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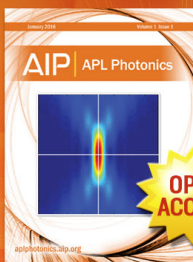
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## Plasma Diffusion in Two Dimensions

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Diffusion of plasma in two dimensions is studied in the guiding center model. It is shown that in this model diffusion always exhibits the anomalous  $1/B$  variation with magnetic field. The velocity correlation function and the diffusion coefficient are calculated in detail using functional probabilities. In addition to the  $1/B$  field dependence, the diffusion coefficient is unusual in that it depends weakly on the size of the system. These theoretical results are compared with those from computer experiments and their significance for real plasma is discussed.

### I. INTRODUCTION

It is well known that a strong magnetic field  $B$  inhibits the diffusion of plasma transverse to the field and that for plasma in local thermal equilibrium the diffusion coefficient is proportional to  $1/B^2$ . However, experiments often show a much larger anomalous diffusion and to describe this an empirical coefficient proportional to  $1/B$  (originally proposed by Bohm<sup>1</sup>) is often invoked. The Bohm diffusion coefficient is

$$D_B = \alpha(c\kappa T/eB),$$

where  $T$  is the plasma temperature and  $\alpha$  is a numerical coefficient, conventionally taken to be  $1/16$ . The origin of anomalous diffusion has been assumed to lie in fluctuating electric fields which exceed the thermal level.

In studying anomalous cross field diffusion and its dependence on magnetic field strength it is tempting to invoke two-dimensional models, particularly in computer simulations which involve computation of orbits for a large number of interacting particles. It is important, therefore, to understand the fundamental behavior of a two-dimensional plasma and the way in which it differs from real systems. In this paper we investigate two-dimensional plasma in the high magnetic field limit, when the Larmor radius is smaller than the Debye length, using the guiding center model in which particles move according to the equation of motion;

$$\mathbf{v} = [(\mathbf{B} \times \nabla\varphi)/B^2]c \quad (1)$$

with the potential  $\varphi$  determined by Poisson's equation. (This model also describes the two-dimensional motion of an incompressible inviscid fluid and some of our results may have hydrodynamic applications, but it is convenient to develop the theory solely in terms of the plasma problem.)

The results of our investigation of diffusion in a two-dimensional guiding center plasma are surprising. It is found that the diffusion coefficient always has the Bohm ( $1/B$ ) variation with magnetic field—even in thermal equilibrium. Far from being anomalous, therefore, a Bohm-like diffusion formula is the classical one!

Furthermore, the diffusion coefficient depends weakly on the size of the system.

As a test of these theoretical predictions a series of numerical experiments has been carried out. In these the orbits for several thousand interacting particles are computed and the diffusion coefficient and velocity correlation function calculated. The results are in excellent agreement with the theory but also bring out the importance of initial conditions and of statistical errors which are not reduced merely by increasing the number of particles.

The theoretical model is developed in the next three sections after which the numerical experiments are described. Finally the significance of our results for real plasmas and for other two-dimensional plasma simulations is discussed.

### II. GENERAL PROPERTIES OF GUIDING CENTER DIFFUSION

We regard the plasma as an assembly of two species of charged particles, which in two dimensions are represented by rods with a charge  $e_i/l$  per unit length ( $e_i = \pm e$ ), immersed in a uniform magnetic field  $\mathbf{B}$ . The guiding center equation of motion for the particles is

$$\dot{\mathbf{R}} = c(\mathbf{B} \times \nabla\varphi)/B^2, \quad (2)$$

and the potential  $\varphi$  is determined by

$$\nabla^2\varphi = 4\pi \sum (e_i/l)\delta(\mathbf{r}-\mathbf{R}_i), \quad (3)$$

where the sum is over all particles. Together with an initial probability distribution these equations completely determine the plasma properties. In thermal equilibrium the appropriate distribution would be

$$W\{\mathbf{R}\} = \mathcal{N} \exp \left[ - \sum_{i \neq j} (e_i e_j / l \kappa T) \log |\mathbf{R}_i - \mathbf{R}_j| \right]. \quad (4)$$

(Throughout this paper  $\mathcal{N}$  denotes the appropriate normalization.)

These equations can be reduced to a universal form by the scale transformations:

$$\mathbf{R} = s\mathbf{X}, \quad \varphi = 4\pi e\psi/l, \quad t = (s^2 B l / 4\pi e c)\tau, \quad (5)$$

in which  $s$  is an arbitrary length scale. In the scaled variables the plasma equations become

$$\frac{d\mathbf{X}_i}{d\tau} = \mathbf{b} \times \nabla_x \psi,$$

$$\nabla_x^2 \psi = \sum_{i=1}^N \left( \frac{e_i}{e} \right) \delta(\mathbf{X} - \mathbf{X}_i), \quad (6)$$

$$W\{X\} = \mathcal{N} \exp\left[-\sum_{i \neq j} (e_i e_j / 4\pi e^2) (1/n_D) \log |\mathbf{X}_i - \mathbf{X}_j|\right],$$

where  $\mathbf{b} = \mathbf{B}/B$  and  $n_D = \kappa Tl / 4\pi e^2$  is the number of particles per square Debye length ( $\lambda^2 = \kappa Tl / 4\pi n e^2$ , see Appendix B).

It is apparent from this form of the problem that the particle dynamics depends only on  $N$  and the sign of the charges,  $e_i$ . The distribution function depends on only one other parameter,  $n_D$ , so that all plasma properties can be expressed in the scaled variables as functions of  $N$  and  $n_D$  alone. Any intensive plasma property (i.e., one which is independent of plasma size) will depend only on  $n_D$ .

To make use of this result in determining the diffusion coefficient we write it in terms of the velocity correlation as

$$D = \int_0^\infty \langle v_i(0)v_i(t) \rangle dt = \frac{ckT}{eBn_D} \int \langle \dot{X}_i(0)\dot{X}_i(\tau) \rangle d\tau, \quad (7)$$

which, in accordance with the remarks above, must take the universal form

$$D = (ckT/eB)f(N, n_D). \quad (8)$$

We see, therefore, that if a diffusion coefficient exists, then even in thermal equilibrium it can only be proportional to  $1/B$ .

If the plasma is not in thermal equilibrium the diffusion coefficient will still be proportional to  $1/B$ , although the other plasma parameters cannot then be reduced to one universal quantity. To confirm the  $1/B$  variation it is only necessary to introduce the transformation

$$t = (mc/eB)\tau \quad (9)$$

into the original equations and to observe that  $B$  then disappears from the problem. Expressing the diffusion coefficient in terms of the velocity correlation as in (7) then shows it to be of the form

$$D = (mc/eB)g \quad (10)$$

where  $g$  may depend on other parameters but must be independent of the magnetic field.

### III. THE ELECTRIC FIELD CORRELATION

By introducing Fourier transforms, taken for convenience in a square of unit area, into Eq. (3) the po-

tential may be written

$$\varphi(\mathbf{r}, t) = 4\pi \sum_j (e_j/l) \sum_{\mathbf{k}} (1/k^2) \exp\{i\mathbf{k} \cdot [\mathbf{R}_j(t) - \mathbf{r}]\}, \quad (11)$$

where  $k_x/2\pi, k_y/2\pi$  take all integer values. The electric field correlation function is therefore

$$Q_{\alpha\beta}(\tau) = \langle E_\alpha(t) E_\beta(t+\tau) \rangle$$

$$= 16\pi^2 \sum_{ij} (e_i e_j / l^2) \sum_{\mathbf{k}} (k_\alpha k_\beta / k^4)$$

$$\times \langle \exp\{i\mathbf{k} \cdot [\mathbf{R}_i(t) - \mathbf{R}_j(t+\tau)]\} \rangle, \quad (12)$$

where the fact that  $Q$  is independent of  $\mathbf{r}$  has been used to simplify the last expression.

Using the guiding center equations of motion (12) becomes

$$Q_{\alpha\beta}(\tau) = 16\pi^2 \sum_{ij} \frac{e_i e_j}{l^2} \sum_{\mathbf{k}} \frac{k_\alpha k_\beta}{k^4} \left\langle \exp\{i\mathbf{k} \cdot [\mathbf{R}_i(t) - \mathbf{R}_j(t)]\} \right.$$

$$\left. \times \exp\left(\frac{-ic}{B} \int_t^{t+\tau} \mathbf{k} \times \mathbf{E} \cdot \mathbf{b} d\tau'\right) \right\rangle. \quad (13)$$

Strictly, the final integral depends on the orbit of the  $j$ th particle but we shall ignore the correlation of  $\mathbf{E}$  with the orbit and write

$$Q_{\alpha\beta}(t) = \sum_{\mathbf{k}} \left\langle E_\alpha(\mathbf{k}, 0) E_\beta^*(\mathbf{k}, 0) \right.$$

$$\left. \times \exp\left(\frac{ic}{B} \mathbf{k} \times \mathbf{b} \int_0^t \cdot \mathbf{E} d\tau\right) \right\rangle. \quad (14)$$

To evaluate the ensemble average we assume that the fluctuating electric field can be represented by a normal distribution (Appendix A). Then, the probability of  $\mathbf{E}(t)$  can be expressed in terms of the correlation  $\langle \mathbf{E} \cdot \mathbf{E} \rangle$  which we wish to calculate. This is equivalent to neglecting all higher cumulants, such as  $(\langle \mathbf{E} \cdot \mathbf{E} \cdot \mathbf{E} \cdot \mathbf{E} \rangle - \langle \mathbf{E} \cdot \mathbf{E} \rangle^2)$ . The normal distribution is also that which maximizes the entropy subject to a given correlation  $\langle \mathbf{E}(t) \mathbf{E}(t+\tau) \rangle$ . (In three dimensions a useful expression for the correlation can be obtained when the influence of the electric field on the particle orbits is entirely neglected.<sup>2</sup> Here, such a crude approximation would merely give a constant value for  $Q$ .)

The normal probability distribution for the function  $\mathbf{E}(t)$  has the form

$$P\{E\} = \mathcal{N} \exp -\frac{1}{2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E_\alpha(\tau_1) Q_{\alpha\beta}^{-1}(\tau_1 - \tau_2)$$

$$\times E_\beta(\tau_2) d\tau_1 d\tau_2, \quad (15)$$

where  $Q_{\alpha\beta}^{-1}$  is the inverse of the correlation function so

that

$$\int Q_{\alpha\beta}^{-1}(\tau_1-\tau)Q_{\beta\gamma}(\tau-\tau_2) d\tau = \delta_{\alpha\gamma} \delta(\tau_1-\tau_2). \quad (16)$$

(Summation over repeated indices is implied.) It is important to appreciate that  $P\{E\}$  is the probability that the whole time history of the electric field shall be given by the function  $\mathbf{E}(t)$ . Equivalently one may divide the time up into a large number of discrete intervals and specify the probability for the whole set of  $\mathbf{E}(t_i)$  as

$$P\{E_i\} = \mathcal{N} \exp\left(-\frac{1}{2} \sum_{ij} E_i Q_{ij}^{-1} E_j\right). \quad (17)$$

Using the weighting functional (15) one finds, after some manipulation that the average

$$\begin{aligned} &\left\langle E_\alpha(\mathbf{k}, 0) E_\beta^*(\mathbf{k}, 0) \exp\left(\frac{ic}{B} \mathbf{k} \times \mathbf{b} \cdot \int_0^t \mathbf{E}(\tau) d\tau\right) \right\rangle \\ &= \langle E_\alpha(\mathbf{k}) E_\beta^*(\mathbf{k}) \rangle \\ &\times \exp\left(-\frac{c^2 \hat{k}_\alpha \hat{k}_\mu}{2B^2} \int_0^t \int_0^t Q_{\alpha\mu}(\tau_1-\tau_2) d\tau_1 d\tau_2\right), \quad (18) \end{aligned}$$

where for brevity  $\hat{\mathbf{k}} \equiv (\mathbf{k} \times \mathbf{b})$ . We therefore obtain the important equation for  $Q$

$$\begin{aligned} Q_{\alpha\beta}(t) &= \sum_{\mathbf{k}} \langle E_\alpha(\mathbf{k}) E_\beta^*(\mathbf{k}) \rangle \\ &\times \exp\left(-\frac{c^2 \hat{k}_\alpha \hat{k}_\mu}{2B^2} \int_0^t \int_0^t Q_{\alpha\mu}(\tau_1-\tau_2) d\tau_1 d\tau_2\right) \quad (19) \end{aligned}$$

which specifies the *time-dependent* correlation function in terms of the *stationary* average field fluctuation. When the fluctuations are isotropic, as in the cases we shall discuss later,  $Q_{\alpha\beta} = Q\delta_{\alpha\beta}$  and Eq. (19) simplifies to

$$\begin{aligned} Q(t) &= (4\pi)^{-1} \int k dk \langle |E^2(k)| \rangle \\ &\times \exp\left(-\frac{c^2 k^2}{2B^2} \int_0^t \int_0^t Q(\tau_1-\tau_2) d\tau_1 d\tau_2\right), \quad (20) \end{aligned}$$

where we have also taken the opportunity to replace the sum over  $k$  by an integral, using the substitution  $(2\pi)^2 \sum \rightarrow \int d\mathbf{k}$ . This will enable us to give simple analytic forms for the diffusion coefficient and correlation function.

For the purposes of calculation Eq. (20) can be put in a much more convenient form by introducing

$$R(t) = \frac{c^2}{2B^2} \int_0^t \int_0^t Q(\tau_1-\tau_2) d\tau_1 d\tau_2. \quad (21)$$

This is given by the differential equation

$$\frac{d^2 R}{dt^2} = \frac{c^2}{4\pi B^2} \int_a^b k dk \langle |E^2(k)| \rangle \exp[-k^2 R(t)] \quad (22)$$

which has a first integral

$$\frac{1}{2} \left(\frac{dR}{dt}\right)^2 = \frac{c^2}{4\pi B^2} \int_a^b \frac{dk}{k} \langle |E^2(k)| \rangle [1 - \exp(-k^2 R)]. \quad (23)$$

This last form is convenient for numerical calculation. The upper and lower limits of integration are the shortest and longest wavelengths in the fluctuating field:  $a \equiv k_{\min}$ ,  $b \equiv k_{\max}$ . The longest wavelength is clearly limited by the size of the system.

The function  $R(t)$  can be computed from (23) when the fluctuating spectrum  $E^2(k)$  is given (see below) and provides complete information on the internal dynamics of the plasma. The correlation function is given by the second derivative of  $R$ ,

$$Q(t) = \frac{B^2}{c^2} \frac{d^2 R}{dt^2}, \quad (24)$$

and the diffusion coefficient by the first derivative

$$D = \lim_{t \rightarrow \infty} \left(\frac{dR}{dt}\right). \quad (25)$$

$R(t)$  itself gives the mean dispersion of a group of diffusing particles:

$$R(t) = \frac{1}{2} [\Delta r(t)]^2. \quad (26)$$

#### IV. THE DIFFUSION COEFFICIENT AND CORRELATION FUNCTION

The diffusion coefficient can be found without the need of a full solution of (23). For  $R(t)$  must be unbounded as  $t \rightarrow \infty$  and, therefore, at sufficiently long times  $a^2 R(t) \gg 1$ . In this limit Eq. (23) immediately gives

$$D^2 = \frac{c^2}{2\pi B^2} \int_a^b \frac{dk}{k} \langle |E^2(k)| \rangle. \quad (27)$$

We shall evaluate this in two cases.

##### A. Thermal Equilibrium

In thermal equilibrium the spectrum of electric field fluctuations is given by (see Appendix B):

$$\langle |E^2(k)| \rangle = (4\pi/l) [\kappa T / (1 + k^2 \lambda^2)] \quad (28)$$

so that

$$D = (c\kappa T / eB) [(2\pi n_D)^{-1} \log(L/2\pi\lambda)]^{1/2} \quad (29)$$

which is the diffusion coefficient for a two-dimensional guiding center plasma in thermal equilibrium. It has the form predicted earlier, proportional to  $\kappa T/B$ , and we now see that the function  $f$  is

$$f(n_D, N) \equiv [(2\pi n_D)^{-1} \log(N/n_D)]^{1/2}. \quad (30)$$

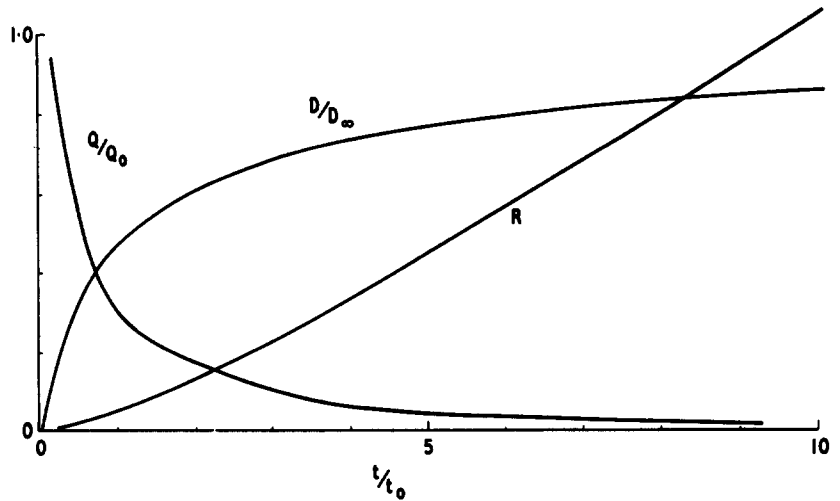


FIG. 1. Theoretical correlation and diffusion for thermal plasma.

A feature of the result is that  $D$  is not an intensive quantity; instead it increases indefinitely (but very slowly) with the size of the system. The correlation function and the dispersion may be computed from Eq. (23). Figure 1 shows typical results. The normalizing factors are

$$Q_0 = 4\pi(nc^2/l^2) \log(k_{max}\lambda) \quad (31)$$

and

$$t_0 = \lambda Bl/\pi^{1/2}cn^{1/2}e, \quad (32)$$

while  $D_\infty$  has the value given by (29). It is interesting to note that the diffusion coefficient (29) does not suffer from a divergence at large wavenumber. The correlation function does diverge at  $t=0$  if  $k_{min} \rightarrow 0$ , but is integrable, and initially has the form

$$Q(t) = Q(0) - Q^2(0)t. \quad (33)$$

**B. Random Distribution**

The process of relaxation to thermal equilibrium may be slow and it is, therefore, of interest to consider other

distributions. Of these, the random distribution is an obvious choice, again particularly relevant to computer experiments where it is difficult to simulate the thermal distribution. The fluctuating spectrum for a random distribution of particles (Appendix B) is

$$\langle |E^2(k)| \rangle = (16\pi^2 n^2/l^2) (1/k^2) \quad (34)$$

and in this case

$$D = ecn^{1/2}L/\pi^{1/2}lB \quad (35)$$

which depends strongly on the size of the system.

The correlation function and dispersion for the random distribution may be computed as before and typical results are shown in Fig. 2. The normalizing factors are now

$$Q_0 = 4\pi(nc^2/l^2) \log(k_{max}/k_{min}), \quad (36)$$

$$t_0 = BlL/(2\pi)^{3/2}cen^{1/2} \quad (37)$$

with  $D_\infty$  given by Eq. (35).

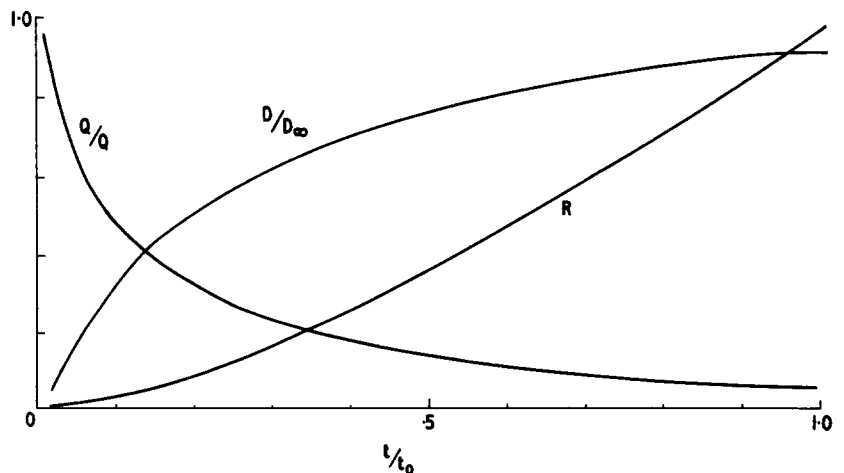


FIG. 2. Theoretical correlation and diffusion for random distribution.

## V. SOME NUMERICAL EXPERIMENTS

### A. The Computer Programs

To check the theory presented above a number of computer experiments have been carried out. In these simulations the orbits of several thousand particles are computed, using Eq. (6), in a square with periodic boundary conditions. From these orbits diffusion coefficients and correlation functions are subsequently computed by appropriate analysis programs.

Two distinct computer models were used, both based on programs described elsewhere. In the first, a modification of GALAXY,<sup>3</sup> the charge of each particle is ascribed to the nearest mesh point and Poisson's equation is solved by a Fourier transform technique in both coordinates. The advantages of this program are that it can accommodate a very large number of particles (over 16 000 in our calculation) and that the Fourier transform technique allows one to retain an optional number of Fourier modes so that the numerical experiment has a direct equivalent to the maximum and minimum wavenumbers introduced in the analytic theory. However, at least in our modified form, GALAXY is less accurate than the alternative program.

The second program VORTEX<sup>4</sup> was originally written to study vortex motion of fluids in two dimensions, but as we have noted the equations for this are identical to the guiding center Eqs. (6) if  $e$  is interpreted as the vortex strength. The VORTEX code uses more accurate integration methods than the modified GALAXY, and the charge density on the mesh is obtained by area weighting. The Poisson equation is solved by a Fourier-transform technique in one space direction and a cyclic reduction in the other. As a result of these more elaborate numerical techniques, the VORTEX code is more accurate than GALAXY, but it cannot deal with more than about 3000 particles and there is no explicit introduction of a cutoff in the Fourier modes as in GALAXY and the analytic theory. However, a comparison of results from GALAXY and from VORTEX, and a direct analytic assessment, show that the area-weighting technique of VORTEX is equivalent to retaining 10–12 exponential Fourier modes in each direction (i.e., 100–150 modes altogether).

### B. Initial Conditions and Fluctuations

A feature of the numerical experiments is their dependence on initial conditions. Although it is easy to simulate a thermal *velocity* distribution, the important *spatial* correlations, essential to our computation, are difficult to simulate correctly. We have, therefore, carried out our computations, and the comparison with theory, for the random distribution of Sec. IVB.

The influence of initial conditions in this type of numerical experiment is reflected in the fact that, even with more than 3000 particles, the statistical accuracy

of the results is poor with wide fluctuations from one run to another. These fluctuations are not reduced by increasing the number of particles. The existence of such statistical fluctuations can be seen from elementary considerations (cf. Appendix B). For if the charge density is

$$\rho_{\mathbf{k}} = \sum_i (e_i/l) \exp(i\mathbf{k} \cdot \mathbf{R}_i), \quad (38)$$

then for a random distribution:

$$\langle |\rho_{\mathbf{k}}|^2 \rangle = N(e^2/l^2) \quad (39)$$

and

$$\langle |\rho_{\mathbf{k}}|^4 \rangle = (2N^2 - N)(e^4/l^4), \quad (40)$$

where  $N$  is the number of particles. The relative fluctuation in  $|\rho_{\mathbf{k}}|^2$  is, therefore,

$$\langle [|\rho_{\mathbf{k}}|^2 - \langle |\rho_{\mathbf{k}}|^2 \rangle]^2 \rangle / \langle |\rho_{\mathbf{k}}|^2 \rangle^2 = (1 - N^{-1}), \quad (41)$$

and does not tend to zero as  $N$  is increased. However, the fluctuations can be reduced by taking the average of a number of independent runs, according to the usual rules for the statistics of independent events.

### C. Diffusion and Correlation Coefficient

Our main numerical experiments were 16 independent runs, using the VORTEX code, each of which followed 3072 particles on a 64×64 mesh. Supplementary experiments using GALAXY were done with up to 16 000 particles. At each time step in the calculation the  $x$  component of velocity of each particle was recorded and subsequent analysis yielded the average velocity correlation function  $Q(\tau)$  and diffusion coefficient

$$D(\tau) = \int_0^\tau Q(\tau) d\tau.$$

Each run occupied about 250 time steps.

## VI. COMPARISON WITH THEORY

The theory has been developed in terms of physical variables, but it was pointed out that the equations of motion become parameter free on introducing one arbitrary scale  $s$  which in the numerical experiments is taken to be  $L/64$  as the programs use a 64×64 mesh on which to solve Poisson's equation. Apart from statistical effects and a weak dependence on the number of modes retained in solving Poisson's equation, all possible experiments are thus embodied in a single calculation. This one calculation therefore provides a complete test of the theory.

Before the comparison can be made however, a minor modification of the theory is required. Because the experiment involves only 10 or 12 Fourier modes in each direction, the replacement of sums over wavenumber by integrals is not sufficiently accurate, and the theory

TABLE I. Comparison of experimental and theoretical results.

	Numerical experiment	Theoretical value (10 modes)	Theoretical value (12 modes)	Theoretical value (integral)
$D(\infty)$	$6.37 \pm 1.82$	6.16	6.10	4.55
$Q(0)$	$0.197 \pm 0.034$	0.207	0.203	0.238
$\tau_{1/2}$	$12.5 \pm 3.0$	12.13	10.4	8.23

must be written using summations rather than integrals. The basic equation, in the dimensionless variables  $X$ ,  $\tau$  is then,

$$\left(\frac{dR}{d\tau}\right)^2 = \frac{2N}{(64)^2} \sum_{k,l} \frac{1}{(k^2+l^2)^2} \{1 - \exp[-(k^2+l^2)R(\tau)]\}. \tag{42}$$

In the same dimensionless variables the initial correlation function  $Q(0)$  and the diffusion coefficient become

$$Q(0) = \frac{N}{64^2} \frac{1}{4\pi^{1/2}} \sum_{k,l} \frac{1}{(k^2+l^2)} \tag{43}$$

and

$$D(\infty) = \frac{1}{4\pi} \left( \frac{N}{\pi} \sum_{k,l} \frac{1}{(k^2+l^2)^2} \right)^{1/2}. \tag{44}$$

In Figs. 3 and 4 we compare the results of the main computer experiments with the analytic theory as given by (42). We have made the comparison in terms of two quantities, the correlation function  $Q(\tau)$  and the running diffusion coefficient

$$D(\tau) = \int_0^\tau Q(\tau') d\tau'.$$

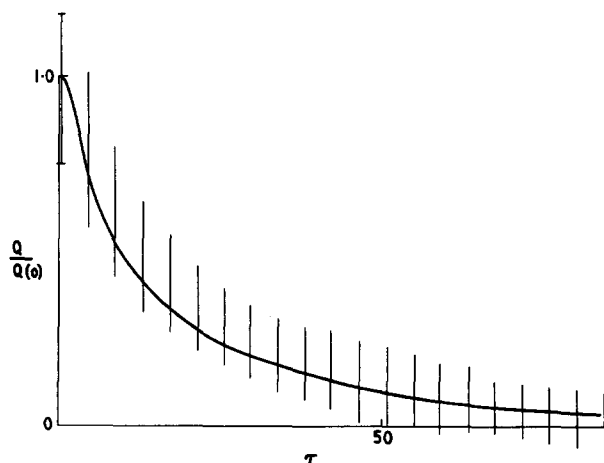


FIG. 3. Experimental and theoretical correlation functions.

The error bars on the results from the computer experiment represent one standard deviation on each side of the mean of 16 runs. The experimental and theoretical value for the salient parameters of these curves are summarized in Table I which provides a concise indication of the over-all agreement between theory and experiment. ( $\tau_{1/2}$  is the time for the correlation to fall to half its initial value.) It can be seen from this and the figures that the theory and computer experiment are in very good agreement, any discrepancies lying well within the statistical errors. In assessing this agreement it should be recalled that there were *no* adjustable parameters or normalizing factors in the theory, which is entirely self-contained and derived from basic principles.

The close agreement of theoretical and experimental values of  $Q_0$  is to be expected and represents a check on the VORTEX program. The agreement of the correlation decay time is the most sensitive test of the over-all theory. The agreement of the asymptotic diffusion coefficients confirms indirectly that the diffusion depends on the size of the system. The other important theoretical result, that the diffusion is proportional to  $1/B$ , is a consequence of the scaling laws alone and is independent of the need for experimental confirmation.

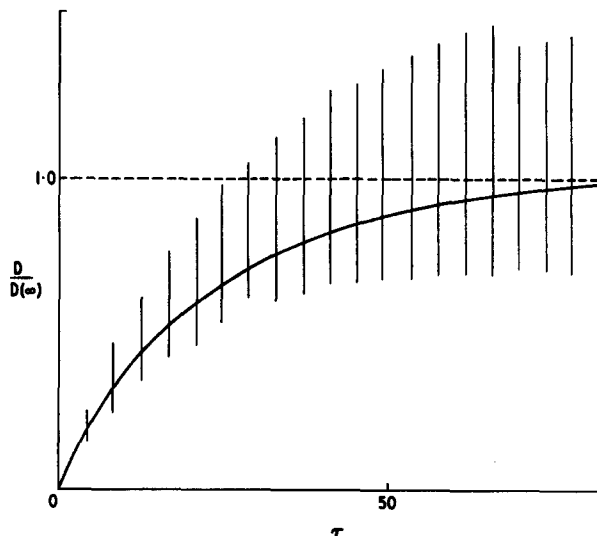


FIG. 4. Experimental and theoretical diffusion.

VII. DISCUSSION

A theory of two-dimensional plasma based on the guiding center equations of motion has been developed and the velocity correlation function and diffusion coefficient, have been calculated. In this model diffusion must always be proportional to  $1/B$  and depends on the size of the system. In thermal equilibrium the size dependence is very weak, the diffusion coefficient being

$$D = \frac{c\kappa T}{eB} \left( \frac{1}{2\pi n\lambda^2} \log \frac{1}{2\pi\lambda} \right)^{1/2}.$$

These results must throw some doubt on the usefulness of two-dimensional computer calculations as a mean of investigating anomalous diffusion in experiments. On the other hand, they make the study of two-dimensional systems an interesting problem in its own right, particularly as the theory yields detailed predictions without arbitrary parameters.

Whether the theory is relevant to real plasma is more speculative. Clearly, an equilibrium plasma differs greatly from our two-dimensional model, but there are circumstances in which real plasma may behave very like a two-dimensional system. This is because equilibrium in a plasma is attained quickly along the lines of force but more slowly perpendicular to them. Furthermore, as all charges on a given flux tube tend to remain together, the flux tubes and the particles on them retain their identity. If, therefore, a plasma acquires an imbalance of charge between various flux tubes, e.g., during its formation, or as a result of passing through an unstable phase, then these tubes would behave exactly as the charged rods of our model. (It may also be noted that two-dimensional behavior depends mainly on the longer wavelength fluctuations, which are the slowest to disperse and for which the guiding center approximation is most accurate.) Unfortunately, it is not possible to predict the charge acquired by each flux tube so the diffusion coefficient in such a situation can only be expressed in terms of the potential fluctuations. Using Eq. (27) it then takes the form

$$D = (\sqrt{2} c/B) \langle \varphi^2 \rangle^{1/2}.$$

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APPENDIX A. THE NORMAL DISTRIBUTION FOR A FUNCTION

The normal distribution for a finite set of independent variables  $X_i$  is

$$P = \mathcal{N} \exp \left[ -\sum \frac{1}{2} (X_i^2 / \sigma_i) \right], \quad (A1)$$

a form which can be justified by the central limit theorem. If the  $X_i$  are not independent but have finite correlations,  $\langle X_i X_j \rangle = \sigma_{ij}$ , then (A1) generalizes to

$$P = \mathcal{N} \exp \left[ -\sum \frac{1}{2} (\sigma_{ij})^{-1} X_i X_j \right]. \quad (A2)$$

If we consider a transition from the sequence of  $\{X_i\}$  to a continuous function  $X(t)$ , then Eq. (A2) goes over to the functional probability

$$P = \mathcal{N} \exp \left( -\frac{1}{2} \int X(t_1) \sigma^{-1}(t_1, t_2) X(t_2) dt_1 dt_2 \right). \quad (A3)$$

For systems which are time invariant,

$$\sigma^{-1}(t_1, t_2) = \sigma^{-1}(t_1 - t_2).$$

Equation (A3) gives, in a formal sense, the probability of the function  $X(t)$  occurring.

A property of Eq. (A3) is that it yields a normal distribution for the probability of  $X$  at any given instant, i.e.,

$$P(X) = \mathcal{N} \exp(-X^2/2\langle X^2 \rangle). \quad (A4)$$

Note the distinction between (A4), the probability that the value  $X$  shall occur at some chosen instant, and (A3), the probability that  $X$  shall be specified by the function  $X(t)$  for all times.

APPENDIX B. SPECTRUM OF FLUCTUATIONS OF TWO-DIMENSIONAL PLASMA

(i) Random Distribution

It is convenient to first treat the case of a random distribution of particles. If  $\rho_{\mathbf{k}}$  is the Fourier transform of the charge distribution and  $r_{\mathbf{k}}^2 \equiv |\rho_{\mathbf{k}}|^2$ , then

$$\langle r_{\mathbf{k}}^2 \rangle \equiv \langle \rho_{\mathbf{k}} \rho_{\mathbf{k}}^* \rangle = \left\langle \sum_{ij} (e_i e_j / l^2) \exp[i\mathbf{k} \cdot (\mathbf{R}_i - \mathbf{R}_j)] \right\rangle \quad (B1)$$

so that for a random distribution

$$\langle r_{\mathbf{k}}^2 \rangle = n e^2 / l^2, \quad (B2)$$

and the electric field fluctuations are

$$\langle |E^2(\mathbf{k})| \rangle = (16\pi^2 n e^2 / l^2) (1/k^2). \quad (B3)$$

It is convenient to write the full distribution for  $r_{\mathbf{k}}^2$ , which is most easily done by considering all its moments. The odd moments vanish and for large  $n$  the even moments are

$$\langle r_{\mathbf{k}}^{2m} \rangle = m! (n e^2 / l^2)^m, \quad (B4)$$

so that the distribution for  $r_{\mathbf{k}}^2$  must be

$$P(r_{\mathbf{k}}^2) dr_{\mathbf{k}}^2 = \mathcal{N} \exp \left[ -\frac{r_{\mathbf{k}}^{2l}}{2n e^2} \right] dr_{\mathbf{k}}^2. \quad (B5)$$



(ii) Thermal Distribution

In thermal equilibrium the probability of any configuration  $\mathbf{R}_i$  is given by the usual Gibbs function of the energy of that configuration, and the problem of finding the fluctuation spectrum is simply that of transforming from a description in terms of the  $\mathbf{R}_i$  to a description in terms of the  $\rho_{\mathbf{k}}$ . As both species are involved the charge density fluctuations alone do not completely specify the configuration, and one must introduce a  $\rho_{\mathbf{k}}^i$  and  $\rho_{\mathbf{k}}^e$  for ions and electrons separately. In a random distribution,  $\rho^i$  and  $\rho^e$  are each distributed according to (B5) and all cross correlations are zero.

To transform from  $\{\mathbf{R}_i\}$  to  $\{\rho_{\mathbf{k}}^e, \rho_{\mathbf{k}}^i\}$  we need the Jacobian of the transformation from  $R \rightarrow \rho$ . However, this is exactly the same quantity as the distribution of the  $\rho_{\mathbf{k}}$  when the  $\mathbf{R}_i$  are uniformly distributed,<sup>5</sup> and this we have already calculated.

The energy may be expressed in terms of the  $\rho_{\mathbf{k}}$  as

$$W = \sum (2\pi l/k^2) |\rho_{\mathbf{k}}|^2 = \sum (2\pi l/k^2) r_{\mathbf{k}}^2, \quad (\text{B6})$$

so that combining this with the Jacobian, the thermal

equilibrium distribution for  $r_{\mathbf{k}}^2$  is

$$P(r_{\mathbf{k}}^2) dr_{\mathbf{k}}^2 = \exp \left[ - \left( \frac{2\pi l}{k^2 \kappa T} + \frac{l^2}{2m e^2} \right) r_{\mathbf{k}}^2 \right] dr_{\mathbf{k}}^2. \quad (\text{B7})$$

Therefore the fluctuations of  $r_{\mathbf{k}}^2$  are

$$\langle r_{\mathbf{k}}^2 \rangle = [k^2 \kappa T / 4\pi l (1 + k^2 \lambda^2)], \quad (\text{B8})$$

where the two-dimensional Debye length is defined by

$$\lambda^2 = \kappa T l / 4\pi n e^2. \quad (\text{B9})$$

The corresponding electric field fluctuations are

$$\langle |E_{\mathbf{k}}|^2 \rangle = 4\pi \kappa T / l (1 + k^2 \lambda^2). \quad (\text{B10})$$

<sup>1</sup>D. Bohm, in *The Characteristics of Electrical Discharges in Magnetic Fields*, edited by A. Guthrie and R. Wakerling (McGraw-Hill, New York, 1949), p. 201.

<sup>2</sup>J. B. Taylor, *Phys. Fluids* **3**, 792 (1960).

<sup>3</sup>J. P. Boris, in *Proceedings of Culham Conference on Computational Physics* (H. M. Stationery Office, London, 1969), Vol. 2, p. 41.

<sup>4</sup>J. P. Christiansen, in *Proceedings of Culham Conference on Computational Physics* (H. M. Stationery Office, London, 1969), Vol. 2, p. 51.1.

<sup>5</sup>S. F. Edwards, *Phil. Mag.* **3**, 119 (1958).