Fluids in Flatland I — { Models / Dual Cascade

— apologies to Edwin Abbott

a) Models

Why 2D?:

— Recall Taylor–Proudman Theorem:

— in rotating fluid, $(\underline{\omega}+2\underline{\Omega})/\rho$ is "frozen in"
$$\frac{d}{dt}\left(\frac{\underline{\omega}+2\underline{\Omega}}{\rho}\right) = \left(\frac{2\underline{\Omega}+\underline{\omega}}{\rho}\right)\cdot\nabla\underline{V}$$

— showed if $\Omega \gg$ other rates in problem

$$2\Omega\frac{\partial\underline{V}}{\partial z} \cong 0, \quad \text{leading order}$$

$\Rightarrow$ ⓝ 2D dynamics

— immediately realize that ⓝ 2D dynamics $\Leftrightarrow$ Rossby Number < 1

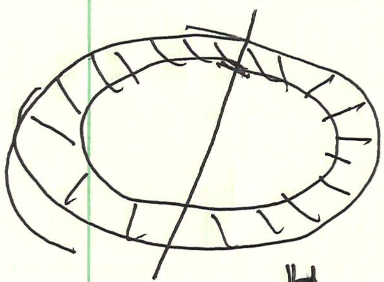
$$\boxed{Ro = V/L\Omega < 1} \rightarrow \text{characteristic of ⓝ 2D dynamics}$$

Ro: $\underline{V}\cdot\nabla\underline{V}$ vs. $2\underline{V}\times\underline{\Omega}$ — Coriolis wins

favors slow, large scale motion in (thin) rotating system — i.e. atmosphere, ocean, etc.

- so, 2D models motivated by rotating thin layers

thickness H

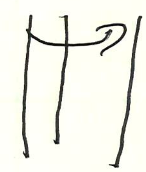


$\frac{H}{R}$

- $H/R \ll 1$, stably stratified in thickness dir. (vertical)
- $\frac{V_{Layer}}{L_{Layer}} < \Omega$

i.e. $\left(L_\perp < \frac{N}{\Omega} H \right)$

- other ⓒ 2D systems
  - shallow water layer
  - magnetized plasma



$\omega,\ V_\perp / L_\perp < \Omega_{ci}$

- (Nonlinear) Vortex tube stretching negligible, i.e.

$$(2\underline{\Omega} + \underline{\omega}) \cdot \nabla \underline{V} \quad \text{and} \quad Ro \ll 1$$

$$\text{vs} \quad \frac{d}{dt}\left(\frac{\underline{\omega} + 2\underline{\Omega}}{\rho}\right) = \left(\frac{2\underline{\Omega} + \cancel{\underline{\omega}}}{\rho}\right) \cdot \nabla \underline{V} \qquad \text{(component)}$$

$\Rightarrow$ ultimately, energy and (potential) enstrophy conserved

$\rightarrow$ essence of ⓒ 2D GFD problem

– Given $Ro < 1$, have fundamental relation between pressure and velocity

(p^* includes centrifugal force)

$$\frac{d\underline{v}}{dt} = -\underline{\nabla}(p^*/\rho) - 2\underline{\Omega} \times \underline{v}$$

$Ro < 1 \Rightarrow$ Geostrophic balance

$$0 \simeq -\underline{\nabla}(p/\rho) - 2\underline{\Omega} \times \underline{w}$$

$$\underline{v}_\perp = \underline{\Omega} \times \nabla(p^*/\rho)/\Omega^2$$

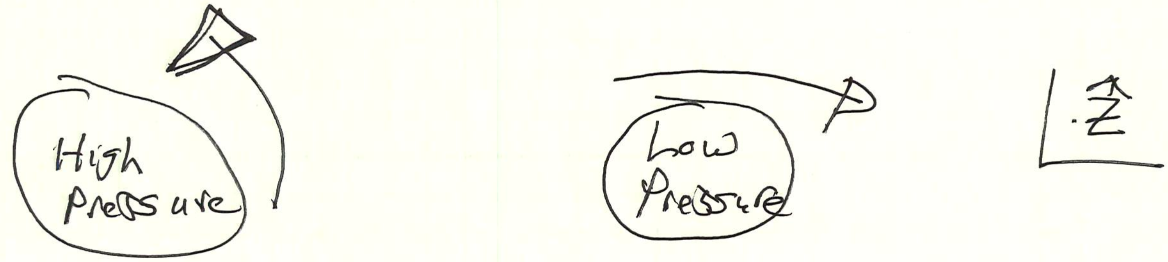
$$\boxed{\underline{v}_\perp = \frac{-\underline{\nabla}(p^*/\rho) \times \hat{z}}{\Omega}}$$

$\hat{z} = \dfrac{\underline{\Omega}}{\Omega}$

more generally $\perp$ to 2D plane.

$$-p^*/\rho \Leftrightarrow \phi$$

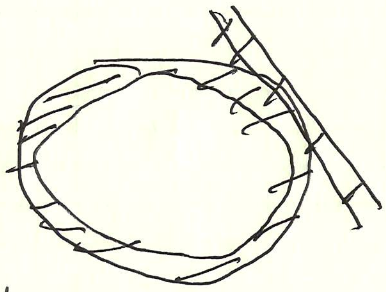
Pressure-as-stream-function:



$\left.\begin{array}{l}\text{clockwise}\\ \text{counter-clockwise}\end{array}\right\}$ Fluid rotation about $\left.\begin{array}{l}\text{Low}\\ \text{High}\end{array}\right\}$ pressure cells.

— β-plane Model

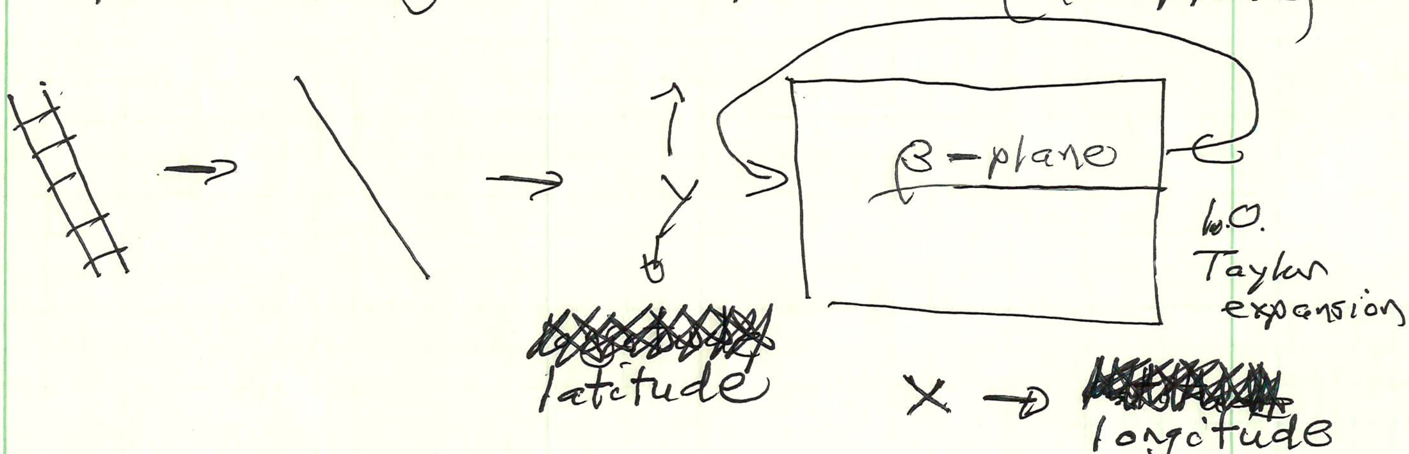
→ Quickie derivation of an important basic equation:



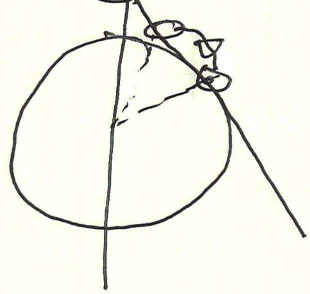
$(N^2 \to \infty)$

- i.e. tangent layer/plane to spherical shell atmosphere
- strongly stably stratified on scale of layer thickness (⊥)

→ so, describe dynamics in (2D!) plane tangent to sphere (β-plane)



→ now, consider displacement of fluid/vortex element:



→ $\underline{\omega} + 2\underline{\Omega}$ frozen in

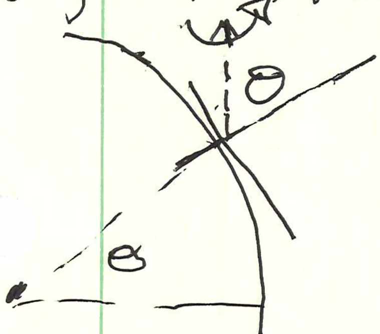
→ $\underset{\text{circulation}}{C} = \int d\underline{S} \cdot (\underline{\omega} + 2\underline{\Omega})$

Point: displacing fluid element implies change in $\int d\underline{s} \cdot \underline{\Omega}$

$\sim \hat{n} \cdot \hat{z} \sim \cos\theta_p$.

$\Rightarrow$ there must be a change in fluid vorticity to conserve circulation, since planetary (vorticity) circulation changed by displacement

So, if $A \equiv$ Area of vortex element:



$$\frac{dG}{dt} = 0$$

$$\frac{d}{dt}\left(A\omega + A\,2\Omega\underbrace{\sin\theta}_{\text{projection factor}}\right) = 0$$

$\Rightarrow$

$$\frac{d\omega}{dt} = -2\Omega\cos\theta\,\frac{d\theta}{dt}$$

$$= -\frac{2\Omega}{R}\cos\theta\,\frac{d}{dt}(R\theta)$$

$$= -\beta\,V_y$$

$\beta = \frac{2\Omega}{R}\cos\theta \rightarrow$ gradient in Coriolis force.

of course $\frac{d}{dt}(R\theta) = \frac{d}{dt} y = v_y$

$\therefore$ $$\boxed{\frac{d\omega}{dt} = -\beta v_y}$$ add dissipation, forcing

$$\left\{\begin{array}{l} \frac{d}{dt} = \partial_t + \underline{v}\cdot\underline{\nabla} \\ \underline{v} = -\frac{\underline{\nabla}\phi}{2\Omega}\times\hat{z} \end{array}\right.$$

$\hat{z} \perp \beta$ plane

$$\underline{\omega} = \omega\hat{z} = (\underline{\nabla}\times\underline{v})\cdot\hat{z}$$
$$= \nabla_\perp^2\phi/2\Omega$$

supports: waves, zonal flows, eddys

inviscid β - plane equation

$$\frac{d}{dt}\nabla_\perp^2\phi = -\beta v_y.$$ (Charney)

add forcing, dissipation.

$R \to \infty$, $\beta \to 0$. (scale)

$$\boxed{\partial_t\nabla^2\phi - \underline{\nabla}\phi\times\hat{z}\cdot\nabla\,\nabla^2\phi = 0}$$

supports eddys.

inviscid 2D Euler Eqn.

$\rightarrow$ simplest incarnation of "2D fluid" (Turbulence) (ie motivates study thereof).

$\rightarrow$ β - plane equation is next simplest

$\Rightarrow$ supports waves, as well as eddies. (Wave Turbulence)

Observe

- in 2D, $\underline{\nabla} \cdot \underline{V}$

$$\partial_t \underline{\omega} = \underline{\nabla} \times \underline{V} \times \underline{\omega}$$

$$\partial_t \underline{\omega} + \underline{V} \cdot \underline{\nabla} \underline{\omega} = \underline{\omega} \cdot \underline{\nabla} \underline{V}$$

$$= 0$$

vorticity advected, no stretching

- can re-write 2D equation:

$$\partial_t \omega_z + \{\omega_z, H\} = 0$$

$$H = \phi$$

conservative Hamiltonian evolution.

similar to Vlasov:

$$\partial_t f + \{f, H\} = 0$$

$$H = \frac{p^2}{2m} + |e|\phi$$

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial x} + \frac{Z}{m} E \frac{\partial f}{\partial v} = 0$$

Brings us to:

$$\boxed{\text{Potential Vorticity}}$$

observe can write equations in conservative form:

$$\frac{d}{dt}\omega = 0$$

and, for β-plane:

$$\frac{d}{dt}(\omega + \beta y) = 0$$

ω → fluid vorticity

βy → planetary vorticity (l.o. in expansion)

$\omega + \beta y \equiv$ a simple example of potential vorticity

- generalized or extended vorticity
- akin phase space density (conserved on orbits)

"GFD = The study of (fluids with) PV"

- "The Fluid Dynamics of PV".

For more on PV:

- for rotating fluid:

$$\frac{d}{dt}\left(\frac{\underline{\omega} + 2\underline{\Omega}}{\rho}\right) = \frac{(\underline{\omega} + 2\underline{\Omega})\cdot \underline{\nabla}\underline{V}}{\rho}$$

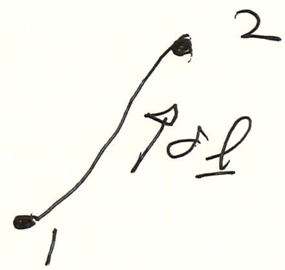
akin: $\frac{d}{dt}\delta\underline{l} = \delta\underline{l}\cdot\underline{\nabla}\underline{V}$

same eqn.

Now, also have passive scalar field:

$$\frac{d\psi}{dt} = 0 \qquad \psi \to \text{scalar field}$$

$$\frac{d}{dt}(\psi_1 - \psi_2) = 0$$



so $\quad \delta\psi = \underline{\nabla}\psi \cdot \delta\underline{l}$

$$\Rightarrow \quad \frac{d}{dt}\left(\underline{\nabla}\psi \cdot \delta\underline{l}\right) = 0$$

and $\quad \overset{\text{satisfies}}{\delta\underline{l}} \Leftrightarrow \overset{\text{must satisfy}}{\frac{\underline{\omega} + 2\underline{\Omega}}{\rho}}$

so

PV conservation

$$\boxed{\frac{d}{dt}\left(\frac{(\underline{\omega} + 2\underline{\Omega}) \cdot \underline{\nabla}\psi}{\rho}\right) = 0}$$

$$\boxed{q = \frac{(\underline{\omega} + 2\underline{\Omega}) \cdot \underline{\nabla}\psi}{\rho}} \qquad \underline{\text{PV.}}$$

$$Q = \int d^3x \, q \quad \to \text{PV charge}$$

$\Rightarrow$ PV conserved $\Leftrightarrow$ symmetry ?

$\Rightarrow$ <u>Particle re-labeling symmetry</u> $\Rightarrow$

PV conserved when particles can be re-labeled without changing the thermodynamic state.

Note

- can derive β plane equation from PV conservation $\left(\begin{array}{l} \nabla\Psi \equiv \eta \\ \nabla\Psi \equiv z \text{ for H-14} \\ \rho = \text{const} \end{array}\right)$

- when is PV not conserved?
  - Baroclinic torque $\nabla p \times \nabla \rho \neq 0$
    → Ertel's Theorem

$V = 1/\rho$ → specific volume.
$\underline{u}$ = velocity

$$\frac{dV}{dt} = V \, \underline{\nabla} \cdot \underline{u}$$

$$\frac{d\underline{u}}{dt} + 2\underline{\Omega} \times \underline{u} = -V \underline{\nabla} p - \underline{\nabla} \Phi$$

so vorticity evolution

$$\frac{d}{dt}(\underline{\omega} + 2\underline{\Omega}) = (\underline{\omega} + 2\underline{\Omega}) \cdot \nabla \underline{u} - (\underline{\omega} + 2\underline{\Omega}) \underline{\nabla} \cdot \underline{u} - \underline{\nabla} V \times \underline{\nabla} p$$

$$\frac{d}{dt}\left[V(\underline{\omega} + 2\underline{\Omega})\right] = V(\underline{\omega} + 2\underline{\Omega}) \cdot \nabla \underline{u} - V(\underline{\nabla} V \times \underline{\nabla} p)$$

so

$$\boxed{\frac{dq}{dt} = (\underline{\nabla} p \times \underline{\nabla} V) \cdot \underline{\nabla} \Psi}$$

q as before

— GFD always weakly compressible

$$\cancel{\frac{\partial\rho}{\partial t}} + \nabla\cdot(\rho \underline{v}) = 0$$

$$\nabla\cdot\underline{v} + \frac{1}{\rho}\frac{d\rho}{dz} v_z = 0$$

$$k > 1/L_\rho .$$

* — if finite thickness shell:
(QG)
(what of 3D?)

— 2D
— β-plane
— QG

$$\frac{d}{dt} q = 0$$

$$q = \nabla_\perp^2 \phi + \beta y + \frac{f_0^2}{\rho} \partial_z \left(\frac{\rho}{N^2} \partial_z \phi \right)$$

$$\begin{cases} f_0 = 2\Omega \sin\theta & \text{— rotation} \\ N^2 = g/L_\rho & \text{— buoyancy.} \end{cases}$$

$$\frac{1}{L^2} \quad \text{vs} \quad \frac{f_0^2}{N^2 H^2} \equiv \frac{1}{L_d^2}$$

deformation radius

$L \sim L_d \rightarrow$ { Relative vorticity and deformation effects contribute equally.

(~ 100 km ocean)
(~ 1000 km atm)

$L < L_d \rightarrow \sim 2D \rightarrow \beta$-plane.

→ 2D Turbulence

- issues: conservation energy and enstrophy
- trends in constrained spectral evolution.
- self-similarity ranges.
- fate of energy

⇒ Issues:

- 2D turbulence ($\nabla \cdot \underline{V} = 0$) emerges as THE generic problem of GFD family.
- β plane, with $\beta \to 0$

$$\partial_t \nabla_\perp^2 \phi + \nabla_\perp \phi \times \hat{z} \cdot \nabla_\perp \nabla_\perp^2 \phi - \nu \nabla_\perp^2 \nabla_\perp^2 \phi$$

$$+ \mu \nabla_\perp^2 \phi = \tilde{f}$$

drag
damp i.c.

any location/scale

Key: - 2 quadratic inviscid invariants

i.e. inviscid conservation of

- energy $\left\langle \frac{(\nabla_\perp \phi)^2}{2} \right\rangle = \left\langle \frac{\underline{V}^2}{2} \right\rangle = \int d^2x \, \frac{(\nabla\phi)^2}{2}$

* - enstrophy $\left\langle \frac{(\nabla_\perp^2 \phi)^2}{2} \right\rangle = \int d^2x \, \frac{(\nabla^2\phi)^2}{2}$

due: → absence of NL vortex tube stretching

i.e.

$$\frac{d\omega}{dt} = \cancel{\omega \cdot \nabla V} \Rightarrow \frac{d}{dt}\langle \omega^2 \rangle = \cancel{\langle \omega \cdot \underline{\omega} \cdot \nabla \underline{V} \rangle}$$

no enstrophy creation

note → all powers $\langle \omega^n \rangle$ conserved

→ only $\langle \omega^2 \rangle$ conserved in finite box.

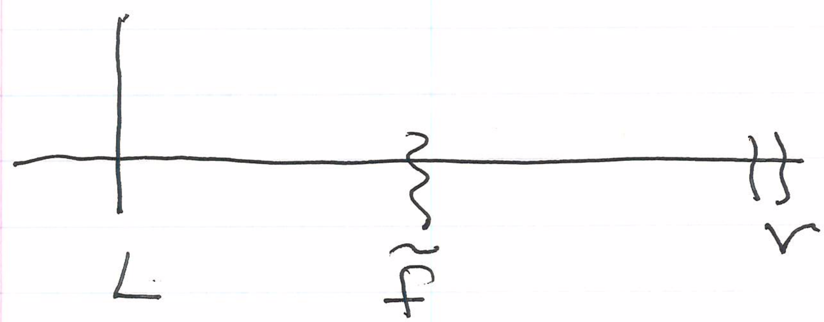
→ clearly incompatible with K41 story.

→ in accord with "negative viscosity" phenomenology from atmospherics

→ excitation to larger scales.

so → Central Problem of 2D Fluids is:

Given forcing at any scale l_f s/t $L \geq l_f \geq l_\nu$



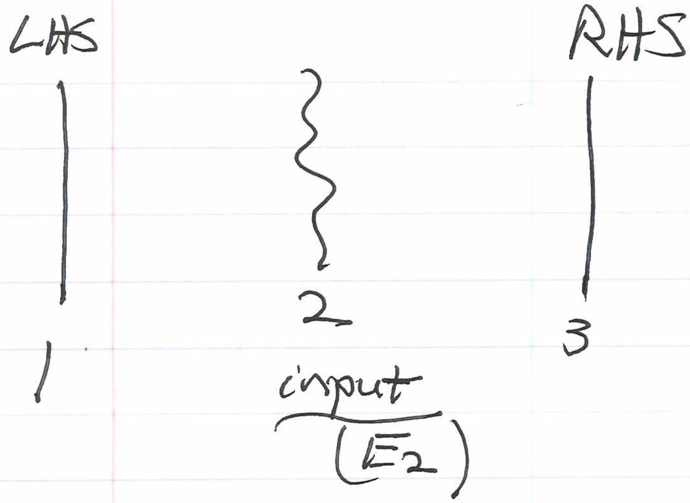
- How does dual conservation of E, Ω constrain self-similar transfer?
- What happens? — What is the Phenomenology?

Some clues:

①

- consider 3 modes (need 3 to conserve quad.)

1	2	3
k_1^2	k_2^2	k_3^2



$$k_1^2 \ll k_2^2 \ll k_3^2$$

Outcome:

$$E_2 = E_1 + E_3$$

$$\Omega_2 = \Omega_1 + \Omega_3 \quad \Leftrightarrow \text{Top!?}$$

$$k_2^2 E_2 = k_1^2 E_1 + k_3^2 E_3$$

$$\therefore \quad \left(\frac{k_3^2 - k_2^2}{k_3^2 - k_1^2}\right) E_2 = E_1$$

$$\left(\frac{k_2^2 - k_1^2}{k_3^2 - k_1^2}\right) E_2 = E_3$$

so $k_1^2 \ll k_2^2 \ll k_3^2$

$E_1 \cong E_2 \rightarrow$ energy accumulates to LHS $\rightarrow$ larger scale

$\Omega_3 \cong \Omega_2 \rightarrow$ enstrophy accumulates to RHS $\rightarrow$ smaller scale

$\Rightarrow$ Needs $\underline{\underline{2}}$ self-similar cascades.

(3) Further,

Rhines
(ex-post-facto)

$\bar{k}$

Consider a spectral 'slug' of turbulence.
How will $\bar{k}$ evolve, given $\partial_t \langle \Delta k^2 \rangle > 0$?

Now,
$$\langle (\Delta k)^2 \rangle = \int dk\, (k-\bar{k})^2 E(k) \Big/ \int dk\, E(k)$$

$$= \frac{\int dk \left(k^2 - 2k\bar{k} + \bar{k}^2\right) E(k)}{\int E(k)\, dk}$$

$$\langle \Delta k^2 \rangle = \Big[\int dk\, k^3 E(k) - 2\bar{k} \int dk\, E(k) + \bar{k}^2 \int dk\, E(k) \Big] \Big/ \int dk\, E(k)$$

$$\int dk\, E(k) = E_0 \rightarrow \text{const}$$

$$\int dk\, E(k) k^2 = \Omega_0 \rightarrow \text{const}$$

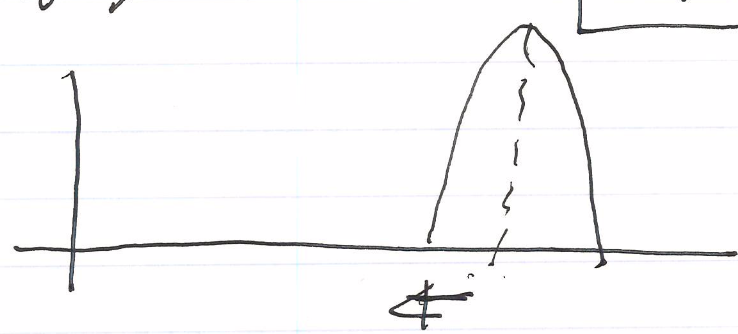
$$\int dk\, k\, E(k) = \bar{k} E_0 \rightarrow \text{centroid}$$

$$\langle (\Delta k)^2 \rangle = \frac{\Omega_0 - 2\bar{k}^2 E_0 + \bar{k}^2 E_0}{E_0}$$

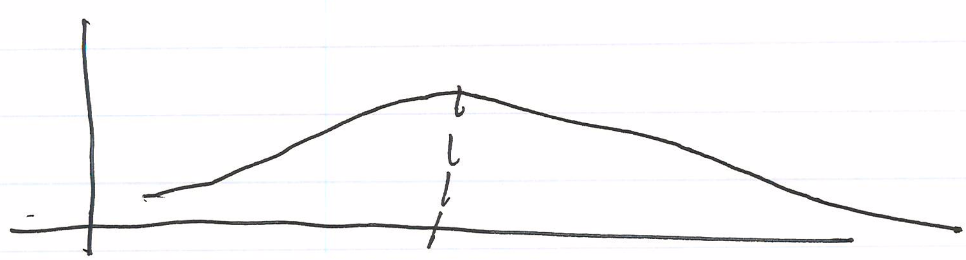
$$= \Omega_0 / E_0 - \bar{k}^2$$

$$\partial_t \langle (\Delta k)^2 \rangle > 0 \Rightarrow \boxed{\partial_t \bar{k} < 0}$$

i.e.



→



Recall:

- in 2D NST, forced at intermediate scale:

$$\overset{E}{\longleftarrow} \;\; \overset{\Omega}{\longrightarrow}$$

$$E_2 = E_1 + E_3$$
$$\Omega_2 = \Omega_1 + \Omega_3$$

$\partial_t \bar{k} < 0$ ⇒ energy accumulates large scale

⇒ Enter Dual Cascade

Dual Self-Similarity Ranges

Spectrum broadens but shifts toward larger scales!

⇒ Energy content shuffled / coupled to larger scale.

∴ again suggestive of energy <u>inverse</u> cascade.

N.B. Similar story for enstrophy
⇒ forward cascade

∴ Enter the <u>Dual Cascade</u>!

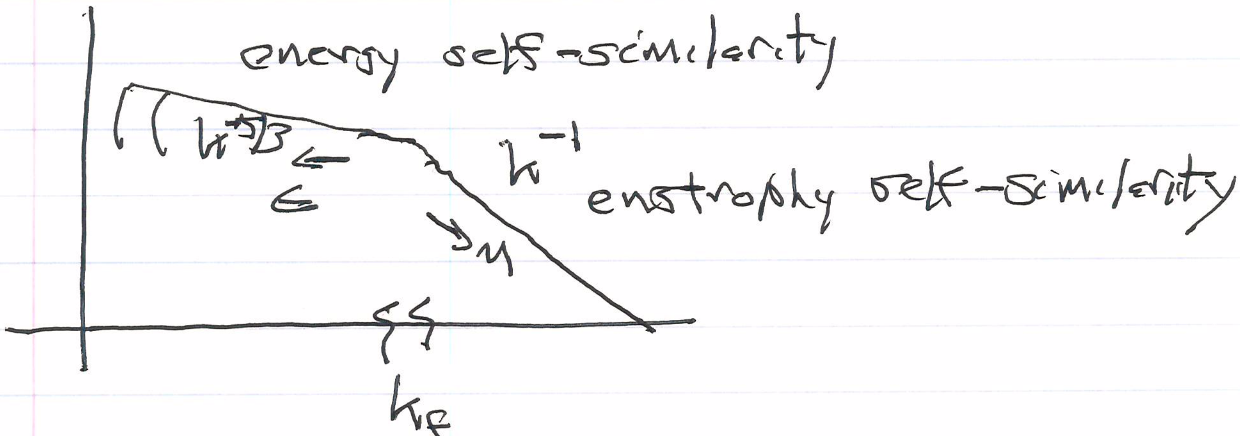
i.e. from forcing, system supports $\underline{\underline{2}}$ self-similarity ranges

- forward enstrophy range / cascade
  - no forward energy flux
  - no energy dissipation by viscosity!

- inverse energy cascade
  - no inverse ~~xxxxx~~ enstrophy flux
  - damping ..., drag etc

Cascade $\equiv$ range self-similar transfer

i.e.



$$\eta = \frac{d}{dt}\langle \omega^2 \rangle \sim \left(v_f/l_f\right)^3$$

n.b.
{ enstrophy dissipation controversial

$$\epsilon = \frac{d}{dt}\langle v^2 \rangle \sim \frac{v_f^3}{l_f}$$

{ not dissipation, necessarily

$$\left(\epsilon k_f^2 \sim \eta\right)$$

— forward — Enstrophy:

$$\langle v^2 \rangle = \left[k\, E(k)\right]$$

DOS

$$1/\tau_k = k\left[k E(k)\right]^{1/2} \sim k\tilde{v}_k$$
$$= \left[k^3 E(k)\right]^{1/2}$$

$$\Omega(k) = k^2 E(k)$$

so $\quad \eta = \dfrac{\langle \omega^2 \rangle}{T} = \left[k^3 E(k)\right]\left[k^3 E(k)\right]^{1/2}$

$\Rightarrow$

$$\left\{ \begin{array}{l} E(k) = \eta^{2/3} k^{-3} \\ \\ \Omega(k) = \eta^{2/3} k^{-1} \end{array} \right.$$

energy spectrum in enstrophy range

enstrophy range

no forward energy flux ($=0$) in enstrophy range

Observe: $\quad 1/\tau_k = \left[k^3 E(k)\right]^{1/2} \rightarrow \eta^{1/3}$

const !?

vs $\quad 1/\tau_\ell \sim \dfrac{\epsilon^{1/3}}{\ell^{2/3}}$ (faster for smaller!)

$\Rightarrow$ tip off that non-local transfer of enstrophy occurs. Corrections.

up-scale transfer

For energy range:

$$\epsilon = \left[k E(k) \right] \left[k^3 E(k) \right]^{1/2}$$

$$= \left[E(k) \right]^{3/2} k^{5/2}$$

$$\boxed{E(k) = \epsilon^{2/3} k^{-5/3}} \rightarrow$$

inverse energy cascade range.

akin 3D, but up scale

N.B.: $1/\tau_k = \left[k^3 E(k) \right]^{1/2}$

$$\cong \epsilon^{1/3} k^{2/3}$$

no inverse enstrophy flux (= 0) in energy range

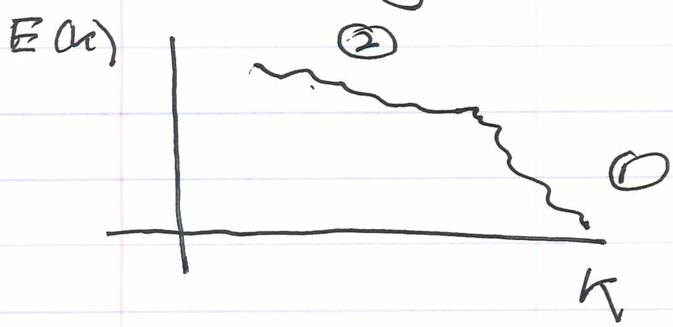
$\Rightarrow$ cascade slows as larger scales approached.

$\Leftrightarrow$ eventually encounters boundary, B.C., etc.

Leads to some questions:

$\rightarrow$ What of particles? particle dispersion?

→ re-visiting Richardson



→ it matters where inserted !!

i.e. $\ell_{1,2}$ → enstrophy range

i.e. $$\frac{d\ell}{dt} = v(\ell)$$

but $$v = (kE)^{1/2}$$
$$= (\eta^{2/3} k^{-2})^{1/2} = \eta^{1/3} \ell$$

$$\frac{d\ell}{dt} = \eta^{1/3} \ell$$

⇒ separation grows exponentially in enstrophy range.

Upon reaching / insertion in energy range

$$\frac{d\ell}{dt} = v(\ell) = \epsilon^{1/3}\ell^{1/3}$$

$\Rightarrow$ $\ell^2 \sim \epsilon t^3$, as 'usual'.

→ Is there anything rigorous to be said? 4/5 analogue

see Celani[10], et al.: (2001 Posted)

$$\langle \delta v^3 \rangle \sim \frac{3}{2}\epsilon \ell$$

(in inverse cascade 'inertial range')

where:

+ → inverse cascade

$\epsilon \equiv$ "dissipation rate"

$$\epsilon = \frac{dE}{dt} \sim \frac{V_F^3}{\ell_F}$$

↓

energy input rate

i.e.

not stationary state.

No analogue for forward enstrophy range ⇒ locality ?

→ Where does the energy go ?

⇒ builds up large scales, encounters friction, etc.

⇒ as no forward energy flux, ϵ_d by viscosity → 0.

→ β-Plane: Turbulence, Waves, Flows

Recall:

$$\partial_t \nabla^2\phi + \nabla\phi \times \hat{z} \cdot \nabla \nabla^2\phi - \nu \nabla^2 \nabla^2 \phi + \mu \nabla^2 \phi = -\beta V_{\text{[redacted]}\, y} + \tilde{f}$$

Ignoring: ν, μ, $\tilde{f}$

$$\partial_t \nabla^2\phi + \nabla\phi \times \hat{z} \cdot \nabla \nabla^2\phi = -\beta V_y$$

⇒ Waves

$$\omega_k = -\beta k_x / k^2 , \qquad V_{gy} = \frac{2\beta k_x k_y}{(k^2)^2}$$

→ Rossby wave

and

⇒ flows

how does large scale order emerge?

$$\left.\begin{array}{l} k_x \to 0 \\ k_y \text{ finite} \\ \omega_k \to 0 \end{array}\right\}$$

Zonal Mode

jets, belts, jet stream

2 new players → waves, flows.

Numerous questions:

2 → how do zonal flows form?
why ⇒ many ways!

1 → how do {waves, flows} modify, interact with inverse cascade?

3 → scale of zonal flows?

4 → implications for atmospheric phenomenology

On zonal flows:

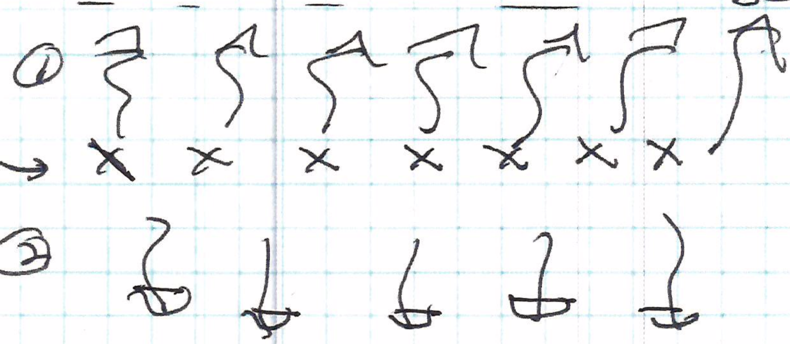
Reynolds stress

- ZFs ubiquitous
- Flows produced by momentum transport
- simplest perspective ⇒ wave propagation!

ii)

(Linear) wave propagation can account for ZF formation!

recall:



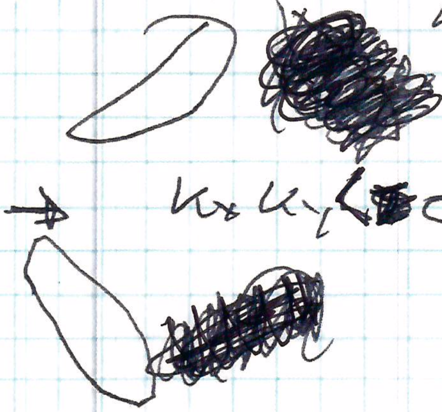
$$\underline{S} = v_{gy}\,\mathcal{E}\,\hat{y} = \frac{2k_xk_y\beta\mathcal{E}}{(k^2)^2}\,\hat{y}$$

beach (absorber) (useful cartoon)

① outgoing waves → $k_xk_y > 0$

$S \sim (+)\,\hat{y}$ ($\omega > 0$)

② $S \sim -\hat{y}$ → $k_xk_y < 0$ ($\omega > 0$)



eddy tilt changes

but

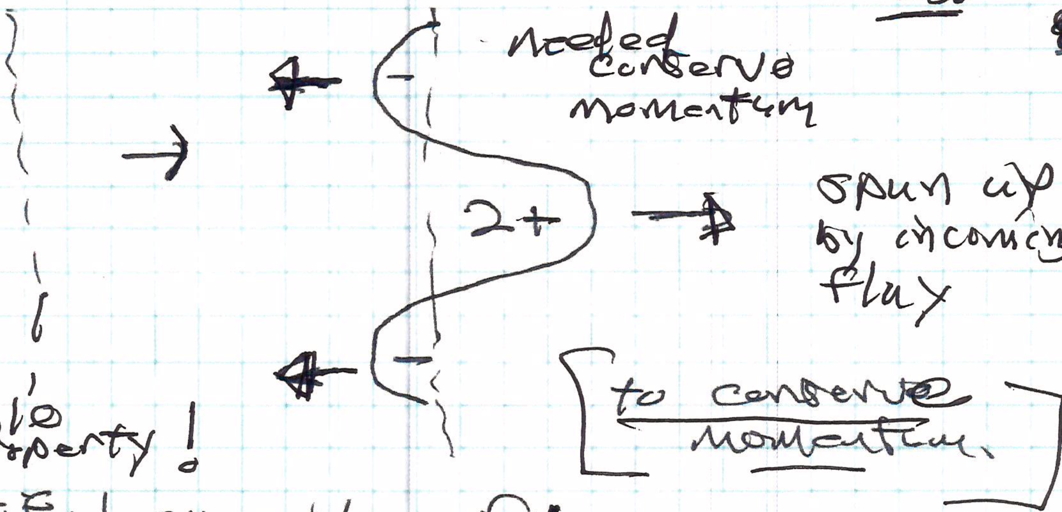
$$\langle \tilde{V}_y \tilde{V}_x \rangle = \sum_k -k_xk_y|\phi_k|^2$$

so ① → $\Pi_{y,x} < 0$

② → $\Pi_{y,x} > 0$

point: outgoing wave energy density flux generates incoming momentum flux!

50



Remarkable property!

→ beautiful example of:

..." the central result that a rapidly rotating flow, when stirred in a localized region, will converge angular momentum into this region" — Isaac Held ('01)

(stirring → spin-up!)

Flows ⟺ energy stirring

→ Wave mechanism requires separation of excitation and dissipation (beach) regions.

→ Requires:

- waves
- vorticity / momentum transport in space

* – irreversibility → outgoing waves

- symmetry breaking, → has direction
- sep. forcing / damping

⇒ Useful to investigate wave theorems for flow production → something general

→ Key observation: PV flux — Why?
(inhomogeneous PV mixing)

$$q = \beta y + \nabla^2\phi$$

recall essence of PV conservation: forces planetary — flow vorticity exchange.

$\langle \tilde{v}_y \tilde{q} \rangle_z$ → zonal avg.

$$= \langle \tilde{v}_y \nabla^2\phi \rangle_z$$

$$= \langle (-\partial_x\phi)(\partial_x^2\phi + \partial_y^2\phi) \rangle_x$$

but: $\langle \partial_x\phi \, \partial_x^2\phi \rangle = \langle \partial_x \left[\frac{(\partial_x\phi)^2}{2}\right] \rangle_x = 0$

symmetry!

$$\langle \tilde{v}_y \tilde{q} \rangle_z = -\langle (\partial_x\phi)\, \partial_y^2\phi \rangle$$

$$= -\partial_y \langle \partial_x\phi \, \partial_y\phi \rangle_x + \langle \partial_{xy}^2\phi \, \partial_y\phi \rangle$$

$$\langle \partial_{xy}^2\phi \, \partial_y\phi \rangle = \langle \partial_x \frac{(\partial_y\phi)^2}{2} \rangle_x = 0.$$

$$= \partial_y \langle \tilde{v}_y \tilde{v}_x \rangle_y$$

Taylor Identity

$$\boxed{\langle \tilde{v}_y \tilde{q} \rangle_z = \partial_y \langle \tilde{v}_y \tilde{v}_x \rangle_z}$$

(contrast 3D) — EP. ⟶ Reynolds force drives flow!

z dropped hereafter.

⇒ Look at potential enstrophy balance.

⇒ Zonally averaged Latitudinal PV flux = zonally averaged Latitudinal Reynolds force ⇒ drives flow.

As Reynolds stress controls flow:

i.e.

$$\rho\left(\frac{\partial V_x}{\partial t} + \underline{V}\cdot\underline{\nabla} V_x\right) = \cancel{-\partial_x P} - \cancel{(2\underline{\Omega}\times\underline{V})_x}$$

cancel → geostrophic balance

so

$$\left[\frac{\partial\langle V_x\rangle}{\partial t} = -\partial_y\langle\tilde{V}_y\tilde{V}_x\rangle + \nu\,\partial_y^2\langle V_x\rangle - \mu\langle V_x\rangle\right.$$

then PV evolution / Potential Enstrophy } necessarily control flow.

⇒ What are essential to ZF generation:

- inhomogeneous PV mixing / transport in space
- translation symmetry in direction of the flow.

(forcing)

Now, consider P.E. balance:

$$\frac{d}{dt} q - \nu \nabla^2 q = 0$$

$$\frac{\partial}{\partial t} \tilde{q} + \underset{\sim}{V} \cdot \nabla \tilde{q} - \nu \nabla^2 \tilde{q} = - \tilde{v}_y \frac{d\langle q \rangle}{dy}$$

potential enstrophy evolution

so

$$\frac{\partial}{\partial t} \left\langle \frac{\tilde{q}^2}{2} \right\rangle + \partial_y \left\langle \tilde{v}_y \frac{\tilde{q}^2}{2} \right\rangle + \nu \langle (\nabla \tilde{q})^2 \rangle$$

Flux of potential enstrophy.

$D_\Omega \rightarrow$ dissipation

$$= - \langle \tilde{v}_y \tilde{q} \rangle \frac{d\langle q \rangle}{dy}$$

[potential enstrophy production (flux - gradient)

$$\left(\frac{d\langle q \rangle}{dy}\right)^{-1} \left[\partial_t \left\langle \frac{\tilde{q}^2}{2} \right\rangle + \partial_y \left\langle \tilde{v}_y \frac{\tilde{q}^2}{2} \right\rangle + \nu \langle (\nabla \tilde{q})^2 \rangle \right]$$

$$= - \langle \tilde{v}_y \tilde{q} \rangle = - \langle \tilde{v}_y \nabla^2 \tilde{\phi} \rangle$$

but Mean (zonal) flow

$$\partial_t \langle v_x \rangle = - \frac{\partial}{\partial y} \langle \tilde{v}_y \tilde{v}_x \rangle - \mu \langle v_x \rangle$$

$$= - \langle \tilde{v}_y \nabla^2 \tilde{\phi} \rangle - \mu \langle v_x \rangle$$

$$-\langle \tilde{v}_y \nabla^2 \tilde{\phi} \rangle = (\partial_t \langle V_x \rangle + \mu \langle V_x \rangle)$$

$\therefore$

WAD

$$\partial_t \left\{ \langle V_x \rangle + \underbrace{\frac{\langle \tilde{\xi}^2 \rangle}{2\, d\langle \xi \rangle / dy}}_{\text{WAD}} \right\} = -\nu \frac{\langle (\nabla \tilde{\xi})^2 \rangle}{d\langle \xi \rangle / dy} - \partial_y \frac{\langle \tilde{v}_y \tilde{\xi}^2 / 2 \rangle}{d\langle \xi \rangle / dy} - \mu \langle V_x \rangle$$

Wave Activity Density

pseudomomentum to

$$\partial_t \left\{ \langle V_x \rangle - \left(\frac{k_x \langle \tilde{\xi}^2 \rangle}{2 k_x\, d\langle \xi \rangle / dy} \right) \right\} = -\mu \langle V_x \rangle - \delta \langle \tilde{\xi}^2 \rangle \Big/ \frac{d\langle \xi \rangle}{dy}$$

absent { – drag, – damping, – mixing (3rd order) } ⇒ { Flow locked to wave momentum density! }

non-acceleration thm!

(Charney – Drazin Thm.)

ZF's ⟺ Wave Momentum Density.

→ ~~Cannot accelerate~~ (or maintain vs drag) zonal flow without changing (decreasing) wave intensity.

Note:
$$\frac{-k_x\langle\tilde{q}^2\rangle}{2k_x\, d\langle q\rangle/dy}$$

$$q = \nabla^2\phi + \beta y$$
absent mean flow,
$$\frac{d\langle q\rangle}{dy} = \beta\ !$$
$$\frac{\langle\tilde{q}^2\rangle}{2} = k^2\mathcal{E}$$

$$\frac{-k_x k^2\mathcal{E}}{k_x\frac{d\langle q\rangle}{dy}} = \frac{k_x\mathcal{E}}{\frac{-k_x\beta}{k^2}} = \frac{k_x\mathcal{E}}{\omega_{\underline{k}}}$$
$\to$ Action Density

$$= k_x\ N_{\underline{k}}$$

$$= P_W.$$
wave momentum density

i.e. a/a/ Adiabatic Invariant

$\Rightarrow$

$$\partial_t\left\{\langle V_x\rangle - P_W\right\} = -\nu\langle V_x\rangle - \delta\langle\tilde{q}^2\rangle\Big/\frac{\partial\langle q\rangle}{\partial y} + \cdots$$