Stoch Fields, cont'd

Exercises (suggested):

i.) Derive the magnetic diffusivity with magnetic drifts. How do these modify D_M? Explain why high energy particles (runaways) are confined longer than thermals.

ii.) Formulate the theory of diffusion due stochastic fields in toroidal geometry using ballooning mode formalism for the fluctuations

iii.) What happens to net cross field transport in a standing spectrum of e.s. and magnetic perturbations. When might transport vanish?, why?

→ Collisional Regime

Here: $l_{ac} < l_{mfp} < l_c$

(short mean free path)

Point: ⇒ $l_{mfp} < l_c$ ⇒ particle random walks parallel and undergoes many kicks in l_c. So parallel motion is diffusive.

→ perpendicular motion is continuous coarse graining/spreading, at $D_\perp \sim \rho_e^2 \nu_{ee} \sim \rho_e^2 \frac{V_{the}}{l_{mfp}}$

So, can write:

$$\langle \delta r^2 \rangle \sim D_M \, l_{c,d}$$

↓

parallel correlation length
(d signifies diffusive regime)

but also note that parallel motion is diffusive, so:

but time set by

$$\chi_\parallel / l_{c,\delta}^2 \sim 1/t$$

$$\Rightarrow \frac{\langle \delta r^2 \rangle}{t} \sim \frac{\chi_\parallel}{l_{c,\delta}^2} D_M l_{c,\delta}$$

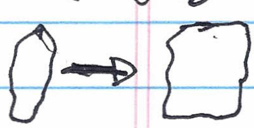
$$\sim D_M \frac{\chi_\parallel}{l_{c,\delta}} \sim D_M \chi_\parallel / l_{c,\delta}$$

$$\boxed{\chi_\perp = D_M \frac{\chi_\parallel}{l_{c,\delta}}} \Rightarrow$$ perpendicular heat conductivity in collisional regime.

Now, what is $l_{c,\delta}$?

Notice $l_{c,\delta}$ is set by competition between 2 processes:

① width δ increases due diffusion (coarse graining)



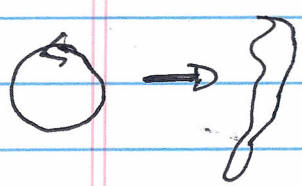
so
$$(d\delta)^2 \sim (D_\perp dt)$$
$$d\delta \sim (D_\perp dt)^{1/2}$$

but
$$\chi_\parallel/(dL)^2 \sim 1/dt$$

$\Rightarrow$
$$d\delta \sim \left(\frac{D_\perp (dL)^2}{\chi_\parallel}\right)^{1/2}$$

$$\boxed{d\delta \sim \left(D_\perp/\chi_\parallel\right)^{1/2} dL}$$

② width shrinks, due stochastic instability and area conservation:



$d\delta/dL = -\delta/l_c$ (exponential decay) then balance at:

$$d\delta \sim \underbrace{(D_\perp/\chi_\parallel)^{1/2}}_{\text{smearing}} dL \sim \underbrace{\frac{\delta}{l_c}}_{\text{thinning}} dL$$

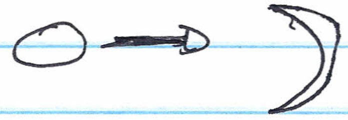
so $\quad \delta \sim l_c (D_\perp/\chi_\parallel)^{1/2}$

N.B.: Can select δ from:

$$\partial_t T - \chi_\parallel \nabla_\parallel^2 T - D_\perp \nabla_\perp^2 T = 0$$

$$\Rightarrow \quad \frac{\chi_\parallel}{l_c^2} \sim \frac{D_\perp}{\delta^2}. \qquad \delta \sim l_c (D_\perp/\chi_\parallel)^{1/2}$$

Finally, need correlation length $l_{c\delta}$ for chunk size δ. Assume set by k_θ



$$k_\theta^{-1} \sim \delta e^{z/l_c}\Big|_{l_{c\delta}} \sim \delta e^{l_{c\delta}/l_c}$$

$$l_{c\delta} \sim l_c \ln\left(1/k_0\delta\right)$$

$$l_{c\delta} \sim l_c \ln\left(\chi_{\parallel}/D_{\perp}\right)^{1/2}$$

$$\boxed{l_{c\delta} \sim l_c \ln\left(\left(\chi_{\parallel}/D_{\perp}\right)^{1/2}/k_0 l_c\right)}$$

$$\Rightarrow \boxed{\chi_{\perp} \sim D_M \, \chi_{\parallel}/l_{c\delta}}$$

Apart from log factor:

$$\chi_{\perp} \sim v_{Th} D_M \underbrace{\left(l_{mfp}/l_c\right)}_{<1}$$

$\Rightarrow$ reduced relative to collisionless values

Lesson: — collisions reduce ($l_{mfp} < l_c$)
reduce χ_{eff} relative to
"collisionless case"

— interplay of perp and
parallel diffusion

— again, critical to knock
particle off field line.

Now, the above calculation requires
thought. It's much more convenient
to crank ~~[illegible]~~ mindlessly.

⇒ Hydro approach: Kadomtsev and
Pogutse (not mindless, but systematic)

Consider heat flux along wiggling fields
i.e.

$$\underline{q} = -\chi_{\parallel} \nabla_{\parallel} T \, \hat{b} - \chi_{\perp} \underline{\nabla_{\perp}} T$$

parallel conduction

perp. conduction

{ tricky in codes

$\chi_{\parallel} \gg \chi_{\perp}$

Here: $\underline{b} = \underline{b_0} + \underline{\tilde{b}}$

$\underline{b_0}$: unperturbed

$\underline{\tilde{b}}$ → fluctuating

$$\nabla_{\parallel} = \partial_z + \underline{\tilde{b}} \cdot \nabla_{\perp}$$

$\underline{\tilde{b}} \cdot \nabla_{\perp}$: piece along wiggling line

So, seek mean radial heat flux

$$\langle q_r \rangle = -\chi_{\parallel} \langle \tilde{b}_r^2 \rangle \partial_r \langle T \rangle \quad (1)$$
$$- \chi_{\parallel} \langle \tilde{b}_r \partial_z \tilde{T} \rangle \quad (2)$$

usual quadratic

$$- \chi_{\parallel} \langle \tilde{b}_r \tilde{b}_r \partial_r \tilde{T} \rangle \quad (3)$$

→ cubic

$$- \chi_{\perp} \nabla_r \langle T \rangle$$

Now $\dfrac{(3)}{(2)} \sim \dfrac{\cancel{\chi_{\parallel}} \cancel{\tilde{b}_r} \tilde{b}_r \cancel{\tilde{T}} / \Delta_r}{\cancel{\chi_{\parallel}} \cancel{\tilde{b}_r} \cancel{\tilde{T}} / l_{ac}}$

$$\sim \frac{\tilde{b}_r \, l_{ac}}{\Delta_r} \sim Ku$$

so cubic nonlinearity dominates for $Ku > 1$.

$Ku < 1 \Rightarrow$ drop cubic.

To compute $\langle Q_r \rangle$, need

- retain ① (usual), and ②
- iterate for $\tilde{T}$ using

$\underline{\nabla} \cdot \underline{Q} = 0$ i.e. as QLT.

Thinking (gasp!) first:

$$\langle Q_r \rangle \cong -\chi_{\parallel} \left[\langle \tilde{b}_r^2 \rangle \partial_r T + \langle \tilde{b}_r \partial_z \tilde{T} \rangle \right] - \chi_{\perp} \nabla_r \langle T \rangle$$

$$\cong -\chi_{\parallel} \left[\langle \tilde{b}_r \, \overbrace{\underline{b} \cdot \nabla T} \rangle \right] - \chi_{\perp} \nabla_r \langle T \rangle$$

linearization:

$\tilde{b}_r \partial_r \langle T \rangle + \partial_z \tilde{T}$

$\therefore$

Point: - need non-zero $b \cdot \nabla T$ fluctuation to drive heat flux

- i.e. temperature can't be constant along line to drive parallel heat flux

- $\underline{\nabla} \cdot \underline{q} = 0 \Rightarrow$ result must imply $\chi_\perp$ dependence!

b)

$$\langle q_r \rangle = -\chi_\parallel \left[\langle \tilde{b}_r^2 \rangle \partial_r \langle T \rangle + \langle \tilde{b}_r \partial_z \tilde{T} \rangle \right]$$

$$- \chi_\perp \nabla_\perp \langle T \rangle$$

$$\nabla \cdot \underline{q} = 0$$

$$\Rightarrow \quad \nabla_\parallel \tilde{q}_\parallel + \nabla_\perp \cdot \tilde{q}_\perp = -\chi_\parallel \partial_z \tilde{b}\, \partial \langle T \rangle / \partial r$$

i.e.

$$q = -\chi_{\parallel}\left[(\partial_z + \tilde{b}\cdot\nabla)(T_0 + \tilde{T})(b_0 + \tilde{b})\right]$$

$$- \chi_{\perp}\nabla_{\perp} T$$

so

$$-\chi_{\parallel}\partial_z^2\tilde{T} - \chi_{\perp}\nabla_{\perp}^2\tilde{T} = -\chi_{\parallel}\partial_z\tilde{b}\frac{\partial\langle T\rangle}{\partial r}$$

$\Rightarrow$

$$\tilde{T}_k = \frac{-\chi_{\parallel} i k_z \tilde{b}_k \;\; \partial\langle T\rangle/\partial r}{\left(\chi_{\parallel}k_z^2 + \chi_{\perp}k_{\perp}^2\right)}$$

so

$$-\chi_{\parallel}\langle\tilde{b}^2\rangle\frac{\partial\langle T\rangle}{\partial r} - \chi_{\parallel}\langle\tilde{b}\,\partial_z\tilde{T}\rangle$$

$$= -\chi_{\parallel}\sum_k\left(-\frac{\chi_{\parallel}k_{\parallel}^2|\tilde{b}_k|^2}{\chi_{\parallel}k_z^2 + \chi_{\perp}k_{\perp}^2} + |\tilde{b}_k|^2\right)\frac{\partial\langle T\rangle}{\partial r}$$

$$= -\chi_{\parallel}\frac{\partial\langle T\rangle}{\partial r}\sum_k\left(\frac{\cancel{-\chi_{\parallel}k_{\parallel}^2}}{\chi_{\parallel}k_z^2 + \chi_{\perp}k_{\perp}^2} + \frac{\cancel{\chi_{\parallel}k_{\parallel}^2} + \chi_{\perp}k_{\perp}^2}{\chi_{\parallel}k_z^2 + \chi_{\perp}k_{\perp}^2}\right)$$

so

$$\boxed{\langle q_r \rangle_{NL} = -\chi_\parallel \frac{\partial \langle T\rangle}{\partial r} \sum_k \frac{\chi_\perp k_\perp^2 |b_k|^2}{\chi_\parallel k_\parallel^2 + \chi_\perp k_\perp^2}}$$

Note explicit dependence on $\chi_\perp$!

so

$$\langle q_r\rangle_{NL} \cong -\cancel{\chi_\parallel} \frac{\partial\langle T\rangle}{\partial r}\int dk_\perp \int dk_z \frac{\chi_\perp k_\perp^2 \langle \tilde b_k^2\rangle}{\cancel{\chi_\parallel}\left(k_z^2 + \frac{\chi_\perp k_\perp^2}{\chi_\parallel}\right)}$$

$$= -\frac{\partial\langle T\rangle}{\partial r}\int dk_\perp \int dk_z \frac{\chi_\perp k_\perp^2 \langle \tilde b^2\rangle_k}{\left(\frac{k_z^2}{(\chi_\perp/\chi_\parallel)k_\perp^2}+1\right)\left(\frac{\chi_\perp}{\chi_\parallel}k_\perp^2\right)}$$

$$= -\frac{\partial\langle T\rangle}{\partial r}\int dk_\perp \frac{k_\perp^2 (\chi_\parallel\chi_\perp)^{1/2}}{\sqrt{k_\perp^2}} \langle \tilde b^2\rangle_k l_{ac}$$

auto correlation

autocorrelation l_{ac} enters via normalization

$\Rightarrow$

$$\langle q_r \rangle_{NL} \cong -\sqrt{\chi_\parallel \chi_\perp}\, \langle \tilde{b}^2 \rangle\, l_{ac} \sqrt{\overline{k_\perp^2}}\, \frac{\partial \langle T \rangle}{\partial r}$$

Note: – need $\nabla_\parallel \tilde{T} \neq -\tilde{b}_r\, \partial \langle T \rangle / \partial r$

$(\widetilde{B \cdot \nabla T} \neq 0)$ for $\perp$ heat flux

– $\langle \tilde{b}^2 \rangle\, l_{ac} \sim D_M$.

$\sqrt{k_\perp^2} \sim 1/\Delta_\perp$

so

$$\langle q_r \rangle \cong -\chi_{\perp\,\text{eff}}\, \partial \langle T \rangle / \partial r \;-\; \chi_\perp\, \partial \langle T \rangle / \partial r$$

$$\chi_{\perp\,\text{eff}} \cong \sqrt{\chi_\parallel \chi_\perp}\, \frac{D_M}{\Delta_\perp}.$$

$$\left(\begin{array}{c} \chi_\parallel \chi_\perp \sim \\ \frac{v_{th_e}^2}{\cancel{\nu}}\, \rho_e^2 \cancel{\nu} \sim D_B \end{array} \right)$$

$$\boxed{\chi_{\perp\,\mathrm{eff}} \simeq \frac{D_B\, D_M}{\Delta_\perp}}$$

- $\chi_{\perp\,\mathrm{eff}}$ scales with Bohm, not Spitzer ($\chi_{\parallel}$)

- kicking off line important, again.

To compare R&R:

$$\chi_\perp \sim \sqrt{\chi_{\parallel}\chi_\perp}\, \frac{\langle \tilde{b}^2 \rangle}{\Delta_\perp} l_{ac}$$

what is $\Delta_\perp$?

Now $\dfrac{\chi_{\parallel}}{l_c^2} \sim \dfrac{\chi_\perp}{\Delta_\perp^2}$ {diffusion set ⊥ scale.

$$\Delta_\perp \sim l_c \sqrt{\chi_\perp/\chi_{\parallel}}$$

enters spectrum (small layer)

$\Rightarrow$

$$\chi_\perp \sim \sqrt{\chi_\parallel \chi_\perp} \; \frac{\langle b^2 \rangle l_{ac}}{l_0 (\chi_\perp/\chi_\parallel)^{1/2}}$$

$$\boxed{\chi_\perp \sim \frac{\chi_\parallel}{l_c} D_M}$$

- so - modulo $k_\perp$, $\Delta_\perp$; agrees with RTR to within log. factor

- $\chi_\perp \sim V_{th} D_M \frac{l_{mfp}}{l_c}$

$\Rightarrow$ covers diffusion in $K_u < 1$ stochastic fields

$\Rightarrow$ Lesson: Take care re: irreversibility!