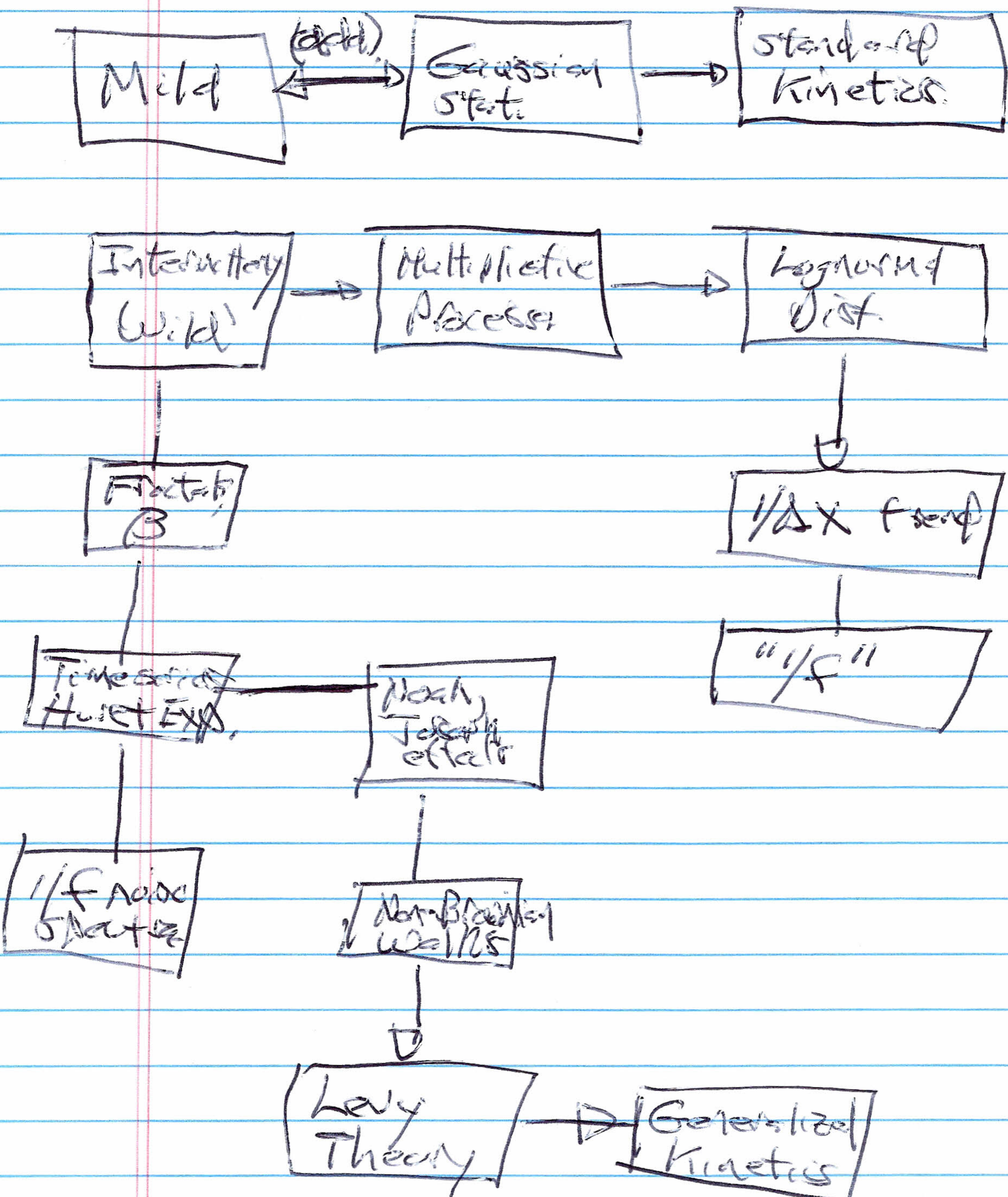
Self-Organized Criticality I: Basics

Recall the flow of ideas:

Randomness



Need generalize kinetics: $\Rightarrow$ Levy Theory

Gaussian $\Rightarrow$ L-stable (Linear combo of 2 elements on dist is on dist.)

$$\underbrace{(a' X_1 + b')}_{\text{on dist.}} + \underbrace{(a'' X_2 + b'')}_{\text{on dist.}} \Rightarrow \underbrace{a X_3 + b}_{\text{on dist.}}$$

$\Rightarrow$ Levy distribution:

$e^{-c|k|^\alpha}$ for pdf, characterized by $0 < \alpha \leq 2$

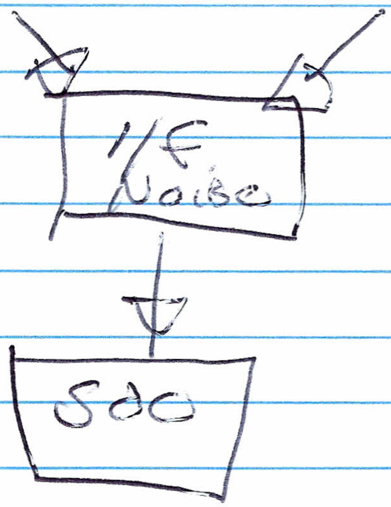
$e^{-ct|k|^\alpha}$ for process $0 < \alpha \leq 2$

and strange kinetics: $\left\{ \begin{array}{l} \text{CTRW} \\ P(\Delta x, \Delta t) \rightarrow P(\Delta x) P(\Delta t) \\ \text{Fractional kinetics.} \end{array} \right.$

So, claim:

$\boxed{\text{Intermittency, lognormal, mult. process}}$ $\rightarrow$ $\leftarrow$ $\boxed{\text{Fractals, Time Series, Levy, Hurst Exponent}}$

cont'd



Re: 1/f noise:

→ what is it?

→ why is 1/f noise ubiquitous?

→ how realize in simple system → SOC, pile?

Now: → 1/f defined by what its not

standard ansatz for correlation function in time:

$$\langle \phi(t_1)\phi(t_2)\rangle = |\hat{\phi}|^2 e^{-|(t_2-t_1)|/\tau_c} = c(\tau)$$

$$= |\hat{\phi}|^2 e^{-\tau/\tau_c}$$

then $S(\omega) = \int_{-\infty}^{+\infty} d\tau \, e^{i\omega\tau} e^{-|\tau|/\tau_c|}$

spectrum

$$\cong \tau_c / \omega^2\tau_c^2 + 1$$

$$= \frac{1/\tau_c}{\omega^2 + 1/\tau_c^2}$$

i.e. – 'usual' case ~~xxxx~~ has

$S(\omega) \sim 1/\omega^2$

scale composed by τ_c

– 1/f has $\rightarrow$ $S(\omega) \sim 1/\omega$

no scale.

– easy to get $1/\omega^2$.

To get 1/f:

$\rightarrow$ consider large number of different random processes each with own τ_c, randomly assigned.

→ power spectrum depends on statistical distribution of τ_c's

$$S(\omega)_{eff} = \int_{\tau_{c_1}}^{\tau_{c_2}} P(\tau_c)\, S_{\tau_c}(\omega)\, d\tau_c$$

$S(\omega)_{eff}$ ↓ effective, observed spectrum

$P(\tau_c)$ ↓ prob. τ_c

$S_{\tau_c}(\omega)$ ↓ spectrum given τ_c

Now if $P(\tau_c)$ (weighting function) is scale invariant, i.e. no characteristic time:

$$P(\tau_c)\, d\tau_c = d\tau_c / \tau_c$$

$$S(\omega)_{eff} = \int_{\tau_{1c}}^{\tau_{2c}} d\tau_c\, P(\tau_c)\, S_{\tau_c}(\omega)\, d\tau_c$$

$$= \int_{\tau_{1c}}^{\tau_{2c}} \frac{d\tau_c}{\cancel{\tau_c}} \frac{\cancel{\tau_c}}{\omega^2 \tau_c^2 + 1} = \frac{\tan^{-1} \omega\tau_c}{\omega} \Bigg|_{\tau_{1c}}^{\tau_{2c}}$$

$$\simeq 1/\omega .$$

→ What Type of Systems Behave Like / Exhibit 1/f Noise ?

→ Self-Organized Criticality / Sandpiles

N.B. Bak, Tang, Wiesenfeld '87

Self-Organized Criticality : An Explanation of 1/f Noise

Abstract:

We show that dynamical systems with spatial d-o-f's naturally evolve into an (SOC state) self-organized critical point. Flicker noise or 1/f noise, can be identified with the dynamics of the critical state. This picture also yields insight into the origin of fractal objects.

Key elements:

- spatial d-o-f's
- flicker (1/f) noise for perturbation about criticality
- fractality.

Some basic ideas/definitions/questions:

- What is SOC? Where might it occur?

→ slowly driven, interaction dominated threshold system
(constructive)

→ system exhibiting power law scaling without tuning
(phenomenological)

Sandpile paradigm, avalanches

Ingredients (constructive defn.):

⇒ Interaction dominated

- many d-o-f's interact
  - → cells
  - → modes

toppling rules couple elements.

c.f. broad distribution of avalanche sizes

- dynamics dominated by mutual interaction of elements, not properties of single d-o-f's.

→ threshold & slowly driven ⇒ large number of quasi-static metastable configurations accessible

⇒ "local comparative rigidity"

→ slow drive reveals effect of threshold

↕

strong drive will not allow relaxation from one configuration (metastable) to another

i.e. strong drive buries "local comparative rigidity".

More generally:

→ under general conditions systems with many interacting constituents exhibit general characteristic behavior, complex but general structure.

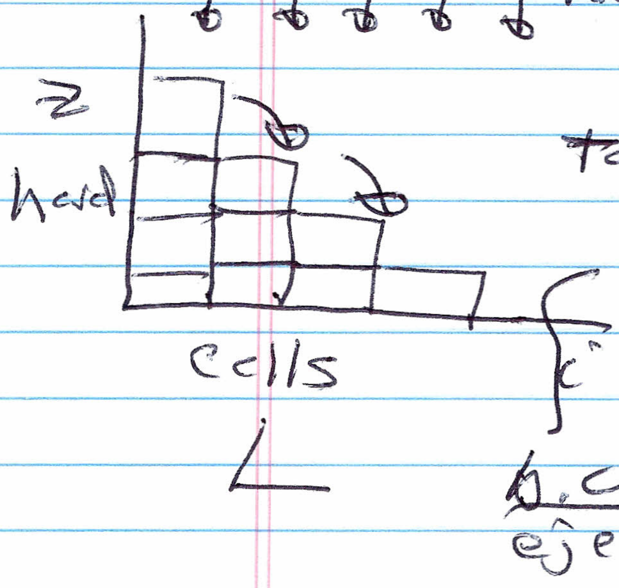
→ "complex" → scale invariant,
no characteristic space/time scale
⇒ distribution of scales
(Zipf)

→ "simple" → power laws
$1/f$, Zipf

→ systems are self-tuning, by relaxation

→ Classic Paradigm: Sandpile (1D)

random deposition → random sprinkling



toppling rules:

$$z_i - z_{i+1} > \Delta z_{crit}$$

topple N over range Δ

$$L/\Delta \sim N \gg 1 \Rightarrow$$ necessary for meaningful collective phenomena.

- emergent collective phenomena
  $\Rightarrow$ avalanches.

- SOC state is running, driven pile near, not at, threshold manifesting avalanche spectrum.

- at SOC:

$$\begin{cases} \ell_c \rightarrow L \\ \tau_c \text{ divergent.} \end{cases}$$

$$\begin{cases} \mathrm{Prob}\,(N_{\mathrm{Toppling}}) \sim 1/N_{\mathrm{Toppling}} \\ \text{a la' Zipf} \end{cases}$$

$$\left\{ \langle N_T(\omega)^2 \rangle \sim 1/\omega \right.$$

$\Rightarrow$ statistical collective phenomena - avalanches - occur

– Aside: short paper topic:

Give a critical analysis of how well SOC/pile models do or do not represent MFE phenomena.

Think about zonal flows, pinch, transitions

– Contrast: Usual externally controlled } criticality

i.e. Ferromagnet → Ginzburg/Landau Theory

$$\mathcal{F} = \int d^3x\, F = \int d^3x \left[\frac{(\nabla M)^2}{2} + a(T-T_c)\frac{M^2}{2} + b\frac{M^4}{4} \right]$$

↓
free energy

$$\frac{\partial M}{\partial t} = c\,\frac{\delta F}{\delta M} = c\nabla^2 M - a'(T-T_c)M - b' M^3$$

$$\frac{\partial M}{\partial t} - c\nabla^2 M = -a'(T-T_c)M - b' M^3$$

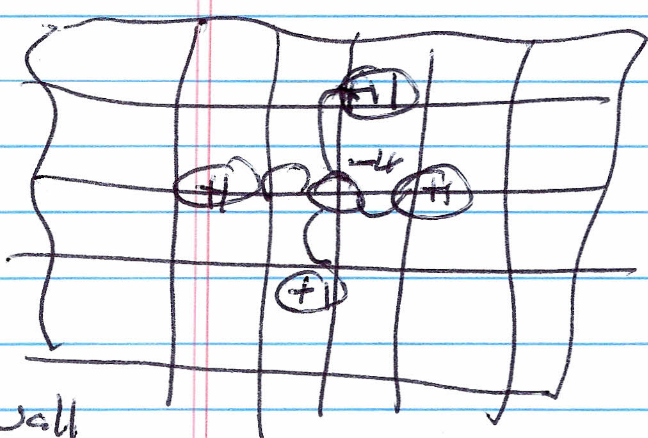
- transition at $T \sim T_c$, $\ell_c \to \infty$, $\tau_c \to \infty$.
- tuning T_c tunes transition

$\Rightarrow$ externally controlled

$\rightarrow$ 2D Example.

2D Cellular Automata Model (BTW)

- Consider synchronous update of:



$Z(x,y) \to Z(x,y) - 4$ if $Z > Z_{crit}$

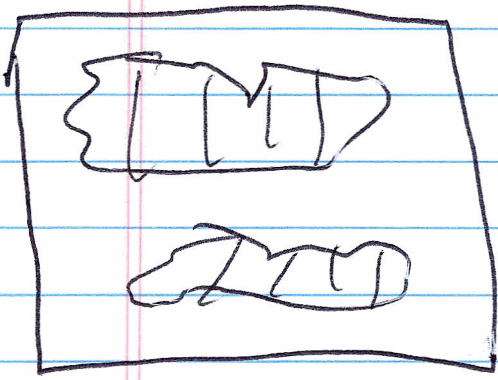
$Z(x \pm 1, y) \to Z(x \pm 1) + 1$

$Z(x, y \pm 1) \to Z(y \pm 1) + 1$

$\rightarrow$ out $\quad Z = 0$ bndry

start by random i.c.'s

$\Rightarrow$ SOC state from random i.c.



- clusters appear
- clusters = points reached by toppling of single site

(i.e. akin percolation)

- can define cluster size via toppling in response to perturbation

Clearly SOC → $l_{cluster} \to \sim L$

$\downarrow$

$l_{toppling}$ range

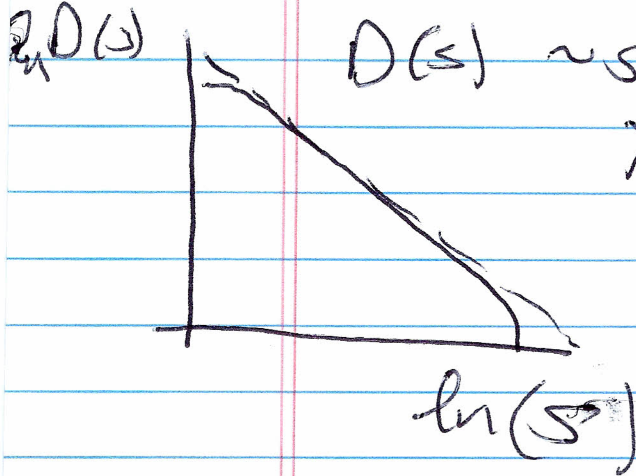
→ percolation threshold.

Related:

$\underset{\#}{\theta}(\overset{\text{size}}{s})$ → distribution of cluster sizes of 2D system

where:

Cluster → # affected sites from a seed toppling, avg.



$D(s) \sim s^{-\tau}$

$\tau = .98$

→ linear over 2 decades

→ And, on examine cluster lifetimes,

– if cluster grows as

$$\left\{ \begin{array}{l} \dfrac{\text{size}}{\text{time}} \sim t^{\gamma} \\ s \sim t^{\gamma+1} \end{array} \right.$$

{ i.e. bigger longer lived clusters grow faster

⇒

size dist

$$\underbrace{D(t)}_{\text{lifetime distribution}} \sim \frac{s}{t}\, D(s)\, \frac{ds}{dt}$$

$$\boxed{t\, \overset{\#}{\underset{\text{time dist}}{D(t)}}\, dt = s\, \overset{\#}{D(s)}\, ds}$$

can count in terms of size, lifetime.

<u>so</u>

$D(t) \sim t^{-a}$

$a \sim .4$ in 2D.