PHYS 273, Winter 2019, Homework 2

Due date: Wednesday, January 30th, 2019

1. *Sequence length.* How much information does the length of a sequence give about the content of a sequence? Suppose that we consider a Bernoulli (1/2) process $\{X_i\}$ i.e., for every i, X_i is 1 with probability 1/2 and 0 with probability 1/2. Stop the process when the first 1 appears. Let N designate this stopping time. Thus, X^N is an element of the set of all finite-length binary sequences $\{0,1\}^* = \{0, 1, 00, 01, 10, 11, 000, \dots\}$. Find $I(N; X^N), H(X^N|N), H(X^N)$.

Now consider a different stopping time. For this part, again assume that $X_i \sim$ Bernoulli(1/2) but stop at time $N = 6$ with probability 1/3 and stop at time $N = 12$ with probability 2/3. Let this stopping time be independent of the sequence $X_1 X_2 \dots X_{12}$. Find $I(N; X^N), H(X^N|N), H(X^N)$.

2. *The value of a question.* Let $X \sim p(x), x = 1, 2, \dots, m$. We are given a set $S \subseteq \{1, 2, \dots, m\}$. We ask whether $X \in S$ and receive the answer

$$Y = \begin{cases} 1 & \text{if } X \in S \\ 0 & \text{if } X \notin S \end{cases} \tag{1}$$

Suppose that $\Pr(X \in S) = \alpha$. Find the decrease in uncertainty $H(X) - H(X|Y)$.

3. *Noisy channel.* Consider three random variables X, Y, Z which can each take values 0 or 1; x and y are independent with $\Pr(X = 0) = p$ and $\Pr(Y = 0) = q$ and

$$z = (x + y) \mod 2$$

a. If $q = 1/2$, what is $\Pr(Z = 0)$? What is $I(Z; X)$?

b. For general p and q, what is $\Pr(Z = 0)$? What is $I(Z; X)$? This is an example of a single-bit noisy channel with $x =$ input, $y =$ noise and $z =$ output.

4. *Mutual information of heads and tails.*

a. A fair coin is flipped. How much information does looking at the visible side of the coin give about the hidden side of the coin? In other words, what is the mutual information between the visible and hidden sides of the coin?

b. A six-sided fair die is rolled. How much information does looking at the number on the top face of the die give about the number on the front face (the side most facing you)?