Lecture 18 (Dec. 2)

- **Canonical perturbation theory**

Suppose

dimensionless

$$H(\vec{q},\vec{p},t) = H_0(\vec{q},\vec{p},t) + \epsilon H_1(\vec{q},\vec{p},t)$$

where $|\epsilon| \ll 1$. Let's implement a type-II CT generated by $S(\vec{q},\vec{P},t)$ (not intended to signify Hamilton's principal function):

$$\tilde{H}(\vec{Q},\vec{P},t) = H(\vec{q},\vec{p},t) + \frac{\partial}{\partial t} S(\vec{q},\vec{P},t)$$

Expand everything in sight in powers of ϵ:

$$\begin{aligned}
q_\sigma &= Q_\sigma + \epsilon q_{1,\sigma} + \epsilon^2 q_{2,\sigma} + \cdots \\
p_\sigma &= P_\sigma + \epsilon p_{1,\sigma} + \epsilon^2 p_{2,\sigma} + \cdots \\
\tilde{H} &= \tilde{H}_0 + \epsilon \tilde{H}_1 + \epsilon^2 \tilde{H}_2 + \cdots \\
S &= \underbrace{q_\sigma P_\sigma}_{\text{identity CT}} + \epsilon S_1 + \epsilon^2 S_2 + \cdots
\end{aligned}$$

Then

$$\begin{aligned}
Q_\sigma = \frac{\partial S}{\partial P_\sigma} &= q_\sigma + \epsilon \frac{\partial S_1}{\partial P_\sigma} + \epsilon^2 \frac{\partial S_2}{\partial P_\sigma} + \cdots \\
&= Q_\sigma + \left(q_{1,\sigma} + \frac{\partial S_1}{\partial P_\sigma}\right)\epsilon + \left(q_{2,\sigma} + \frac{\partial S_2}{\partial P_\sigma}\right)\epsilon^2 + \cdots
\end{aligned}$$

We also have

$$p_\sigma = \frac{\partial S}{\partial q_\sigma} = P_\sigma + \epsilon \frac{\partial S_1}{\partial q_\sigma} + \epsilon^2 \frac{\partial S_2}{\partial q_\sigma} + \dots$$

$$= P_\sigma + \epsilon p_{1,\sigma} + \epsilon^2 p_{2,\sigma} + \dots$$

Thus we conclude, order by order in ϵ,

$$q_{k,\sigma} = -\frac{\partial S_k}{\partial P_\sigma} \quad , \quad p_{k,\sigma} = \frac{\partial S_k}{\partial q_\sigma}$$

Next, expand the Hamiltonian:

$$\tilde{H}(\vec{Q},\vec{P},t) = H_0(\vec{q},\vec{p},t) + \epsilon H_1(\vec{q},\vec{p},t) + \frac{\partial S}{\partial t}$$

$$= H_0(\vec{Q},\vec{P},t) + \frac{\partial H_0}{\partial Q_\sigma}(q_\sigma - Q_\sigma) + \frac{\partial H_0}{\partial P_\sigma}(p_\sigma - P_\sigma) + \dots$$

$$+ \epsilon H_1(\vec{Q},\vec{P},t) + \epsilon \frac{\partial}{\partial t} S_1(\vec{Q},\vec{P},t) + O(\epsilon^2)$$

$$= H_0(\vec{Q},\vec{P},t) + \left(-\frac{\partial H_0}{\partial Q_\sigma}\frac{\partial S_1}{\partial P_\sigma} + \frac{\partial H_0}{\partial P_\sigma}\frac{\partial S_1}{\partial Q_\sigma} + \frac{\partial S_1}{\partial t} + H_1 \right)\epsilon + O(\epsilon^2)$$

Notice we are writing $q_\sigma = Q_\sigma + (q_\sigma - Q_\sigma) = Q_\sigma - \epsilon \frac{\partial S_1}{\partial P_\sigma} + \cdots$

so, e.g.

$$S_1(\vec{q},\vec{P},t) = S_1(\vec{Q},\vec{P},t) + (q_\sigma - Q_\sigma)\frac{\partial S_1}{\partial Q_\sigma} + \dots$$

$$= S_1(\vec{Q},\vec{P},t) - \frac{\partial S_1(\vec{Q},\vec{P},t)}{\partial P_\sigma}\,\frac{\partial S_1(\vec{Q},\vec{P},t)}{\partial Q_\sigma}\,\epsilon + O(\epsilon^2)$$

Thus, we have

$$\tilde{H}(\vec{Q},\vec{P},t) = H_0(\vec{Q},\vec{P},t) + \left(H_1 + \{S_1,H_0\} + \frac{\partial S_1}{\partial t}\right)\epsilon + \mathcal{O}(\epsilon^2)$$
$$= \tilde{H}_0(\vec{Q},\vec{P},t) + \epsilon\tilde{H}_1(\vec{Q},\vec{P},t) + \mathcal{O}(\epsilon^2)$$

We therefore conclude

$$\tilde{H}_0(\vec{Q},\vec{P},t) = H_0(\vec{Q},\vec{P},t)$$
$$\tilde{H}_1(\vec{Q},\vec{P},t) = \left[H_1 + \{S_1,H_0\} + \frac{\partial S_1}{\partial t}\right]_{\vec{Q},\vec{P},t}$$

We are left with a single equation in two unknowns, i.e. $\tilde{H}_1$ and S_1. The problem is underdetermined. We could at this point demand $\tilde{H}_1 = 0$, but this is just one of many possible choices. Similar story in QM:

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = (\hat{H}_0 + \epsilon\hat{H}_1)|\psi\rangle$$

Now define $|\psi\rangle \equiv e^{i\hat{S}/\hbar}|\chi\rangle$ with $\hat{S} = \epsilon\hat{S}_1 + \epsilon^2\hat{S}_2 + \dots$. Then find

$$i\hbar\frac{\partial}{\partial t}|\chi\rangle = \hat{H}_0|\chi\rangle + \epsilon\left(\hat{H}_1 + \frac{1}{i\hbar}\underbrace{[\hat{S}_1,\hat{H}_0]}_{\text{commutator}} + \frac{\partial\hat{S}_1}{\partial t}\right)|\chi\rangle + \dots$$
$$\equiv \hat{\tilde{H}}|\chi\rangle$$

Typically we choose $\hat{S}_1$ such that the $\mathcal{O}(\epsilon)$ term vanish. But this isn't the only possible choice. (Note here the correspondence $\{A,B\} \leftrightarrow \frac{1}{i\hbar}[\hat{A},\hat{B}]$.)

- **CPT for $n=1$ systems**

Here we demonstrate the implementation of CPT in a general $n=1$ system. We will need to deal with resonances when $n>1$, which we discuss later on.

We assume $H(q,p) = H_0(q,p) + \epsilon H_1(q,p)$ is time-independent. Let (ϕ_0, J_0) be AAV for H_0, so that

$$\tilde{H}_0(J_0) = H_0(q(\phi_0,J_0), p(\phi_0,J_0))$$

We define

$$\tilde{H}_1(\phi_0,J_0) \equiv H_1(q(\phi_0,J_0), p(\phi_0,J_0))$$

We assume that $\tilde{H} = \tilde{H}_0 + \epsilon \tilde{H}_1$ is integrable, which for $n=1$ is indeed always the case. [Reminder: $H(q,p)=E$ means all motion takes place on the one-dimensional level sets of $H(q,p)$.] Thus there must be a CT taking $(\phi_0,J_0) \to (\phi,J)$, where

$$\tilde{H}(\phi_0(\phi,J), J_0(\phi,J)) = E(J)$$

We solve by a type-II CT:

$$S(\phi_0,J) = \underbrace{\phi_0 J}_{\text{identity CT}} + \epsilon S_1(\phi_0,J) + \epsilon^2 S_2(\phi_0,J) + \cdots$$

Then

$$J_0 = \frac{\partial S}{\partial \phi_0} = J + \epsilon \frac{\partial S_1}{\partial \phi_0} + \epsilon^2 \frac{\partial S_2}{\partial \phi_0} + \cdots$$

$$\phi = \frac{\partial S}{\partial J} = \phi_0 + \epsilon \frac{\partial S_1}{\partial J} + \epsilon^2 \frac{\partial S_2}{\partial J} + \cdots$$

We also write

$$E(J) = E_0(J) + \epsilon E_1(J) + \epsilon^2 E_2(J) + \dots$$
$$= \tilde{H}_0(J_0) + \epsilon \tilde{H}_1(\phi_0, J_0) \quad \text{(no higher order terms)}$$

Now we expand $\tilde{H}(\phi_0, J_0) = \tilde{H}(\phi_0, J + \underbrace{(J_0 - J)}_{\delta J})$ in powers of $(J_0 - J)$:

$$\tilde{H}(\phi_0, J_0) = \tilde{H}_0(J) + \frac{\partial \tilde{H}_0}{\partial J}(J_0 - J) + \frac{1}{2}\frac{\partial^2 \tilde{H}_0}{\partial J^2}(J_0 - J)^2$$
$$+ \epsilon \tilde{H}_1(\phi_0, J) + \epsilon \left.\frac{\partial \tilde{H}_1}{\partial J}\right|_{\phi_0}(J_0 - J) + \dots$$

Substitute

$$J_0 - J = \epsilon \frac{\partial S_1}{\partial \phi_0} + \epsilon^2 \frac{\partial S_2}{\partial \phi_0} + \dots$$

and collect terms to obtain

$$\tilde{H}(\phi_0, J_0) = \tilde{H}_0(J) + \left(\tilde{H}_1 + \frac{\partial \tilde{H}_0}{\partial J}\frac{\partial S_1}{\partial \phi_0}\right)\epsilon$$
$$+ \left(\frac{\partial \tilde{H}_0}{\partial J}\frac{\partial S_2}{\partial \phi_0} + \frac{1}{2}\frac{\partial^2 \tilde{H}_0}{\partial J^2}\left(\frac{\partial S_1}{\partial \phi_0}\right)^2 + \frac{\partial \tilde{H}_1}{\partial J}\frac{\partial S_1}{\partial \phi_0}\right)\epsilon^2 + \dots$$

where all terms on the RHS are expressed in terms of ϕ_0 and J. We may now read off

(0) $E_0(J) = \tilde{H}_0(J)$

(1) $E_1(J) = \tilde{H}_1(\phi_0, J) + \dfrac{\partial \tilde{H}_0}{\partial J}\dfrac{\partial S_1(\phi_0, J)}{\partial \phi_0}$

(2) $E_2(J) = \dfrac{\partial \tilde{H}_0}{\partial J}\dfrac{\partial S_2(\phi_0, J)}{\partial \phi_0} + \dfrac{1}{2}\dfrac{\partial^2 H_0}{\partial J^2}\left(\dfrac{\partial S_1(\phi_0, J)}{\partial \phi_0}\right)^2 + \dfrac{\partial \tilde{H}_1(\phi_0, J)}{\partial J}\dfrac{\partial S_1(\phi_0, J)}{\partial \phi_0}$

But the RHS should be independent of ϕ_0! How can this be? **We use the freedom in the functions $S_k(\phi_0, J)$ to make it so.** Let's see just how this works.

Each of the expressions on the RHSs must be equal to its average over ϕ_0 if it is to be independent of ϕ_0:

$$\langle f(\phi_0) \rangle = \int_0^{2\pi} \frac{d\phi_0}{2\pi} f(\phi_0)$$

The averages $\langle RHS(\phi_0, J) \rangle$ are taken **at fixed J** and **not** at fixed J_0. We must have that

$$S_k(\phi_0, J) = \sum_{l=-\infty}^{\infty} S_{k,l}(J)\, e^{il\phi_0}$$

Thus

$$\left\langle \frac{\partial S_k}{\partial \phi_0} \right\rangle = \frac{1}{2\pi} \left\{ S_k(2\pi, J) - S_k(0, J) \right\} = 0$$

Now let's implement this in our hierarchy. Consider the level (1) equation,

$$E_1(J) = \tilde{H}_1(\phi_0, J) + \underbrace{\frac{\partial \tilde{H}_0}{\partial J}}_{\nu_0(J)} \frac{\partial S_1(\phi_0, J)}{\partial \phi_0}$$

Taking the average,

$$E_1(J) = \langle \tilde{H}_1(\phi_0, J) \rangle + \frac{\partial \tilde{H}_0}{\partial J} \underbrace{\left\langle \frac{\partial S_1(\phi_0, J)}{\partial \phi_0} \right\rangle}_{\text{this vanishes}}$$

$$= \langle \tilde{H}_1 \rangle$$

Thus,

$$\langle \tilde{H}_1 \rangle = \tilde{H}_1 + \nu_0(J)\frac{\partial S_1}{\partial \phi_0} \Rightarrow \frac{\partial S_1(\phi_0, J)}{\partial \phi_0} = \frac{\langle \tilde{H}_1 \rangle_J - \tilde{H}_1(\phi_0, J)}{\nu_0(J)}$$

If we Fourier decompose

$$\tilde{H}_1(\phi_0, J) = \sum_{l=-\infty}^{\infty} \tilde{H}_{1,l}(J)\, e^{il\phi_0}$$

then we obtain

$$\times\,(1-\delta_{l,0})$$

$$l \neq 0 : \; il\, S_{1,l}(J) = \tilde{H}_{1,l}(J) \Rightarrow S_{1,l}(J) = -\frac{i}{l}\tilde{H}_{1,l}(J)$$

We are free to set $S_{1,0}(J) \equiv 0$ (why?).

Now that we've got the hang of the logic here, let's go to second order:

$$E_2(J) = \underbrace{\frac{\partial \tilde{H}_0}{\partial J}}_{\nu_0(J)} \underbrace{\frac{\partial S_2(\phi_0, J)}{\partial \phi_0}}_{\text{averages to zero}} + \frac{1}{2}\underbrace{\frac{\partial^2 H_0}{\partial J^2}}_{\partial \nu_0/\partial J}\Bigg(\underbrace{\frac{\partial S_1(\phi_0, J)}{\partial \phi_0}}_{\frac{\partial S_1}{\partial \phi_0} = \frac{\langle \tilde{H}_1\rangle - \tilde{H}_1}{\nu_0}}\Bigg)^2 + \frac{\partial \tilde{H}_1(\phi_0, J)}{\partial J}\underbrace{\frac{\partial S_1(\phi_0, J)}{\partial \phi_0}}_{\frac{\partial S_1}{\partial \phi_0} = \frac{\langle \tilde{H}_1\rangle - \tilde{H}_1}{\nu_0}}$$

Taking the average,

$$E_2 = \frac{1}{2}\frac{\partial \nu_0}{\partial J}\left\langle \left(\frac{\langle \tilde{H}_1 \rangle - \tilde{H}_1}{\nu_0}\right)^2 \right\rangle + \left\langle \frac{\partial \tilde{H}_1}{\partial J}\left(\frac{\langle \tilde{H}_1 \rangle - \tilde{H}_1}{\nu_0}\right)\right\rangle$$

which yields, after some work,

$$\frac{\partial S_2}{\partial \phi_0} = \frac{1}{\nu_0^2}\Bigg\{ \left\langle \frac{\partial \tilde{H}_1}{\partial J}\right\rangle \langle \tilde{H}_1 \rangle - \left\langle \frac{\partial \tilde{H}_1}{\partial J}\tilde{H}_1 \right\rangle - \frac{\partial \tilde{H}_1}{\partial J}\langle \tilde{H}_1 \rangle + \frac{\partial \tilde{H}_1}{\partial J}\tilde{H}_1 + \frac{1}{2}\frac{\partial \ln \nu_0}{\partial J}\left(\langle \tilde{H}_1^2 \rangle - 2\langle \tilde{H}_1 \rangle^2 + 2\langle \tilde{H}_1 \rangle \tilde{H}_1 - \tilde{H}_1^2\right)\Bigg\}$$

and the energy to second order is

$$E(J) = \tilde{H}_0 + \epsilon \langle \tilde{H}_1 \rangle + \frac{\epsilon^2}{\nu_0}\left\{ \left\langle \frac{\partial \tilde{H}_1}{\partial J} \right\rangle \langle \tilde{H}_1 \rangle - \left\langle \frac{\partial \tilde{H}_1}{\partial J} \tilde{H}_1 \right\rangle + \frac{1}{2}\frac{\partial \ln \nu_0}{\partial J}\left(\langle \tilde{H}_1^2 \rangle - \langle \tilde{H}_1 \rangle^2 \right) \right\} + \mathcal{O}(\epsilon^3)$$

Note that we don't need $S(\phi_0, J)$ to obtain $E(J)$, though of course we do need it to obtain (ϕ_0, J_0) in terms of (ϕ, J). The perturbed frequencies are $\nu(J) = \partial E/\partial J$. For the full motion, we need

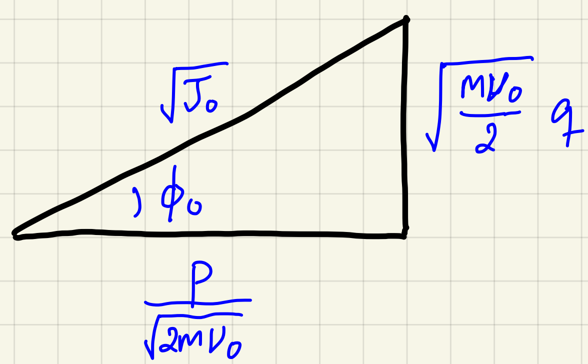
$$(\phi, J) \longrightarrow (\phi_0, J_0) \longrightarrow (q, p)$$

- **Example : quartic oscillator**

The Hamiltonian is

$$H(q,p) = \underbrace{\frac{p^2}{2m} + \frac{1}{2} m \nu_0^2 q^2}_{H_0} + \underbrace{\frac{\alpha}{4} \epsilon q^4}_{\epsilon H_1}$$

Recall the AAV for the SHO :



$$J_0 = \frac{p^2}{2m\nu_0} + \frac{1}{2} m \nu_0 q^2 = \frac{H_0}{\nu_0}$$

$$\phi_0 = \tan^{-1}\left(\frac{m\nu_0 q}{p}\right)$$

$$q = \left(\frac{2J_0}{m\nu_0}\right)^{1/2} \sin\phi_0$$

$$p = \sqrt{2J_0 m \nu_0}\, \cos\phi_0$$

Thus, we have

$$\tilde{H}(\phi_0, J_0) = \nu_0 J_0 + \frac{\alpha}{4}\epsilon\left(\sqrt{\frac{2J_0}{m\nu_0}}\,\sin\phi_0\right)^4$$

$$= \underbrace{\nu_0 J_0}_{\tilde{H}_0(J_0)} + \underbrace{\epsilon\left(\frac{\alpha}{m^2\nu_0^2}\right) J_0^{\,2}\sin^4\phi_0}_{\tilde{H}_1(\phi_0, J_0)}$$

We therefore have

$\langle \sin^4\phi_0 \rangle = \frac{3}{8}$

$$E_1(J) = \langle \tilde{H}_1(\phi_0, J) \rangle$$

$$= \frac{\alpha J^2}{m^2\nu_0^2}\int_0^{2\pi}\frac{d\phi_0}{2\pi}\sin^4\phi_0 = \frac{3\alpha J^2}{8m^2\nu_0^2}$$

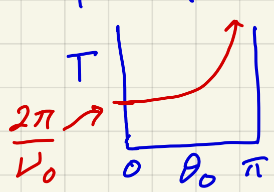
The frequency, to order ϵ, is then

$$\nu(J) = \frac{\partial}{\partial J}(E_0 + \epsilon E_1) = \nu_0 + \frac{3\epsilon\alpha J}{4m^2\nu_0^2} + \mathcal{O}(\epsilon^2)$$

To this order, we may replace J above by $J_0 = \frac{1}{2}m\nu_0 A^2$, where A = amplitude of oscillations. Thus,

$$\nu(A) = \nu_0 + \frac{3\epsilon\alpha A^2}{8m\nu^2} + \mathcal{O}(\epsilon^2)$$

pendulum:



Only for the linear oscillator $\ddot{q} = -\nu_0^{\,2} q$ is the oscillation frequency independent of the amplitude.

Next, let's work through the CT $(\phi_0, J_0) \to (\phi, J)$.

We have

$$v_0 \frac{\partial S_1}{\partial \phi_0} = \frac{\alpha J^2}{m^2 v_0^2}\left(\frac{3}{8} - \sin^4\phi_0\right)$$

$$\Rightarrow \quad S_1(\phi_0, J) = \frac{\alpha J^2}{8m^2 v_0^3}\left(3+2\sin^2\phi_0\right)\sin\phi_0\cos\phi_0$$

and

$$\phi = \phi_0 + \epsilon \frac{\partial S_1}{\partial J} + \mathcal{O}(\epsilon^2)$$

$$= \phi_0 + \frac{\epsilon\alpha J}{4m^2 v_0^3}\left(3+2\sin^2\phi_0\right)\sin\phi_0\cos\phi_0 + \mathcal{O}(\epsilon^2)$$

$$J_0 = J + \epsilon \frac{\partial S_1}{\partial \phi_0}$$

$$= J + \frac{\epsilon\alpha J^2}{8m^2 v_0^3}\left(4\cos(2\phi_0) - \cos(4\phi_0)\right) + \mathcal{O}(\epsilon^2)$$

To lowest nontrivial order we may invert to obtain

$$J = J_0 - \frac{\epsilon\alpha J_0^2}{8m^2 v_0^3}\left(4\cos(2\phi_0) - \cos(4\phi_0)\right) + \mathcal{O}(\epsilon^2)$$

With $q = (2J_0/mv_0)^{1/2}\sin\phi_0$ and $p = (2mv_0 J_0)^{1/2}\cos\phi_0$, we can obtain (q,p) in terms of (ϕ, J).

- **$n>1$: degeneracies and resonances**

Generalizing the CPT formalism to $n>1$ is straightforward. We have $S = S(\vec{\phi}_0, \vec{J})$, so with $\alpha \in \{1, \ldots, n\}$,

$$J_0^\alpha = \frac{\partial S}{\partial \phi_0^\alpha} = J^\alpha + \epsilon \frac{\partial S_1}{\partial \phi_0^\alpha} + \epsilon^2 \frac{\partial S_2}{\partial \phi_0^\alpha} + \dots$$

$$\phi^\alpha = \frac{\partial S}{\partial J^\alpha} = \phi_0^\alpha + \epsilon \frac{\partial S_1}{\partial J^\alpha} + \epsilon^2 \frac{\partial S_2}{\partial J^\alpha} + \dots$$

and

$$E_0(\vec{J}) = \tilde{H}_0(\vec{J})$$

$$E_1(\vec{J}) = \tilde{H}_1(\vec{\phi}_0, \vec{J}) + \nu_0^\alpha(\vec{J}) \frac{\partial S_1(\vec{\phi}_0, \vec{J})}{\partial \phi_0^\alpha}$$

$$E_2(\vec{J}) = \nu_0^\alpha(\vec{J}) \frac{\partial S_2(\phi_0, \vec{J})}{\partial \phi_0^\alpha} + \frac{1}{2} \frac{\partial \nu_0^\alpha(\vec{J})}{\partial J^\beta} \frac{\partial S_1(\vec{\phi}_0, \vec{J})}{\partial J^\alpha} \frac{\partial S_1(\vec{\phi}_0, \vec{J})}{\partial J^\beta}$$
$$+ \frac{\partial \tilde{H}_1(\vec{\phi}_0, \vec{J})}{\partial J^\alpha} \frac{\partial S_1(\vec{\phi}_0, \vec{J})}{\partial J^\alpha}$$

where $\nu_0^\alpha(\vec{J}) = \partial \tilde{H}_0(\vec{J}) / \partial J^\alpha$. Now we average:

$$\langle f(\vec{\phi}_0, \vec{J}) \rangle = \int_0^{2\pi} \frac{d\phi_0^1}{2\pi} \dots \int_0^{2\pi} \frac{d\phi_0^n}{2\pi} f(\vec{\phi}_0, \vec{J})$$

The equation for $S_1(\vec{\phi}_0, \vec{J})$ is

$$\nu_0^\alpha \frac{\partial S_1(\vec{\phi}_0, \vec{J})}{\partial \phi_0^\alpha} = \langle \tilde{H}_1(\vec{\phi}_0, \vec{J}) \rangle - \tilde{H}_1(\vec{\phi}_0, \vec{J})$$
$$= -\sum_{\vec{l} \in \mathbb{Z}^n}' V_{\vec{l}}(\vec{J}) e^{i \vec{l} \cdot \vec{\phi}_0}$$

where $V_{\vec{l}}(\vec{J}) = \tilde{H}_{1, \vec{l}}(\vec{J})$, i.e. $\tilde{H}_1(\vec{\phi}_0, \vec{J}) = \sum_{\vec{l}} V_{\vec{l}}(\vec{J}) e^{i \vec{l} \cdot \vec{\phi}_0}$

The prime on the sum means $\vec{\ell} = (0, 0, \ldots, 0)$ is excluded. The solution is

$$S_1(\vec{\phi}_0, \vec{J}) = -i \sum_{\vec{\ell} \in \mathbb{Z}^n}' \frac{V_{\vec{\ell}}(\vec{J})}{\vec{\ell} \cdot \vec{\nu}_0(\vec{J})} e^{i \vec{\ell} \cdot \vec{\phi}_0}$$

When the **resonance condition**

$$\vec{\ell} \cdot \vec{\nu}_0(\vec{J}) = 0$$

pertains (with $\vec{\ell} \neq 0$), the denominator vanishes and CPT breaks down. One can always find such an $\vec{\ell}$ whenever two or more of the frequencies $\nu_0^\alpha(\vec{J})$ have a rational ratio. Suppose for example that $\nu_0^2(\vec{J})/\nu_0^1(\vec{J}) = r/s$ with $r, s \in \mathbb{Z}$ relatively prime. Then $r\nu_0^1 = s\nu_0^2$ and with $\vec{\ell} = (r, -s, 0, \ldots, 0)$, we have $\vec{\ell} \cdot \vec{\nu}_0 = 0$. Even if all the frequency ratios are irrational, for large enough $|\vec{\ell}|$ we can make $|\vec{\ell} \cdot \vec{\nu}_0|$ as small (but finite) as we please. In §15.9, we'll see how any given resonance may be **removed** canonically. We're just looking at things the wrong way at the moment.