PHYSICS 200A : MIDTERM EXAMINATION

Normative time limit: two hours (consecutive!)

Submission deadline: Friday, Nov. 6, 2:00 pm PST (via Gradescope)

You are allowed to consult the online PHYS 200A course materials.

[**1**] Consider a two-body central force,

$$U(r) = -\frac{k}{r} + \frac{4kb^{1/2}}{3r^{3/2}} \quad ,$$

where $k > 0$ and where $b > 0$ has dimensions of length.

(a) Sketch the effective potential $U_{\text{eff}}(r)$. Prove that there is only one circular orbit. `[14 points]`

(b) Find the radius r_0 of the circular orbit. Is this orbit stable? `[12 points]`

(c) For small perturbations about the circular orbit, find the frequency of the perturbation. You may express your answer in terms of quantities such as r_0 itself. `[12 points]`

(d) Find the geometric equation of the shape of the perturbed orbit. Is the perturbed orbit closed? `[12 points]`

Note: I found it useful to define a dimensionless length $s \equiv r/a$ with $a \equiv \ell^2/\mu k$.

[**2**] Consider three masses $m_1 = m$, $m_2 = 2m$, and $m_3 = 3m$ connected by two springs $k_{12} = k$ (unstretched length $4a$) and $k_{23} = 2k$ (unstretched length a). The motion is frictionless and along a horizontal line (*i.e.* gravity does not enter this problem).

(a) Choose as generalized coordinates the positions $x_{1,2,3}$ of the masses. Write the Lagrangian. `[8 points]`

(b) Find the canonical momenta $p_{1,2,3}$ and the forces $F_{1,2,3}$.`[7 points]`

(c) Find the T and V matrices. `[7 points]`

(d) Find the characteristic polynomial $P(\omega^2) = \mathsf{det}(\omega^2\,\mathrm{T} - \mathrm{V})$. This expression can be reduced to one involving a single parameter by writing $\widetilde{P}(\lambda) \equiv P(\omega^2)/m^3$ where $\lambda \equiv \omega^2/\omega_0^2$, with $\omega_0^2 \equiv k/m$. Show that $\widetilde{P}(\lambda)$ is a cubic which factorizes into a product of λ and a quadratic $a\lambda^2 + b\lambda + c$. `[7 points]`

(e) Solve for the nonzero roots $\lambda_\pm$. The corresponding eigenfrequencies are then given by $\omega_\pm = \sqrt{\lambda_\pm}\,\omega_0$. `[7 points]`

(f) Find expressions for three normal mode eigenvectors. (This means that you can express their components in terms of the values for $\lambda_\pm$.) You don't have to normalize them. `[7 points]`

(g) Show that your zero mode agrees with the conclusions from Noether's theorem. `[7 points]`