PHYSICS 200A : CLASSICAL MECHANICS
SOLUTION SET #7

[1] A particle of mass m moves in one dimension subject to the potential

$$U(x) = \frac{k}{\sin^2(x/a)} \quad .$$

(a) Obtain an integral expression for Hamilton's characteristic function.

(b) Under what conditions may action-angle variables be used?

(c) Assuming that action-angle variables are permissible, determine the frequency of oscillation by the action-angle method.

(d) Check your result for the oscillation frequency in the limit of small oscillations.

[2] Consider one-dimensional motion in the potential $V(x) = -V_0 \operatorname{sech}^2(x/a)$ with $V_0 > 0$.

(a) Sketch the potential $V(x)$. Over what range of energies may action-angle variables be used?

(b) Find the action J and the Hamiltonian $H(J)$.

(c) Find the angle variable ϕ in terms of x and the energy E.

(d) Find the Solution for $x(t)$ by first solving for the motion of the action-angle variables.

Helpful mathematical identities :

$$\int\limits_0^{\bar{u}(E)} du\, \sqrt{E + V_0 \operatorname{sech}^2 u} = \frac{\pi}{2}\left(\sqrt{V_0} - \sqrt{-E}\right) \quad \text{if} \quad -V_0 < E < 0$$

$$\int du\, \left(E + V_0 \operatorname{sech}^2 u\right)^{-1/2} = \begin{cases} (-E)^{-1/2} \sin^{-1}\left(\sqrt{\frac{-E}{V_0+E}}\, \sinh u\right) & \text{if} \quad -V_0 < E < 0 \\ E^{-1/2} \sinh^{-1}\left(\sqrt{\frac{E}{V_0+E}}\, \sinh u\right) & \text{if} \quad E > 0 \end{cases}$$

where $\bar{u}(E) = \cosh^{-1}\sqrt{V_0/(-E)}$ in the first integral.

[3] A particle of mass m moves in the potential $U(q) = A\,|q|$. The Hamiltonian is thus

$$H_0(q,p) = \frac{p^2}{2m} + A\,|q| \quad ,$$

where A is a constant.

(a) List all independent conserved quantities.

(b) Show that the action variable J is related to the energy E according to $J = \beta\, E^{3/2}/A$, where β is a constant, involving m. Find β.

(c) Find $q = q(\phi, J)$ in terms of the action-angle variables.

(d) Find $H_0(J)$ and the oscillation frequency $\nu_0(J)$.

(e) The system is now perturbed by a quadratic potential, so that

$$H(q,p) = \frac{p^2}{2m} + A\,|q| + \epsilon\, B\, q^2 \quad ,$$

where ϵ is a small dimensionless parameter. Compute the shift $\Delta\nu$ to lowest nontrivial order in ϵ, in terms of ν_0 and constants.

[4] Consider the nonlinear oscillator described by the Hamiltonian

$$H(q,p) = \frac{p^2}{2m} + \tfrac{1}{2}kq^2 + \tfrac{1}{4}\epsilon a q^4 + \tfrac{1}{4}\epsilon b p^4 \quad ,$$

where ε is small.

(a) Find the perturbed frequencies $\nu(J)$ to lowest nontrivial order in ϵ.

(b) Find the perturbed frequencies $\nu(A)$ to lowest nontrivial order in ϵ, where A is the amplitude of the q motion.

(c) Find the relationships $\phi = \phi(\phi_0, J_0)$ and $J = J(\phi_0, J_0)$ to lowest nontrivial order in ϵ.