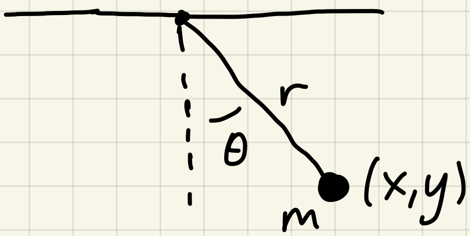
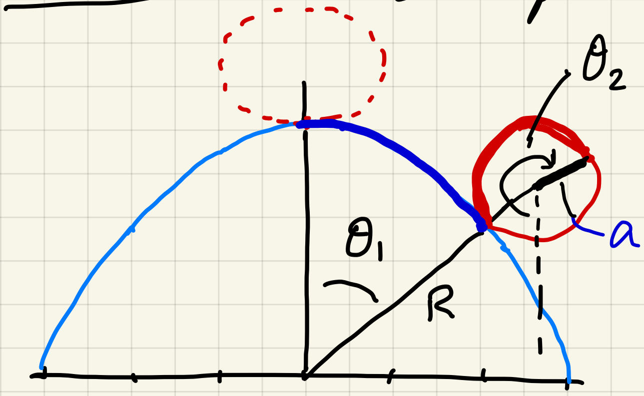
Lecture 4 (Oct. 14)

Today's lecture is about constraints. Examples:



constraint: $r = \ell$

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) = \frac{1}{2}m\ell^2\dot{\theta}^2$$



"no slip" condition: $R\theta_1 = a(\theta_2 - \theta_1)$

$$\Rightarrow \theta_2 = \left(1 + \frac{R}{a}\right)\theta_1$$

In these cases the constraint equations may easily be solved exactly and the number of generalized coordinates thereby reduced: $\{r, \theta\} \to \{\theta\}$, $\{\theta_1, \theta_2\} \to \{\theta_1\}$

In other cases the constraint equations are nonlinear or differential and they can't by solved to eliminate redundant degrees of freedom.

Constrained extremization of functions: Lagrange multipliers

Task: extremize $F(x_1, \dots, x_n)$ subject to k constraints of the form $G_j(x_1, \dots, x_n) = 0$ with $j \in \{1, \dots, k\}$. We want to find solutions $\vec{x}^*$ such that $\vec{\nabla}F(\vec{x}^*)$ is linearly dependent on the k vectors $\{\vec{\nabla}G_j(\vec{x}^*)\}$.

That is,

$$\text{(1)} \qquad \vec{\nabla} F + \sum_{j=1}^{k} \lambda_j \vec{\nabla} G_j = 0 \qquad \text{(n equations)}$$

where the $\{\lambda_j\}$ are all real. This means that any displacement $d\vec{x}$ relative to $\vec{x}^*$ would result in a violation of one or more of the constraint equations. Eqn. (1) provides n equations for the $(n+k)$ quantities $\{x_1, \ldots, x_n; \lambda_1, \ldots, \lambda_k\}$. The remaining k equations are the constraints $G_j(x_1, \ldots, x_n) = 0$.

Equivalently, construct the function

$$F^*(x_1, \ldots, x_n; \lambda_1, \ldots, \lambda_k) \equiv F(x_1, \ldots, x_n) + \sum_{j=1}^{k} \lambda_j G_j(x_1, \ldots, x_n)$$

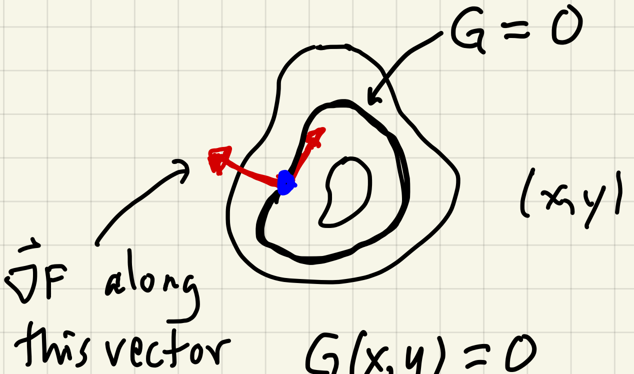
and freely extremize F^* over all its variables:

$$dF^* = \sum_{\sigma=1}^{n} \left(\frac{\partial F}{\partial x_\sigma} + \sum_{j=1}^{k} \lambda_j \frac{\partial G_j}{\partial x_\sigma} \right) dx_\sigma + \sum_{j=1}^{k} G_j \, d\lambda_j \equiv 0$$

This results in the $(n+k)$ equations

$$\frac{\partial F}{\partial x_\sigma} + \sum_{j=1}^{k} \lambda_j \frac{\partial G_j}{\partial x_\sigma} = 0 \qquad (\sigma = 1, \ldots, n)$$

$$G_j = 0 \qquad (j = 1, \ldots, k)$$



usually we set $\vec{\nabla} F = 0 \Rightarrow$ n eqns in n unknowns $\{x_1, \ldots, x_n\}$ but in general these sol$^{\underline{ns}}$ will not satisfy $G_j(\vec{x}) = 0 \;\; \forall j$

Example

Extremize the volume of a cylinder of height h and radius a subject to the constraint

$$G(a,h) = 2\pi a + \frac{h^2}{b} - \ell = 0 \qquad (b, \ell \text{ fixed})$$

Thus, we define

$$V^*(a,h;\lambda) = \pi a^2 h + \lambda\left(2\pi a + \frac{h^2}{b} - \ell\right)$$

$V = \pi a^2 h$

① $$\frac{\partial V^*}{\partial a} = 2\pi a h + 2\pi\lambda = 0$$

② $$\frac{\partial V^*}{\partial h} = \pi a^2 + \frac{2}{b}\lambda h = 0$$

③ $$\frac{\partial V^*}{\partial \lambda} = 2\pi a + \frac{h^2}{b} - \ell = 0$$

Thus ① gives $\lambda = -ah$, whence ② yields

$$\pi a^2 - \frac{2}{b} a h^2 = 0 \Rightarrow a = \frac{2}{\pi b} h^2$$

Finally, ③ gives

$$\frac{4}{b} h^2 + \frac{h^2}{b} = \ell \Rightarrow h = \sqrt{\frac{b\ell}{5}}$$

and therefore $a = \dfrac{2\ell}{5\pi}$ and $\lambda = -\dfrac{2}{5^{3/2}\pi} b^{1/2} \ell^{3/2}$

Thus, the extremal volume is

$$V^* = \pi a^2 h = \frac{4}{5^{5/2}\pi} b^{1/2} \ell^{5/2}$$

Constraints and variational calculus

Consider the following class of functionals:

$$F[\vec{y}(x)] = \int_{x_L}^{x_R} dx \; L(\vec{y}, \vec{y}\,', x)$$

Here $\vec{y}(x)$ may stand for a vector of functions $\{y_\sigma(x)\}$.
We consider two classes of constraints:

① Integral constraints: these are of the form

$$\int_{x_L}^{x_R} dx \; N_j(\vec{y}, \vec{y}\,', x) = C_j \quad , \quad j \in \{1, \ldots, k\}$$

② Holonomic constraints: these take the form

$$G_j(\vec{y}, x) = 0 \quad \text{on} \quad x \in [x_L, x_R]$$

Integral constraints

Here we introduce a separate multiplier λ_j for each integral constraint. That is, we extremize the extended functional

$$F^*[\vec{y}(x)\,;\,\vec{\lambda}] = \int_{x_L}^{x_R} dx \; L(\vec{y}, \vec{y}\,', x) + \sum_{j=1}^{k} \lambda_j \int_{x_L}^{x_R} dx \; N_j(\vec{y}, \vec{y}\,', x)$$

$$\equiv \int_{x_L}^{x_R} dx \; L^*(\vec{y}, \vec{y}\,', x\,;\,\vec{\lambda})$$

$$L^*(\vec{y}, \vec{y}\,', x\,;\,\vec{\lambda}) \equiv L(\vec{y}, \vec{y}\,', x) + \sum_j \lambda_j N_j(\vec{y}, \vec{y}\,', x)$$

This results in the following set of equations:

$$\frac{\partial L}{\partial y_\sigma} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'_\sigma}\right) + \sum_{j=1}^{k} \lambda_j \left\{\frac{\partial N_j}{\partial y_\sigma} - \frac{d}{dx}\left(\frac{\partial N_j}{\partial y'_\sigma}\right)\right\} = 0$$

$\sigma \in \{1, \dots, n\}$

$$\int_{x_L}^{x_R} dx\, N_j(\vec{y}, \vec{y}\,', x) = C_j$$

$j \in \{1, \dots, k\}$

Note that n of these are second order ODEs.
We have assumed that $\vec{y}(x_L)$ and $\vec{y}(x_R)$ are fixed.

Holonomic constraints

Now extremize

$$F[\vec{y}(x)] = \int_{x_L}^{x_R} dx\, L(\vec{y}, \vec{y}\,', x) \quad , \quad \vec{y}(x) = \{y_1(x), \dots, y_n(x)\}$$

subject to the k conditions

$$G_j(\vec{y}(x), x) = 0 \quad , \quad j \in \{1, \dots, k\}$$

Again, construct the extended functional $L^*(\vec{y}, \vec{y}\,', x; \vec{\lambda})$

$$F^*[\vec{y}(x), \vec{\lambda}(x)] = \int_{x_L}^{x_R} dx \left\{ \overbrace{L(\vec{y}, \vec{y}\,', x) + \sum_{j=1}^{k} \lambda_j\, G_j(\vec{y}, x)}^{L^*(\vec{y}, \vec{y}\,', x; \vec{\lambda})} \right\}$$

and freely extremize wrt the $(n+k)$ functions

$$\{y_1(x), \dots, y_n(x);\ \lambda_1(x), \dots, \lambda_k(x)\}$$

This results in n second order ODEs plus k algebraic constraints:

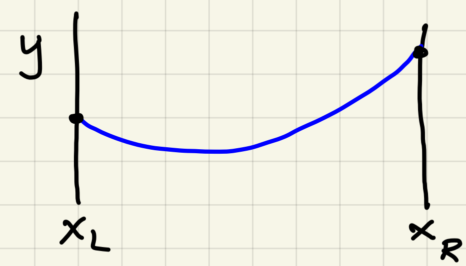
$$\frac{d}{dx}\left(\frac{\partial L}{\partial y'_\sigma}\right) - \frac{\partial L}{\partial y_\sigma} = \sum_{j=1}^{k} \lambda_j \frac{\partial G_j}{\partial y_\sigma} \quad , \quad \sigma \in \{1, \dots, n\}$$

$$G_j = 0 \quad , \quad j \in \{1, \dots, k\}$$

Each of these equations holds for all $x \in [x_L, x_R]$.

Examples

① hanging rope of fixed length



The potential energy functional is

$$U[y(x)] = \rho g \int_{x_L}^{x_R} ds\, y \quad ; \quad ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + (y')^2}\, dx$$

The length is

$$C[y(x)] = \int_{x_L}^{x_R} ds = \int_{x_L}^{x_R} dx \sqrt{1 + (y')^2}$$

Thus we form

$$U^*[y(x), \lambda] = \int_{x_L}^{x_R} dx \underbrace{(\rho g y + \lambda)\sqrt{1 + (y')^2}}_{L^*(y, y', x; \lambda)}$$

Since $\partial L^* / \partial x = 0$, the "Hamiltonian" is conserved:

$$H = y' \frac{\partial L^*}{\partial y'} - L^* = - \frac{\rho g y + \lambda}{\sqrt{1 + (y')^2}} = \text{constant}$$

Thus,

$$\frac{dy}{dx} = \pm \frac{1}{H}\sqrt{(\rho g y+\lambda)^2 - H^2}$$

Integrate to get

$$y(x) = -\frac{\lambda}{\rho g} + \frac{H}{\rho g}\cosh\left(\frac{\rho g}{H}(x-a)\right)$$

where a is a constant of integration.
The constants λ, H, and a are fixed by the conditions $y(x_L) = y_L$, $y(x_R) = y_R$, and by the fixed length constraint $\int_{x_L}^{x_R} dx\, \sqrt{1+(y')^2} = C$.

Constraints in Lagrangian Mechanics

We write our system of constraints in differential form:

$$\sum_{\sigma=1}^{n} g_{j\sigma}(q,t)\, dq_\sigma + h_j(q,t)\, dt = 0 \quad , \quad \begin{matrix}\sigma \in \{1,\dots,n\} \\ j \in \{1,\dots,k\}\end{matrix}$$

where $q = \{q_1,\dots,q_n\}$. If the partial derivatives satisfy the conditions

$$\frac{\partial g_{j\sigma}}{\partial q_{\sigma'}} = \frac{\partial g_{j\sigma'}}{\partial q_\sigma} \quad , \quad \frac{\partial g_{j\sigma}}{\partial t} = \frac{\partial h_j}{\partial q_\sigma}$$

then the k differentials may be integrated to yield k holonomic constraints $G_j(q,t) = 0$, with

$$g_{j\sigma} = \frac{\partial G_j}{\partial q_\sigma} \quad \text{and} \quad h_j = \frac{\partial G_j}{\partial t}$$

One *may* then be able to eliminate redundant degrees of freedom directly.

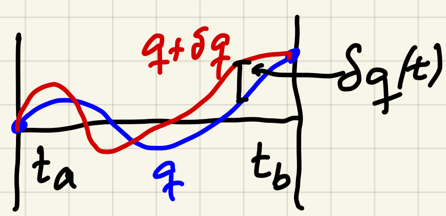
The action functional is

$$S[q(t)] = \int_{t_a}^{t_b} dt \, L(q, \dot{q}, t) \quad ; \quad \delta q_\sigma(t_a) = \delta q_\sigma(t_b) = 0$$

Its variation is

$$\delta S = \int_{t_a}^{t_b} dt \sum_{\sigma=1}^{n} \left\{ \frac{\partial L}{\partial q_\sigma} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right) \right\} \delta q_\sigma(t)$$

Since the $\{\delta q_\sigma(t)\}$ are no longer all independent, we cannot infer that the term in curly brackets vanishes for each σ. What are the constraints on the $\{\delta q_\sigma(t)\}$? Since they occur in *zero time* we call them "virtual displacements", and setting $\delta t = 0$ we have the conditions



$$\sum_{\sigma=1}^{n} g_{j\sigma}(q, t) \, \delta q_\sigma(t) = 0$$

Now we may relax the constraint by introducing k Lagrange multipliers $\lambda_j(t)$ at each time, and write

$$\sum_{\sigma=1}^{n} \left\{ \frac{\partial L}{\partial q_\sigma} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right) + \sum_{j=1}^{k} \lambda_j(t) \, g_{j\sigma}(q, t) \right\} \delta q_\sigma(t) = 0$$

We may set each of the bracketed terms to zero.

Thus, we obtain a set of $(n+k)$ equations:

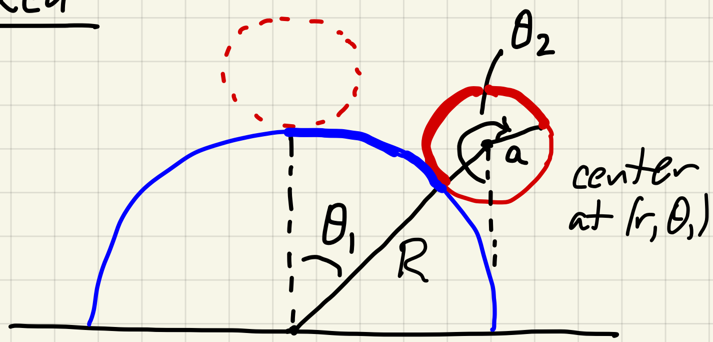
$$\underbrace{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\sigma}\right)}_{\dot{p}_\sigma} - \underbrace{\frac{\partial L}{\partial q_\sigma}}_{F_\sigma} = \underbrace{\sum_{j=1}^{k} \lambda_j(t)\, g_{j\sigma}(q,t)}_{Q_\sigma \,=\, \text{force of constraint}} \quad , \quad \sigma \in \{1,\dots,n\}$$

and

$$\sum_{\sigma=1}^{n} g_{j\sigma}(q,t)\,\dot{q}_\sigma + h_j(q,t) = 0 \quad , \quad j \in \{1,\dots,k\}$$

- Please read § 3.16.8 on constraints and conservation laws!

Example: Two cylinders, one fixed



Constraints:

1) contact : $r = R + a$
2) no slip : $R\theta_1 = a(\theta_2 - \theta_1)$

$$g_{j\sigma} = \begin{array}{cl} \begin{pmatrix} 1 & 0 & 0 \\ 0 & R+a & -a \end{pmatrix} & \begin{array}{l} \leftarrow \text{contact } (j=1) \\ \leftarrow \text{no slip } (j=2) \end{array} \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ r & \theta_1 & \theta_2 \end{array} & \end{array} \quad , \quad h_j = 0$$

Lagrangian :

mass of rolling cylinder

$$L = T - U = \frac{1}{2}M(\dot{r}^2 + r^2\dot{\theta}_1^2) + \frac{1}{2}I\dot{\theta}_2^2 - Mgr\cos\theta_1$$

rotational inertia of rolling cylinder

$$\underset{\substack{\shortparallel \\ 1}}{g_{1r}}\,\dot{r} + \underbrace{g_{1\theta_1}\dot{\theta}_1 + g_{1\theta_2}\dot{\theta}_2 + h_1}_{\text{all vanish}} = 0$$

i.e. $\dot{r} = 0 \longrightarrow r = R + a$

$n = 3$ equations of motion:

$$r: \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = M\ddot{r} - Mr\dot{\theta}_1^2 + Mg\cos\theta_1 = \lambda_1 = Q_r$$

$$\theta_1: \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} = Mr^2\ddot{\theta}_1 + 2Mr\dot{r}\dot{\theta}_1 - Mgr\sin\theta_1 = \lambda_2(R+a) = Q_{\theta_1}$$

$$\theta_2: \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} = I\ddot{\theta}_2 = -\lambda_2 a = Q_{\theta_2}$$

$$\cancel{\lambda_1 g_{1\theta_1}} + \lambda_2 \underbrace{g_{2\theta_1}}_{R+a}$$

$k = 2$ equations of constraint:

$$\left.\begin{aligned} &\text{contact}: \quad \dot{r} = 0 \\ &\text{no slip}: \quad R\dot{\theta}_1 - a(\dot{\theta}_2 - \dot{\theta}_1) = 0 \end{aligned}\right\} \xrightarrow[\text{integrate}]{} \left\{\begin{aligned} &r = R + a \\ &\theta_2 = \left(1 + \frac{R}{a}\right)\theta_1 \end{aligned}\right.$$

Now we have 5 equations in 5 unknowns $\{r, \theta_1, \theta_2, \lambda_1, \lambda_2\}$. We've already integrated the constraints so we may eliminate r and θ_2, yielding

$$-M(R+a)\dot{\theta}_1^2 + Mg\cos\theta_1 = \lambda_1$$

$$M(R+a)^2\ddot{\theta}_1 - Mg(R+a)\sin\theta_1 = \lambda_2(R+a)$$

$$I\left(1 + \frac{R}{a}\right)\ddot{\theta}_1 = -\lambda_2 a$$

We can read now read off the result $\lambda_2 = -\frac{I}{a^2}(R+a)\ddot{\theta}_1$

Substituting this into the second of these equations gives

$$\left(M + \frac{I}{a^2}\right)(R+a)^2\ddot{\theta}_1 - Mg(R+a)\sin\theta_1 = 0$$

Multiply this by $\dot{\theta}_1$ and then integrate to obtain...

$$\dot{\theta}_1\ddot{\theta}_1 = \frac{d}{dt}\left(\frac{1}{2}\dot{\theta}_1^2\right), \quad \dot{\theta}_1\sin\theta_1 = \frac{d}{dt}(-\cos\theta_1)$$

$$\frac{1}{2} M\left(1+\frac{I}{Ma^2}\right)\dot{\theta}_1^2 + \frac{Mg}{R+a}\cos\theta_1 = \frac{Mg}{R+a}\cos\theta_1^0$$

where we assume the upper cylinder is released from rest (i.e. $\dot{\theta}_1^0 = 0$) at $\theta_1 = \theta_1^0$. Finally, we may use this to express $\dot{\theta}_1^2$ in terms of θ_1, and stick the result into the first equation, resulting in

$$Q_r = \frac{Mg}{1+\alpha}\left\{(3+\alpha)\cos\theta_1 - 2\cos\theta_1^0\right\}$$

where $\alpha = I/Ma^2$ is dimensionless, with $\alpha \in [0,1]$

$\alpha = 0$: all mass of rolling cylinder at its center

$\alpha = 1$: all mass of rolling cylinder at its edge

When Q_r vanishes, the cylinders lose contact (the normal force of the bottom cylinder on the top one can only be positive). This happens for

$$\theta_1^* = \cos^{-1}\left(\frac{2\cos\theta_1^0}{3+\alpha}\right) = \text{detachment angle}$$

Note θ_1^* is an increasing function of α, i.e. larger rotational inertia I delays detachment. Physics here is that kinetic energy gain is split between translational and rotational motions.

Note also: $\dot{\theta}_1 = \left(\frac{2g}{R+a}\right)^{1/2}(\cos\theta_1^0 - \cos\theta_1)$

$$dt = \left(\frac{R+a}{2g}\right)^{1/2}\frac{d\theta_1}{\sqrt{\cos\theta_1^0 - \cos\theta_1}} \rightarrow \text{integrate for } \theta_1(t)$$