- Satellites and spacecraft

Recall: $\tau = \dfrac{2\pi}{\sqrt{GM_E}} (R_E + h)^{3/2}$ $\quad (m_s << M_E)$

LEO = "Low Earth Orbit" $(h << R_E = 6.37 \times 10^6 \text{m})$

So find $\tau_{LEO} = 1.4$ hr.

Problem: $h_p = 200$ km, $\quad h_a = 7200$ km

$$a = \tfrac{1}{2}(R_E + h_p + R_E + h_a) = 10071 \text{ km}$$

$$\tau_{sat} = (a/R_E)^{3/2} \cdot \tau_{LEO} \simeq 2.65 \text{ hr}$$

- Read §§ 4.5 and 4.6

## Lecture 6 (Oct. 21)

- A **rigid body** is a collection of point particles whose separations $|\vec{r}_i - \vec{r}_j|$ are all fixed in magnitude. Six independent coordinates are required to specify completely the position and orientation of a rigid body. For example, the location of the first particle $(i)$ is specified by $\vec{r}_i$, which is three coordinates. The second $(j)$ is then specified by a direction unit vector $\hat{n}_{ij}$, which requires two additional coordinates (polar and azimuthal angle). Finally, a third particle, $k$, is then fixed by its angle relative to the $\hat{n}_{ij}$ axis. Thus, six generalized coordinates in all are required.

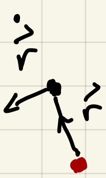
Usually, one specifies three CM coordinates $\vec{R}$, and three orientational coordinates (e.g. the Euler angles). The equations of motion are then

$$\vec{P} = \sum_i m_i \dot{\vec{r}}_i \quad , \quad \dot{\vec{P}} = \vec{F}^{ext} \quad \text{(external force)}$$

$$\vec{L} = \sum_i m_i \vec{r}_i \times \dot{\vec{r}}_i \quad , \quad \dot{\vec{L}} = \vec{N}^{ext} \quad \text{(external torque)}$$

- Inertia tensor

Suppose a point within a rigid body is fixed. This eliminates the translational motion. If we measure distances relative to this fixed point, then in an inertial frame,



$$\frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r} \quad ; \quad \vec{\omega} = \text{angular velocity}$$

The kinetic energy is then

$$T = \frac{1}{2} \sum_i m_i \left(\frac{d\vec{r}_i}{dt}\right)^2 = \frac{1}{2} \sum_i (\vec{\omega} \times \vec{r}_i) \cdot (\vec{\omega} \times \vec{r}_i)$$

$$= \frac{1}{2} \sum_i m_i \left[\omega^2 \vec{r}_i^{\,2} - (\vec{\omega} \cdot \vec{r}_i)^2\right] \equiv \frac{1}{2} I_{\alpha\beta} \omega_\alpha \omega_\beta$$

where $I_{\alpha\beta}$ is the **inertia tensor**,

3x3 real symmetric matrix ⇒ 6 DOF

$$I_{\alpha\beta} = \sum_i m_i \left[\vec{r}_i^{\,2} \delta^{\alpha\beta} - r_i^\alpha r_i^\beta\right] \quad \text{(discrete)}$$

$$= \int d^d r \, \rho(\vec{r}) \left[\vec{r}^{\,2} \delta^{\alpha\beta} - r^\alpha r^\beta\right] \quad \text{(continuous)}$$

Diagonal elements of $I_{\alpha\beta}$ are <u>moments</u> of inertia, while off-diagonal elements are <u>products</u> of inertia.

- coordinate transformations

$\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ = orthonormal basis ; $\hat{e}_\alpha \cdot \hat{e}_\beta = \delta_{\alpha\beta}$

Orthogonal basis transformation :

$$\hat{e}'_\alpha = R_{\alpha\mu}\hat{e}_\mu \quad ; \quad \hat{e}'_\alpha \cdot \hat{e}'_\beta = R_{\alpha\mu}R_{\beta\nu}\hat{e}_\mu \cdot \hat{e}_\nu = (R^T R)_{\alpha\beta} = \delta_{\alpha\beta}$$

Let $\vec{A} = A^\mu \hat{e}_\mu$ be a vector with $A^\alpha$ the components. Then

$$\vec{A} = A^\mu \hat{e}_\mu = A^\mu R_{\alpha\mu}\hat{e}'_\alpha \Rightarrow \underbrace{A'^\alpha = R_{\alpha\mu}A^\mu}_{\text{coordinate transformation}}$$

How does the inertia tensor transform?

$$I'_{\alpha\beta} = \int d^3r' \rho'(r')\left[\vec{r}'^2 \delta^{\alpha\beta} - r'^\alpha r'^\beta\right]$$

$$= \int d^3r \rho(\vec{r})\left[\vec{r}^2 \delta^{\alpha\beta} - R_{\alpha\mu} r^\mu R_{\beta\nu} r^\nu\right]$$

$$= R_{\alpha\mu} I_{\mu\nu} R^T_{\nu\beta} \, , \quad \text{since } \rho'(\vec{r}') = \rho(\vec{r})$$

i.e. $\vec{v}' = R\vec{v}$ is the transformation rule for vectors, and $I' = RIR^T$ the rule for tensors. For scalars, $s' = s$. Note $\vec{\omega}$ is a vector, as is $\vec{L}$, but

$$T = \frac{1}{2}\omega_\alpha I_{\alpha\beta}\omega_\beta \quad \text{is a scalar}$$

Note : $T = \frac{1}{2} R^T_{\alpha\mu}\omega'_\mu I_{\alpha\beta} R^T_{\beta\nu}\omega'_\nu = \frac{1}{2}\omega'_\mu (R_{\mu\alpha} I_{\alpha\beta} R^T_{\beta\nu})\omega'_\nu$

$$= \frac{1}{2}\omega'_\mu I'_{\mu\nu}\omega'_\nu = T' \qquad (\vec{\omega} = R^T\vec{\omega}')$$

- The case of no fixed point

If there is no fixed point, choose CM as instantaneous origin for the body-fixed frame:

$$\vec{R} = \frac{1}{M}\sum_i m_i \vec{r}_i = \frac{1}{M}\int d^3r\, \rho(\vec{r})\, \vec{r}$$

$$M = \sum_i m_i = \int d^3r\, \rho(\vec{r}) = \text{total mass}$$

Then

$$T = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}I_{\alpha\beta}\,\omega^\alpha\omega^\beta$$

$$L_\alpha = \epsilon_{\alpha\beta\gamma} M R^\beta \dot{R}^\gamma + I_{\alpha\beta}\,\omega^\beta$$

- Parallel axis theorem

Suppose we have $I_{\alpha\beta}$ in a body-fixed frame. Now shift the origin from $O$ to $\vec{d}$. A mass at position $\vec{r}_i$ is located at $\vec{r}_i - \vec{d}$ as a result. Thus,

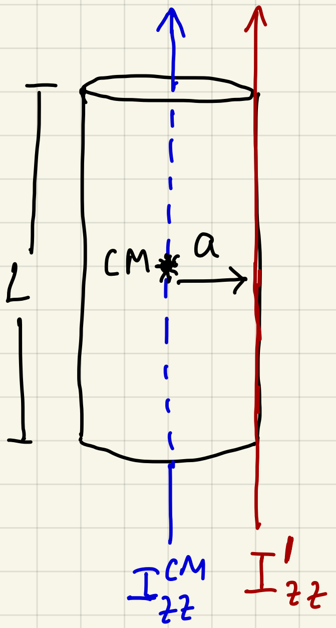
$$I_{\alpha\beta}(\vec{d}) = \sum_i m_i \left[ (\vec{r}_i^{\,2} - 2\vec{d}\cdot\vec{r}_i + \vec{d}^{\,2})\,\delta^{\alpha\beta} - (r_i^\alpha - d^\alpha)(r_i^\beta - d^\beta) \right]$$

If $\vec{r}_i$ in the original frame is wrt the CM, then $\sum_i m_i \vec{r}_i = 0$, and we have

$$I_{\alpha\beta}(\vec{d}) = I_{\alpha\beta}^{CM} + M(\vec{d}^{\,2}\delta^{\alpha\beta} - d^\alpha d^\beta)$$

Since we are only translating the origin, the coordinate axes remain parallel. Hence this result is known as the parallel axis theorem.

Example : uniform cylinder of radius $a$, height $L$



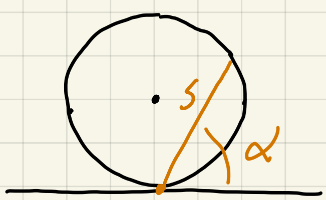

With origin at CM,

$$I^{CM}_{zz} = \int d^3r\, \rho(\vec{r})\,(\underbrace{x^2+y^2}_{\vec{r}^{\,2}-z^2})$$

$$= 2\pi\rho L \int_0^a dr_\perp\, r_\perp^3 = \frac{\pi}{2}\rho L a^4$$

$$= \frac{1}{2}Ma^2 \quad \text{since} \quad M = \pi a^2 L\rho$$

Displace origin to surface : $\vec{d} = a\hat{\rho}$




Distance $s$ ranges from $0$ to $s_0$, with

$$a^2 = (s_0\cos\alpha)^2 + (s_0\sin\alpha - a)^2$$

$$= s_0^2 + a^2 - 2as_0\sin\alpha \Rightarrow s_0 = 2a\sin\alpha$$

Thus, $I'_{zz} = \rho L \int_0^\pi d\alpha \int_0^{2a\sin\alpha} ds\, s^3 = \dfrac{M}{\pi a^2}\cdot 4a^4 \cdot \underbrace{\int_0^\pi d\alpha\, \sin^4\alpha}_{3\pi/8}$

$$I'_{zz} = \frac{3}{2}Ma^2$$

Using parallel axis theorem : $\vec{d} = a\hat{x}$

$$I'_{zz} = I^{CM}_{zz} + M(\vec{d}^{\,2}\delta^{zz} - d^z d^z)$$

$$= \frac{1}{2}Ma^2 + Ma^2 = \frac{3}{2}Ma^2 \checkmark$$

No need for trigonometry or integration!

- Read § 8.3.1 (inertia tensor for right triangle)

- Planar mass distributions:

If $\rho(x,y,z) = \sigma(x,y)\,\delta(z)$, then $I_{xz} = I_{yz} = 0$

Furthermore,

$$I_{xx} = \int dx \int dy\, \sigma(x,y)\, y^2$$

$$I_{yy} = \int dx \int dy\, \sigma(x,y)\, x^2$$

$$I_{xy} = -\int dx \int dy\, \sigma(x,y)\, xy$$

$$I = \begin{pmatrix} I_{xx} & I_{xy} & 0 \\ I_{xy} & I_{yy} & 0 \\ 0 & 0 & I_{xx}+I_{yy} \end{pmatrix}$$

and $I_{zz} = I_{xx} + I_{yy}$. Only 3 parameters.

- Principal axes of inertia

In general, if you have a symmetric matrix and you diagonalize it, good things will happen.

Recall that basis transformation $\hat{e}'_\alpha = R_{\alpha\mu}\hat{e}_\mu$ entails the transformation rules for vectors and tensors,

$$\vec{A}' = R\vec{A} \quad , \quad I' = RIR^T$$

i.e. $\quad A'^{\alpha} = R_{\alpha\mu}A^{\mu} \quad , \quad I'_{\alpha\beta} = R_{\alpha\mu}\, I_{\mu\nu}\, R^T_{\nu\beta}$

Since $I = I^T$ is symmetric, we can find a new orthonormal basis $\{\hat{e}'_\mu\}$ with respect to which $I'$ is diagonal. Dropping the primes, we have that in a diagonal basis,

$$I = \mathrm{diag}(I_1, I_2, I_3) \quad , \quad \vec{L} = (I_1\omega_1, I_2\omega_2, I_3\omega_3)$$

$$T = \tfrac{1}{2}\omega_\alpha I_{\alpha\beta}\omega_\beta = \tfrac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right)$$

How to diagonalize $I_{\alpha\beta}$ (or any real symmetric matrix):

1) Find the diagonal elements of $I'$, which are the eigenvalues of $I$, by solving $P(\lambda) \equiv \det(\lambda \cdot \mathbb{1} - I) = 0$. If $I_{\alpha\beta}$ is of rank $n$, $P(\lambda)$ is a polynomial in $\lambda$ of order $n$.

2) For each eigenvalue $\lambda_a$ $(a = 1, \ldots, n)$, solve the $n$ equations

$$\sum_{\nu=1}^{n} I_{\mu\nu} \psi_\nu^a = \lambda_a \psi_\mu^a$$

where $\psi_\mu^a$ is the $\mu^{th}$ component of the $a^{th}$ eigenvector $\vec{\psi}^a$. Since $(\lambda_a \cdot \mathbb{1} - I)$ is degenerate, the above equations are linearly dependent, and we may solve for the $(n-1)$ ratios $\{\psi_2^a / \psi_1^a, \ldots, \psi_n^a / \psi_1^a\}$.

3) Since $I_{\alpha\beta}$ is real and symmetric, its eigenfunctions corresponding to distinct eigenvalues are necessarily orthogonal. Eigenvectors corresponding to degenerate eigenvalues may be chosen to be orthogonal via the Gram–Schmidt procedure. Finally, the eigenvectors are normalized, thus

$$\langle \vec{\psi}^a | \vec{\psi}^b \rangle = \sum_{\mu=1}^{n} \psi_\mu^a \psi_\mu^b = \delta^{ab}$$

4) The matrix elements of $R$ are then given by $R_{a\mu} = \psi_\mu^a$, i.e. the $a^{th}$ row of $R$ is the eigenvector $\psi_\mu^a$, which is the $a^{th}$ column of $R^T$.

5) The eigenvectors are complete and orthonormal.

completeness : $\sum_a \psi^a_\mu \psi^a_\nu = R_{a\mu} R_{a\nu} = (R^T R)_{\mu\nu} = \delta_{\mu\nu}$

orthogonality : $\sum_\mu \psi^a_\mu \psi^b_\mu = R_{a\mu} R_{b\mu} = (R R^T)_{ab} = \delta_{ab}$

See § 8.4 Eqns. 8.32 – 8.38 for an example

- Euler's equations

We choose our coordinate axes such that $I_{\alpha\beta}$ is diagonal. Such a choice $\{\hat{e}_\alpha\}$ are called **principal axes of inertia.** We further choose the origin to be located at the CM. Thus

$$\vec{\omega} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}, \quad I = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix}, \quad \vec{L} = I\vec{\omega} = \begin{pmatrix} I_1\omega_1 \\ I_2\omega_2 \\ I_3\omega_3 \end{pmatrix}$$

The equations of motion are then

in body-fixed frame

$$\vec{N}^{ext} = \left(\frac{d\vec{L}}{dt}\right)_{inertial} = \left(\frac{d\vec{L}}{dt}\right)_{body} + \vec{\omega} \times \vec{L}$$

in inertial frame

$$= I\dot{\vec{\omega}} + \vec{\omega} \times (I\vec{\omega})$$

Here we have used the important relation

$$\left(\frac{d\vec{A}}{dt}\right)_{inertial} = \left(\frac{d\vec{A}}{dt}\right)_{body} + \vec{\omega} \times \vec{A} ,$$

valid for any vector $\vec{A}$. Let's derive this important result.

- Interlude : accelerated coordinate systems (§ 7.1)

Consider an inertial frame with fixed coordinate axes $\hat{e}_\mu$, and a rotating frame with axes $\hat{e}'_\mu$, where $\mu \in \{1, \ldots, d\}$. The two frames share a common origin which is fixed within the body.

Any vector $\vec{A}$ may be written as

$$\vec{A} = \sum_\mu A_\mu \hat{e}_\mu = \sum_\mu A'_\mu \hat{e}'_\mu$$

Thus in the inertial frame

$$\hat{e}'_\alpha(t) = R_{\alpha\mu}(t)\,\hat{e}_\mu$$

$$\left(\frac{d\vec{A}}{dt}\right)_{\text{inertial}} = \sum_\mu \frac{dA_\mu}{dt}\,\hat{e}_\mu$$

$$= \underbrace{\sum_\mu \frac{dA'_\mu}{dt}\,\hat{e}'_\mu}_{\text{this is } (d\vec{A}/dt)_{\text{body}}} + \sum_\mu A'_\mu \frac{d\hat{e}'_\mu}{dt}$$

What is $d\hat{e}'_\mu/dt$ ? Since the basis $\{\hat{e}'_\nu\}$ is complete, we may expand

$$d\hat{e}'_\mu = \sum_\nu d\Omega_{\mu\nu}\,\hat{e}'_\nu \iff d\Omega_{\mu\nu} = d\hat{e}'_\mu \cdot \hat{e}'_\nu$$

But $d\big(\underbrace{\hat{e}'_\mu \cdot \hat{e}'_\nu}_{\delta_{\mu\nu}}\big) = d\hat{e}'_\mu \cdot \hat{e}'_\nu + \hat{e}'_\mu \cdot d\hat{e}'_\nu = d\Omega_{\mu\nu} + d\Omega_{\nu\mu} = 0$

Thus, $d\Omega_{\mu\nu}$ is a real, antisymmetric, infinitesimal $d \times d$ matrix.

A $d \times d$ real antisymmetric matrix has $\frac{1}{2}d(d-1)$ independent entries. For $d=3$, we may write

$$d\Omega_{\mu\nu} = \sum_{\sigma} \epsilon_{\mu\nu\sigma}\, d\Omega_{\sigma}$$

and we define $\omega_\sigma \equiv d\Omega_\sigma / dt$. This yields

$$\frac{d\hat{e}'_\mu}{dt} = \vec{\omega} \times \hat{e}'_\mu$$

and we have

$$\left(\frac{d\vec{A}}{dt}\right)_{\text{inertial}} = \left(\frac{d\vec{A}}{dt}\right)_{\text{body}} + \vec{\omega} \times \vec{A}$$

is valid for any vector $\vec{A}$. We may then write

$$\left.\frac{d}{dt}\right|_{\text{inertial}} = \left.\frac{d}{dt}\right|_{\text{body}} + \vec{\omega} \times$$

so long as we apply this to vectors only. Applied to the vector $\vec{\omega}$ itself, this yields $\dot{\vec{\omega}}_{\text{inertial}} = \dot{\vec{\omega}}_{\text{body}}$.

Applied twice,

$$\left.\frac{d^2\vec{A}}{dt}\right|_{\text{inertial}} = \left.\frac{d^2\vec{A}}{dt}\right|_{\text{body}} + \frac{d\vec{\omega}}{dt} \times \vec{A} + \underbrace{2\vec{\omega} \times \left.\frac{d\vec{A}}{dt}\right|_{\text{body}}}_{\text{Coriolis}} + \underbrace{\vec{\omega} \times (\vec{\omega} \times \vec{A})}_{\text{centrifugal}}$$

This formula contains the description of centrifugal and Coriolis forces, which you can read about in chapter 7 of the notes. But for now, back to rigid body dynamics...

Euler's equations along body-fixed principal axes:

$$\left(\frac{d\vec{L}}{dt}\right)_{\text{inertial}} = \left(\frac{d\vec{L}}{dt}\right)_{\text{body}} + \vec{\omega}\times\vec{L} = I\dot{\vec{\omega}} + \vec{\omega}\times(I\vec{\omega}) = \vec{N}^{ext}$$

Component by component,

$$\begin{aligned} I_1\dot{\omega}_1 &= (I_2 - I_3)\,\omega_2\omega_3 + N_1^{ext} \\ I_2\dot{\omega}_2 &= (I_3 - I_1)\,\omega_3\omega_1 + N_2^{ext} \\ I_3\dot{\omega}_3 &= (I_1 - I_2)\,\omega_1\omega_2 + N_3^{ext} \end{aligned}$$

These three equations are coupled and nonlinear. The components $N_\alpha^{ext}$ must be evaluated along the body-fixed principal axes. The simplest case is when there is no net external torque, which is the case when a body moves in free space, but also in a uniform gravitational field:

$$\vec{N}^{ext} = \sum_i \vec{r}_i \times (m_i\vec{g}) = \left(\sum_i m_i\vec{r}_i\right)\times\vec{g}$$

In a body fixed frame with the origin at the CM, the term in parentheses vanishes, hence $\vec{N}^{ext} = 0$, and

$$\dot{\omega}_1 = \left(\frac{I_2 - I_3}{I_1}\right)\omega_2\omega_3\,, \quad \dot{\omega}_2 = \left(\frac{I_3 - I_1}{I_2}\right)\omega_3\omega_1\,, \quad \dot{\omega}_3 = \left(\frac{I_1 - I_2}{I_3}\right)\omega_1\omega_2$$

- Torque-free symmetric tops:

Suppose $I_1 = I_2 \neq I_3$. Then $\dot{\omega}_3 = 0$, hence $\omega_3 = \text{const.}$ The remaining two equations are

$$\dot{\omega}_1 = \left(\frac{I_1 - I_3}{I_1}\right)\omega_3\omega_2 \quad , \quad \dot{\omega}_2 = \left(\frac{I_3 - I_1}{I_1}\right)\omega_3\omega_1$$

hence $\dot{\omega}_1 = -\Omega\omega_2$, $\dot{\omega}_2 = +\Omega\omega_1$, with $\Omega = \left(\frac{I_3 - I_1}{I_1}\right)\omega_3$.

Thus,

$$\omega_1(t) = \omega_\perp \cos(\Omega t + \delta) \, , \quad \omega_2(t) = \omega_\perp \sin(\Omega t + \delta) \, , \quad \omega_3(t) = \omega_3$$

where $\omega_\perp$ and $\delta$ are constants of integration. Therefore, in the body-fixed frame, $\vec{\omega}(t)$ precesses about $\hat{e}_3$ ($\equiv \hat{e}_3^{\text{body}}$) with frequency $\Omega$ at an angle $\lambda = \tan^{-1}(\omega_\perp/\omega_3)$. For the earth, this is called the **Chandler wobble**, and $\lambda \simeq 6 \times 10^{-7}$ rad, meaning that the north pole moves by about four meters during the wobble. Again for earth, $(I_3 - I_1)/I_1 \approx \frac{1}{305}$, hence the precession period is predicted to be about 305 days. In fact, the period of the Chandler wobble is about 14 months, which is a substantial discrepancy, attributed to the mechanical properties of the earth (elasticity and fluidity): the earth isn't solid!

- Asymmetric tops

In principal, we may invoke energy and angular momentum conservation,

$$E = \frac{1}{2} I_1 \omega_1^2 + \frac{1}{2} I_2 \omega_2^2 + \frac{1}{2} I_3 \omega_3^2$$

$$\vec{L}^2 = I_1^2 \omega_1^2 + I_2^2 \omega_2^2 + I_3^2 \omega_3^2$$

and obtain $\omega_1$ and $\omega_2$ in terms of $\omega_3$. Then

$$\dot{\omega}_3 = \left( \frac{I_1 - I_2}{I_3} \right) \omega_1 \omega_2$$

becomes a nonlinear first order ODE. Using Lagranges method and extremizing the energy at fixed $L^2$, we obtain the following:

| conditions | energy $E$ | extremum classification $I_i < I_j < I_k$ | | | | | |
|---|---|---|---|---|---|---|---|
| | | 123 | 213 | 132 | 312 | 231 | 321 |
| $\omega_2 = \omega_3 = 0$ | $\frac{1}{2} I_1 \omega_1^2 = \frac{L^2}{2I_1}$ | MAX | SP | MAX | SP | MIN | MIN |
| $\omega_1 = \omega_3 = 0$ | $\frac{1}{2} I_2 \omega_2^2 = \frac{L^2}{2I_2}$ | SP | MAX | MIN | MIN | MAX | SP |
| $\omega_1 = \omega_2 = 0$ | $\frac{1}{2} I_3 \omega_3^2 = \frac{L^2}{2I_3}$ | MIN | MIN | SP | MAX | SP | MAX |

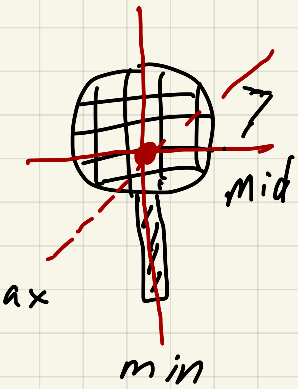
We can then analyze the nonlinear ODE $\dot{\omega}_3 = f(\omega_3)$. This is somewhat unpleasant.

We can however easily linearize the equations of motion about a known solution. For example, $\omega_1 = \omega_2 = 0$ and $\omega_3 = \omega_0$ is a solution of Euler's equations. Let us then write $\vec{\omega} = \omega_0 \hat{e}_3 + \delta\vec{\omega}$. Then

$$\delta\dot{\omega}_1 = \left(\frac{I_2 - I_3}{I_1}\right)\omega_0\,\delta\omega_2 + \mathcal{O}(\delta\omega_2\,\delta\omega_3)$$

$$\delta\dot{\omega}_2 = \left(\frac{I_3 - I_1}{I_2}\right)\omega_0\,\delta\omega_1 + \mathcal{O}(\delta\omega_1\,\delta\omega_3)$$

$$\delta\dot{\omega}_3 = 0 + \mathcal{O}(\delta\omega_1\,\delta\omega_2)$$




Thus, we have $\delta\ddot{\omega}_1 = -\Omega^2\delta\omega_1$ and $\delta\ddot{\omega}_2 = -\Omega^2\delta\omega_2$ with

$$\Omega^2 = \frac{(I_3 - I_1)(I_3 - I_2)}{I_1 I_2}\,\omega_0^2$$

The solution is $\delta\omega_1(t) = \epsilon\cos(\Omega t + \eta)$, in which case

$$\delta\omega_2(t) = \omega_0^{-1}\frac{I_1}{I_2 - I_3}\,\delta\dot{\omega}_1 = \left(\frac{I_1(I_3 - I_1)}{I_2(I_3 - I_2)}\right)^{1/2}\epsilon\sin(\Omega t + \delta)$$

If $\Omega \in \mathbb{R}$, $\delta\omega_1(t)$ and $\delta\omega_2(t)$ are harmonic functions with period $2\pi/\Omega$. This is the case when $I_3 > I_{1,2}$ or $I_3 < I_{1,2}$. But if $I_3$ is in the middle, i.e. $I_1 < I_3 < I_2$ or $I_2 < I_3 < I_1$, then $\Omega^2 < 0$, $\Omega \in i\mathbb{R}$, and the behavior is exponential, i.e. $\vec{\omega}(t) = \omega_0\hat{e}_3$ is <u>unstable</u>.

- Read § 8.5.1 (example problem for Euler's equations)

- Euler's angles

The dimension of the orthogonal group $O(n)$ is

$$\dim O(n) = \frac{1}{2} n(n-1)$$

Thus in dimension $n=2$, a rotation is specified by a single parameter, i.e. the planar angle. In $n=3$ dimensions, we require three parameters in order to specify a general rotation, i.e. a general orientation of an object with respect to some fiducial orientation. These three parameters are often taken to be **Euler's angles** $\{\phi, \theta, \psi\}$.

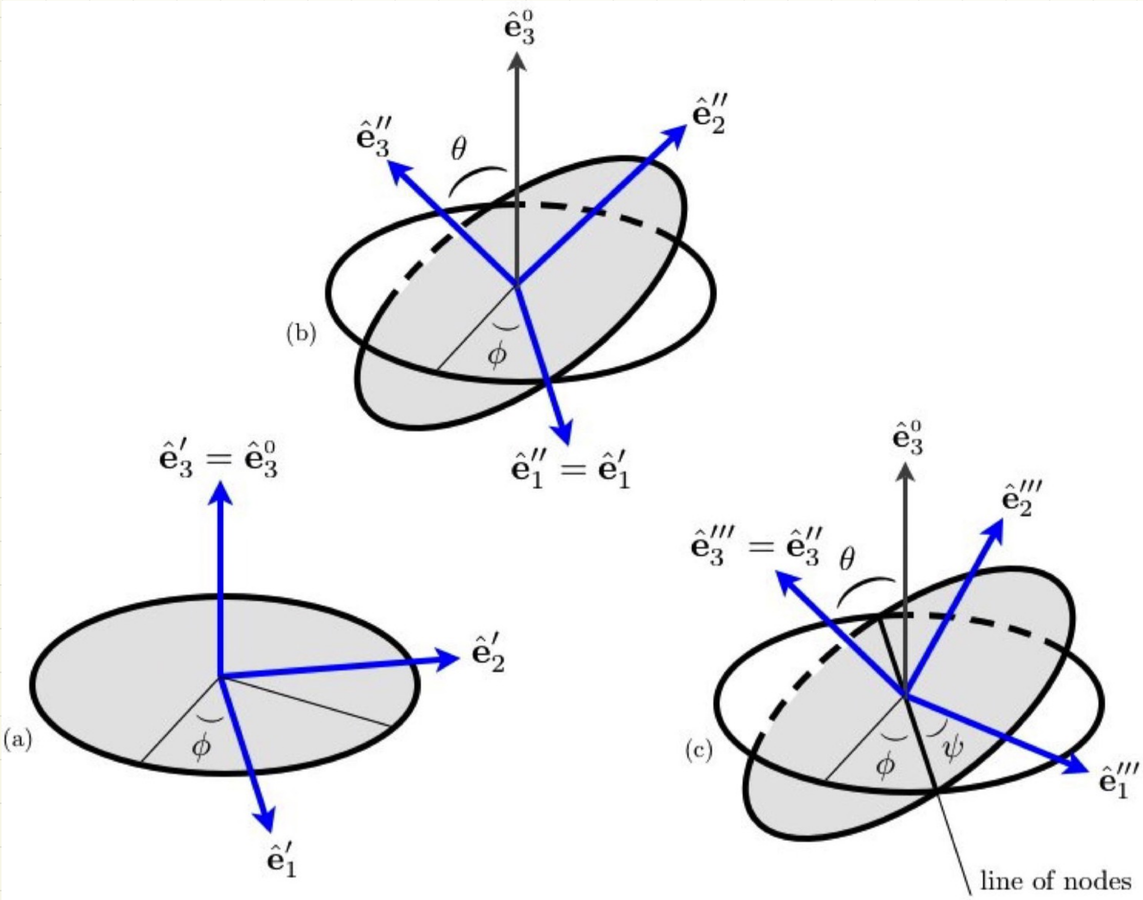
- General rotation matrix $R(\phi,\theta,\psi) \in SO(3)$:

Start with an orthonormal triad $\{\hat{e}^0_\mu\}$. We first rotate by $\phi$ about the $\hat{e}^0_3$ axis:

$$\hat{e}'_\mu = R_{\mu\nu}(\phi, \hat{e}^0_3)\,\hat{e}^0_\nu \quad ; \quad R(\phi, \hat{e}^0_3) = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The next step is to rotate by $\theta$ about $\hat{e}'_1$:

$$\hat{e}''_\mu = R_{\mu\nu}(\theta, \hat{e}'_1)\,\hat{e}'_\nu \quad ; \quad R(\theta, \hat{e}'_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}$$




Constructing a general rotation in $SO(3)$ using Euler's angles $\{\phi, \theta, \psi\}$

Finally, rotate by $\psi$ about $\hat{e}_3''$:

$$\hat{e}_\mu \equiv \hat{e}_\mu''' = R_{\mu\nu}(\psi, \hat{e}_3'')\,\hat{e}_\nu'' \quad ; \quad R(\psi, \hat{e}_3'') = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Multiply the three matrices to get $\hat{e}_\mu = R_{\mu\nu}(\phi,\theta,\psi)\,\hat{e}_\nu^0$ with

$$R(\phi,\theta,\psi) = \begin{pmatrix} \cos\psi\cos\phi - \sin\psi\cos\theta\sin\phi & \cos\psi\sin\phi + \sin\psi\cos\theta\cos\phi & \sin\psi\sin\theta \\ -\sin\psi\cos\theta - \cos\psi\cos\theta\sin\phi & -\sin\psi\sin\phi + \cos\psi\cos\theta\cos\phi & \cos\psi\sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{pmatrix}$$

See the figure at the top of this page.

Next we relate the components of $\vec{\omega}$ to the derivatives $\{\dot{\phi}, \dot{\theta}, \dot{\psi}\}$. This is accomplished by writing

$$\vec{\omega} = \dot{\phi}\,\hat{e}_\phi + \dot{\theta}\,\hat{e}_\theta + \dot{\psi}\,\hat{e}_\psi$$

where (consult previous figure)

$$\hat{e}_\phi = \sin\theta \sin\psi\, \hat{e}_1 + \sin\theta\cos\psi\, \hat{e}_2 + \cos\theta\, \hat{e}_3 = \hat{e}^0_3$$
$$\hat{e}_\theta = \cos\psi\, \hat{e}_1 - \sin\psi\, \hat{e}_2 \quad (\text{"line of nodes"})$$
$$\hat{e}_\psi = \hat{e}_3$$

We may now read off

$$\omega_1 = \vec{\omega}\cdot\hat{e}_1 = \dot{\theta}\sin\theta\sin\psi + \dot{\theta}\cos\psi$$
$$\omega_2 = \vec{\omega}\cdot\hat{e}_2 = \dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi$$
$$\omega_3 = \vec{\omega}\cdot\hat{e}_3 = \dot{\phi}\cos\theta + \dot{\psi}$$

Note that:

$\dot{\phi} \leftrightarrow$ precession , $\dot{\theta} \leftrightarrow$ nutation , $\dot{\psi} \leftrightarrow$ axial rotation

In spinning tops, axial rotation is sufficiently fast that it appears to us as a blur. We can, however, discern precession and nutation. The rotational kinetic energy is then

$$T_{rot} = \tfrac{1}{2} I_1 \left(\dot{\theta}\sin\theta\sin\psi + \dot{\theta}\cos\psi\right)^2 + \tfrac{1}{2} I_2 \left(\dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi\right)^2 + \tfrac{1}{2} I_3 \left(\dot{\phi}\cos\theta + \dot{\psi}\right)^2$$

The canonical momenta are then

$$p_\phi = \frac{\partial T}{\partial \dot\phi}\ , \qquad p_\theta = \frac{\partial T}{\partial \dot\theta}\ , \qquad p_\psi = \frac{\partial T}{\partial \dot\psi}$$

and the angular momentum vector is

$$\vec{L} = p_\phi \hat{e}_\phi + p_\theta \hat{e}_\theta + p_\psi \hat{e}_\psi$$

Note that we don't need to specify the reference frame when writing $\vec{L}$ – only for time-derivatives of vectors must we specify inertial or body-fixed frame.

- Torque-free symmetric top: $\vec{N}^{ext} = 0$

  Let $I_1 = I_2$. Then

$$T = \frac{1}{2} I_1 \left(\dot\theta^2 + \sin^2\theta\, \dot\phi^2\right) + \frac{1}{2} I_3 \left(\cos\theta\, \dot\phi + \dot\psi\right)^2$$

The potential is $U = 0$ so the Lagrangian is $L = T$.
Since $\phi$ and $\psi$ are cyclic in $L$, their momenta are conserved:

$$p_\phi = \frac{\partial L}{\partial \dot\phi} = I_1 \sin^2\theta\, \dot\phi + I_3 \cos\theta \left(\cos\theta\, \dot\phi + \dot\psi\right)$$

$$p_\psi = \frac{\partial L}{\partial \dot\psi} = I_3 \left(\cos\theta\, \dot\phi + \dot\psi\right)$$

Since $p_\psi = I_3 \omega_3$, we have $\omega_3 = \text{const.}$, as we have already derived from Euler's equations.

Let's solve for the motion. Note that $\vec{L}$ is conserved in the inertial frame, i.e. $(\dot{\vec{L}})_{inertial} = 0$. We choose $\hat{e}_3^\circ = \hat{e}_\phi = \vec{L}$. From $\hat{e}_\phi \cdot \hat{e}_\psi = \cos\theta$, we have $p_\psi = \vec{L} \cdot \hat{e}_\psi = L\cos\theta$ and conservation of $p_\psi$ thus entails $\dot{\theta} = 0$. From

$$\dot{p}_\theta = I_1 \ddot{\theta} = \frac{\partial L}{\partial \theta} = (I_1 \cos\theta\, \dot{\phi} - p_\psi) \sin\theta\, \dot{\phi}$$

and $\dot{\theta} = 0$, we conclude $\dot{\phi} = p_\psi / I_1 \cos\theta$. Now, from the equation for $p_\psi$, we have

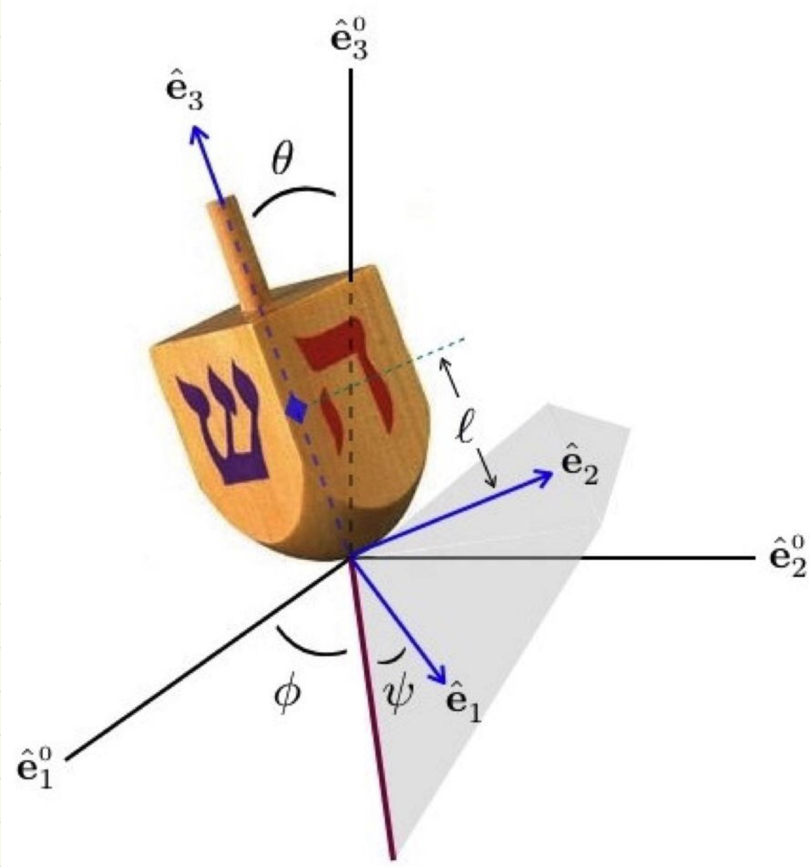
$$\dot{\psi} = \frac{p_\psi}{I_3} - \cos\theta\, \dot{\phi} = \left(\frac{1}{I_3} - \frac{1}{I_1}\right) p_\psi = \left(\frac{I_3 - I_1}{I_3}\right) \omega_3$$

as we had derived from Euler's equations.

- Symmetric top with one point fixed:
  Now gravity exerts a torque. The Lagrangian is

$$L = \frac{1}{2} I_1 (\dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2) + \frac{1}{2} I_3 (\cos\theta\, \dot{\phi} + \dot{\psi})^2 - Mgl\cos\theta$$

where $l$ is the distance from the fixed point to the CM. Let us now analyze the motion of this system.




The dreidl (Yid. דריידל, Heb. סביבון = spinner) is a symmetric top. Fourfold rotational symmetry is good enough to guarantee $I_1 = I_2$ and $I_{12} = 0$.

We have that $\phi$ and $\psi$ are still cyclic, so

$$p_\phi = \frac{\partial L}{\partial \dot\phi} = I_1 \sin^2\theta\, \dot\phi + I_3 \cos\theta \left(\cos\theta\, \dot\phi + \dot\psi\right)$$

$$p_\psi = \frac{\partial L}{\partial \dot\psi} = I_3 \left(\cos\theta\, \dot\phi + \dot\psi\right)$$

are again conserved. Thus,

$$\dot\phi = \frac{p_\phi - p_\psi \cos\theta}{I_1 \sin^2\theta}, \quad \dot\psi = \frac{p_\psi}{I_3} - \frac{(p_\phi - p_\psi\cos\theta)\cos\theta}{I_1 \sin^2\theta}$$

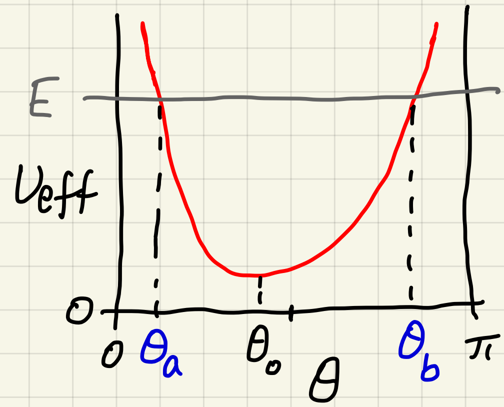
Energy $E = T + U$ is conserved:

$$E = \frac{1}{2} I_1 \dot\theta^2 + \underbrace{\frac{(p_\phi - p_\psi\cos\theta)^2}{2 I_1 \sin^2\theta} + \frac{p_\psi^2}{2 I_3} + Mg\ell\cos\theta}_{\text{effective potential } U_{\text{eff}}(\theta)}$$

Again :

$$E = \frac{1}{2} I_1 \dot{\theta}^2 + \overbrace{\frac{(p_\phi - p_\psi \cos\theta)^2}{2 I_1 \sin^2\theta} + \frac{p_\psi^2}{2 I_3} + Mg\ell\cos\theta}^{U_{eff}(\theta)}$$

Straightforward analysis (see lecture notes, ch. 8, p. 18) reveals that $U_{eff}(\theta)$ has a single minimum at $\theta_0$ $[0, \pi]$, and that $U_{eff}(\theta)$ diverges as $\theta \to 0$ and $\theta \to \pi$. Thus, the equation of motion,



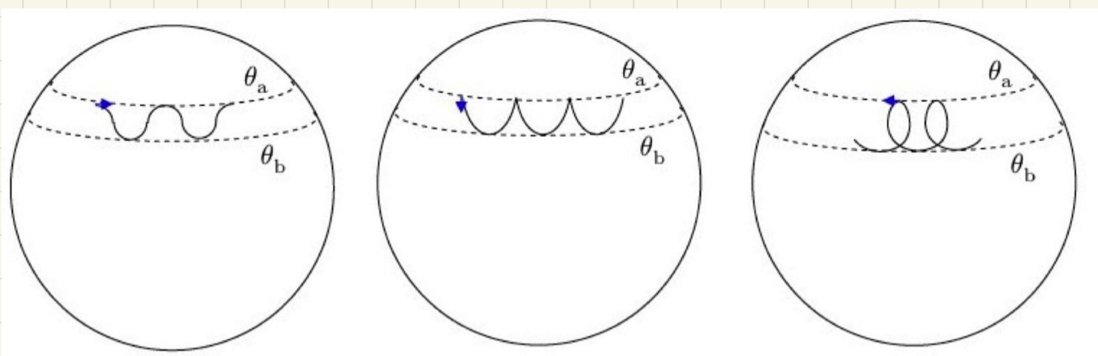

$$I_1 \ddot{\theta} = - U'_{eff}(\theta)$$

yields two **turning points**, which we label $\theta_a$ and $\theta_b$, satisfying $E = U_{eff}(\theta_{a,b})$. Now we have already derived the result

$$\dot{\phi} = \frac{p_\phi - p_\psi \cos\theta}{I_1 \sin^2\theta}$$

Thus we conclude that if $p_\psi \cos\theta_b < p_\phi < p_\psi \cos\theta_a$ then $\dot{\phi}$ will change sign when $\theta$ reaches $\theta^* = \cos^{-1}(p_\phi / p_\psi)$. This leads to two types of motion, as shown below

Note that $\hat{e}_3 = \sin\theta \sin\phi\, \hat{e}_1^0 - \sin\theta\cos\phi\, \hat{e}_2^0 + \cos\theta\, \hat{e}_3^0$.




$p_\phi > p_\psi \cos\theta_a$ $\qquad$ $p_\phi = p_\psi \cos\theta_a$ $\qquad$ $p_\psi \cos\theta_b < p_\phi < p_\psi \cos\theta_a$

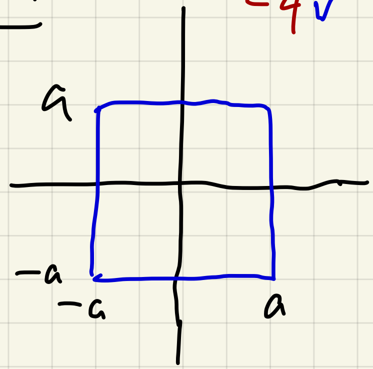
$\phi$ : precession

$\theta$ : nutation

$\psi$ : axial angle

## Scratch

$C_{4v}$




$$\rho(x,y) = \rho(-y,x)$$

$$I_{\alpha\beta} = \int_{-a}^{a} dx \int_{-a}^{a} dy\, \rho(x,y) \left[ \vec{r}^{\,2} \delta^{\alpha\beta} - r^{\alpha} r^{\beta} \right]$$

$$(x,y) \rightarrow (-y,x) \rightarrow (-x,-y),\ (y,-x) \rightarrow (x,y)$$

$$\pi/2 \qquad \pi \qquad 3\pi/2 \qquad 2\pi$$

$$I_{xx} = \int_{-a}^{a} dx \int_{-a}^{a} dy\, \rho(x,y)\, y^2 = \int_{-a}^{a} dx' \int_{-a}^{a} dy'\, \rho(y',-x')\, x'^2$$

$$x \equiv -y', \quad y \equiv x'$$

$$\downarrow$$

$$\int_{-a}^{a} dx \int_{-a}^{a} dy\, \underbrace{\rho(y,-x)}_{\rho(x,y)}\, x^2 = I_{yy}$$

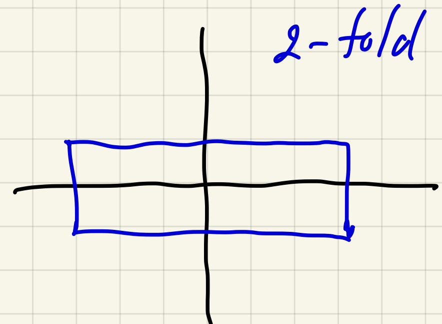
$$I_{xy} = -\int_{-a}^{a} dx \int_{-a}^{a} dy\, \rho(x,y)\, xy$$

$$\downarrow$$

$$-\int_{-a}^{a} dx' \int_{-a}^{a} dy'\, \rho(-y',x')\, (-y'x')$$

$$\downarrow$$

$$-I_{xy}$$



2-fold symmetry

$C_{2v}$

$$I_{cube} = \frac{1}{6} M a^2 \cdot \mathbb{1}$$

$I_{xx} < I_{yy}$

$I_{xy} = 0$

2 (isotropic)

3 (cubic)

$$C_{\alpha\beta\mu\nu} = C_{\beta\alpha\mu\nu} = C_{\alpha\beta\nu\mu} = C_{\mu\nu\alpha\beta} \longrightarrow 21 \text{ components}$$