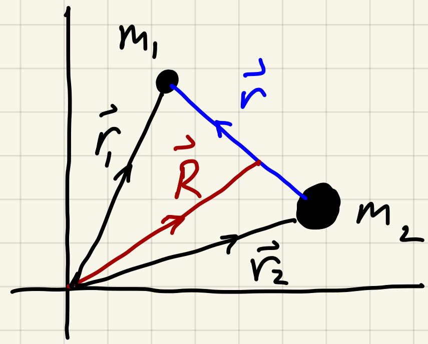
Lecture 5 (Oct. 19)

Two body central force problem:

$$L = T - U = \frac{1}{2} m_1 \dot{\vec{r}}_1^{\,2} + \frac{1}{2} m_2 \dot{\vec{r}}_2^{\,2} - U(|\vec{r}_1 - \vec{r}_2|)$$

① Change to CM and relative coordinates:

$$\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} \quad , \quad \vec{r} = \vec{r}_1 - \vec{r}_2$$



Invert to obtain:

$$\vec{r}_1 = \vec{R} + \frac{m_2}{m_1 + m_2} \vec{r} \quad , \quad \vec{r}_2 = \vec{R} - \frac{m_1}{m_1 + m_2} \vec{r}$$

Substitute in $L(\vec{r}_1, \vec{r}_2, \dot{\vec{r}}_1, \dot{\vec{r}}_2)$:

$$L(\vec{R}, \dot{\vec{R}}, \vec{r}, \dot{\vec{r}}) = \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2} \mu \dot{\vec{r}}^2 - U(r)$$

$\left\{ \text{decoupled CM and relative motion!} \right.$

where $M = m_1 + m_2$ (total mass)

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \quad \text{(reduced mass)}$$

NB : $m_1 \ll m_2 \Rightarrow \mu = m_1 - \frac{m_1^2}{m_2} + \cdots$

$m_1 = m_2 = m \Rightarrow \mu = \frac{1}{2} m$

② Integrate CM eqns of motion:

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\vec{R}}} = \frac{\partial L}{\partial \vec{R}} \Rightarrow M \ddot{\vec{R}} = 0 \quad , \quad \vec{P} = \frac{\partial L}{\partial \dot{\vec{R}}} = M \dot{\vec{R}} = \text{const}$$

$$\vec{R}(t) = \vec{R}(0) + \dot{\vec{R}}(0)\, t$$

③ Relative coordinate problem

$$L_{rel} = \frac{1}{2}\mu\dot{\vec{r}}^2 - U(r)$$

Continuous rotational symmetry $\Rightarrow$

$$\vec{l} = \vec{r}\times\vec{p} = \mu\vec{r}\times\dot{\vec{r}} \quad \text{conserved}$$

Since $\vec{r}\cdot\vec{l} = 0$, all motion $\vec{r}(t)$ is confined to the plane perpendicular to $\vec{l}$. Choose 2D polar coordinates (r,ϕ) in this plane. The relative coordinate Lagrangian is then

$$L_{rel} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) - U(r)$$

Since the coordinate ϕ is cyclic, the angular momentum $l = \mu r^2\dot{\phi}$ is conserved. And since $\partial L/\partial t = 0$, $H = \dot{r}\frac{\partial L}{\partial \dot{r}} + \dot{\phi}\frac{\partial L}{\partial \dot{\phi}} - L$ is conserved. Find

$$H = E = T + U = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\phi}^2 - U(r)$$
$$= \frac{1}{2}\mu\dot{r}^2 + U_{eff}(r)$$

where

$$U_{eff}(r) = \frac{l^2}{2\mu r^2} + U(r)$$

We can now solve to obtain radial motion $r(t)$, and then obtain ϕ by integrating $\dot{\phi} = l/\mu r^2(t)$.

Specifically, from $E_{rel} = \frac{1}{2}\mu\dot{r}^2 + U_{eff}(r)$, we have

$$\dot{r} = \frac{dr}{dt} = \pm\sqrt{\frac{2}{\mu}\left(E - U_{eff}(r)\right)} \Rightarrow$$

$$\left.\begin{array}{l} + \text{ for } dr > 0 \\ - \text{ for } dr < 0 \end{array}\right\} \quad dt = \pm\sqrt{\frac{\mu}{2}}\, \frac{dr}{\sqrt{E - \frac{\ell^2}{2\mu r^2} - U(r)}}$$

Integrate to get $t(r)$. In principle this is possible. This introduces a constant of integration $r_0 = r(t=0)$

Next, with $r(t)$ in hand, integrate

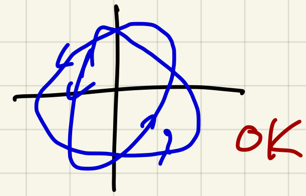
$$\dot{\phi} = \frac{d\phi}{dt} = \frac{\ell^2}{2\mu r^2} \Rightarrow d\phi = \frac{\ell}{\mu}\frac{dt}{r^2(t)}$$

to get $\phi(t)$. This introduces a second constant, $\phi_0 = \phi(t=0)$. Now we have the complete motion of the system, $\{r(t), \phi(t)\}$ with four constants of integration: E, ℓ, r_0, ϕ_0. Recall that the three-dimensional motion is confined to a plane perpendicular to $\vec{\ell}$, so its direction $\hat{\ell}$ accounts for two additional constants of integration. Overall, there are 12 such constants:

$$\vec{R}(0)\ (\times 3),\ \dot{\vec{R}}(0)\ (\times 3),\ E_{rel},\ \vec{\ell}\ (\times 3),\ r_0,\ \phi_0$$

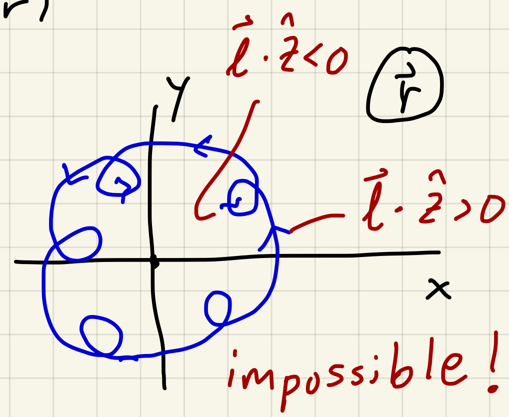
which is expected given two coupled second order equations of motion for the six quantities $\vec{r}_1, \vec{r}_2$.

- Geometric equation of the orbit



The 2nd order ODE for $r(t)$ is

$$\mu \ddot{r} = -\frac{\partial U_{eff}}{\partial r} = \frac{\ell^2}{\mu r^3} - U'(r)$$



Since $l = \mu r^2 \frac{d\phi}{dt}$ is conserved,

$$\frac{d}{dt} = \frac{\ell}{\mu r^2}\frac{d}{d\phi}$$

Therefore

$$\mu\left(\frac{\ell}{\mu r^2}\frac{d}{d\phi}\right)\left(\frac{\ell}{\mu r^2}\frac{d}{d\phi}\right) r = \frac{\ell^2}{\mu r^3} - U'(r)$$

$$\frac{\ell^2}{\mu r^4}\frac{d^2 r}{d\phi^2} - \frac{2\ell^2}{\mu r^5}\left(\frac{dr}{d\phi}\right)^2 = \frac{\ell^2}{\mu r^3} - U'(r)$$

$$\Rightarrow \quad \frac{d^2 r}{d\phi^2} - \frac{2}{r}\left(\frac{dr}{d\phi}\right)^2 = r + \frac{\mu r^4}{\ell^2} F(r)$$

where $F(r) = -U'(r)$ is the radial force. Using energy conservation, we can write

$$E = \frac{1}{2}\mu \dot{r}^2 + U_{eff}(r)$$

$$= \frac{\ell^2}{2\mu r^2}\left(\frac{dr}{d\phi}\right)^2 + U_{eff}(r)$$

to obtain

$$d\phi = \pm \frac{\ell}{\sqrt{2\mu}} \frac{dr}{r^2\sqrt{E - U_{eff}(r)}}$$

It is sometimes convenient to write the equation

$$r'' - \frac{2}{r}(r')^2 = \frac{\mu r^4}{\ell^2} F(r) + r \qquad \left(r' = \frac{dr}{d\phi} \text{ etc.}\right)$$

in terms of the variable $s \equiv 1/r$. Then

$$\frac{d^2 s}{d\phi^2} + s = -\frac{\mu}{\ell^2 s^2} F(s^{-1})$$

Suppose for example that $r(\phi) = r_0 e^{\kappa\phi}$, i.e. a logarithmic spiral. Then $s(\phi) = s_0 e^{-\kappa\phi}$, and

$$(\kappa^2+1)s = -\frac{\mu}{\ell^2 s^2} F(s^{-1})$$

$$F(s^{-1}) = -\frac{\ell^2}{\mu}(\kappa^2+1)s^3 \iff F(r) = -\frac{\ell^2}{\mu}(\kappa^2+1)\frac{1}{r^3}$$

This corresponds to a potential $U(r) = -\dfrac{C}{r^3}$ $(C>0)$ with

$$\kappa = \left(\frac{\mu C}{\ell^2} - 1\right)^{1/2}$$

Thus, the general shape of the orbit for $\ell^2 > \mu C > 0$ is

$a, b \in \mathbb{R}$
2 real const.

$$r(\phi) = \frac{1}{a e^{\kappa\phi} + b e^{-\kappa\phi}}$$

spiral orbit for $a = 0$ or $b = 0$

When $\mu C > \ell^2 > 0$, let $\bar{\kappa} = \left(1 - \dfrac{\mu C}{\ell^2}\right)^{1/2}$, in which case

$A \in \mathbb{C}$
1 complex const.

$$r(\phi) = \frac{1}{A e^{i\bar{\kappa}\phi} + A^* e^{-i\bar{\kappa}\phi}}$$

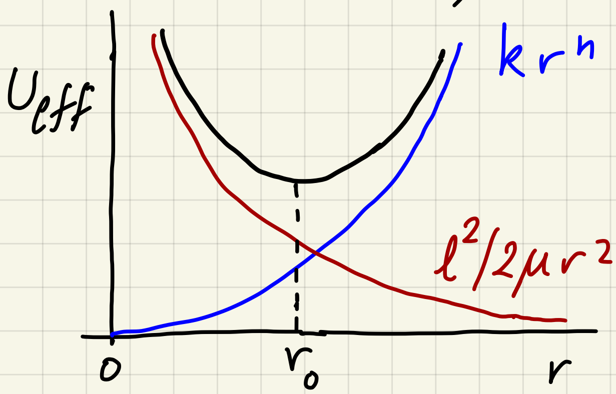
orbit is unbound, with $r(\phi) = \infty$ when

$\kappa\phi = (n+\tfrac{1}{2})\pi - \arg A$

• Almost circular orbits

A circular orbit $r(t) = r_0$ requires $U'_{eff}(r_0) = 0$.

For a homogeneous attractive potential $U(r) = kr^n$ with $k>0$, $n>0$, we have:

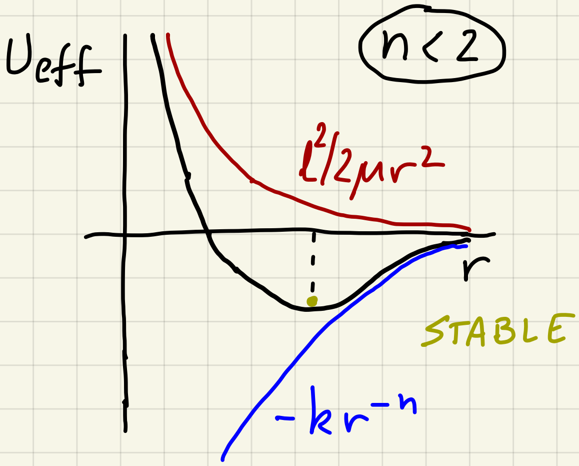


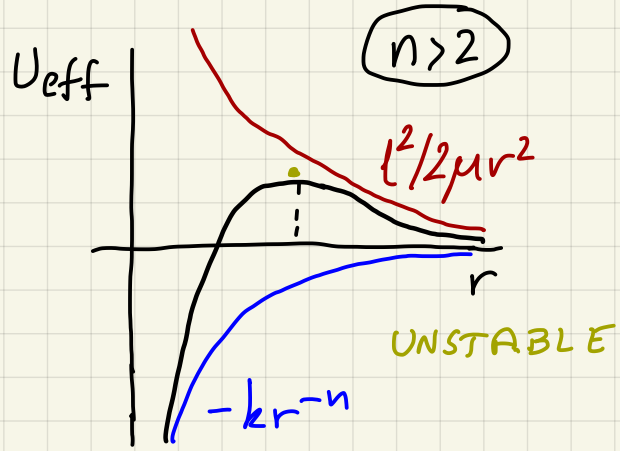
$$U_{eff} = \frac{\ell^2}{2\mu r^2} + kr^n$$

$$U'_{eff} = -\frac{\ell^2}{\mu r^3} + nkr^{n-1} \equiv 0$$

$$r_0 = (\ell^2/n\mu k)^{1/(n+2)}$$

For $U(r) = -kr^{-n}$ with $k>0$, $n>0$, we have





$$U_{eff} = \frac{\ell^2}{2\mu r^2} - \frac{k}{r^n} \quad , \quad U'_{eff} = -\frac{\ell^2}{\mu r^3} + \frac{nk}{r^{n+1}}$$

$$r_0 = \left(\frac{n\mu k}{\ell^2}\right)^{1/(n-2)}$$

If we write $r = r_0 + \eta$ with $|\eta| \ll r_0$, then

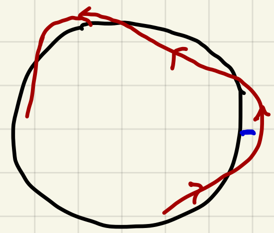
$$\mu\ddot{\eta} = -U''_{eff}(r_0)\,\eta \Rightarrow \ddot{\eta} = -\omega^2\eta \text{ with } \omega^2 = \frac{U''_{eff}(r_0)}{\mu}$$

We can also use

$$\frac{d^2r}{d\phi^2} - \frac{2}{r}\left(\frac{dr}{d\phi}\right)^2 = \frac{\mu r^4}{\ell^2} F(r) + r$$

and linearize in η with $r = r_0 + \eta$. This yields

$$\eta'' = \underbrace{\left[\frac{\mu r_0^4}{\ell^2} F(r_0) + r_0\right]}_{= -\frac{\mu r_0^4}{\ell^2} U'_{eff}(r_0) = 0} + \left(\underbrace{\frac{4\mu r_0^3}{\ell^2} F(r_0)}_{=4} + \frac{\mu r_0^4}{\ell^2} F'(r_0) - 1\right)\eta + \mathcal{O}(\eta^2)$$



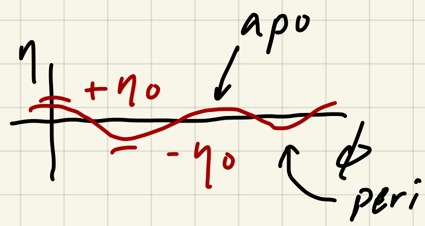
and hence

$$\eta''(\phi) = -\beta^2 \eta(\phi)$$

with $$\beta^2 = 3 - \frac{\mu r_0^4}{\ell^2} F'(r_0) = 3 - \left.\frac{d\ln F}{d\ln r}\right|_{r_0}$$

The solution is

$$\eta(\phi) = \eta_0 \cos\left[\beta(\phi - \delta_0)\right]$$



where η_0 and ϕ_0 set the initial conditions. Note that $\eta(\phi) = +\eta_0$ for $\phi = \phi_n \equiv 2\pi\beta^{-1} n + \delta_0$. This is called apoapsis (farthest point). The condition for periapsis (closest point) occurs for $\phi = \phi_n + \pi\beta^{-1}$. The difference,

$$\Delta\phi = \phi_{n+1} - \phi_n - 2\pi = 2\pi(\beta^{-1} - 1)$$

is the angle by which the apsides (i.e. periapsis and apoapsis) precess during each cycle. If $\beta > 1$, the apsides advance, (come sooner) while if $\beta < 1$ the apsides recede (later).

If $\beta = \frac{p}{q} \in \mathbb{Q}$ is a rational number, then the orbit is <u>closed</u> and will retrace itself every q revolutions.

- Example: $U(r) = -k r^{-\alpha}$ with $k>0$, $n>0$. Then

$$U'_{eff}(r) = -\frac{\ell^2}{\mu r^3} + \frac{\alpha k}{r^{\alpha+1}} \Rightarrow r_0 = \left(\frac{\ell^2}{\alpha \mu k}\right)^{1/(2-\alpha)}$$

We then have $\beta^2 = 3 - \left.\frac{d\ln F}{d\ln r}\right|_{r_0} = 2-\alpha$. These orbits are stable only for $\alpha<2$. For $\alpha>2$ the circular orbit is unstable and $r(t)$ either falls to the force center or escapes to infinity. In either case, for $\alpha>2$ the orbit is <u>unbound</u>. ($r \to \infty$ or $r \to 0$ whence $p_r \to \infty$). In order that small perturbations about a stable orbit be <u>closed</u>, we must have $\alpha = 2 - (p/q)^2$.

- Fun fact: If we consider <u>nonlinear</u> perturbations of a circular orbit, the <u>only</u> values of β which yield a closed orbit are $\beta^2 = 1$ (Kepler problem, $\alpha = 1$) and $\beta^2 = 4$ (harmonic oscillator, $\alpha = -2$). See §14.7.1.

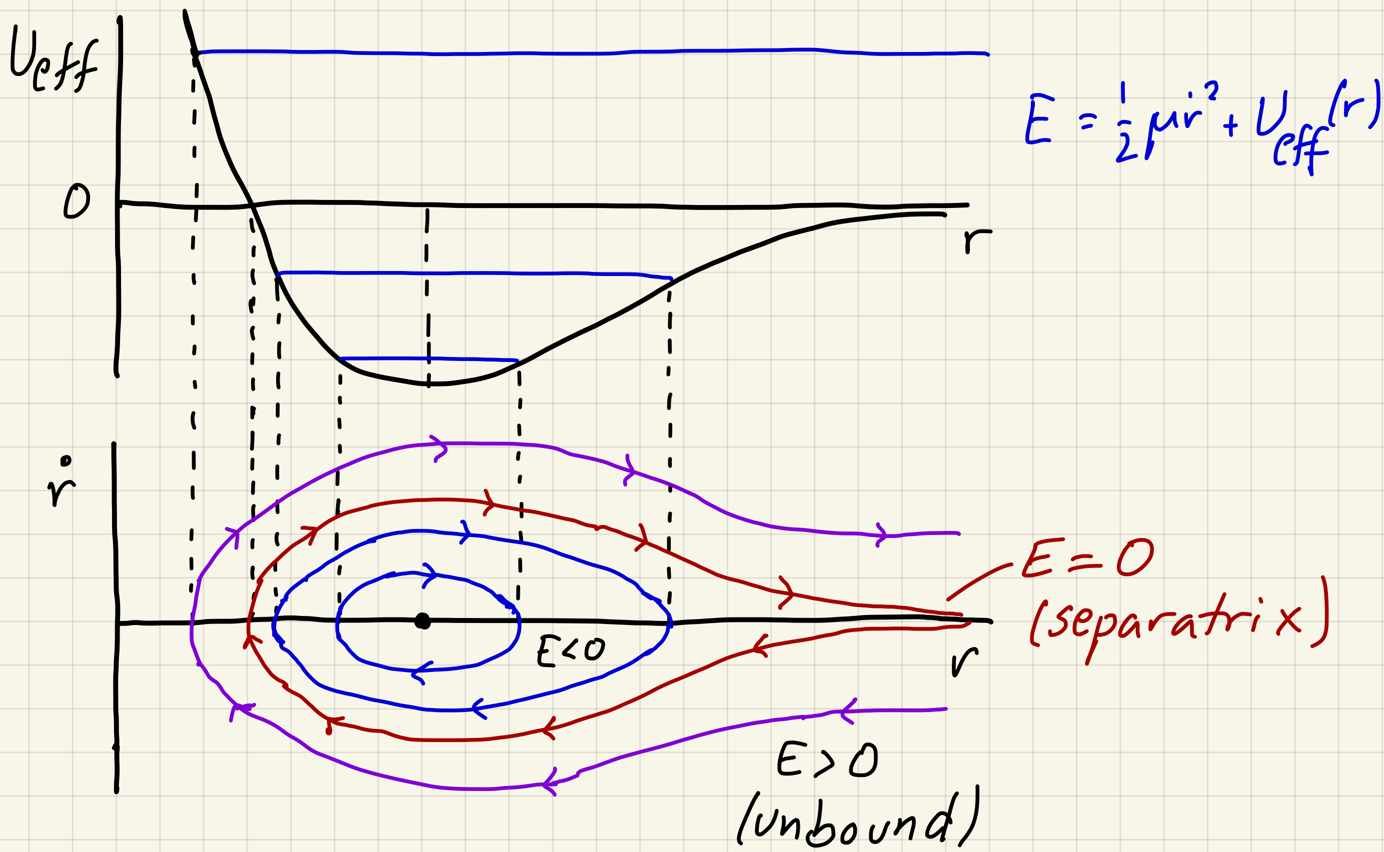
- Read §4.3: "Precession in a Soluble Model"

$$F = -\frac{k}{r} + \frac{C}{r^2} \Rightarrow r(\phi) = \frac{r_0}{1-\epsilon \cos\beta\phi}, \quad \beta = \left(1 + \frac{\mu C}{\ell^2}\right)^{1/2}$$

$$\epsilon^2 = 1 + \frac{2E(\ell^2+\mu C)}{\mu k^2} = \text{eccentricity}, \quad E = \text{energy}$$ (see Fig 4.3)

- The Kepler Problem : $U(r) = -\frac{k}{r}$, $k = G m_1 m_2 = GM\mu$

Effective potential and phase curves :



From $F(r) = -kr^{-2}$, we have, with $s = 1/r$,

$$s''(\phi) + s = -\frac{\mu}{\ell^2 s^2} F(s^{-1}) = \frac{\mu k}{\ell^2} = \text{const.}$$

Thus, $s(\phi) = \frac{\mu k}{\ell^2} - C\cos(\phi - \phi_0)$, i.e.

$$r(\phi) = \frac{r_0}{1 - \epsilon\cos(\phi - \phi_0)}$$

with $r_0 = \frac{\ell^2}{\mu k}$ and $\epsilon \equiv C r_0$. Since $r(\phi) = r(\phi + 2\pi n)$, the bound Kepler orbits (circles, ellipses) are closed.

– Laplace – Runge – Lenz vector

Define $\vec{A} \equiv \vec{p} \times \vec{\ell} - \mu k \hat{r}$ $\quad \left(\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \text{unit vector} \right)$

Then :

$$\frac{d\vec{A}}{dt} = \dot{\vec{p}} \times \vec{\ell} + \vec{p} \times \overbrace{\dot{\vec{\ell}}}^{=0} - \mu k \frac{\dot{\vec{r}}}{\vec{r}} + \mu k \frac{\dot{r}\vec{r}}{r^2}$$

$$= -\frac{k\vec{r}}{r^3} \times (\mu \vec{r} \times \dot{\vec{r}}) - \mu k \frac{\dot{\vec{r}}}{\vec{r}} + \mu k \frac{\dot{r}\vec{r}}{r^2}$$

interlude : $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - (\vec{c}\ \vec{a})\vec{b}$

$$\frac{d\vec{A}}{dt} = -\frac{\mu k}{r^3} \left[\cancel{\vec{r}(\underbrace{\vec{r} \cdot \dot{\vec{r}}}_{r\dot{r}})} - \cancel{\dot{\vec{r}}(\underbrace{\vec{r} \cdot \vec{r}}_{r^2})} \right] - \cancel{\mu k \frac{\dot{\vec{r}}}{r}} + \cancel{\mu k \frac{\dot{r}\vec{r}}{r^2}} = 0$$

Thus, $\vec{A}$ is a conserved vector lying in the plane of the motion. If we assume apoapsis occurs at $\phi = \phi_0$,

$$\vec{A} \cdot \vec{r} = -A r \cos(\phi - \phi_0) = \ell^2 - \mu k r$$

and

$$r(\phi) = \frac{\ell^2}{\mu k - A \cos(\phi - \phi_0)} = \frac{a(1-\epsilon^2)}{1 - \epsilon \cos(\phi - \phi_0)}$$

where

$$\epsilon = \frac{A}{\mu k} \quad , \quad a(1-\epsilon^2) = \frac{\ell^2}{\mu k}$$

From $\vec{A}^2 = 2\mu \ell^2 \left(E + \frac{\mu k^2}{2\ell^2} \right)$, we find

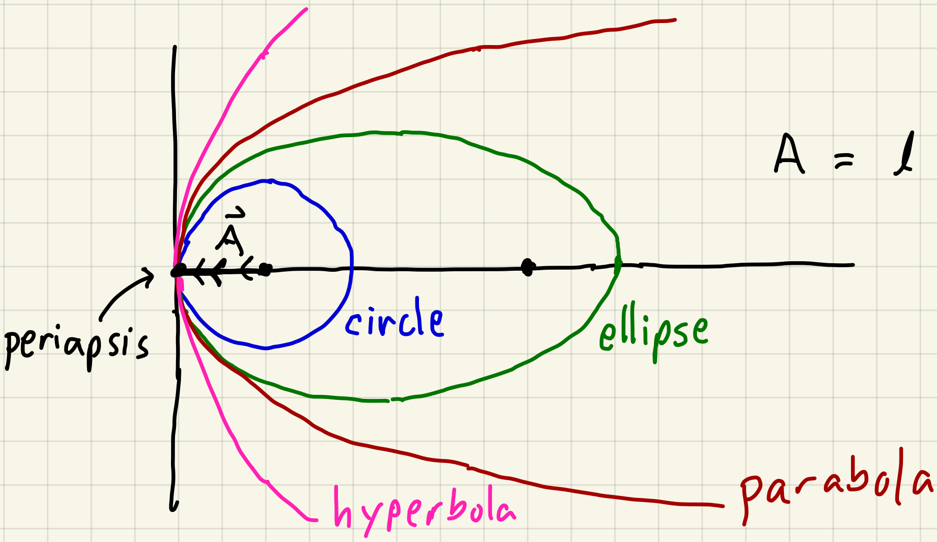
$$a = -\frac{k}{2E} \quad , \quad \epsilon^2 = 1 + \frac{2E\ell^2}{\mu k^2}$$

One can now show (§ 4.4.3) that Keplerian orbits are <u>conic sections</u> :

$$r(\phi) = \frac{a(1-\epsilon^2)}{1-\epsilon\cos(\phi-\phi_0)} \quad , \quad a = -\frac{k}{2E} \quad , \quad \epsilon^2 = 1 + \frac{2E\ell^2}{\mu k^2}$$

Note $\epsilon^2 > 0$ since $E_0 = -\frac{\mu k^2}{2\ell^2}$ is the energy of the (stable) circular orbit.

- circle : $E = -\frac{\mu k^2}{2\ell^2}$, $\epsilon = 0$, $a = \frac{\ell^2}{\mu k} = r_0$
- ellipse : $-\frac{\mu k^2}{2\ell^2} < E < 0$, $0 < \epsilon < 1$, semimajor axis length $a = -\frac{k}{2E}$, seminor $b = a\sqrt{1-\epsilon^2}$
- parabola : $E = 0$, $\epsilon = 1$, $a(1-\epsilon^2) = \frac{\ell^2}{\mu k} = r_0$
  focus lies at force center
- hyperbola : $E > 0$, $\epsilon > 1$, $\phi = \phi_0 + \cos^{-1}(1/\epsilon) \Rightarrow r(\phi) = \infty$
  Force center is closest (attractive) or furthest (repulsive) focus.



$$A = \ell\sqrt{2\mu\left(E + \frac{\mu k^2}{2\ell^2}\right)} = \mu k \epsilon$$

($A = 0$ for circles)

- Period of bound Kepler orbits (circles, ellipses)

Since $\ell = \mu r^2 \dot{\phi} = 2\mu \dot{\Sigma}$, where $d\Sigma = \frac{1}{2} r^2 d\phi$ is the differential area enclosed, the period is

$$\tau = \frac{2\mu}{\ell} \Sigma = \frac{2\mu}{\ell} \underbrace{\pi a^2 \sqrt{1-\epsilon^2}}_{\text{area of ellipse/circle}}$$

Now $\epsilon^2 = 1 + \frac{2E\ell^2}{\mu k^2}$ and $a = -\frac{k}{2E}$, so eliminating $E \Rightarrow$

$$E = -\frac{k}{2a} \quad \Rightarrow \quad 1 - \epsilon^2 = \frac{\ell^2}{\mu k a}$$

and we conclude $\tau = 2\pi (\mu a^3/k)^{1/2} = 2\pi (a^3/GM)^{1/2}$ since $k = G m_1 m_2 = G M \mu$. Equivalently,

$$\frac{a^3}{\tau^2} = \frac{GM}{4\pi^2} = \text{const.}$$

For planets orbiting the sun, $\frac{a^3}{\tau^2} = \left(1 + \frac{m_p}{M_\odot}\right) \frac{G M_\odot}{4\pi^2} \approx \frac{G M_\odot}{4\pi^2}$

Note $m_p / M_\odot \lesssim 10^{-3}$ even for Jupiter.

- Escape velocity : threshold for <u>energy</u> is $E = 0$

$$E = 0 = \frac{1}{2} \mu v_{esc}^2(r) - \frac{G m_1 m_2}{r}$$

$$\Rightarrow v_{esc}(r) = \sqrt{\frac{2GM}{r}}$$

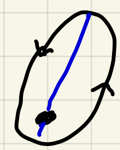
On earth's surface, $g = \frac{G M_E}{R_E^2} \Rightarrow v_{esc,E} = \sqrt{2 g R_E}$
$= 11.2 \text{ km/s}$

- Satellites and spacecraft

Recall: $\tau = \frac{2\pi}{\sqrt{GM_E}} (R_E + h)^{3/2}$ $(m_s << M_E)$

LEO = "Low Earth Orbit" $(h << R_E = 6.37 \times 10^6 m)$

So find $\tau_{LEO} = 1.4$ hr.



Problem: $h_p = 200$ km, $h_a = 7200$ km

$$a = \frac{1}{2}(R_E + h_p + R_E + h_a) = 10071 \text{ km}$$

$$\tau_{sat} = (a/R_E)^{3/2} \cdot \tau_{LEO} \simeq 2.65 \text{ hr}$$

- Read §§ 4.5 and 4.6

<u>Lecture 6 (Oct. 21)</u>

- A **rigid body** is a collection of point particles whose separations $|\vec{r}_i - \vec{r}_j|$ are all fixed in magnitude. <u>Six</u> independent coordinates are required to specify completely the position and orientation of a rigid body. For example, the location of the first particle (i) is specified by $\vec{r}_i$, which is three coordinates. The second (j) is then specified by a direction unit vector $\hat{n}_{ij}$, which requires two additional coordinates (polar and azimuthal angle). Finally, a third particle, k, is then fixed by its angle relative to the $\hat{n}_{ij}$ axis. Thus, six generalized coordinates in all are required.

Usually, one specifies three CM coordinates $\vec{R}$, and three orientational coordinates (e.g. the Euler angles). The equations of motion are then

$$\vec{P} = \sum_i m_i \dot{\vec{r}}_i \quad , \quad \dot{\vec{P}} = \vec{F}^{ext} \quad \text{(external force)}$$

$$\vec{L} = \sum_i m_i \vec{r}_i \times \dot{\vec{r}}_i \quad , \quad \dot{\vec{L}} = \vec{N}^{ext} \quad \text{(external torque)}$$

- Inertia tensor

Suppose a point within a rigid body is fixed. This eliminates the translational motion. If we measure distances relative to this fixed point, then in an inertial frame,

$$\frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r} \quad ; \quad \vec{\omega} = \text{angular velocity}$$

The kinetic energy is then

$$T = \frac{1}{2} \sum_i m_i \left(\frac{d\vec{r}_i}{dt} \right)^2 = \frac{1}{2} \sum_i (\vec{\omega} \times \vec{r}_i) \cdot (\vec{\omega} \times \vec{r}_i)$$

$$= \frac{1}{2} \sum_i m_i \left[\omega^2 \vec{r}_i^{\,2} - (\vec{\omega} \cdot \vec{r}_i)^2 \right] \equiv \frac{1}{2} I_{\alpha\beta} \omega_\alpha \omega_\beta$$

where $I_{\alpha\beta}$ is the **inertia tensor**,

$$I_{\alpha\beta} = \sum_i m_i \left[\vec{r}_i^{\,2} \delta^{\alpha\beta} - r_i^{\alpha} r_i^{\beta} \right] \quad \text{(discrete)}$$

3x3 real symmetric matrix $\Rightarrow$ 6 DOF

$$= \int d^d r \, \rho(\vec{r}) \left[\vec{r}^{\,2} \delta^{\alpha\beta} - r^{\alpha} r^{\beta} \right] \quad \text{(continuous)}$$

Diagonal elements of $I_{\alpha\beta}$ are <u>moments</u> of inertia, while off-diagonal elements are <u>products</u> of inertia.