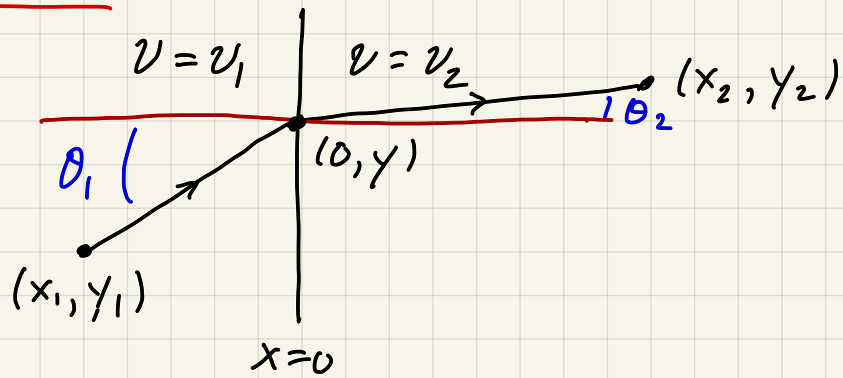
200A Lecture 1

Snell's law:



$$T(y) = \frac{1}{v_1}\sqrt{x_1^2 + (y-y_1)^2} + \frac{1}{v_2}\sqrt{x_2^2 + (y_2-y)^2}$$

$$\frac{dT}{dy} = \frac{1}{v_1}\frac{y-y_1}{\sqrt{x_1^2+(y-y_1)^2}} - \frac{1}{v_2}\frac{y_2-y}{\sqrt{x_2^2+(y_2-y)^2}} \equiv 0$$

$$= \frac{\sin\theta_1}{v_1} - \frac{\sin\theta_2}{v_2} \equiv 0$$

Thus with $v_j = c/n_j$ we have $n_1 \sin\theta_1 = n_2 \sin\theta_2$

Now consider a sequence of slabs with differing v_j.

We must have

$$\frac{\sin\theta_j}{v_j} = \frac{\sin\theta_{j+1}}{v_{j+1}} \xrightarrow[\text{limit}]{\text{continuum}} \frac{\sin\theta(x)}{v(x)} = P = \text{constant}$$

We'll see that P corresponds to conserved momentum in mechanics. Note that

$$\sin\theta(x) = \frac{y'(x)}{\sqrt{1+[y'(x)]^2}} = P\,v(x)$$

which yields

$$y' = \frac{Pv}{\sqrt{1-P^2v^2}} \Rightarrow y(x) = y(x_0) + \int_{x_0}^{x} ds\, \frac{P v(s)}{\sqrt{1-P^2 v^2(s)}}$$

$$\frac{d}{dx}\frac{y'}{v\sqrt{1+(y')^2}} = \frac{y''}{v\sqrt{1+(y')^2}} - \frac{y'^2 y''}{v(1+(y')^2)^{3/2}} - \frac{v' y'}{v^2\sqrt{1+(y')^2}}$$

$$= \frac{1}{v\left[1+(y')^2\right]^{3/2}}\left\{y'' - \frac{v'}{v}\left(1+(y')^2\right)y'\right\} = 0$$

Thus,

$$y'' - (\ln v)'\left[1+(y')^2\right]y' = 0$$

Of course this may be integrated once to yield

$$\frac{y'(x)}{\sqrt{1+[y'(x)]^2}} = P\,v(x)$$

Functional calculus

- Functions : eat numbers, excrete numbers

  e.g. $f:\mathbb{R}\to\mathbb{R}$, $f(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4$

  extremization : demand $df=0$ to lowest order in dx

$$f(x^*+dx) = f(x^*) + \underbrace{f'(x^*)\,dx + \frac{1}{2}f''(x^*)(dx)^2 + \ldots}_{df}$$

Thus, $df=0$ in $dx\to 0$ limit says $f'(x^*)=0$, i.e. if $f'(x^*)=0$ then x^* is an extremum. To second order,

$$f''(x^*)>0 \Rightarrow \text{minimum}\ ,\quad f''(x^*)<0 \Rightarrow \text{maximum}\ ,$$

$$f''(x^*)=0 \Rightarrow \text{inflection}$$

$$f : \mathbb{R}^n \to \mathbb{R}$$

Multivariable functions: $f(\overbrace{x_1, \ldots, x_n}^{\vec{x}})$

$$f(\vec{x}^* + d\vec{x}) = f(\vec{x}^*) + \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}\bigg|_{\vec{x}^*} dx_j + \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_k}\bigg|_{\vec{x}^*} dx_j\, dx_k + \cdots$$

Extremum $\Rightarrow$ $\dfrac{\partial f}{\partial x_j}\bigg|_{\vec{x}^*} = 0 \quad \forall\, j = 1, \ldots, n$

Hessian matrix: $H_{jk} = \dfrac{\partial^2 f}{\partial x_j \partial x_k}\bigg|_{\vec{x}^*}$ real, symmetric

eigenvalues of H: $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$

All $\lambda_j > 0 \Rightarrow \vec{x}^*$ local minimum

All $\lambda_j < 0 \Rightarrow \vec{x}^*$ local maximum

Some positive, some negative eigenvalues $\Rightarrow \vec{x}^*$ inflection pt

- **Functionals**: functionals eat functions, excrete numbers

Typically, functionals are integrals, e.g.

$$F[y(x)] = \int_{x_L}^{x_R} dx \left\{ \frac{1}{2} K \left(\frac{dy}{dx}\right)^2 + \frac{1}{2} a y^2 + \frac{1}{4} b y^4 \right\}$$

Consider a class of functionals of the form

$$F[y(x)] = \int_{x_L}^{x_R} dx\; L(y, y', x)$$

where $L(y, y', x)$ is a specified function of three variables, e.g.

$$L = \frac{1}{2} K (y')^2 + \frac{1}{2} a y^2 + \frac{1}{4} b y^4$$

Note this class may be extended to

$$G[y(x)] = \int_{x_L}^{x_R} dx\; L(y, y', y'', x)$$

Etc.

We now compute the functional variation by computing

$$\delta F = F[y(x)+\delta y(x)] - F[y(x)]$$

$$= \int_{x_L}^{x_R} dx \{ L(y'+\delta y', y+\delta y, x) - L(y', y, x) \}$$

$$= \int_{x_L}^{x_R} dx \left\{ \frac{\partial L}{\partial y'} \delta y' + \frac{\partial L}{\partial y} \delta y + \dots \right\} \qquad \delta y' = \frac{d}{dx} \delta y$$

$$= \int_{x_L}^{x_R} dx \left\{ \frac{d}{dx}\left(\frac{\partial L}{\partial y'} \delta y \right) + \left[\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'} \right) \right] \delta y \right\}$$

$$= \left. \frac{\partial L}{\partial y'} \right|_{x_R} \delta y(x_R) - \left. \frac{\partial L}{\partial y'} \right|_{x_L} \delta y(x_L) + \int_{x_L}^{x_R} dx \left[\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'} \right) \right] \delta y$$

Suppose $y(x)$ is fixed at the endpoints, in which case

$$\delta y(x_L) = \delta y(x_R) = 0$$

Then since $\delta y(x)$ elsewhere on $[x_L, x_R]$ is arbitrary, we conclude that

$$\frac{\delta F}{\delta y(x)} = \left[\frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'} \right) \right]_x = 0 \quad \forall \; x \in [x_L, x_R]$$

Since $L = L(y', y, x)$, the above equation is a second order ODE, known as the Euler-Lagrange equation. NB: If $y(x_{L,R})$ are not fixed, then we also require

$$\left. \frac{\partial L}{\partial y'} \right|_{x_{L,R}} = 0 \quad \text{as well as} \quad \frac{\partial L}{\partial y} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'} \right) = 0$$

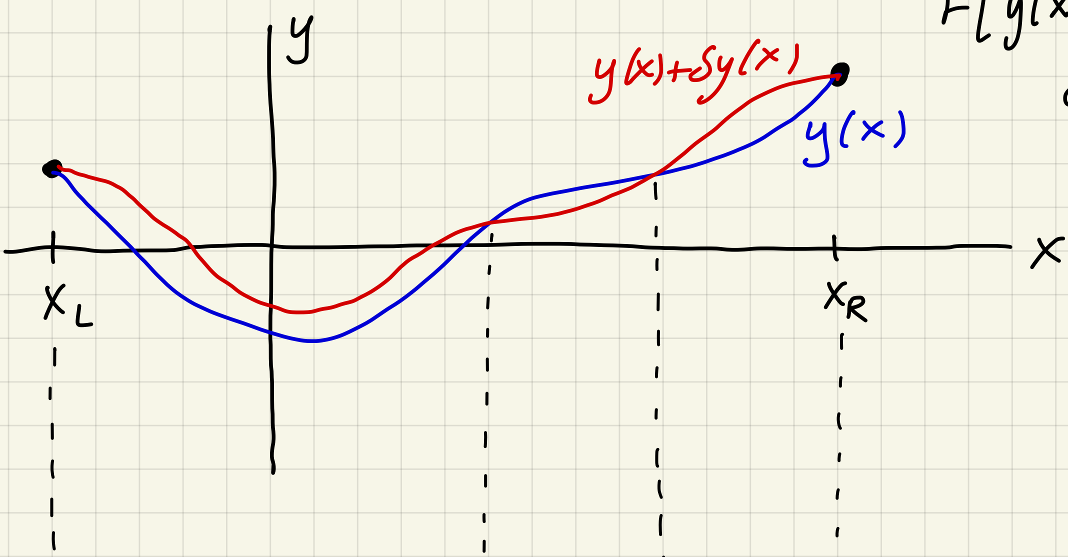
in order that $\delta F = 0$.

Graphical representation:

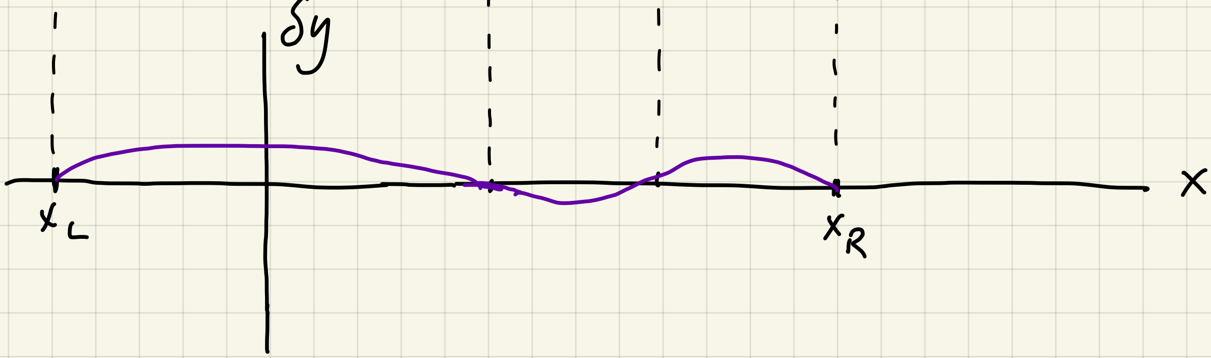
$F[y(x)] = F$

$F[y(x)+\delta y(x)] = F + \delta F$

$\delta y(x_{L,R}) \equiv 0$



The variation $\delta y(x)$ resembles the following



$$\delta F[y(x)] = F[y(x)+\delta y(x)] - F[y(x)]$$

$$\delta y' = \frac{d}{dx}\delta y = \delta \frac{dy}{dx} \quad , \text{ i.e. } [\delta, d] = 0$$

$$\frac{\partial L}{\partial y'}\delta y' = \frac{\partial L}{\partial y'}\frac{d}{dx}\delta y = \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\delta y\right) - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right)\delta y$$

$$\frac{d}{dx}\frac{\partial L}{\partial y'} : \quad \frac{d}{dx} = \frac{\partial}{\partial x} + y''\frac{\partial}{\partial y'} + y'\frac{\partial}{\partial y}$$

We now consider two important special cases:

① $\frac{\partial L}{\partial y} = 0$, i.e. $L(y, y', x)$ independent of y

Then EL eqn says $\cancel{\frac{\partial L}{\partial y}} - \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) = 0$,

which may be integrated once to yield $\frac{\partial L}{\partial y'} = P$, where P = constant. This is then a first order ODE in $y(x)$. Example: $L = \frac{1}{v(x)}\sqrt{1+(y')^2}$. Then

$$P = \frac{\partial L}{\partial y'} = \frac{y'}{v\sqrt{1+(y')^2}} \equiv \frac{1}{v_0}$$

(momentum conservation in mechanics)

$$\Rightarrow \frac{dy}{dx} = \frac{v(x)}{\sqrt{v_0^2 - v^2(x)}} \quad \text{with } v_0 \equiv 1/P$$

② $\frac{\partial L}{\partial x} = 0$, i.e. $L(y, y', x)$ independent of x

Define $H \equiv y' \frac{\partial L}{\partial y'} - L$. Then

(energy conservation in mechanics)

$$\frac{dH}{dx} = \frac{d}{dx}\left\{y'\frac{\partial L}{\partial y'} - L\right\}$$

$$= \cancel{y''\frac{\partial L}{\partial y'}} + y'\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) - \cancel{\frac{\partial L}{\partial y'}y''} - \frac{\partial L}{\partial y}y' - \cancel{\frac{\partial L}{\partial x}}$$

$$= y'\left[\frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) - \frac{\partial L}{\partial y}\right] = 0 \quad \text{if EL satisfied}$$

Thus, $\frac{\partial L}{\partial x} = 0 \Rightarrow \frac{dH}{dx} = 0 \Rightarrow H$ is constant

$$y'\frac{\partial L}{\partial y'} - L = H \quad \text{again a first order ODE}$$

③ If $L(y,y',x) = L_0(y,y',x) + \frac{d}{dx}\Delta(y,x)$, then

$$F[y(x)] = \int_{x_L}^{x_R} dx\, L_0(y,y',x) + \Delta(y(x_R),x_R) - \Delta(y(x_L),x_L)$$

If $\delta y(x_{L,R}) = 0$ (fixed endpoints), then the Δ term makes no contribution to the EL eqns, which are then

$$\frac{\partial L_0}{\partial y} - \frac{d}{dx}\left(\frac{\partial L_0}{\partial y'}\right) = 0$$

- Functional Taylor series :

$$\begin{aligned} F[y+\delta y] = F[y] &+ \int_{x_L}^{x_R} dx_1\, K_1(x_1)\,\delta y(x_1) \\ &+ \frac{1}{2!}\int_{x_L}^{x_R} dx_1 \int_{x_L}^{x_R} dx_2\, K_2(x_1,x_2)\,\delta y(x_1)\,\delta y(x_2) \\ &+ \frac{1}{3!}\int_{x_L}^{x_R} dx_1 \int_{x_2}^{x_R} dx_2 \int_{x_L}^{x_R} dx_3\, K_3(x_1,x_2,x_3)\,\delta y(x_1)\,\delta y(x_2)\,\delta y(x_3) \\ &+ \mathcal{O}(\delta y^4) \end{aligned}$$

Thus,

$$K_n(x_1,\ldots,x_n) = \frac{\delta^n F}{\delta y(x_1)\cdots\delta y(x_n)} = n^{th} \text{ functional derivative}$$

- Examples : §3.3 in the lecture notes
- More on functionals : §3.4

} READ!

Mechanics

Hamilton's principle : $\delta S = 0$ where

$$S[q(t)] = \int_{t_1}^{t_2} dt \, L(q, \dot{q}, t) = \text{action functional}$$

with $q = \{q_1, \ldots, q_n\}$ = set of generalized coordinates

The function $L(q, \dot{q}, t)$ is the Lagrangian, and is given by $L = T - U$, where T = kinetic energy and U = potential energy. Typically $T = T(q, \dot{q})$ is a quadratic form in the generalized velocities $\{\dot{q}_\sigma\}$, i.e. $T(q, \dot{q}) = T_{\sigma\sigma'}(q)\, \dot{q}_\sigma \dot{q}_{\sigma'}$. For example

$$T = \frac{1}{2} m \dot{\vec{x}}^{\,2} = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right) \qquad \text{Cartesian } (x, y, z)$$

$$\frac{m}{2} \begin{array}{c} \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{array} \right) \\ \begin{array}{ccc} \dot{r} & \dot{\theta} & \dot{\phi} \end{array} \end{array} \begin{array}{c} \dot{r} \\ \dot{\theta} \\ \dot{\phi} \end{array} = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta \, \dot{\phi}^2 \right) \qquad \text{polar } (r, \theta, \phi)$$

The potential energy U is most often a function of q, but $U = U(q, \dot{q})$ applies, e.g., for charged particles in a magnetic field, where

$$U(\vec{x}, \dot{\vec{x}}) = \underbrace{q}_{\text{charge}} \overbrace{\phi(\vec{x})}^{\text{scalar potential}} - \frac{q}{c} \underbrace{\vec{A}(\vec{x})}_{\text{vector potential}} \cdot \frac{d\vec{x}}{dt}$$

Free particle $\Rightarrow$ $L = \frac{1}{2} m \vec{v}^{\,2}$ (§3.6.3)

- NB : In general $L = \frac{1}{2} T_{\sigma\sigma'}(q, t)\, \dot{q}_\sigma \dot{q}_{\sigma'} - U(q, \dot{q}, t)$