→ Turbulence

→ a crucial example in scaling and self-similarity is turbulence

→ self-similarity:

~ phenomenon 'looks the same' over a range of scales ℓ

$$\underset{\text{inner}}{\ell_i} < \ell < \underset{\text{outer}}{\ell_o}$$

~ 'looks same':

$$\rho(r,t) = \rho(r/R(t))$$

$$R(t) = \sigma t^{2/5}$$

$$t \to \alpha t$$

$\Rightarrow$ $r \to \alpha^{2/5} r$ leaves ρ invariant

~ power law dependence is symptom

i.e. $\delta V \sim \epsilon^{1/3} \ell^{1/3}$

$$\ell \to \alpha \ell \Leftrightarrow \delta V \to \alpha^{1/3} \delta V.$$

Some examples:

① cascade: hierarchical fragmentation - "shattering" → 3D fluid turbulence

② aggregation: ("inverse cascade")
→ colloidal aggregation, a la Schmoluchowski

③ Fractals and β-model:
→ meaning of dimension
→ fractals

① Fluid Turbulence (c.f.: Frisch)

What is it?

- spatio-temporal 'disorder'

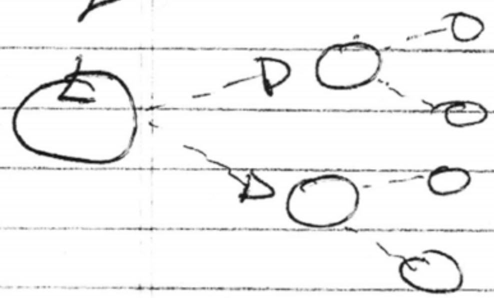
- broad range of space-time scales

- ⊛ power transfer thru broad range scales

- ⊛ energy dissipation

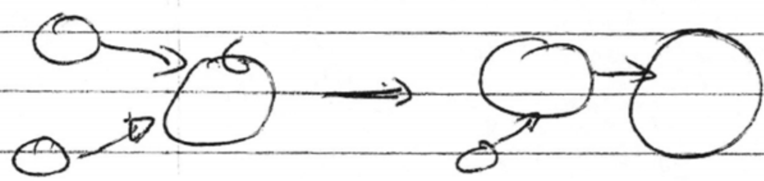
- can 'view' as consisting of sequence of basic interactions

i.e.



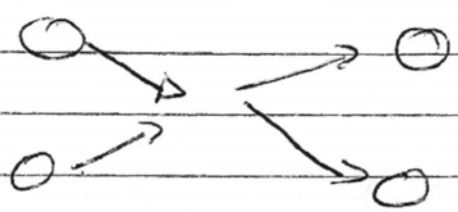
cascade
→ fragmentation sequence
→ # eddys increase.

vs. aggregation / inverse cascade



aggregates decreases
size increases

vs. plain vanilla collision



particles conserved

More characteristics:

- decay of large scales
- irreversible mixing
- can be intermittent/bursty

Key parameter: $Re = v(L)\, L/\nu$

Reynolds #

$$l_o/l_i \sim Re^{\alpha} \qquad \alpha = 3/4$$

For atmospheric turbulence: BL on hot day

$Re \sim 10^8$

$l_{out} \sim$ few km

$l_{in} \sim$ few mm

Laws (Empirical)

- recall

$$F_d \sim C_D A \rho V^2$$

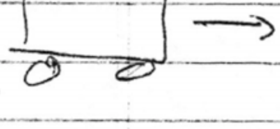
$C_D = C_D(Re)$ Flat in turbulent regime

$\Rightarrow$

- Finite Energy Dissipation Rate

If, in experiment on turbulent flow, all control parameters kept the same except viscosity, which is lowered as much as possible, energy dissipation per mass dE/dt approaches a finite limit

Simple Terms: Energy dissipation is due to viscosity yet does not depend explicitly on it!



recall $F_d \sim c_D \rho S_A u^2$

$$dE/dt \sim F_d u \sim \underbrace{c_D \rho}_{\text{const}} S_A u^3$$

$$d\mathcal{E}/dt \sim u^3/\ell$$

so $\boxed{\frac{dE}{dt} \sim u^3/\ell} \equiv \epsilon$

macroscopic length scale

Where does energy go?

$\Rightarrow$ viscous dissipation!

i.e. imagine large scale forcing

$\nabla \cdot \underline{v} = 0$

advection $\rightarrow$ no net effect

$$\partial_t \langle \tilde{v}^2 \rangle + \langle \nabla \cdot (\underline{v}\,\underline{v}) \rangle = -\nu \langle (\nabla \underline{v})^2 \rangle - \langle \nabla \cdot (\underline{v} P) \rangle + \langle \underline{f} \cdot \underline{v} \rangle$$

pressure – no net effect

st $\Rightarrow$ $\nu \langle (\nabla \underline{v})^2 \rangle = \langle \underline{f} \cdot \underline{v} \rangle$

Now, necessarily $\langle \underline{f} \cdot \underline{v} \rangle = \epsilon$

so

$$\epsilon = \nu \langle (\nabla v)^2 \rangle \quad \rightarrow \text{ balance}$$

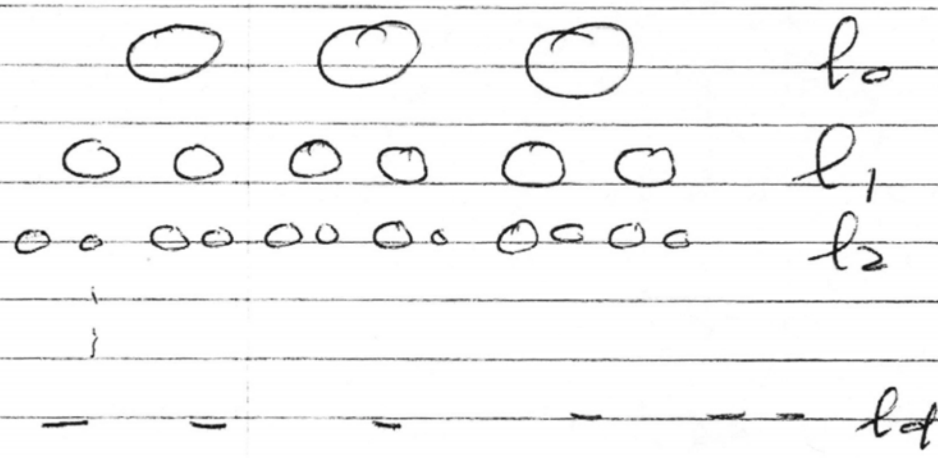
$\downarrow$ indep ν $\Rightarrow$

$$(\nabla v)_{rms} \sim 1/\nu^{1/2}$$

$\Rightarrow$ turbulence forms <u>singular</u> velocity gradients

$\Rightarrow$ must necessarily access small scales.

<u>How</u>: Cascade $\rightarrow$ hierarchical fragmentation



~ again empirical $\Rightarrow$ broad range of scales, with no gaps

How described? → structure functions!

$$\delta v(\ell) = \left| \left(\underline{v}(\underline{r}+\underline{\ell}) - \underline{v}(\underline{r})\right) \cdot \frac{\underline{\ell}}{|\ell|} \right|$$

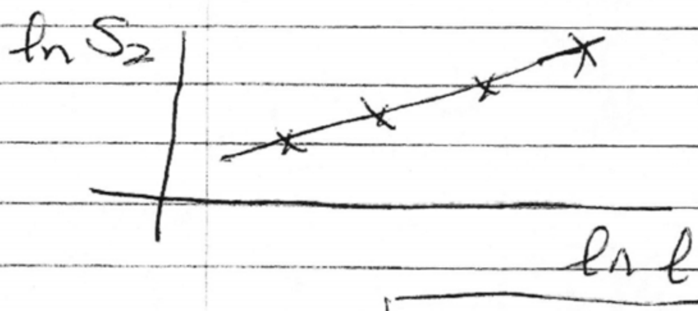
difference in velocity across scale ℓ

$$\Rightarrow \langle \delta v(\ell)^2 \rangle \quad \ldots \quad \langle \delta v(\ell)^n \rangle$$

related energy distribution in scale

⇒ 2/3 Law (Empirical)

$$S_2(\ell) = \langle (\delta v(\ell))^2 \rangle \sim \ell^{2/3}$$



→ Rigorous: $\boxed{\langle (\delta v(\ell))^3 \rangle = -\frac{4}{5}\,\epsilon\,\ell}$

4/5 Law.

~ energetics

→ What's the Story?

– K41 (Kolmogorov Phenomenology)

Ideas:

– Flux of energy in scale space from l_0 (input/integral scale) to l_d (dissipation scale – set by ν).

– energy flux is same at all scales between l_0, l_d ⟺ self-similarity

– energy dissipation – set as $\nu \to 0$ but not $= 0$

– symmetry breaking of stirring, etc. lost ⇒ symmetry restored.

Ingredients / Players

→ exciton → eddy

→ l: scale parameter, eddy scale

→ $v(\ell)$ $\quad v(\ell) \sim \langle \delta v_{\parallel}(\ell)^2 \rangle^{1/2}$

$$\delta v_{\parallel} \sim \left[v(\underline{r}+\underline{\ell}) - v(\underline{r}) \right] \cdot \frac{\underline{\ell}}{\ell}$$

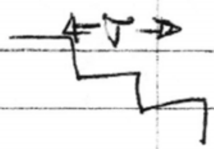
velocity increment on ℓ

→ V_0 : rms eddy fluctuation (large scale dominates)

$V(\ell_0) \sim V_0$

→ $\tau(\ell)$: eddy transfer / life time / turn-over rate

⇒ characteristic scale of transfer in cascade step



Now, self-similarity ⇒ constant flow-thru rate:

$\epsilon = v(\ell)^2 / \tau(\ell)$.

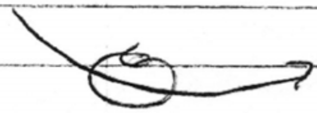
$\tau(\ell)$?

$T(\ell)$:

- dimensionally → 'lifetime' of structure of scale ℓ
→ time to distort out of existance.

For scale ℓ, which ℓ' affect lifetime?

- $\ell' \gg \ell$ (?)



advect eddy, but don't distort it

⇒ irrelevant - physics not change under random Galilean boost.

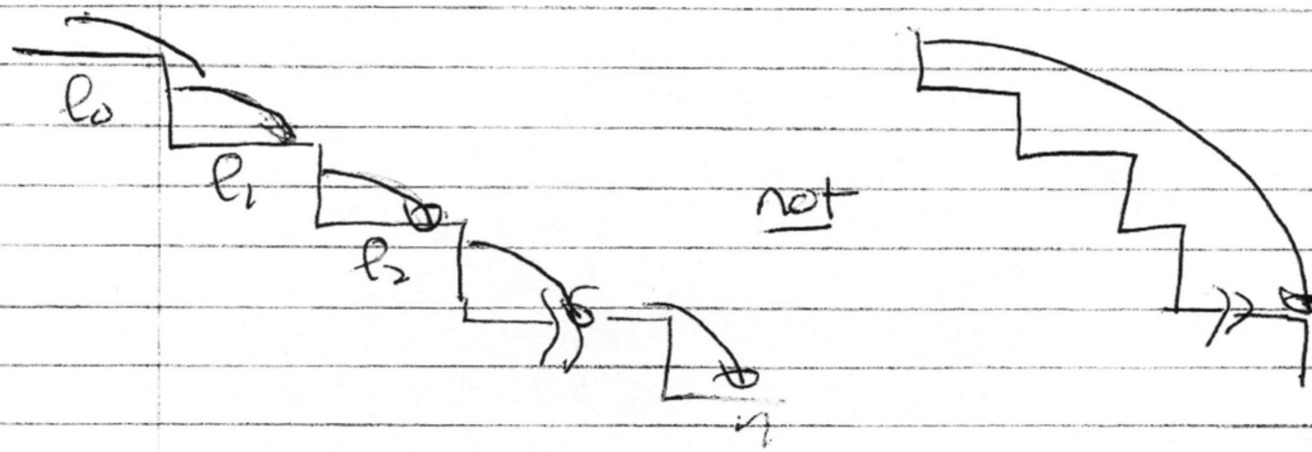
⇒ violates symmetry restoration

- scales $\ell' \ll \ell$
~ irrelevant, as very little energy / shear in such eddys

$\Rightarrow$

- strongest interaction on $\ell' \sim \ell$. Comparable scales distort one another

cascade ⓘ local in scale!



so $\quad T(\ell) \sim \ell / v(\ell)$

$$\Rightarrow \quad \epsilon \sim \frac{v(\ell)^2}{T(\ell)} \sim \frac{v(\ell)^3}{\ell}$$

i.e.

$$\frac{v_0^3}{\ell_0} \sim \frac{v(\ell)^3}{\ell}$$

$$\boxed{v(\ell) \sim (\epsilon \ell)^{1/3}}$$

$$\boxed{\begin{aligned} V(\ell)^2 &\sim \epsilon^{2/3}\,\ell^{2/3} \\ &\sim V_0^2\,(\ell/\ell_0)^{2/3} \end{aligned}}$$

- Power law
- recovers '2/3 Law'
- dependence on ℓ_0, V_0 only via ϵ.

For k spectrum:

if $\quad E(k) = |V(k)|^2$

s/t $\quad E = \int dk\, |V(k)|^2 = \int dk\, E(k)$

i.e. absorb density of states

then

$$V(\ell)^2 = \int_{k_{\ell_{n-1}}}^{k_{\ell_n}} dk\, E(k)$$

$$v(\ell)^2 \sim \epsilon^{2/3} \ell^{2/3} \sim \epsilon^{2/3} k_\ell^{-2/3}$$

$$\boxed{E(k) \sim \epsilon^{2/3} k^{-5/3}}$$

Kolmogorov spectrum.

N.B.: $1/\tau(\ell) \sim v(\ell)/\ell \sim \epsilon^{1/3}/\ell^{2/3}$

transfer rate increases as scale decreases.

finite time to end!

i.e. total time:

$$T = \sum_{n=0}^{\infty} \tau_n$$

$$= \sum_{n=0}^{\infty} \frac{\ell_0}{v_0} \left(\frac{\ell_n}{\ell_0}\right)^{2/3}$$

$\ell_n/\ell_0 \sim \alpha^n$ $\qquad$ $\alpha < 1$

$$T = \sum_{n=0}^{\infty} \frac{\ell_0}{v_0} \alpha^{2n/3}$$

$$T \sim \frac{l_0}{v_0} \Big/ 1 - \alpha^{2/3}$$

$\Rightarrow$ τ_0 sets cascade time

— cascade can go thru ∞ # steps in finite time

— hence analogy with "shattering"

$\rightarrow$ For dissipation scale l_d

— occurs when viscous diffusion kicks in and cuts-off cascade

— $1/\tau(l) \sim \nu/l^2$ $\rightarrow$ diffusive and NL time scales cross

— $\epsilon^{1/3}/l^{2/3} \sim \nu/l^2$

$$\boxed{l_d \sim \nu^{3/4}/\epsilon^{1/4}}$$

$\rightarrow$ dissp^n scale

→ Finally

$$\# \text{DOF's} \sim \left(\ell_0/\ell_i\right)^3$$
$$\sim \left(\ell_0/\ell_d\right)^3$$
$$\sim \left(Re^{3/4}\right)^3 \sim Re^{9/4}.$$

For $\ell_0 \sim 1\,\text{km}$, $\ell_d \sim 1\,\text{mm}$ $\Rightarrow N \sim 10^{18}$.

N.B.: What is missing?