

Isothermal Sphere

Consider an ideal gas which is self-gravitating and isothermal

$$p = \frac{\rho k_B T}{m} \quad \text{ideal gas} \quad k_B \text{ Boltzmann constant}$$

p pressure

T temperature

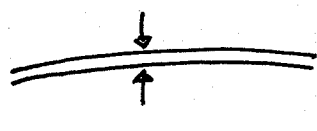
m mass per particle

ρ density

Equation of hydrostatic support:

$$(1) \frac{dp}{dr} = \frac{k_B T}{m} \frac{d\rho}{dr} = -\rho \frac{GM(r)}{r^2}$$

proof:

 dr dp net pressure on thin spherical shell

$M(r)$ total mass within r

$$-\underbrace{\rho}_{\text{mass of shell}} 4\pi r^2 dr \frac{G \cdot M(r)}{r^2} \quad \text{gravitational force on spherical shell}$$

balancing force from pressure: $4\pi r^2 \cdot dp$

$$4\pi r^2 dp = -4\pi r^2 \rho \frac{G \cdot M(r)}{r^2} dr$$

$$(1) \frac{dp}{dr} = -\rho \frac{G \cdot M(r)}{r^2} \quad \checkmark$$

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho$$

multiply (1) by $\frac{r^2 m}{\int k_B T}$ and take $\frac{d}{dr}$ on

both sides:

$$r^2 \frac{1}{\int} \frac{d\mathcal{S}}{dr} = - \frac{m G M(r)}{k_B T}$$

$$\frac{d}{dr} \ln \rho = \frac{1}{\int} \frac{d\mathcal{S}}{dr}$$

$$(2) \quad \frac{d}{dr} \left(r^2 \frac{d \ln \rho}{dr} \right) = - \frac{G \cdot m}{k_B T} 4\pi r^2 \rho$$

Independently, introduce now a simple DF f for a steady-state spherical distribution:

$$(3) \quad f(\mathcal{E}) = \frac{\mathcal{S}_1}{(2\pi \sigma^2)^{3/2}} e^{\frac{\mathcal{E}}{\sigma^2}} = \frac{\mathcal{S}_1}{(2\pi \sigma^2)^{3/2}} \exp\left(\frac{\psi - \frac{1}{2} v^2}{\sigma^2}\right)$$

$$\mathcal{E} = -\phi + \phi_0 = \psi - \frac{1}{2} v^2$$

$$\psi = -\phi + \phi_0$$

↑
vanishes at infinity

$\mathcal{E} > 0$ for bound particles

$$\rho = 4\pi G \int f d^3V$$

$$\rho = \rho_1 e^{\frac{\psi}{\sigma^2}}$$

(4) $\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\psi}{dr} \right) = -4\pi G \cdot \rho$ Poisson Equation

(5) $\frac{d}{dr} \left(r^2 \frac{d \ln \rho}{dr} \right) = - \frac{4\pi G}{\sigma^2} r^2 \rho$ using $\rho = \rho_1 e^{\frac{\psi}{\sigma^2}}$

choosing $\sigma^2 = \frac{k_B T}{m}$ Eq. 5 becomes identical to

Eq. 2 of isothermal self-gravitating ideal gas

Physical explanation: The distribution of velocities at each point in the stellar-dynamical isothermal system (sphere) is the Maxwell distribution

$$F(v) = N e^{-\frac{1}{2} \frac{v^2}{\sigma^2}}$$

Kinetic theory, however, tells us that this is also the equilibrium Maxwell-Boltzmann distribution which would emerge if the stars were allowed to bounce elastically off each other like the molecules of a gas. Therefore, if the DF of a system is given by Eq. 3 it is a matter of indifference whether the particles of the system collide with one another, or not.

The mean-square speed of stars at a point in the isothermal sphere is

$$\overline{v^2} = \frac{\int_0^\infty \exp\left(\frac{\psi - \frac{1}{2}v^2}{\sigma^2}\right) v^4 dv}{\int_0^\infty \exp\left(\frac{\psi - \frac{1}{2}v^2}{\sigma^2}\right) \cdot v^2 dv} = 2\sigma^2 \frac{\int_0^\infty x^4 e^{-x^2} dx}{\int_0^\infty x^2 e^{-x^2} dx} = 3\sigma^2$$

$\overline{v^2} = 3\sigma^2$ independent of position

The dispersion in any one component of velocity, for example, $(\overline{v_r^2})^{\frac{1}{2}}$ is equal to σ

It is easy to find one solution of Eq. 5 :

$\rho = C \cdot r^{-b}$ Ansatz

$\frac{d}{dr} \ln \rho = -b \frac{1}{r}$

$\frac{d}{dr} \left(r^2 \frac{d}{dr} \ln \rho \right) = -b \rightarrow -\frac{4\pi G}{\sigma^2} C \cdot r^{2-b}$
 $b = 2$

$\rho(r) = \frac{\sigma^2}{2\pi G \cdot r^2} \quad C = \frac{2\sigma^2}{4\pi G}$

singular isothermal sphere

We want solutions which are well behaved at origin.

$$\tilde{\rho} = \frac{\rho}{\rho_0}, \quad \tilde{r} = \frac{r}{r_0} \quad \text{rescaled variables}$$

$$r_0 = \sqrt{\frac{9\sigma^2}{4\pi G \rho_0}}$$

King radius where density falls to $0.5013 \rho_0$

ρ_0 central density

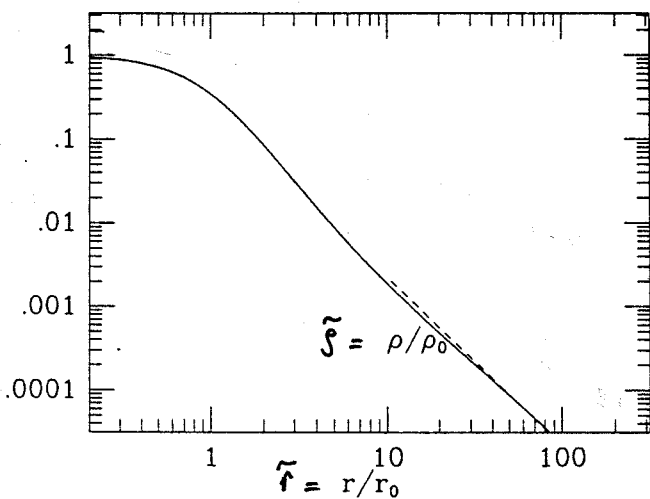
Eq. 5 becomes

$$(6) \quad \frac{d}{d\tilde{r}} \left(\tilde{r}^2 \frac{d \ln \tilde{\rho}}{d\tilde{r}} \right) = -g \tilde{r}^2 \tilde{\rho}$$

or

$$(7) \quad \frac{d}{d\tilde{r}} \left[\tilde{r}^2 \frac{d(\psi/\sigma^2)}{d\tilde{r}} \right] = -g \tilde{r}^2 \exp \left[\frac{\psi(r) - \psi(0)}{\sigma^2} \right]$$

numerical integration with $\tilde{\rho}(0) = 1$ and $\frac{d\tilde{\rho}}{d\tilde{r}} = 0$ boundary conditions:



dashed line is the density profile of singular isothermal sphere

Total mass is infinite

The circular speed at r is given by

$$(8) \quad v_c^2(r) = \frac{G \cdot M(r)}{r}$$

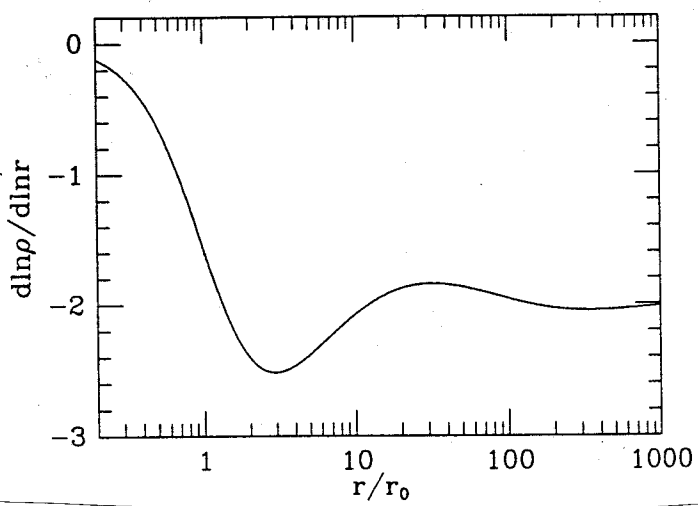
From Eq. (1)

$$v_c^2 \frac{d\rho}{dr} = -\int \frac{GM}{r^2} = -\int v_c^2 \frac{1}{r}$$

$$-v_c^2 d \ln \rho = v_c^2 d \ln r$$

$$v_c^2 = -v_c^2 \frac{d \ln \rho}{d \ln r}$$

numerically:



← -2 asymptotically

for large $\frac{r}{r_0} \quad v_c = \sqrt{2} \sigma$

We would like to modify now isothermal sphere in a minimal fashion to make the total mass finite

King Model

$$(9) \quad f_K(\epsilon) = \begin{cases} \rho_1 (2\pi\sigma^2)^{-\frac{3}{2}} \left(e^{\frac{\epsilon}{\sigma^2}} - 1 \right) & \epsilon > 0 \\ 0 & \epsilon \leq 0 \end{cases}$$

$$\rho_K(\psi) = \frac{4\pi\rho_1}{(2\pi\sigma^2)^{\frac{3}{2}}} \int_0^{\sqrt{2\psi}} \left[\exp\left(\frac{\psi - \frac{1}{2}v^2}{\sigma^2}\right) - 1 \right] v^2 dv$$

$$(10) \quad = \rho_1 \left[e^{\frac{\psi}{\sigma^2}} \cdot \operatorname{erf}\left(\frac{\sqrt{\psi}}{\sigma}\right) - \sqrt{\frac{4\psi}{\pi\sigma^2}} \left(1 + \frac{2\psi}{3\sigma^2} \right) \right]$$

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt \quad \operatorname{erf}(0) = 0$$

$$\operatorname{erf}(\infty) = 1$$

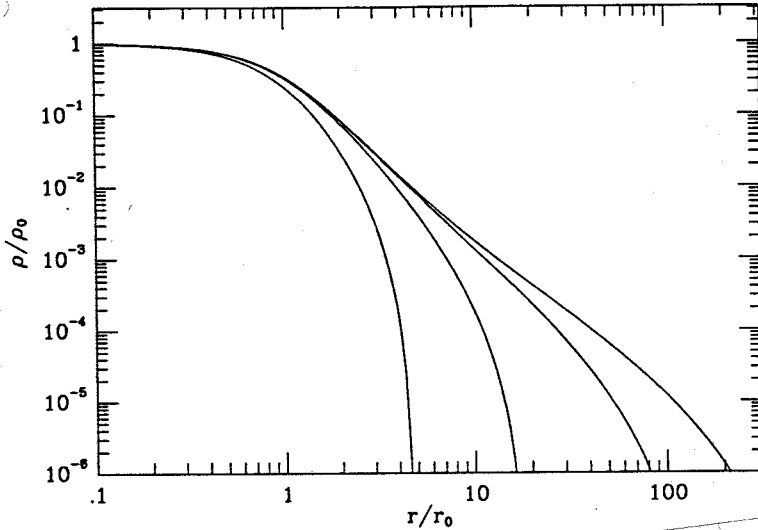
$$\lim_{x \rightarrow \infty} (1 - \operatorname{erf}(x)) = \frac{e^{-x^2}}{\sqrt{\pi} x} \quad \operatorname{erf}(-z) = -\operatorname{erf}(z)$$

Poisson equation :

$$\frac{d}{dr} \left(r^2 \frac{d\psi}{dr} \right) = -4\pi G \cdot \rho_1 \cdot r^2 \left[e^{\frac{\psi}{\sigma^2}} \cdot \operatorname{erf}\left(\frac{\sqrt{\psi}}{\sigma}\right) - \sqrt{\frac{4\psi}{\pi\sigma^2}} \left(1 + \frac{2\psi}{3\sigma^2} \right) \right]$$

$$(11) \quad \left. \begin{array}{l} \psi(0) \\ \frac{d\psi}{dr} = 0 \end{array} \right\} \text{boundary conditions}$$

King models form a single sequence as a function of $\psi(0)/\sigma^2$. In the limit $\psi(0)/\sigma^2 \rightarrow \infty$, the sequence goes over into the isothermal sphere.



$\frac{\psi(0)}{\sigma^2} = 12, 9, 6, 3$
sequence

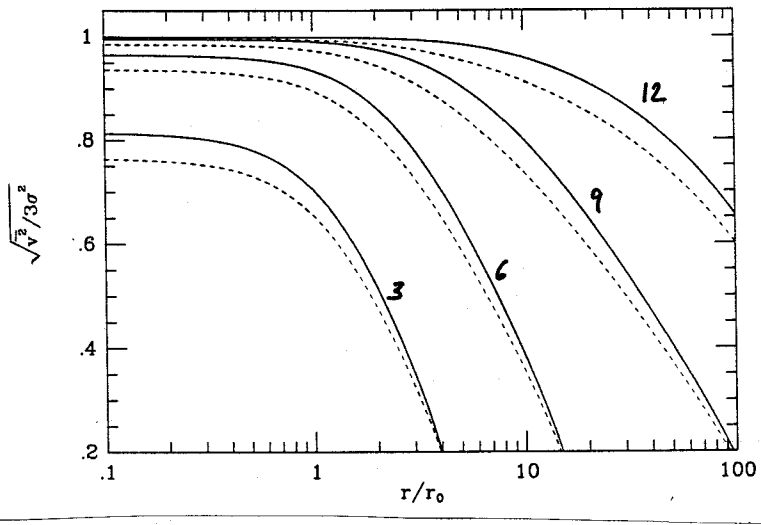
The parameter σ is not exactly the actual velocity dispersion $(\overline{v^2})^{1/2}$ of the stars.

$$\overline{v^2} = 3 \overline{v_r^2}$$

$$\overline{v^2}(r) = \frac{J_2}{J_0}$$

$$J_n = \int_0^{\sqrt{2\psi}} \left[\exp\left(\frac{\psi - \frac{1}{2}v^2}{\sigma^2}\right) - 1 \right] v^{n+2} dv$$

numerical velocity profile of King sequence:



$$\frac{\psi(0)}{\sigma^2} = 12, 9, 6, 3$$