain a cubic equation. Since $x - x^*$ must be a factor, we can divide out by this monomial and obtain a quadratic equation for x_1 and x_2. We find

$$
x_{1,2} = \frac{1 + r \pm \sqrt{(r + 1)(r - 3)}}{2r} \, .
\tag{1.25}
$$

How stable is this 2-cycle? We find

$$\frac{d}{dx}g^{(2)}(x)\bigg|_{x_{1,2}} = r^2(1-2x_1)(1-2x_2) = -r^2 + 2r + 4 \ . \tag{1.26}$$

The condition that the 2-cycle be stable is then

$$-1 < r^2 - 2r - 4 < 1 \qquad \Longrightarrow \qquad r \in \left[3\, , 1+\sqrt{6}\right]\ . \tag{1.27}$$

At $r = 1 + \sqrt{6} = 3.4494897\ldots$ there is a bifurcation to a 4-cycle, as can be seen in fig. 1.2.

In the 1970s, Mitchell Feigenbaum described how this system exhibits an increasingly dense cascade of period doubling transitions in which a 2^n-cycle becomes unstable and is replaced by a 2^{n+1}-cycle at $r = r_n$. The value $n = \infty$ is reached for a finite value $r_\infty = 3.5699456\ldots$. We will study this system in more detail in chapter 17.

1.1.5 Dynamical systems

A *dynamical system* in n variables is a set of n coupled ordinary differential equations. It's general form can be written as

$$\frac{d\boldsymbol{\varphi}}{dt} = \boldsymbol{V}(\boldsymbol{\varphi}) \tag{1.28}$$

where[3]

$$\boldsymbol{\varphi} = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_n \end{pmatrix} \quad , \qquad \boldsymbol{V}(\boldsymbol{\varphi}) = \begin{pmatrix} V_1(\varphi_1, \ldots, \varphi_n) \\ V_2(\varphi_1, \ldots, \varphi_n) \\ \vdots \\ V_n(\varphi_1, \ldots, \varphi_n) \end{pmatrix} \tag{1.29}$$

In general $\boldsymbol{\varphi} \in \mathcal{M}$ lives on a *manifold* $\mathcal{M}$, which is an n-dimensional topological space which is locally diffeomorphic to $\mathbb{R}^n$. But for our purposes we can ignore all the fancy math vernacular and just consider $\boldsymbol{\varphi} \in \mathbb{R}^n$ is some n-tuple of real numbers[4]. The vector $\boldsymbol{V}(\boldsymbol{\varphi})$ is called the *velocity vector* at the point $\boldsymbol{\varphi}$. As $\boldsymbol{V}(\boldsymbol{\varphi})$ specifies a vector at each point $\boldsymbol{\varphi} \in \mathcal{M}$, we call $\boldsymbol{V}$ a *vector field*. The solution $\boldsymbol{\varphi}(t)$ to these coupled ODEs, subject to some set of initial conditions $\boldsymbol{\varphi}(0)$, is what we mean by the motion of the system, also called an *integral curve*. Thus, an integral curve is a set of points $\{t, \boldsymbol{\varphi}(t)\} \in \mathbb{R} \times \mathcal{M}$. The collection of points $\{\boldsymbol{\varphi}(t)\,|\,t \in \mathbb{R}\}$ is a curve in $\mathcal{M}$ itself, known as a *phase curve*. The difference is that a phase curve does not include the time coordinate. (See fig. 1.3.)

There's a helpful theorem which says that if $\boldsymbol{V}(\boldsymbol{\varphi})$ is a smooth vector field over some open set $\mathcal{D} \subset \mathcal{M}$, then for any $\boldsymbol{\varphi}(0) \in \mathcal{D}$ the initial value problem (*i.e.* the dynamical system plus its initial conditions) has a solution on some finite time interval $t \in [-\tau, +\tau]$, and furthermore that solution is unique. Moreover, this solution may be extended forward and backward in time either indefinitely or until $\boldsymbol{\varphi}(t)$ reaches the boundary of $\mathcal{D}$. A corollary of this theorem guarantees that different trajectories never intersect. Some

---

[3]It is important that the dynamical system as defined here is *autonomous*, *i.e.* $\boldsymbol{V}(\boldsymbol{\varphi})$ is a function only of the coordinates $\{\varphi_1, \ldots, \varphi_n\}$ and not on t itself - at least not explicitly.

[4]The mathy language just means that we could consider $\boldsymbol{\varphi}$ to live on a torus, or on the surface of a sphere, or on some complicated twisty higher dimensional space with lots of holes and handles.
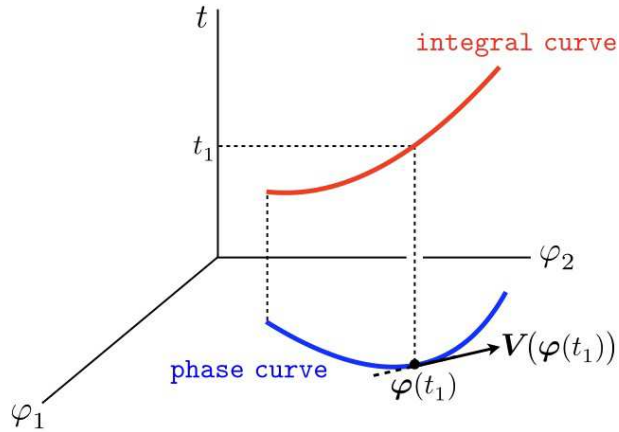
Figure 1.3: Integral curve *vs.* phase curve.

aspects of dynamical systems in low dimensions (*i.e.* $n = 1$ and $n = 2$) are discussed in chs. 11 through 13 of these lecture notes.

Note that any n^{th} order ODE, of the general form

$$\frac{d^n x}{dt^n} = F\left(x, \frac{dx}{dt}, \dots, \frac{d^{n-1}x}{dt^{n-1}} \right) \quad , \tag{1.30}$$

may be represented by the first order system $\dot{\boldsymbol{\varphi}} = \boldsymbol{V}(\boldsymbol{\varphi})$. To see this, define $\varphi_k \equiv d^{k-1}x/dt^{k-1}$, with $k = 1, \dots, n$. Thus, for $j < n$ we have $\dot{\varphi}_j = \varphi_{j+1}$, with $\dot{\varphi}_n = F$. In other words,

$$\frac{d}{dt} \overbrace{\begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_{n-1} \\ \varphi_n \end{pmatrix}}^{\dot{\boldsymbol{\varphi}}} = \overbrace{\begin{pmatrix} \varphi_2 \\ \vdots \\ \varphi_n \\ F(\varphi_1, \dots, \varphi_n) \end{pmatrix}}^{\boldsymbol{V}(\boldsymbol{\varphi})} \quad . \tag{1.31}$$

Fixed points

A *fixed point* of a dynamical system is a point $\boldsymbol{\varphi}^*$ such that $\boldsymbol{V}(\boldsymbol{\varphi}^*) = 0$. Thus, if we start at time zero with $\boldsymbol{\varphi}(0) = \boldsymbol{\varphi}^*$, the system will remain at that point in phase space. But suppose we deviate just a teensy bit from the fixed point. We write $\boldsymbol{\varphi}(t) = \boldsymbol{\varphi}^* + \boldsymbol{\epsilon}(t)$. Since

$$V_j(\boldsymbol{\varphi}^* + \boldsymbol{\epsilon}) = \sum_{k=1}^{n} \left. \frac{\partial V_j}{\partial \varphi_k} \right|_{\boldsymbol{\varphi}^*} \epsilon_k + \mathcal{O}(\boldsymbol{\epsilon}^2) \quad , \tag{1.32}$$

we have to lowest order in $\boldsymbol{\epsilon}$ the system

$$\frac{d\epsilon_j}{dt} = \sum_{k=1}^{n} M_{jk}\, \epsilon_k + \mathcal{O}(\boldsymbol{\epsilon}^2) \quad , \qquad M_{jk} = \left. \frac{\partial V_j}{\partial \varphi_k} \right|_{\boldsymbol{\varphi}^*} \quad . \tag{1.33}$$

The matrix M is real but not necessarily symmetric, so its eigenvalues can either be pure real or occur in complex conjugate pairs. The fixed point φ^* is then stable if all the eigenvalues of M have negative real parts. In this case, the vector $\epsilon(t)$ collapses to zero exponentially at late times. Formally, the solution of the linearized dynamics is given by

$$\epsilon(t) = \exp(Mt)\,\epsilon(0)\quad. \tag{1.34}$$

In general, the right eigenvectors of M will not be the same as the left eigenvectors of M. Indeed it may be that M has fewer than n linearly independent eigenvectors – such is the case when M has nontrivial *Jordan blocks*, which is a nongeneric state of affairs. Assuming that M does have n linearly independent right eigenvectors R^α_j and n linearly independent left eigenvectors L^α_j, where R^α_j is the jth component of the αth right eigenvector. Thus,

$$\sum_{j=1}^{n} L^\alpha_j M_{jk} = \lambda_\alpha\,L^\alpha_k \qquad,\qquad \sum_{k=1}^{n} M_{jk} R^\alpha_k = \lambda_\alpha\,R^\alpha_j \quad, \tag{1.35}$$

as well as the orthonormality and completeness relations

$$\sum_{j=1}^{n} L^\alpha_j\,R^\beta_j = \delta^{\alpha\beta} \qquad,\qquad \sum_{\alpha=1}^{n} R^\alpha_j\,L^\alpha_k = \delta_{jk} \quad. \tag{1.36}$$

Furthermore, we may decompose M into its eigenvectors as follows:

$$M_{jk} = \sum_{\alpha=1}^{n} \lambda_\alpha\,R^\alpha_j\,L^\alpha_k \quad. \tag{1.37}$$

Thus, if we write $\epsilon(t)$ in terms of the right eigenvectors of M, *i.e.*

$$\epsilon_j(t) = \sum_{\alpha=1}^{n} C_\alpha(t)\,R^\alpha_j \quad, \tag{1.38}$$

then

$$C_\alpha(t) = C_\alpha(0)\,\exp(\lambda_\alpha t) \quad. \tag{1.39}$$

Thus, for $\text{Re}\,(\lambda_\alpha) > 0$, $C_\alpha(t)$ grows with increasing time, indicating that the fixed point is unstable. A stable fixed point therefore requires $\text{Re}\,(\lambda_\alpha) < 0$ for all $\alpha \in \{1,\dots,n\}$.

Attractors, strange attractors, and dynamical chaos

An *attractor* of a dynamical system $\dot{\varphi} = V(\varphi)$ is the set of φ values that the system evolves to after a sufficiently long time. For $n = 1$ the only possible attractors are stable fixed points. For $n = 2$, we have, generically, two different classes of stable fixed points, called stable nodes and stable spirals. But there are also stable *limit cycles*, which are one-dimensional curves along which the motion is trapped. For $n > 2$ the situation is qualitatively different, and a fundamentally new type of set, the *strange attractor*, emerges.

A strange attractor is basically a bounded set on which nearby orbits diverge exponentially (*i.e.* there exists at least one positive Lyapunov exponent). To envision such a set, consider a flat rectangle, like

a piece of chewing gum. Now fold the rectangle over, stretch it, and squash it so that it maintains its original volume. Keep doing this. Two points which started out nearby to each other will eventually, after a sufficiently large number of folds and stretches, grow far apart. Formally, a strange attractor is a *fractal*, and may have *noninteger Hausdorff dimension.* (We won't discuss fractals and Hausdorff dimension here.)

The Lorenz Model

The canonical example of a strange attractor is found in the Lorenz model. E. N. Lorenz, in a seminal paper from the early 1960's, reduced the essential physics of the coupled *partial* differential equations describing Rayleigh-Benard convection (a fluid slab of finite thickness, heated from below – in Lorenz's case a model of the atmosphere warmed by the ocean) to a set of twelve coupled nonlinear *ordinary* differential equations. Lorenz's intuition was that his weather model should exhibit recognizable patterns over time. What he found instead was that in some cases, changing his initial conditions by a part in a thousand rapidly led to totally different behavior. This *sensitive dependence on initial conditions* is a hallmark of chaotic systems.

The essential physics/mathematics of Lorenz's $n = 12$ system is elicited by the reduced $n = 3$ system,

$$\dot{X} = -\sigma X + \sigma Y$$
$$\dot{Y} = rX - Y - XZ \qquad\qquad\qquad (1.40)$$
$$\dot{Z} = XY - bZ \;,$$

where σ, r, and b are all real and positive. Here t is the familiar time variable (appropriately scaled), and (X, Y, Z) represent linear combinations of physical fields, such as global wind current and poleward temperature gradient. These equations possess a symmetry under $(X, Y, Z) \to (-X, -Y, Z)$, but what is most important is the presence of nonlinearities in the second and third equations.

Typically the system is studied for fixed σ and b as a function of the single control parameter r. Clearly $(X, Y, Z) = (0, 0, 0)$ is a fixed point for all $\{\sigma, b, r\}$. It is quite easy to show that this fixed point is stable provided $0 < r < 1$. For $r > 1$, a new pair of solutions emerges, with

$$X^* = Y^* = \pm\sqrt{b(r-1)} \qquad , \qquad Z^* = r - 1 \quad . \qquad\qquad (1.41)$$

One can then show that these fixed points are stable for $r \in \big[1, r_\mathrm{c}\big]$, where

$$r_\mathrm{c} = \frac{\sigma(\sigma + b + 3)}{\sigma - b - 1} \;. \qquad\qquad\qquad (1.42)$$

These fixed points correspond to steady convection in the fluid model.

The Lorenz system has commonly been studied with $\sigma = 10$ and $b = \frac{8}{3}$. For these parameters, one has $r_\mathrm{c} = \frac{470}{19} \approx 24.74$. In addition to the new pair of fixed points, a strange attractor appears for $r > r_\mathrm{s} \simeq 24.06$. The capture by the strange attractor is shown in Fig. 1.4. In the narrow interval $r \in [24.06, 24.74]$ there are then *three* stable attractors, two of which correspond to steady convection and the third to chaos. Over this interval, there is also hysteresis. *I.e.* starting with a convective state for
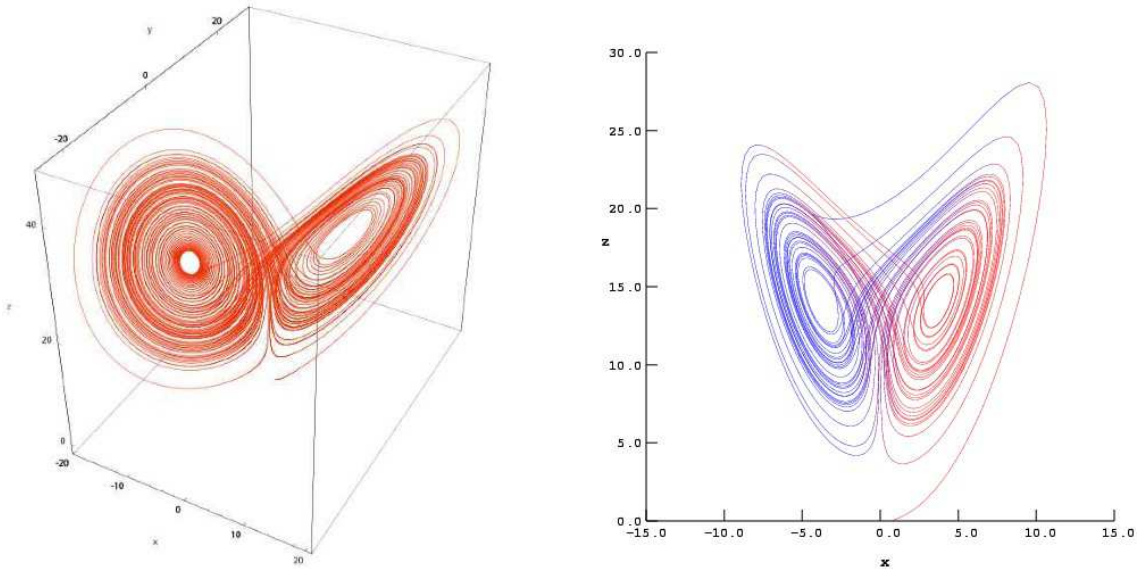
Figure 1.4: Left: Evolution of the Lorenz equations for $\sigma = 10$, $b = \frac{8}{3}$, and $r = 28$, with initial conditions $(X_0, Y_0, Z_0) = (0, 1, 0)$, showing the 'strange attractor'. Right: The Lorenz attractor, projected onto the (X, Z) plane. (Source: Wikipedia)
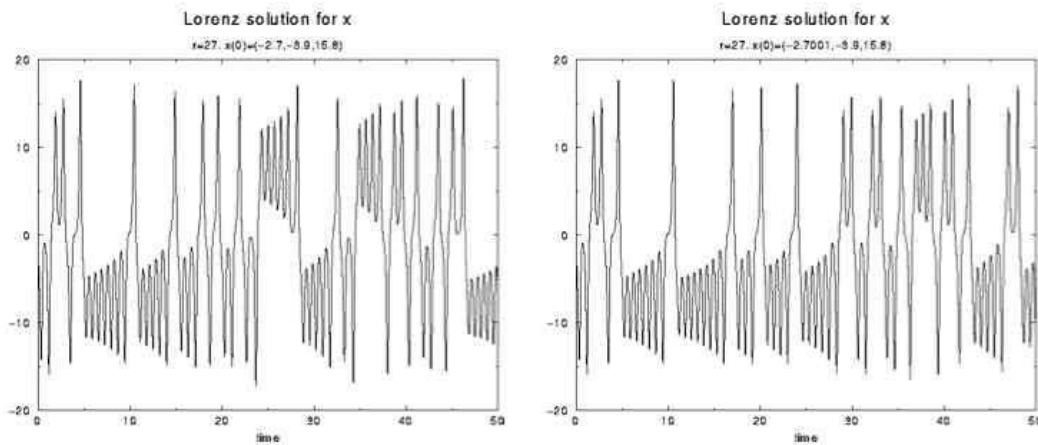
Figure 1.5: $X(t)$ for the Lorenz equations with $\sigma = 10$, $b = \frac{8}{3}$, $r = 28$, and initial conditions $(X_0, Y_0, Z_0) = (-2.7, -3.9, 15.8)$, and initial conditions $(X_0, Y_0, Z_0) = (-2.7001, -3.9, 15.8)$.

$r < 24.06$, the system remains in the convective state until $r = 24.74$, when the convective fixed point becomes unstable. The system is then driven to the strange attractor, corresponding to chaotic dynamics. Reversing the direction of r, the system remains chaotic until $r = 24.06$, when the strange attractor loses its own stability. Fig. 1.5 shows the chaotic evolution of the coordinate $X(t)$ for the case where $r = 28$. Note how, for the chosen parameters, $X(t)$ spends time oscillating about $X \approx -8$ and $X \approx +8$, but jumps randomly between these two regions, sometimes executing a single excursional spike into the opposite region.

Dynamical systems with $n = 1$

The simplest class of dynamical systems are those for which phase space is one-dimensional, *i.e.* $n = 1$. We then have

$$\frac{du}{dt} = f(u) \quad , \tag{1.43}$$

where there is a single coordinate u and the velocity function is $f(u)$. The dynamics are exceedingly simple to describe graphically. Simply sketch the function $f(u)$ *versus* u. In regions where $f(u) > 0$, $\dot{u} > 0$ and u moves to the right, *i.e.* to greater values. In regions where $f(u) < 0$, $\dot{u} < 0$ and u moves to the left. At any point $f(u) = 0$, the motion stops and $\dot{u} = 0$. Such a point is called a *fixed point* of the dynamics. Suppose $f(u^*) = 0$ and we write $u = u^* + \varepsilon$ with $|\varepsilon| \ll 1$. Then

$$\frac{d\varepsilon}{dt} = f(u^* + \varepsilon) = f'(u^*)\,\varepsilon + \tfrac{1}{2}f''(u^*)\,\varepsilon^2 + \mathcal{O}(\varepsilon^3) \quad . \tag{1.44}$$

Working to lowest nontrivial order, we see that if $f'(u^*) < 0$ then $\varepsilon(t)$ will collapse to zero exponentially (stable fixed point), but if $f'(u^*) > 0$ then $\varepsilon(t)$ will grow (unstable fixed point) until eventually we are no longer justifies in dropping higher order terms in the Taylor expansion. The fate of $u(t)$ is thus to be attracted to the first stable fixed point encountered, or to flow off to infinity.

A particularly simple example is the *logistic equation*,

$$\dot{N} = rN\left(1 - \frac{N}{K}\right) \quad , \tag{1.45}$$

with $r > 0$, which has the solution

$$N(t) = \frac{KN_0}{N_0 + \left(K - N_0\right)\exp(-rt)} \quad , \tag{1.46}$$

where the initial conditions are given by $N(0) \equiv N_0$. Note that $N = 0$ is an unstable fixed point and $N = K$ is a stable fixed point. Regardless of the initial value, as $t \to \infty$, $N(t)$ approaches the SFP, $N(+\infty) = K$. Conversely, if we run time backwards we approach the UFP, $N(-\infty) = 0$.

Note that in our discussion of the one-dimensional map $x_{n+1} = g(x_n)$ in §1.1.4, whether or not a fixed point x^* was stable or unstable depended on whether $|g'(x^*)|$ was greater or less than 1. Do you understand the difference between the two?

1.1.6   One-dimensional mechanics : simple examples

Ballistic motion

We now consider the second order ordinary differential equation

$$\frac{d^2x}{dt^2} = a_0 \quad , \tag{1.47}$$

which describes a particle undergoing constant acceleration a_0. Some notation:

$$\dot{x} \equiv \frac{dx}{dt} \quad , \quad \ddot{x} \equiv \frac{d^2x}{dt^2} \quad , \quad \dddot{\dot{x}} = \frac{d^7x}{dt^7} \quad , \quad etc. \tag{1.48}$$

Defining $v \equiv \dot{x}$, we then have $\dot{v} = a_0$, which we can integrate to obtain $v(t) = v(0) + a_0 t$. We now have

$$\dot{x} = \frac{dx}{dt} = v(0) + a_0 t \quad , \tag{1.49}$$

which we integrate to obtain the motion of the system,

$$x(t) = x(0) + v(0)\, t + \tfrac{1}{2} a_0 t^2 \quad . \tag{1.50}$$

Simple harmonic motion

Consider next the second order ODE

$$\frac{d^2x}{dt^2} = -\omega^2 x \quad , \tag{1.51}$$

i.e. $\ddot{x} = -\omega^2 x$. With $v \equiv \dot{x}$ we may write this as two coupled first order ODEs, *viz.*

$$\frac{d}{dt} \overbrace{\begin{pmatrix} x \\ v \end{pmatrix}}^{\boldsymbol{\varphi}} = \begin{pmatrix} v \\ -\omega^2 x \end{pmatrix} = \overbrace{\begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}}^{M} \overbrace{\begin{pmatrix} x \\ v \end{pmatrix}}^{\boldsymbol{\varphi}} \quad , \tag{1.52}$$

i.e. $\dot{\boldsymbol{\varphi}} = M\boldsymbol{\varphi}$. This is a linear set of coupled first order ODEs in the components of the vector $\boldsymbol{\varphi}$. In terms of the components, $\dot{x} = v$ and $\dot{v} = -\omega^2 x$. Provided the matrix M is time-independent[5], we can solve $\dot{\boldsymbol{\varphi}} = M\boldsymbol{\varphi}$ as if $\boldsymbol{\varphi}$ were a simple scalar:

$$\boldsymbol{\varphi}(t) = \exp(Mt)\, \boldsymbol{\varphi}(0) \quad . \tag{1.53}$$

But what do we mean by the exponential of the matrix Mt? We give meaning to the expression $\exp(Mt)$ through its Taylor expansion:

$$\exp(Mt) = \mathbf{1} + Mt + \tfrac{1}{2} M^2 t^2 + \tfrac{1}{6} M^3 t^3 + \dots \quad . \tag{1.54}$$

Notice that

$$M^2 = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix} = \begin{pmatrix} -\omega^2 & 0 \\ 0 & -\omega^2 \end{pmatrix} = -\omega^2 \, \mathbf{1} \quad . \tag{1.55}$$

Thus, $M^{2k} = (-\omega^2)^k \, \mathbf{1}$ and $M^{2k+1} = (-\omega^2)^k M$, which entails

$$\begin{aligned} \exp(Mt) &= \sum_{k=0}^{\infty} \frac{M^{2k}\, t^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{M^{2k+1}\, t^{2k+1}}{(2k+1)!} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k\, (\omega t)^{2k}}{(2k)!} \, \mathbf{1} + \frac{1}{\omega} \sum_{k=0}^{\infty} \frac{(-1)^k\, (\omega t)^{2k+1}}{(2k+1)!}\, M \\ &= \cos(\omega t)\, \mathbf{1} + \omega^{-1} \sin(\omega t)\, M = \begin{pmatrix} \cos(\omega t) & \omega^{-1}\sin(\omega t) \\ -\omega\sin(\omega t) & \cos(\omega t) \end{pmatrix} \quad . \end{aligned} \tag{1.56}$$

---

[5]More precisely, provided that $M(t)$ commutes with $M(t')$ for all t and t'.

Thus, the motion is

$$\varphi(t) = \begin{pmatrix} x(t) \\ v(t) \end{pmatrix} = \begin{pmatrix} \cos(\omega t) & \omega^{-1}\sin(\omega t) \\ -\omega\sin(\omega t) & \cos(\omega t) \end{pmatrix} \begin{pmatrix} x_0 \\ v_0 \end{pmatrix} \quad , \tag{1.57}$$

which is to say

$$\begin{aligned} x(t) &= \cos(\omega t)\, x_0 + \omega^{-1}\sin(\omega t)\, v_0 \\ v(t) &= -\omega\sin(\omega t)\, x_0 + \cos(\omega t)\, v_0 \end{aligned} \quad . \tag{1.58}$$

One can now check explicitly that $\dot{x}(t) = v(t)$ and $\dot{v}(t) = -\omega^2 x(t)$.

Uniform force with linear frictional damping

We consider motion in the $\hat{z}$ direction in the presence of a uniform gravitational field and frictional damping. The equation of motion is

$$m\frac{d^2 z}{dt} = -mg - \gamma\frac{dz}{dt} \tag{1.59}$$

which may be rewritten as a first order equation for $v = \dot{z}$, viz.

$$\frac{dv}{v + mg/\gamma} = -\frac{\gamma}{m}dt \tag{1.60}$$

$$d\log(v + mg/\gamma) = -(\gamma/m)dt \quad .$$

Integrating then gives

$$\log\left(\frac{v(t) + mg/\gamma}{v(0) + mg/\gamma}\right) = -\gamma t/m \tag{1.61}$$

$$v(t) = -\frac{mg}{\gamma} + \left(v(0) + \frac{mg}{\gamma}\right)e^{-\gamma t/m} \quad .$$

Note that the solution to the first order ODE $m\dot{v} = -mg - \gamma v$ entails one constant of integration, $v(0)$.

One can further integrate to obtain the motion

$$z(t) = z(0) + \frac{m}{\gamma}\left(v(0) + \frac{mg}{\gamma}\right)(1 - e^{-\gamma t/m}) - \frac{mg}{\gamma}t \quad . \tag{1.62}$$

The solution to the *second* order ODE $m\ddot{z} = -mg - \gamma\dot{z}$ thus entails *two* constants of integration: $v(0)$ and $z(0)$. Notice that as t goes to infinity the velocity tends towards the asymptotic value $v = -v_\infty$, where $v_\infty = mg/\gamma$. This is known as the *terminal velocity*. Indeed, solving the equation $\dot{v} = 0$ gives $v = -v_\infty$. The initial velocity is effectively "forgotten" on a time scale $\tau \equiv m/\gamma$.

Electrons moving in solids under the influence of an electric field also achieve a terminal velocity. In this case the force is not $F = -mg$ but rather $F = -eE$, where $-e$ is the electron charge ($e > 0$) and E is the electric field. The terminal velocity is then obtained from

$$v_\infty = eE/\gamma = e\tau E/m \quad . \tag{1.63}$$

The *current density* is a product:

$$\text{current density} = (\text{number density}) \times (\text{charge}) \times (\text{velocity}) \quad,$$

thus

$$j = n \cdot (-e) \cdot (-v_\infty) = \frac{ne^2\tau}{m} E \quad. \tag{1.64}$$

The ratio j/E is called the *conductivity* of the metal, σ. According to our theory, $\sigma = ne^2\tau/m$. This is one of the most famous equations of solid state physics! The dissipation is caused by electrons scattering off impurities and lattice vibrations ("phonons"). In high purity copper at low temperatures ($T \lesssim 4\,\text{K}$), the *scattering time* τ is about a nanosecond ($\tau \approx 10^{-9}\,\text{s}$).

Uniform force with quadratic frictional damping

At higher velocities, the frictional damping is proportional to the *square* of the velocity. The frictional force is then $F_\text{f} = -cv^2 \operatorname{sgn}(v)$, where $\operatorname{sgn}(v)$ is the *sign* of v: $\operatorname{sgn}(v) = +1$ if $v > 0$ and $\operatorname{sgn}(v) = -1$ if $v < 0$. (Note one can also write $\operatorname{sgn}(v) = v/|v|$ where $|v|$ is the *absolute value*.) Why all this trouble with $\operatorname{sgn}(v)$? Because it is important that the frictional force *dissipate* energy, and therefore that F_f be *oppositely directed* with respect to the velocity v. We will assume that $v < 0$ always, hence $F_\text{f} = +cv^2$.

Notice that there is a terminal velocity, since setting $\dot{v} = -g + (c/m)v^2 = 0$ gives $v = \pm v_\infty$, where $v_\infty = \sqrt{mg/c}$. One can write the equation of motion as

$$\frac{dv}{dt} = \frac{g}{v_\infty^2}(v^2 - v_\infty^2) \tag{1.65}$$

and using

$$\frac{1}{v^2 - v_\infty^2} = \frac{1}{2v_\infty}\left\{ \frac{1}{v - v_\infty} - \frac{1}{v + v_\infty} \right\} \tag{1.66}$$

we obtain

$$\begin{aligned}
\frac{dv}{v^2 - v_\infty^2} &= \frac{1}{2v_\infty}\frac{dv}{v - v_\infty} - \frac{1}{2v_\infty}\frac{dv}{v + v_\infty} \\
&= \frac{1}{2v_\infty}d\log\left(\frac{v_\infty - v}{v_\infty + v}\right) = \frac{g}{v_\infty^2}\,dt \quad.
\end{aligned} \tag{1.67}$$

Assuming $v(0) = 0$, we integrate to obtain

$$\log\left(\frac{v_\infty - v(t)}{v_\infty + v(t)}\right) = \frac{2gt}{v_\infty} \tag{1.68}$$

which may be massaged to give the final result

$$v(t) = -v_\infty \tanh(gt/v_\infty) \quad. \tag{1.69}$$

Recall that the *hyperbolic tangent* function $\tanh(x)$ is given by

$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \quad . \tag{1.70}$$

Thus, as in the previous example, as $t \to \infty$ one has $v(t) \to -v_\infty$, *i.e.* $v(\infty) = -v_\infty$.

Digression: To gain an understanding of the constant c, consider a flat surface of area S moving through a fluid at velocity v ($v > 0$). During a time Δt, all the fluid molecules inside the volume $\Delta V = S \cdot v \Delta t$ will have executed an elastic collision with the moving surface. Since the surface is assumed to be much more massive than each fluid molecule, the center of mass frame for the surface-molecule collision is essentially the frame of the surface itself. If a molecule moves with velocity u is the laboratory frame, it moves with velocity $u - v$ in the center of mass (CM) frame, and since the collision is elastic, its final CM frame velocity is reversed, to $v - u$. Thus, in the laboratory frame the molecule's velocity has become $2v - u$ and it has suffered a change in velocity of $\Delta u = 2(v - u)$. The total momentum change is obtained by multiplying Δu by the total mass $M = \varrho \Delta V$, where ϱ is the mass density of the fluid. But then the total momentum imparted to the fluid is

$$\Delta P = 2(v - u) \cdot \varrho \, S \, v \, \Delta t \tag{1.71}$$

and the force on the fluid is

$$F = \frac{\Delta P}{\Delta t} = 2S \, \varrho \, v(v - u) \quad . \tag{1.72}$$

Now it is appropriate to average this expression over the microscopic distribution of molecular velocities u, and since on average $\langle u \rangle = 0$, we obtain the result $\langle F \rangle = 2S\varrho v^2$, where $\langle \cdots \rangle$ denotes a microscopic average over the molecular velocities in the fluid. (There is a subtlety here concerning the effect of fluid molecules striking the surface from either side – you should satisfy yourself that this derivation is sensible!) Newton's Third Law then states that the frictional force imparted to the moving surface by the fluid is $F_{\mathrm{f}} = -\langle F \rangle = -cv^2$, where $c = 2S\varrho$. In fact, our derivation is too crude to properly obtain the numerical prefactors, and it is better to write $c = \mu \varrho S$, where μ is a dimensionless constant which depends on the *shape* of the moving object.

1.1.7   Stochastic differential equation: Langevin's equation

Consider a particle of mass m subjected to both dissipation as well as external forcing with both a constant and a random fluctuating component. We'll examine this system in one dimension to gain an understanding of the essential physics. We write

$$\dot{v} + \gamma v = g + \zeta(t) \quad . \tag{1.73}$$

Here, v is the particle's velocity, γ is the damping rate due to friction, $g = F/m$ is the acceleration due to the constant external force, and $\zeta(t)$ is a *stochastic random force* (per unit mass). This equation, known as the *Langevin equation*, describes a ballistic particle in a uniform force field being buffeted by random forcing events. The Langevin equation is an example of a *stochastic differential equation* (*i.e.* a *stochastic dynamical system*), *i.e.* a differential equation where the evolution depends on one or more random functions. Stochastic differential equations are found in many areas of statistical physics and in the mathematical

theory of finance as well, where they describe the time evolution of financial instruments. In the current context, think of a particle of dust as it moves in the atmosphere, in which case $|g|$ would then represent the acceleration due to gravity and $\zeta(t)$ the random acceleration due to collisions with the air molecules. For a sphere of radius a moving in a fluid of dynamical viscosity η, hydrodynamics gives $\gamma = 6\pi\eta a/m$. It is illustrative to compute γ in some setting. Consider a micron sized droplet ($a = 10^{-4}$ cm) of some liquid of density $\rho \sim 1.0\,\mathrm{g/cm^3}$ moving in air at $T = 20°$ C. The viscosity of air is $\eta = 1.8 \times 10^{-4}\,\mathrm{g/cm \cdot s}$ at this temperature[6]. If the droplet density is constant, then $\gamma = 9\eta/2\rho a^2 = 8.1 \times 10^4\,\mathrm{s^{-1}}$, hence the time scale for viscous relaxation of the particle is $\tau = \gamma^{-1} = 12\,\mu$s. We should stress that the viscous damping on the particle is of course also due to the fluid (*e.g.*, air) molecules, in some average 'coarse-grained' sense. The random component $\zeta(t)$ thus represents the fluctuations with respect to this average.

We can easily integrate this equation:

$$\frac{d}{dt}\left(v\,e^{\gamma t}\right) = g\,e^{\gamma t} + \zeta(t)\,e^{\gamma t} \quad \Rightarrow \quad v(t) = v(0)\,e^{-\gamma t} + \gamma^{-1}g\left(1 - e^{-\gamma t}\right) + \int_0^t ds\,\zeta(s)\,e^{\gamma(s-t)} \tag{1.74}$$

Note that the solution $v(t)$ depends on the random function $\zeta(t)$[7]. We can therefore only compute averages in order to characterize the motion of the system. One important feature of the above solution is that we see the system "loses memory" of its initial condition $u(0)$ on a time scale γ^{-1}.

The first average we will compute is that of u itself. In so doing, we assume that $\zeta(t)$ has zero mean: $\langle\zeta(t)\rangle = 0$. Then

$$\langle v(t)\rangle = v(0)\,e^{-\gamma t} + \gamma^{-1}g\left(1 - e^{-\gamma t}\right) \quad . \tag{1.75}$$

On the time scale γ^{-1}, the initial conditions $v(0)$ are effectively forgotten, and asymptotically for $t \gg \gamma^{-1}$ we have $\langle v(t)\rangle \to \gamma^{-1}g$, which is the terminal velocity.

Next, consider

$$\langle v^2(t)\rangle = \langle v(t)\rangle^2 + \int_0^t ds_1 \int_0^t ds_2\,e^{\gamma(s_1-t)}\,e^{\gamma(s_2-t)}\,\langle\zeta(s_1)\,\zeta(s_2)\rangle \quad . \tag{1.76}$$

We now need to know the *autocorrelator* $\langle\zeta(s_1)\,\zeta(s_2)\rangle$ of the random function $\zeta(s)$. We assume that this is a function only of the time difference $\Delta s = s_1 - s_2$, *viz.*

$$\langle\zeta(s_1)\,\zeta(s_2)\rangle = \phi(s_1 - s_2) \quad . \tag{1.77}$$

The function $\phi(s)$ is the autocorrelation function of the random force. A macroscopic object moving in a fluid is constantly buffeted by fluid particles over its entire perimeter. These different fluid particles are almost completely uncorrelated, hence $\phi(s)$ is basically nonzero except on a very small time scale τ_ϕ, which is the time a single fluid particle spends interacting with the object. We can take $\tau_\phi \to 0$ and approximate $\phi(s) \approx \Gamma\,\delta(s)$. As we shall now see, we can determine the value of the constant Γ from equilibrium thermodynamic considerations.

---

[6]The cgs unit of viscosity is the *Poise* (P). $1\,\mathrm{P} = 1\,\mathrm{g/cm \cdot s}$.
[7]Mathematically, we say that $v(t)$ is a *functional* of $\zeta(s)$.

With this form for $\phi(s)$, we can easily calculate the equal time velocity autocorrelation:

$$\langle v^2(t)\rangle = \langle v(t)\rangle^2 + \Gamma \int_0^t ds \, e^{2\gamma(s-t)} = \langle v(t)\rangle^2 + \frac{\Gamma}{2\gamma}\left(1 - e^{-2\gamma t}\right) \quad . \tag{1.78}$$

Consider the case where $F = 0$. We demand that the object thermalize at fluid temperature T at late times $t \gg \gamma^{-1}$, when $\langle v(t)\rangle \to 0$ and the particle has effectively forgotten all about its initial conditions. Thus, we impose the equipartition condition

$$\left\langle \tfrac{1}{2}Mv^2(t)\right\rangle = \tfrac{1}{2}k_{\mathrm{B}}T \qquad \Rightarrow \qquad \Gamma = \frac{2\gamma k_{\mathrm{B}}T}{M} \quad . \tag{1.79}$$

This fixes the value of Γ. We can now compute the general momentum autocorrelator:

$$\langle v(t)\, v(t')\rangle - \langle v(t)\rangle\langle v(t')\rangle = \int_0^t ds \int_0^{t'} ds' \, e^{\gamma(s-t)} \, e^{\gamma(s'-t')} \left\langle \zeta(s)\,\zeta(s')\right\rangle = \frac{\Gamma}{2\gamma}\, e^{-\gamma|t-t'|} \quad , \tag{1.80}$$

which is valid for $|t - t'|$ finite, and in the limit where t and t' each tend to infinity.

Since we have in eqn. 1.74 the full solution for the velocity $u(t)$, we can use it to compute the position $x(t) = x(0) + \int_0^t ds\, v(s)$ and its statistical properties. Let's compute the position $x(t)$. We find

$$x(t) = \langle x(t)\rangle + \int_0^t ds \int_0^s ds_1 \, \zeta(s_1)\, e^{\gamma(s_1-s)} \quad , \tag{1.81}$$

where

$$\langle x(t)\rangle = x(0) + \gamma^{-1}\left(v(0) - \gamma^{-1}g\right)\left(1 - e^{-\gamma t}\right) + \gamma^{-1}gt \quad . \tag{1.82}$$

Note that for $\gamma t \ll 1$ we have $\langle x(t)\rangle = x(0) + v(0)\,t + \tfrac{1}{2}gt^2 + \mathcal{O}(t^3)$, as is appropriate for ballistic particles moving under the influence of a constant force. This long time limit of course agrees with our earlier evaluation for the terminal velocity, $\langle v(\infty)\rangle \equiv v_\infty = \gamma^{-1}g$. We next compute the position autocorrelation:

$$\langle x(t)\, x(t')\rangle - \langle x(t)\rangle\langle x(t')\rangle = \int_0^t ds \int_0^{t'} ds' \, e^{-\gamma(s+s')} \int_0^s ds_1 \int_0^{s'} ds_1' \, e^{\gamma(s_1+s_2)} \left\langle \zeta(s_1)\,\zeta(s_2)\right\rangle$$

$$= \frac{\Gamma}{\gamma^2}\, \mathsf{min}(t,t') + \mathcal{O}(1) \quad .$$

In particular, at late times the equal time autocorrelator is

$$\langle x^2(t)\rangle - \langle x(t)\rangle^2 = \frac{\Gamma t}{\gamma^2} \equiv 2Dt \quad , \tag{1.83}$$

up to terms of order unity. Here, $D = \Gamma/2\gamma^2 = k_{\mathrm{B}}T/\gamma m$ is the *diffusion constant*. For a liquid droplet of radius $a = 1\,\mu\mathrm{m}$ moving in air at $T = 293\,\mathrm{K}$, for which $\eta = 1.8 \times 10^{-4}\,\mathrm{P}$, we have

$$D = \frac{k_{\mathrm{B}}T}{6\pi\eta a} = \frac{(1.38 \times 10^{-16}\,\mathrm{erg/K})\,(293\,\mathrm{K})}{6\pi\,(1.8 \times 10^{-4}\,\mathrm{P})\,(10^{-4}\,\mathrm{cm})} = 1.19 \times 10^{-7}\,\mathrm{cm}^2/\mathrm{s} \quad . \tag{1.84}$$

This result presumes that the droplet is large enough compared to the intermolecular distance in the fluid that one can adopt a continuum approach and use the Navier-Stokes equations, and then assuming a laminar flow.

If we consider molecular diffusion, the situation is quite a bit different. The diffusion constant is then $D = \ell^2/2\tau$, where ℓ is the mean free path and τ is the collision time. Elementary kinetic theory gives that the mean free path ℓ, collision time τ, number density n, and total scattering cross section σ are related by[8] $\ell = \bar{v}\tau = 1/\sqrt{2}\,n\sigma$, where $\bar{v} = \sqrt{8k_{\mathrm{B}}T/\pi m}$ is the average particle speed. Approximating the particles as hard spheres, we have $\sigma = 4\pi a^2$, where a is the hard sphere radius. At $T = 293\,\mathrm{K}$, and $p = 1\,\mathrm{atm}$, we have $n = p/k_{\mathrm{B}}T = 2.51 \times 10^{19}\,\mathrm{cm}^{-3}$. Since air is predominantly composed of N_2 molecules, we take $a = 1.90 \times 10^{-8}\,\mathrm{cm}$ and $m = 28.0\,\mathrm{amu} = 4.65 \times 10^{-23}\,\mathrm{g}$, which are appropriate for N_2. We find an average speed of $\bar{v} = 471\,\mathrm{m/s}$ and a mean free path of $\ell = 6.21 \times 10^{-6}\,\mathrm{cm}$. Thus, $D = \frac{1}{2}\ell\bar{v} = 0.146\,\mathrm{cm}^2/\mathrm{s}$. Though much larger than the diffusion constant for large droplets, this is still too small to explain certain common experiences. Suppose we set the characteristic distance scale at $d = 10\,\mathrm{cm}$ and we ask how much time a point source would take to diffuse out to this radius. The answer is $\Delta t = d^2/2D = 343\,\mathrm{s}$, which is between five and six minutes. Yet if someone in the next seat emits a foul odor, you detect the offending emission in on the order of a second. What this tells us is that diffusion isn't the only transport process involved in these and like phenomena. More important are *convection* currents which distribute the scent much more rapidly.

1.1.8 Newton's laws of motion

Aristotle held that objects move because they are somehow impelled to seek out their natural state. Thus, a rock falls because rocks belong on the earth, and flames rise because fire belongs in the heavens. To paraphrase Wolfgang Pauli, such notions are so vague as to be "not even wrong." It was only with the publication of Newton's *Principia* in 1687 that a theory of motion which had detailed predictive power was developed.

Newton's three Laws of Motion may be stated as follows:

> I. A body remains in uniform motion unless acted on by a force.
>
> II. Force equals rate of change of momentum: $\boldsymbol{F} = d\boldsymbol{p}/dt$.
>
> III. Any two bodies exert equal and opposite forces on each other.

Newton's First Law states that a particle will move in a straight line at constant (possibly zero) velocity if it is subjected to no forces. Now this cannot be true in general, for suppose we encounter such a "free" particle and that indeed it is in uniform motion, so that $\boldsymbol{r}(t) = \boldsymbol{r}(0) + \boldsymbol{v}(0)t$. Now $\boldsymbol{r}(t)$ is measured in some coordinate system, and if instead we choose to measure $\boldsymbol{r}(t)$ in a different coordinate system whose origin $\boldsymbol{R}$ moves according to the function $\boldsymbol{R}(t)$, then in this new "frame of reference" the position of our particle will be

$$\boldsymbol{r}'(t) = \boldsymbol{r}(t) - \boldsymbol{R}(t)$$
$$= \boldsymbol{r}(0) + \boldsymbol{v}(0)\,t - \boldsymbol{R}(t) \quad . \tag{1.85}$$

---

[8]The scattering time τ is related to the particle density n, total scattering cross section σ, and mean speed $\bar{v}$ through the relation $n\sigma\bar{v}_{\mathrm{rel}}\tau = 1$, which says that on average one scattering event occurs in a cylinder of cross section σ and length $\bar{v}_{\mathrm{rel}}\tau$. Here $\bar{v}_{\mathrm{rel}} = \sqrt{\bar{v}}$ is the mean relative speed of a pair of particles.

If the acceleration $d^2\boldsymbol{R}/dt^2$ is nonzero, then merely by shifting our frame of reference we have apparently falsified Newton's First Law – a free particle does *not* move in uniform rectilinear motion when viewed from an accelerating frame of reference. Thus, together with Newton's Laws comes an assumption about the existence of frames of reference – called *inertial frames* – in which Newton's Laws hold. A transformation from one frame $\mathcal{K}$ to another frame $\mathcal{K}'$ which moves at constant velocity $\boldsymbol{V}$ relative to $\mathcal{K}$ is called a *Galilean transformation*. The equations of motion of classical mechanics are *invariant* (do not change) under Galilean transformations.

At first, the issue of inertial and noninertial frames is confusing. Rather than grapple with this, we will try to build some intuition by solving mechanics problems assuming we *are* in an inertial frame. The earth's surface, where most physics experiments are done, is *not* an inertial frame, due to the centripetal accelerations associated with the earth's rotation about its own axis and its orbit around the sun. In this case, not only is our coordinate system's origin – somewhere in a laboratory on the surface of the earth – accelerating, but the coordinate axes themselves are rotating with respect to an inertial frame. The rotation of the earth leads to fictitious "forces" such as the Coriolis force, which have large-scale consequences. For example, hurricanes, when viewed from above, rotate counterclockwise in the northern hemisphere and clockwise in the southern hemisphere. Later on in the course we will devote ourselves to a detailed study of motion in accelerated coordinate systems.

Newton's "quantity of motion" is the momentum $\boldsymbol{p}$, defined as the product $\boldsymbol{p} = m\boldsymbol{v}$ of a particle's mass m (how much stuff there is) and its velocity (how fast it is moving). In order to convert the Second Law into a meaningful equation, we must know how the force $\boldsymbol{F}$ depends on the coordinates (or possibly velocities) themselves. This is known as a *force law*. Examples of force laws include:

$$
\begin{aligned}
\text{Constant force}: \quad & \boldsymbol{F} = -m\boldsymbol{g} \\
\text{Hooke's Law}: \quad & F = -kx \\
\text{Gravitation}: \quad & \boldsymbol{F} = -GMm\,\hat{\boldsymbol{r}}/r^2 \\
\text{Lorentz force}: \quad & \boldsymbol{F} = q\,\boldsymbol{E} + q\,\frac{\boldsymbol{v}}{c} \times \boldsymbol{B} \\
\text{Fluid friction } (v \text{ small}): \quad & \boldsymbol{F} = -b\,\boldsymbol{v} \quad .
\end{aligned}
$$

Note that for an object whose mass does not change we can write the Second Law in the familiar form $\boldsymbol{F} = m\boldsymbol{a}$, where $\boldsymbol{a} = d\boldsymbol{v}/dt = d^2\boldsymbol{r}/dt^2$ is the acceleration. Most of our initial efforts will lie in using Newton's Second Law to solve for the motion of a variety of systems.

The Third Law is valid for the extremely important case of *central forces* which we will discuss in great detail later on. Newtonian gravity – the force which makes the planets orbit the sun – is a central force. One consequence of the Third Law is that in free space two isolated particles will accelerate in such a way that $\boldsymbol{F}_1 = -\boldsymbol{F}_2$ and hence the accelerations are parallel to each other, with

$$
\frac{a_1}{a_2} = -\frac{m_2}{m_1} \quad , \tag{1.86}
$$

where the minus sign is used here to emphasize that the accelerations are in opposite directions. We can also conclude that the *total momentum* $\boldsymbol{P} = \boldsymbol{p}_1 + \boldsymbol{p}_2$ is a constant, a result known as the *conservation of momentum*.

Aside : inertial *vs.* gravitational mass

In addition to postulating the Laws of Motion, Newton also deduced the gravitational force law, which says that the force $\boldsymbol{F}_{ij}$ exerted by a particle i by another particle j is

$$\boldsymbol{F}_{ij} = -Gm_i m_j \frac{\boldsymbol{r}_i - \boldsymbol{r}_j}{|\boldsymbol{r}_i - \boldsymbol{r}_j|^3} \quad , \tag{1.87}$$

where G, the *Cavendish constant* (first measured by Henry Cavendish in 1798), takes the value

$$G = (6.6726 \pm 0.0008) \times 10^{-11} \mathrm{N} \cdot \mathrm{m}^2/\mathrm{kg}^2 \quad . \tag{1.88}$$

Notice Newton's Third Law in action: $\boldsymbol{F}_{ij} + \boldsymbol{F}_{ji} = 0$. Now a very important and special feature of this "inverse square law" force is that a spherically symmetric mass distribution has the same force on an external body as it would if all its mass were concentrated at its center. Thus, for a particle of mass m near the surface of the earth, we can take $m_i = m$ and $m_j = M_\mathrm{e}$, with $\boldsymbol{r}_i - \boldsymbol{r}_j \simeq R_\mathrm{e}\hat{\boldsymbol{r}}$ and obtain

$$\boldsymbol{F} = -mg\hat{\boldsymbol{r}} \equiv -m\boldsymbol{g} \tag{1.89}$$

where $\hat{\boldsymbol{r}}$ is a radial unit vector pointing from the earth's center and $g = GM_\mathrm{e}/R_\mathrm{e}^2 \simeq 9.8\,\mathrm{m/s}^2$ is the acceleration due to gravity at the earth's surface. Newton's Second Law now says that $\boldsymbol{a} = -\boldsymbol{g}$, *i.e.* objects accelerate as they fall to earth. However, it is not *a priori* clear why the *inertial mass* which enters into the definition of momentum should be the same as the *gravitational mass* which enters into the force law. Suppose, for instance, that the gravitational mass took a different value, m'. In this case, Newton's Second Law would predict

$$\boldsymbol{a} = -\frac{m'}{m}\boldsymbol{g} \tag{1.90}$$

and unless the ratio m'/m were *the same number* for *all* objects, then bodies would fall with *different accelerations*. The experimental fact that bodies in a vacuum fall to earth at the same rate demonstrates the equivalence of inertial and gravitational mass, *i.e.* $m' = m$.

1.1.9 Crossed electric and magnetic fields

Consider now a three-dimensional example of a particle of charge q moving in mutually perpendicular $\boldsymbol{E}$ and $\boldsymbol{B}$ fields. We'll throw in gravity for good measure. We take $\boldsymbol{E} = E\hat{\boldsymbol{x}}$, $\boldsymbol{B} = B\hat{\boldsymbol{z}}$, and $\boldsymbol{g} = -g\hat{\boldsymbol{z}}$. The equation of motion is Newton's 2nd Law again:

$$m\ddot{\boldsymbol{r}} = m\boldsymbol{g} + q\boldsymbol{E} + \frac{q}{c}\dot{\boldsymbol{r}} \times \boldsymbol{B} \quad . \tag{1.91}$$

The RHS (right hand side) of this equation is a vector sum of the forces due to gravity plus the Lorentz force of a moving particle in an electromagnetic field. In component notation, we have

$$m\ddot{x} = qE + \frac{qB}{c}\dot{y}$$
$$m\ddot{y} = -\frac{qB}{c}\dot{x} \tag{1.92}$$
$$m\ddot{z} = -mg \quad .$$

The equations for coordinates x and y are coupled, while that for z is independent and may be immediately solved to yield

$$z(t) = z(0) + \dot{z}(0)\, t - \tfrac{1}{2} g t^2 \quad . \tag{1.93}$$

The remaining equations may be written in terms of the velocities $v_x = \dot{x}$ and $v_y = \dot{y}$:

$$\begin{aligned} \dot{v}_x &= \omega_{\mathrm{c}}(v_y + u_{\mathrm{D}}) \\ \dot{v}_y &= -\omega_{\mathrm{c}}\, v_x \quad , \end{aligned} \tag{1.94}$$

where $\omega_{\mathrm{c}} = qB/mc$ is the *cyclotron frequency* and $u_{\mathrm{D}} = cE/B$ is the *drift speed* for the particle. As we shall see, these are the equations for a harmonic oscillator. The solution is

$$\begin{aligned} v_x(t) &= v_x(0)\,\cos(\omega_{\mathrm{c}}t) + \big(v_y(0) + u_{\mathrm{D}}\big)\,\sin(\omega_{\mathrm{c}}t) \\ v_y(t) &= -u_{\mathrm{D}} + \big(v_y(0) + u_{\mathrm{D}}\big)\cos(\omega_{\mathrm{c}}t) - v_x(0)\,\sin(\omega_{\mathrm{c}}t) \quad . \end{aligned} \tag{1.95}$$

Integrating again, the full motion is given by:

$$\begin{aligned} x(t) &= x(0) + A\,\sin\delta + A\,\sin(\omega_{\mathrm{c}}t - \delta) \\ y(r) &= y(0) - u_{\mathrm{D}}\, t - A\cos\delta + A\,\cos(\omega_{\mathrm{c}}t - \delta) \quad , \end{aligned} \tag{1.96}$$

where

$$A = \frac{1}{\omega_{\mathrm{c}}}\sqrt{\dot{x}^2(0) + \big(\dot{y}(0) + u_{\mathrm{D}}\big)^2} \quad , \quad \delta = \tan^{-1}\left(\frac{\dot{y}(0) + u_{\mathrm{D}}}{\dot{x}(0)}\right) \quad . \tag{1.97}$$

Thus, in the full solution of the motion there are *six* constants of integration:

$$x(0) \quad , \quad y(0) \quad , \quad z(0) \quad , \quad A \quad , \quad \delta \quad , \quad \dot{z}(0) \quad . \tag{1.98}$$

Of course instead of A and δ one may choose as constants of integration $\dot{x}(0)$ and $\dot{y}(0)$.

Pause for reflection

In mechanical systems, for each coordinate, or "degree of freedom," there exists a corresponding second order ODE. The full solution of the motion of the system entails two constants of integration for each degree of freedom.

1.2   Motion in One Space Dimension

1.2.1   Equations of motion for potential systems

For one-dimensional mechanical systems, Newton's second law reads

$$m\ddot{x} = F(x) \quad . \tag{1.99}$$

A system is *conservative* if the force is derivable from a potential: $F = -dU/dx$. The total energy,

$$E = T + U = \tfrac{1}{2}m\dot{x}^2 + U(x) \quad , \tag{1.100}$$

is then conserved. This may be verified explicitly:

$$\frac{dE}{dt} = \frac{d}{dt}\left[\tfrac{1}{2}m\dot{x}^2 + U(x)\right] = \left[m\ddot{x} + U'(x)\right]\dot{x} = 0 \quad . \tag{1.101}$$

Conservation of energy allows us to reduce the equation of motion from second order to first order:

$$\frac{dx}{dt} = \pm\sqrt{\frac{2}{m}\left(E - U(x)\right)} \quad . \tag{1.102}$$

Note that the constant E is a constant of integration. The $\pm$ sign above depends on the direction of motion. Points $x(E)$ which satisfy

$$E = U(x) \quad \Rightarrow \quad x(E) = U^{-1}(E) \quad , \tag{1.103}$$

where U^{-1} is the inverse function, are called *turning points*. When the total energy is E, the motion of the system is bounded by the turning points, and confined to the region(s) $U(x) \leq E$. We can integrate eqn. 1.102 to obtain

$$t(x) - t(x_0) = \pm\sqrt{\frac{m}{2}}\int_{x_0}^{x}\frac{dx'}{\sqrt{E - U(x')}} \quad . \tag{1.104}$$

This is to be inverted to obtain the function $x(t)$. Note that there are now *two* constants of integration, E and x_0. Since

$$E = E_0 = \tfrac{1}{2}mv_0^2 + U(x_0) \quad , \tag{1.105}$$

we could also consider x_0 and v_0 as our constants of integration, writing E in terms of x_0 and v_0. Thus, there are two *independent* constants of integration.

For motion confined between two turning points $x_\pm(E)$, the period of the motion is given by

$$T(E) = \sqrt{2m}\int_{x_-(E)}^{x_+(E)}\frac{dx'}{\sqrt{E - U(x')}} \quad . \tag{1.106}$$

1.2.2 The simple harmonic oscillator

In the case of the harmonic oscillator, we have $U(x) = \tfrac{1}{2}kx^2$, hence

$$\frac{dt}{dx} = \pm\sqrt{\frac{m}{2E - kx^2}} \quad . \tag{1.107}$$

The turning points are $x_\pm(E) = \pm\sqrt{2E/k}$, for $E \geq 0$. To solve for the motion, let us substitute

$$x = \sqrt{\frac{2E}{k}} \sin\theta \quad . \tag{1.108}$$

We then find

$$dt = \sqrt{\frac{m}{k}} \, d\theta \quad , \tag{1.109}$$

with solution

$$\theta(t) = \theta_0 + \omega t \quad , \tag{1.110}$$

where $\omega = \sqrt{k/m}$ is the harmonic oscillator frequency. Thus, the motion of the system is given by

$$x(t) = \sqrt{\frac{2E}{k}} \, \sin(\omega t + \theta_0) \qquad , \qquad v(t) = \sqrt{\frac{2E}{m}} \, \cos(\omega t + \theta_0) \quad . \tag{1.111}$$

Note the two constants of integration, E and θ_0.

1.2.3   One-dimensional mechanics as a dynamical system

Rather than writing the equation of motion as a single second order ODE, we can instead write it as two coupled first order ODEs, *viz.*

$$\frac{dx}{dt} = v$$
$$\frac{dv}{dt} = \frac{1}{m} F(x) \quad . \tag{1.112}$$

This may be written in matrix-vector form, as

$$\frac{d}{dt} \begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} v \\ \frac{1}{m} F(x) \end{pmatrix} \quad . \tag{1.113}$$

This is an example of a *dynamical system*, described by the general form

$$\frac{d\boldsymbol{\varphi}}{dt} = \boldsymbol{V}(\boldsymbol{\varphi}) \quad , \tag{1.114}$$

where $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_n)$ is an n-dimensional vector in *phase space*. For the model of eqn. 1.113, we evidently have $n = 2$. The object $\boldsymbol{V}(\boldsymbol{\varphi})$ is called a *vector field*. It is itself a vector, existing at every point in phase space, $\mathbb{R}^n$. Each of the components of $\boldsymbol{V}(\boldsymbol{\varphi})$ is, in general, a function of *all* n components of $\boldsymbol{\varphi}$:

$$V_j = V_j(\varphi_1, \ldots, \varphi_n) \qquad (j = 1, \ldots, n) \quad . \tag{1.115}$$

Solutions to the equation $\dot{\boldsymbol{\varphi}} = \boldsymbol{V}(\boldsymbol{\varphi})$ are called *integral curves*. Each such integral curve $\boldsymbol{\varphi}(t)$ is uniquely determined by n constants of integration, which may be taken to be the initial value $\boldsymbol{\varphi}(0)$. The collection of all integral curves is known as the *phase portrait* of the dynamical system.

In plotting the phase portrait of a dynamical system, we need to first solve for its motion, starting from arbitrary initial conditions. In general this is a difficult problem, which can only be treated numerically. But for conservative mechanical systems in $d = 1$, it is a trivial matter! The reason is that energy conservation completely determines the phase portraits. The velocity becomes a unique double-valued function of position, $v(x) = \pm\sqrt{\frac{2}{m}\big(E - U(x)\big)}$. The phase curves are thus curves of constant energy.

1.2.4 Sketching phase curves

To plot the phase curves,

  (i) Sketch the potential $U(x)$.

 (ii) Below this plot, sketch $v(x; E) = \pm\sqrt{\frac{2}{m}\big(E - U(x)\big)}$.

(iii) When E lies at a local extremum of $U(x)$, the system is at a *fixed point*.

   (a) For E slightly above E_{min}, the phase curves are ellipses.

   (b) For E slightly below E_{max}, the phase curves are (locally) hyperbolae.

   (c) For $E = E_{\mathrm{max}}$ the phase curve is called a *separatrix*[9].

 (iv) When $E > U(\infty)$ or $E > U(-\infty)$, the motion is *unbounded*.

  (v) Draw arrows along the phase curves: to the right for $v > 0$ and left for $v < 0$.

The period of the orbit $T(E)$ has a simple geometric interpretation. The area $\mathcal{A}$ in phase space enclosed by a bounded phase curve is

$$\mathcal{A}(E) = \oint_E dx\, v = \sqrt{\frac{8}{m}} \int_{x_-(E)}^{x_+(E)} dx' \sqrt{E - U(x')} \quad . \tag{1.116}$$

Thus, the period is proportional to the rate of change of $\mathcal{A}(E)$ with E:

$$T = m\,\frac{\partial \mathcal{A}}{\partial E} \quad . \tag{1.117}$$

---

[9]We might as well define separatrices to be phase curves for energies corresponding to *both* local minima as well as local maxima. For $E = E_{\mathrm{min}}$, there is a phase curve corresponding to the point $(x^*, 0)$, where x^* is the location of the local minimum in $U(x)$. For E just below E_{min}, there is no phase curve in the vicinity of x^*, while for E just above E_{min}, the phase curves in the vicinity of x^* are ellipses. When $U(x^*) = E_{\mathrm{max}}$ is a local maximum, the phase curves in the vicinity of x^* are hyperbolae. Precisely at $x = x^*$, the phase curves cross in a diabolical point resembling the letter X. Thus, in both cases corresponding to $E = E_{\mathrm{min}}$ and $E = E_{\mathrm{max}}$, *the separatrix phase curves are not (one-dimensional) manifolds*. At $E = E_{\mathrm{min}}$, the phase curve corresponds to a point, which is zero-dimensional, while at $E = E_{\mathrm{max}}$, the phase curve contains a diabolical point, at which the curve is also no longer locally homeomorphic to $\mathbb{R}^1$. For all other energies, the phase-curves are 1-manifolds, corresponding to the image of the map $t \mapsto \varphi(t)$ from the time manifold $\mathbb{R}$ to the n-dimensional phase space manifold $\mathcal{M}^n$ (typically $\mathbb{R}^n$).
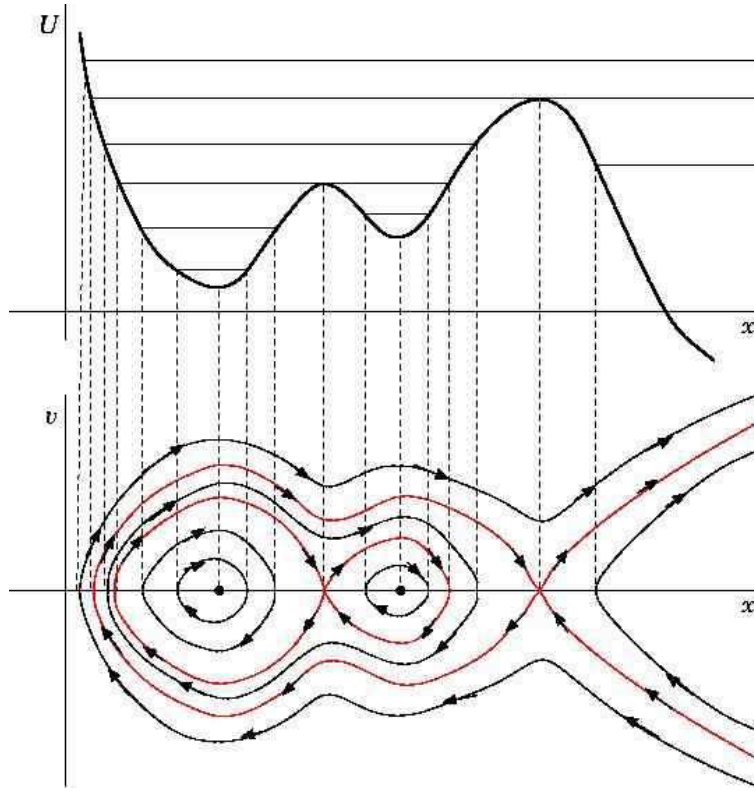
Figure 1.6: A potential $U(x)$ and the corresponding phase portraits (with separatrices in red).

1.2.5 Linearized dynamics in the vicinity of a fixed point

A fixed point (x^*, v^*) of the dynamics satisfies $U'(x^*) = 0$ and $v^* = 0$. Taylor's theorem then allows us to expand $U(x)$ in the vicinity of x^*:

$$U(x) = U(x^*) + U'(x^*) (x - x^*) + \tfrac{1}{2}U''(x^*) (x - x^*)^2 + \tfrac{1}{6}U'''(x^*) (x - x^*)^3 + \ldots \quad . \tag{1.118}$$

Since $U'(x^*) = 0$ the linear term in $\delta x = x - x^*$ vanishes. If δx is sufficiently small, we can ignore the cubic, quartic, and higher order terms, leaving us with

$$U(\delta x) \approx U_0 + \tfrac{1}{2}k\,(\delta x)^2 \quad , \tag{1.119}$$

where $U_0 = U(x^*)$ and $k = U''(x^*)$. The solutions to the motion in this potential are:

$$U''(x^*) > 0 \;:\; \delta x(t) = \delta x_0 \, \cos(\omega t) + \frac{\delta v_0}{\omega}\,\sin(\omega t) \tag{1.120}$$
$$\delta v(t) = -\omega\,\delta x_0\,\sin(\omega t) + \delta v_0\,\cos(\omega t)$$

and

$$U''(x^*) < 0 \;:\; \delta x(t) = \delta x_0 \, \cosh(\gamma t) + \frac{\delta v_0}{\gamma}\,\sinh(\gamma t) \tag{1.121}$$
$$\delta v(t) = \gamma\,\delta x_0\,\sinh(\gamma t) + \delta v_0\,\cosh(\gamma t) \quad ,$$

where $\omega = \sqrt{k/m}$ for $k > 0$ and $\gamma = \sqrt{-k/m}$ for $k < 0$. The energy is

$$E = U_0 + \tfrac{1}{2}m\,(\delta v_0)^2 + \tfrac{1}{2}k\,(\delta x_0)^2 \quad . \tag{1.122}$$

For a separatrix, we have $E = U_0$ and $U''(x^*) < 0$. From the equation for the energy, we obtain $\delta v_0 = \pm\gamma\,\delta x_0$. Let's take $\delta v_0 = -\gamma\,\delta x_0$, so that the initial velocity is directed toward the unstable fixed point (UFP). *I.e.* the initial velocity is negative if we are to the right of the UFP ($\delta x_0 > 0$) and positive if we are to the left of the UFP ($\delta x_0 < 0$). The motion of the system is then

$$\delta x(t) = \delta x_0 \, \exp(-\gamma t) \quad . \tag{1.123}$$

The particle gets closer and closer to the unstable fixed point at $\delta x = 0$, but it takes an infinite amount of time to actually get there. Put another way, the time it takes to get from δx_0 to a closer point $\delta x < \delta x_0$ is

$$t = \gamma^{-1} \log\left(\frac{\delta x_0}{\delta x}\right) \quad . \tag{1.124}$$

This diverges logarithmically as $\delta x \to 0$. Generically, then, *the period of motion along a separatrix is infinite.*

Linearization for general dynamical systems

Linearizing in the vicinity of such a fixed point, we wrote $\delta x = x - x^*$ and $\delta v = v - v^*$, obtaining

$$\frac{d}{dt}\begin{pmatrix} \delta x \\ \delta v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -m^{-1}\,U''(x^*) & 0 \end{pmatrix} \begin{pmatrix} \delta x \\ \delta v \end{pmatrix} + \dots, \tag{1.125}$$

This is a *linear* equation, which we can solve completely. The result for a general n-component dynamical system $\dot{\boldsymbol{\varphi}} = \boldsymbol{V}(\boldsymbol{\varphi})$ is given in eqn. 1.33. The linearized dynamics in the vicinity of a fixed point $\boldsymbol{\varphi}^*$, where $\boldsymbol{V}(\boldsymbol{\varphi}^*) = 0$, is given by $\dot{\boldsymbol{\varphi}} = M\boldsymbol{\varphi}$, where the components of the $n \times n$ matrix M are given by $M_{jk} = (\partial V_j/\partial \varphi_k)\big|_{\varphi*}$.

Consider now the general linear equation $\dot{\boldsymbol{\varphi}} = M\boldsymbol{\varphi}$, where M is a fixed real matrix, *i.e.* one which is independent of time t. Formally, the solution is $\boldsymbol{\varphi}(t) = \exp(Mt)\,\boldsymbol{\varphi}(0)$. Now whenever we have a problem involving matrices, we should instantly start thinking about eigenvalues and eigenvectors. Invariably, the eigenvalues and eigenvectors will prove to be useful, if not essential, in solving the problem. The eigenvalue equation is

$$M\,\boldsymbol{\psi}_\alpha = \lambda_\alpha\,\boldsymbol{\psi}_\alpha \quad . \tag{1.126}$$

Here $\boldsymbol{\psi}_\alpha$ is the αth *right eigenvector*[10] of M. The eigenvalues are roots of the characteristic equation, *i.e.* solutions to the equation $P(\lambda) = 0$, where $P(\lambda) = \det(\lambda \cdot \mathbb{I} - M)$. Let's expand $\boldsymbol{\varphi}(t)$ in terms of the right eigenvectors of M:

$$\boldsymbol{\varphi}(t) = \sum_\alpha C_\alpha(t)\,\boldsymbol{\psi}_\alpha \quad . \tag{1.127}$$

---

[10]If M is symmetric, the right and left eigenvectors are the same. If M is not symmetric, the right and left eigenvectors differ, although the set of corresponding eigenvalues is the same. We assume that the matrix M has no nontrivial Jordan blocks.
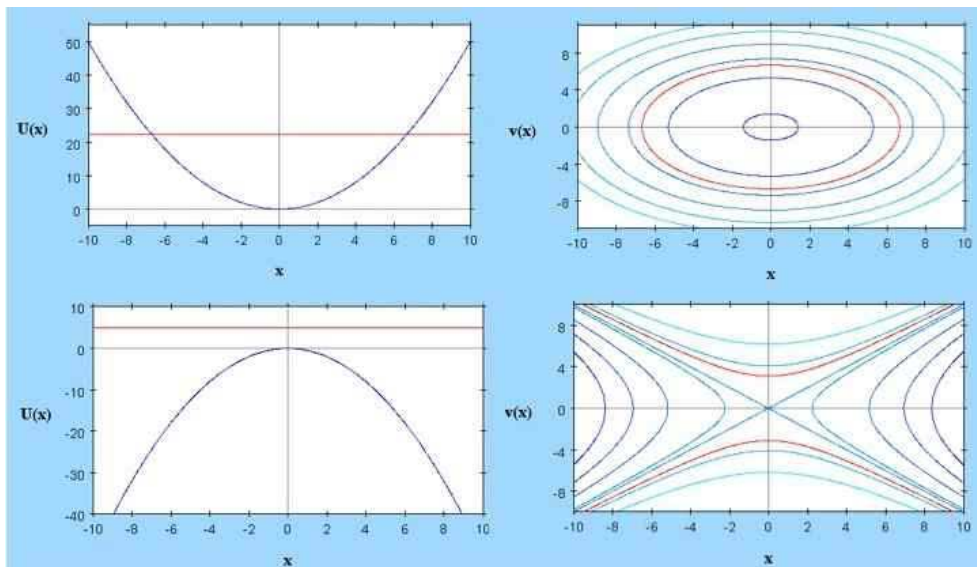
Figure 1.7: Phase curves in the vicinity of centers and saddles.

Assuming, for the purposes of this discussion, that M is nondegenerate, *i.e.* its eigenvectors span $\mathbb{R}^n$, the dynamical system can be written as a set of *decoupled* first order ODEs for the coefficients $C_\alpha(t)$:

$$\dot{C}_\alpha = \lambda_\alpha\, C_\alpha \quad , \tag{1.128}$$

with solutions

$$C_\alpha(t) = C_\alpha(0)\, \exp(\lambda_\alpha t) \quad . \tag{1.129}$$

If $\operatorname{Re}(\lambda_\alpha) > 0$, $C_\alpha(t)$ flows off to infinity, while if $\operatorname{Re}(\lambda_\alpha) < 0$, $C_\alpha(t)$ flows to zero. If $|\lambda_\alpha| = 1$, then $C_\alpha(t)$ oscillates with frequency $\operatorname{Im}(\lambda_\alpha)$.

For a two-dimensional matrix, it is easy to show – an exercise for the reader – that

$$P(\lambda) = \lambda^2 - T\lambda + D \quad , \tag{1.130}$$

where $T = \operatorname{Tr}(M)$ and $D = \det(M)$. The eigenvalues are then

$$\lambda_\pm = \tfrac{1}{2}T \pm \tfrac{1}{2}\sqrt{T^2 - 4D} \quad . \tag{1.131}$$

We'll defer study of the general case. For now, we focus on our conservative mechanical system of eqn. 1.125. The trace and determinant of the above matrix are $T = 0$ and $D = m^{-1}\, U''(x^*)$. Thus, there are only two (generic) possibilities: *centers*, when $U''(x^*) > 0$, and *saddles*, when $U''(x^*) < 0$. Examples of each are shown in fig. 1.6.
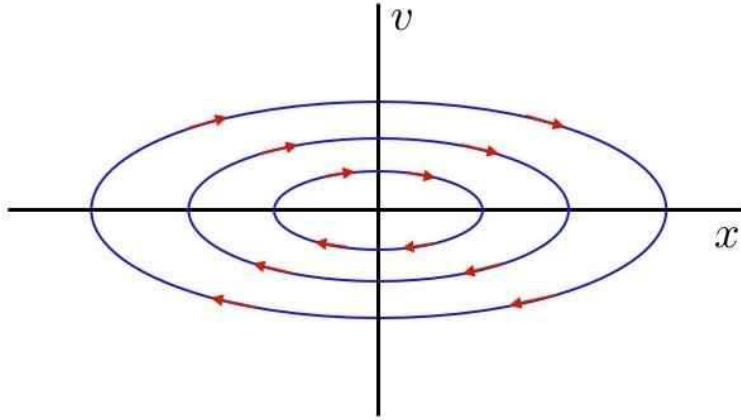
Figure 1.8: Phase curves for the harmonic oscillator.

1.3 Examples of Conservative One-Dimensional Systems

1.3.1 Harmonic oscillator

The potential energy of the harmonic oscillator in $d = 1$ dimension is $U(x) = \frac{1}{2}kx^2$. The equation of motion is

$$m\frac{d^2x}{dt^2} = -\frac{dU}{dx} = -kx \quad , \tag{1.132}$$

where m is the mass and k the force constant (of a spring). With $v = \dot{x}$, this may be written as the $N = 2$ system,

$$\frac{d}{dt}\begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}\begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} v \\ -\omega^2 x \end{pmatrix} \quad , \tag{1.133}$$

where $\omega = \sqrt{k/m}$ has the dimensions of frequency (inverse time). The solution is well known:

$$x(t) = x_0 \cos(\omega t) + \frac{v_0}{\omega} \sin(\omega t)$$
$$v(t) = v_0 \cos(\omega t) - \omega\, x_0 \sin(\omega t) \quad . \tag{1.134}$$

The phase curves are ellipses:

$$\omega_0\, x^2(t) + \omega_0^{-1}\, v^2(t) = C \quad , \tag{1.135}$$

where C is a constant, independent of time. A sketch of the phase curves and of the phase flow is shown in fig. 1.8. Note that the x and v axes have different dimensions.

Energy is conserved:

$$E = \tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2 \quad . \tag{1.136}$$

Therefore we may find the length of the semimajor and semiminor axes by setting $v = 0$ or $x = 0$, which gives

$$x_{\text{max}} = \sqrt{\frac{2E}{k}} \quad , \quad v_{\text{max}} = \sqrt{\frac{2E}{m}} \quad . \tag{1.137}$$

The area of the elliptical phase curves is thus

$$\mathcal{A}(E) = \pi\, x_{\text{max}}\, v_{\text{max}} = \frac{2\pi E}{\sqrt{mk}} \quad . \tag{1.138}$$

The period of motion is therefore

$$T(E) = m\,\frac{\partial \mathcal{A}}{\partial E} = 2\pi\sqrt{\frac{m}{k}} \quad , \tag{1.139}$$

which is independent of E.

1.3.2   Pendulum

Next, consider the simple pendulum, composed of a mass point m affixed to a massless rigid rod of length ℓ. The potential is $U(\theta) = -mg\ell\cos\theta$, hence

$$m\ell^2\,\ddot{\theta} = -\frac{dU}{d\theta} = -mg\ell\sin\theta \quad . \tag{1.140}$$

This is equivalent to

$$\frac{d}{dt}\begin{pmatrix}\theta\\ \omega\end{pmatrix} = \begin{pmatrix}\omega\\ -\omega_0^2\sin\theta\end{pmatrix} \quad , \tag{1.141}$$

where $\omega = \dot{\theta}$ is the angular velocity, and where $\omega_0 = \sqrt{g/\ell}$ is the natural frequency of small oscillations.

The conserved energy is

$$E = \tfrac{1}{2}\,m\ell^2\,\dot{\theta}^2 + U(\theta) \quad . \tag{1.142}$$

Assuming the pendulum is released from rest at $\theta = \theta_0$,

$$\frac{2E}{m\ell^2} = \dot{\theta}^2 - 2\omega_0^2\cos\theta = -2\omega_0^2\cos\theta_0 \quad . \tag{1.143}$$

The period for motion of amplitude θ_0 is then

$$T(\theta_0) = \frac{\sqrt{8}}{\omega_0}\int\limits_0^{\theta_0}\frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}} = \frac{4}{\omega_0}\,\mathrm{K}\!\left(\sin^2\tfrac{1}{2}\theta_0\right) \quad , \tag{1.144}$$

where $\mathrm{K}(z)$ is the complete elliptic integral of the first kind. Expanding $\mathrm{K}(z)$, we have

$$T(\theta_0) = \frac{2\pi}{\omega_0}\left\{1 + \tfrac{1}{4}\sin^2\left(\tfrac{1}{2}\theta_0\right) + \tfrac{9}{64}\sin^4\left(\tfrac{1}{2}\theta_0\right) + \ldots\right\} \quad . \tag{1.145}$$

For $\theta_0 \to 0$, the period approaches the usual result $2\pi/\omega_0$, valid for the linearized equation $\ddot{\theta} = -\omega_0^2\,\theta$. As $\theta_0 \to \frac{\pi}{2}$, the period diverges logarithmically.

The phase curves for the pendulum are shown in fig. 1.9. The small oscillations of the pendulum are essentially the same as those of a harmonic oscillator. Indeed, within the small angle approximation,
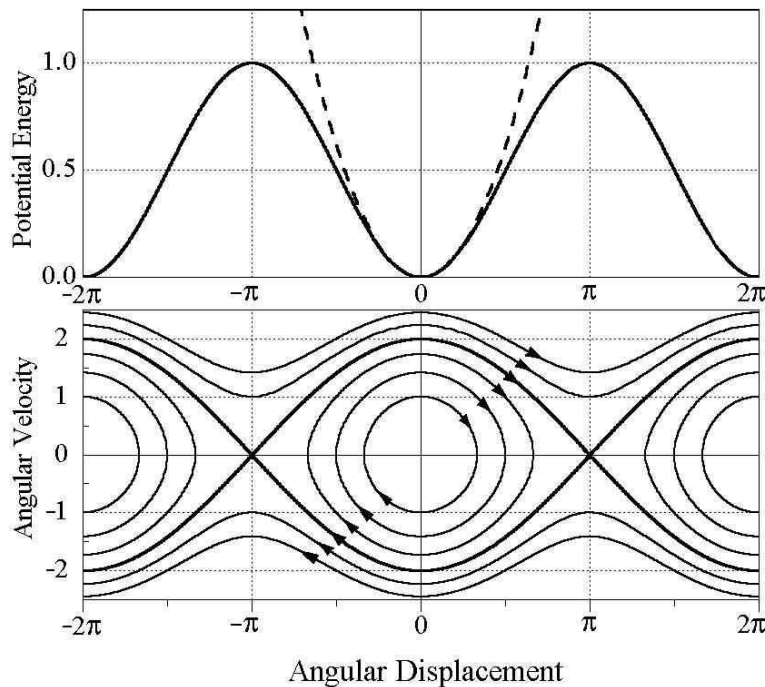
Figure 1.9: Phase curves for the simple pendulum. The *separatrix* divides phase space into regions of rotation and libration.

$\sin\theta \approx \theta$, and the pendulum equations of motion are exactly those of the harmonic oscillator. These oscillations are called *librations*. They involve a back-and-forth motion in real space, and the phase space motion is contractable to a point, in the topological sense. However, if the initial angular velocity is large enough, a qualitatively different kind of motion is observed, whose phase curves are *rotations*. In this case, the pendulum bob keeps swinging around in the same direction, because, as we'll see in a later lecture, the total energy is sufficiently large. The phase curve which separates these two topologically distinct motions is called a *separatrix*.

1.3.3   Other potentials

Using a phase plotter[11] it is possible to explore the phase curves for a wide variety of potentials. Three examples are shown in the following pages. The first is the effective potential for the Kepler problem,

$$U_{\text{eff}}(r) = -\frac{k}{r} + \frac{\ell^2}{2\mu r^2} \quad , \tag{1.146}$$

about which we shall have much more to say when we study central forces. Here r is the separation between two gravitating bodies of masses m_1 and m_2, $\mu = m_1 m_2/(m_1 + m_2)$ is the 'reduced mass', ℓ is the angular momentum perpendicular to the fixed plane of the motion, and $k = Gm_1 m_2$ where G is the

---

[11]The phase plotter used here was written by Benjamin Schmidel.

Cavendish constant. We can then write

$$U_{\text{eff}}(r) = U_0 \left\{ -\frac{1}{x} + \frac{1}{2x^2} \right\} \quad , \tag{1.147}$$

where $x \equiv r/a$ is the radial coordinate measured in units of $a \equiv \ell^2/\mu k$ (which has dimensions of length), and where $U_0 \equiv k/a = \mu k^2/\ell^2$. Thus, if distances are measured in units of a and the potential in units of U_0, the dimensionless potential may be written in dimensionless form as $\mathcal{U}(x) = -\frac{1}{x} + \frac{1}{2x^2}$.

The second is the hyperbolic secant potential,

$$U(x) = -U_0 \,\text{sech}^2(x/a) \quad , \tag{1.148}$$

which, in dimensionless form, is $\mathcal{U}(x) = -\text{sech}^2(x)$, after measuring distances in units of a and potential in units of U_0.

The final example is

$$U(x) = U_0 \left\{ \cos\left(\frac{x}{a}\right) + \frac{x}{2a} \right\} \quad . \tag{1.149}$$

Again measuring x in units of a and U in units of U_0, we arrive at $\mathcal{U}(x) = \cos(x) + \frac{1}{2}x$.
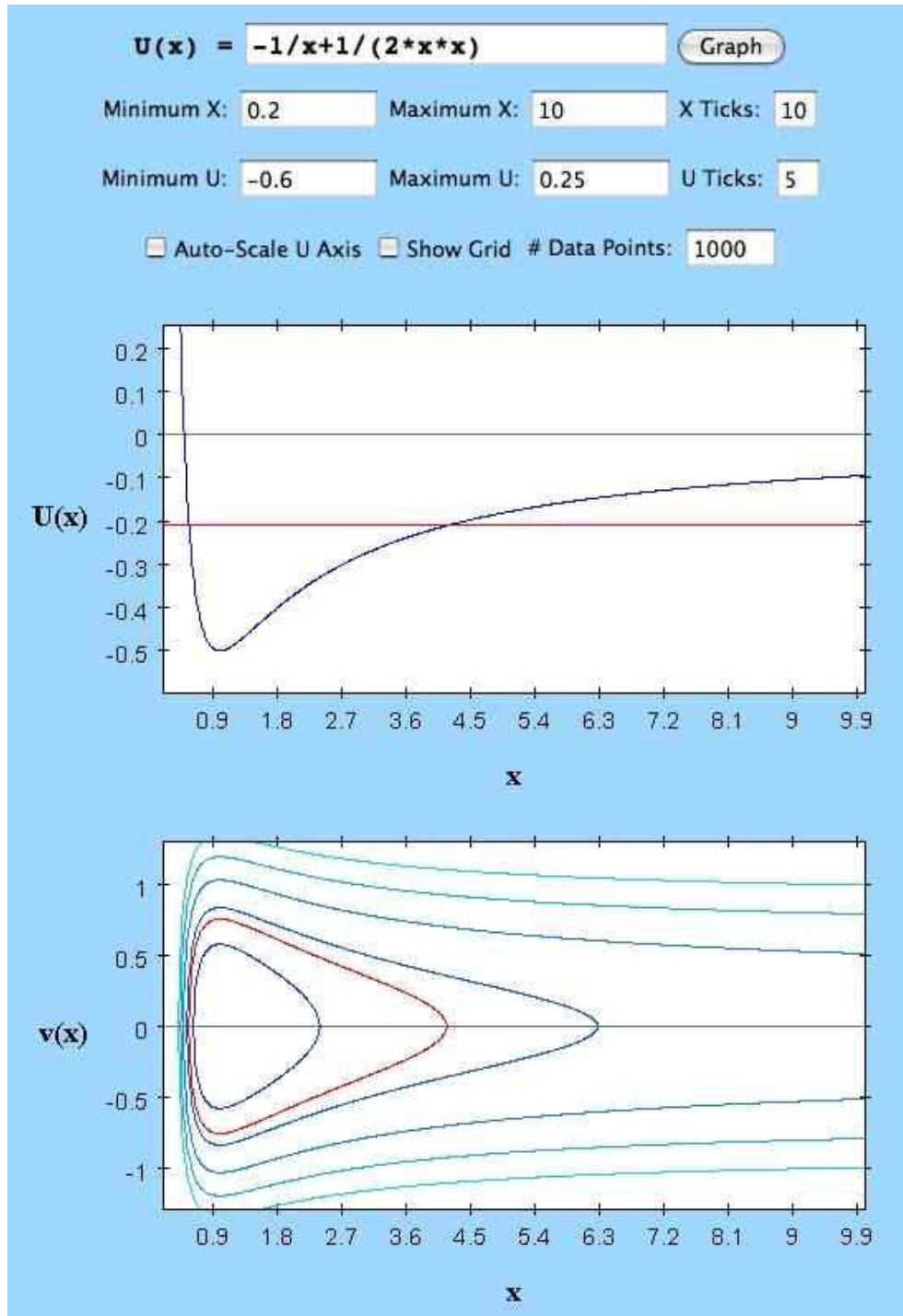
Figure 1.10: Phase curves for the Kepler effective potential $U(x) = -x^{-1} + \frac{1}{2}x^{-2}$.

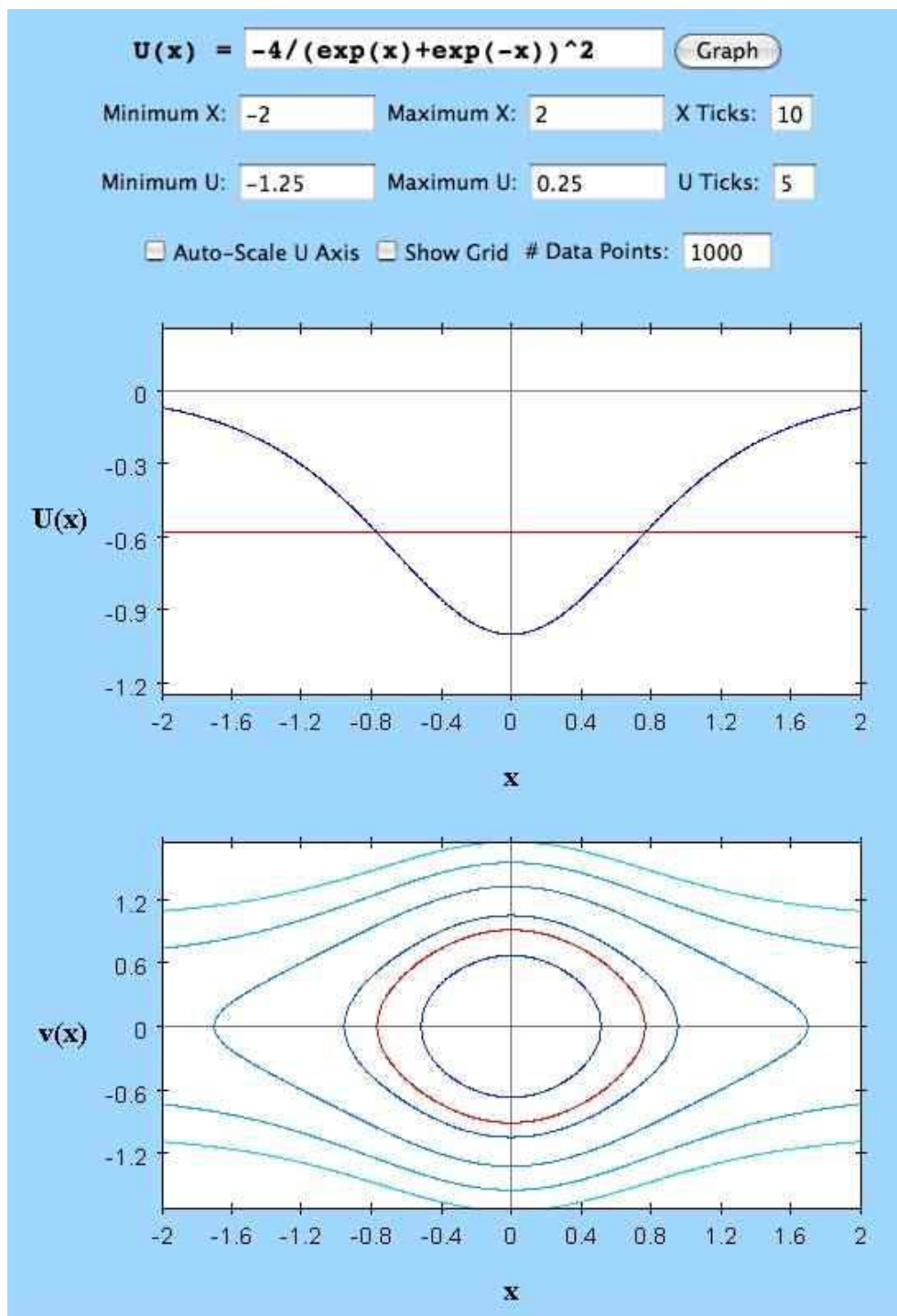Figure 1.11: Phase curves for the potential $U(x) = -\operatorname{sech}^2(x)$.
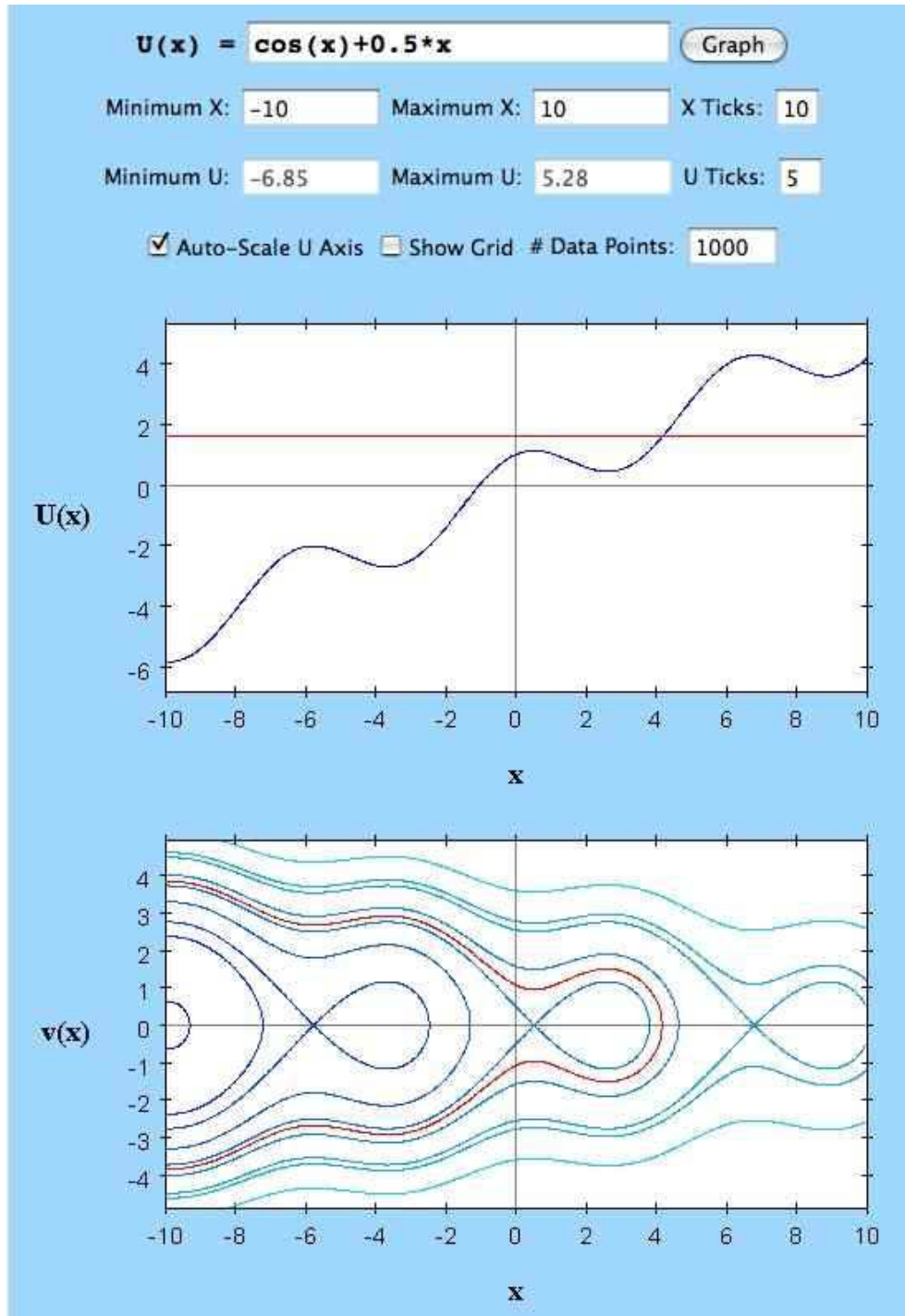
Figure 1.12: Phase curves for the potential $U(x) = \cos(x) + \frac{1}{2}x$.