Physics 235: Lecture Notes 3–4

Transport channels, particle transport, ambipolarity, and towards large Kubo number

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(Dutifully) summarized by Robin Heinonen

featuring fine art by Robin Heinonen

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1 Introduction

Previously, we have discussed the onset of Hamiltonian chaos by the overlapping of resonant structures. This torus-centric topic led us naturally to transport due to stochastic magnetic fields in another donut of interest, the tokamak. We introduced the notion of the Kubo number, which, roughly speaking, is a ratio of nonlinear terms to linear ones. More precisely, it is given by

$$\mathrm{Ku} \equiv \frac{\ell_{ac}}{\Delta_r}\frac{\tilde{B}_r}{B_0}. \tag{1}$$

Here, ℓ_{ac} is the (parallel) autocorrelation length beyond which field lines decohere from perturbations, and Δ_r is the *radial* correlation length of the scattering field. Ku is then the ratio of the (coherent) radial excursion of field lines from $\hat{z}$, $\delta r \simeq \ell_{ac}\tilde{B}_r/B_0$, to Δ_r.

For small Ku, the fields are scattered diffusively since there are many small radial "kicks" per Δ_r. We can define a magnetic diffusivity D_M which gives the mean-square radial excursion of the field lines per parallel length, i.e.

$$\langle \delta r^2 \rangle \sim D_M z. \tag{2}$$

According to quasilinear theory, we can compute

$$D_M = \sum_{\mathbf{k}} \pi \left| \frac{\tilde{B}_{r,\mathbf{k}}}{B_0} \right|^2 \delta(k_\parallel) \simeq \left\langle \left(\frac{\tilde{B}_r}{B_0} \right)^2 \right\rangle \ell_{ac}. \tag{3}$$

Finally, we computed heat transport coefficients in the small-Kubo regime in both the collisional and collisionless cases. A key takeaway in both cases was that the perpendicular thermal diffusivity is proportional to the magnetic diffusivity and the electron thermal speed, $\chi_\perp \sim v_{th} D_M$, and this perpendicular diffusion is due to the irreversible kicking of particles off field lines. Note that this result differs strongly from the Spitzer result [1] $\chi \sim \ell_{mfp} v_{th}$.

The dependence of thermal transport on particle motion motivates the investigation of other transport channels, such as particles and momentum, and their effects on ambipolarity. This discussion will include enlightening connections to zonal flows.

We will then begin to turn our attention to the case of Ku > 1, since, tragically, life isn't always so easy. We will study the example of the 2D guiding center plasma in thermal equilibrium, which will serve as a gentle introduction to resonance broadening theory. Resonance broadening theory is perhaps the simplest and most intuitive approach to including memory effects induced by strong nonlinear scattering. Ultimately, though, it is still a diffusive approximation, and we will finally challenge this perspective on the large Kubo regime with a brief discussion that will point towards percolation theory.

2 Particle transport

Consider the problem of *particle* transport in a stochastic magnetic field (as usual, take $B_z = B_0\hat{z}$ to be large). It is important to distinguish between two cases:

1. The self-consistent case, i.e. the fluctuating field $\tilde{B}$ is produced by a fluctuating current $\tilde{J}_\parallel$ in the plasma in a way that is consistent with Maxwell's equations, via electromagnetic instabilities.

2. $\tilde{B}$ is produced by some external mechanism (a coil). A relevant example is resonant magnetic perturbations (RMP), which are used in tokamaks to suppress ELMs. Note that the plasma response is significant here; generally, we will have something like $\tilde{B} \sim \tilde{B}_{ext}/\epsilon$ for some screening factor ϵ.

We will focus on the first case in what follows, though we note that the key challenge in the second is to calculate the response (both magnetic field and parallel current) in the plasma.

(N.B. there are several calculational errors in Pat's notes on the following, which we rectify here.) A natural starting point is the electron drift kinetic equation. We have

$$\partial_t f + v_\parallel \partial_\parallel f + v_\parallel \frac{\tilde{\mathbf{B}}}{B_0} \cdot \nabla_\perp f + (\text{terms we don't care about}) = 0. \tag{4}$$

We proceed by integrating over velocity space and averaging over the symmetry (poloidal and toroidal) directions. The first term simply becomes $\partial_t \langle n_e \rangle$. The second term becomes

$$\frac{1}{L_z L_\theta} \int d^3v \int d\theta dz\, v_\parallel \frac{\tilde{\mathbf{B}}}{B_0} \cdot \nabla_\perp f = -\frac{1}{L_z L_\theta} \int d^3v \int d\theta dz\, f \nabla_\perp \cdot \left(v_\parallel \frac{\tilde{\mathbf{B}}}{B_0} \right) \tag{5}$$

$$= -\langle n_e \rangle \partial_r \left\langle \tilde{v}_\parallel \frac{\tilde{B}_r}{B_0} \right\rangle, \tag{6}$$

where in the last step we have used periodicity in θ and ignored the triple correlator $\langle \tilde{v}_\parallel \tilde{n}_e \tilde{B} \rangle$. Finally, identifying the electron motion with the electron current $\tilde{v}_\parallel = -n_0|e|\tilde{J}_{\parallel_e}$, we have

$$\partial_t \langle n_e \rangle - \langle n_e \rangle \partial_r \left\langle \frac{\tilde{B}_r}{B_0} \frac{\tilde{J}_{\parallel,e}}{-n_0|e|} \right\rangle + \cdots = 0. \tag{7}$$

However, we must also enforce consistency with Ampère's law

$$\hat{z} \cdot (\nabla \times \tilde{\mathbf{B}}) = \hat{z} \cdot (\nabla \times \nabla \times \tilde{\mathbf{A}}) = -\nabla_\perp^2 \tilde{A}_\parallel = \frac{4\pi}{c} \tilde{J}_\parallel = \frac{4\pi}{c} (\tilde{J}_{\parallel,i} + \tilde{J}_{\parallel,e}), \tag{8}$$

(ignoring variation along $\hat{z}$). Thus,

$$\tilde{J}_{\parallel,e} = -\frac{c}{4\pi} A_\parallel - n_0 |e| \tilde{v}_{\parallel,i}, \tag{9}$$

and an ion contribution will enter the calculation.

We now have

$$\partial_t \langle n_e \rangle - \langle n_e \rangle \partial_r \cdot \left\langle \frac{\tilde{B}_r}{B_0} \left(\frac{c}{4\pi n_0 |e|} \nabla_\perp^2 \tilde{A}_\parallel + \tilde{v}_{\parallel,i} \right) \right\rangle + \cdots = 0. \tag{10}$$

Finally, we use $\tilde{B}_r = \partial_\theta \tilde{A}_\parallel$ to write

$$\partial_t \langle n_e \rangle = \frac{c}{4\pi |e| B_0} \partial_r \left\langle (\partial_\theta \tilde{A}_\parallel)(\nabla_\perp^2 \tilde{A}_\parallel) \right\rangle + \langle n_e \rangle \partial_r \left\langle \frac{\tilde{B}_r}{B_0} \tilde{v}_{\parallel,i} \right\rangle + \ldots . \tag{11}$$

We've now identified two terms: one due to the autocorrelation of the fluctuating magnetic field, and one due to the magnetic "flutter" of the ion flow. We now invoke a trick known as the Taylor identity, after G.I. Taylor [2], to simplify the former term (the Taylor identity is commonly used to relate zonal flow production to the Reynolds stress). We have

$$\left\langle (\partial_\theta \tilde{A}_\parallel)(\nabla_\perp^2 \tilde{A}_\parallel) \right\rangle = \left\langle (\partial_\theta \tilde{A}_\parallel)(\partial_\theta^2 \tilde{A}_\parallel + \partial_r^2 \tilde{A}_\parallel) \right\rangle \tag{12}$$

$$= \left\langle \frac{1}{2} \partial_\theta \left((\partial_\theta \tilde{A}_\parallel)^2 \right) + \partial_r (\partial_\theta \tilde{A}_\parallel \partial_r \tilde{A}_\parallel) - \frac{1}{2} \partial_\theta \left((\partial_r \tilde{A}_\parallel)^2 \right) \right\rangle . \tag{13}$$

The first and third terms vanish due to periodicity. This leaves $\partial_r \left\langle \tilde{B}_r \tilde{B}_\theta \right\rangle$, the derivative of the magnetic stress. Hence,

$$\partial_t \langle n_e \rangle = \frac{c}{4\pi |e| B_0} \partial_r^2 \left\langle \tilde{B}_r \tilde{B}_\theta \right\rangle + \langle n_e \rangle \partial_r \left\langle \frac{\tilde{B}_r}{B_0} \tilde{v}_{\parallel,i} \right\rangle + \ldots . \tag{14}$$

Thus, we have related the electron particle transport to the magnetic stress due to stochastic fields. This suggests the fluctuating fields may have an effect on zonal flows, which are ultimately a charge transport phenomenon. Also note that there is no explicit dependence on the (small) electron inertia, and that a combination of fluctuating parallel ion flows and magnetic tilt changes the electron density. For more on this last idea, see [3].

3 Zonal flows and ambipolarity

Let us say more on the topic of zonal flows, which are "benign" in the sense that they don't lead to radial transport and thus are the darlings of plasma physicists everywhere. ZFs are the result of radial charge separation, which leads to a charge polarization flux and a Reynolds force. We will see that stochastic magnetic fields will impact zonal flow formation by inducing charge transport and a Maxwell force.

The total flux of charge is given by

$$\Gamma_{charge} = |e|(\langle \tilde{v}_r \tilde{n}_i \rangle - \langle \tilde{v}_r \tilde{n}_e \rangle) + \langle \tilde{v}_r \tilde{Q}_{pol} \rangle, \tag{15}$$

where the first two terms are due to guiding center motion and Q_{pol} is the polarization charge density. In absence of the polarization term, quasineutrality $\tilde{n}_i = \tilde{n}_e$ guarantees ambipolarity, i.e. $\Gamma_{charge} = 0$. However, polarization charge flux introduces a new means of charge transport, breaking ambipolarity.

To see how this results in ZF formation, recall the (2D) vorticity evolution equation, in the absence of a stochastic field, from drift wave models such as Hasegawa-Mima:

$$\partial_t \nabla^2 \phi + \mathbf{v}_E \cdot \nabla \nabla^2 \phi = \nu \nabla^2 \nabla^2 \phi. \tag{16}$$

Here $\mathbf{v}_E = \nabla \phi \times \hat{z}$ is the $E \times B$ drift, so $\nabla \times \mathbf{v}_E = \nabla^2 \phi$ is the vorticity. Note that nonzero $\nabla^2 \phi$ is equivalent to charge separation according to Gauss's law:

$$\nabla^2 \phi = -4\pi Q_{pol}. \tag{17}$$

Equation (16) is electrostatic and assumes $\nabla \cdot \mathbf{J} = \nabla \cdot (\mathbf{J}_{pol} + \mathbf{J}_{\parallel} + \mathbf{J}_{PS}) = 0$. Since ZFs have toroidal and poloidal wavenumbers $m, n = 0$, the parallel and Pfirsch-Schlüter terms are zero here, and $\partial_r J_{r,pol} = 0$.

Taking zonal averages $\langle \cdots \rangle$, writing $\phi = \langle \phi \rangle + \tilde{\phi}$, and neglecting dissipation, we have

$$\partial_t \partial_r^2 \langle \phi \rangle + \partial_r \langle \tilde{v}_{E,r} \nabla^2 \tilde{\phi} \rangle = 0, \tag{18}$$

where we have used the fact that $\tilde{v}_E$ is divergence-free. This equation relates the time evolution of the mean polarization charge density to the divergence of the flux of the polarization charge. Noting the second term is identical in form to the second term of Eq. (11) (since $\tilde{v}_{E,r} = \partial_\theta \tilde{\phi}$), we can apply the Taylor identity to find

$$\langle \tilde{v}_{E,r} \nabla^2 \tilde{\phi} \rangle = \partial_r \langle \partial_\theta \tilde{\phi} \partial_r \tilde{\phi} \rangle = -\partial_r \langle \tilde{v}_{E,r} \tilde{v}_{E,\theta} \rangle. \tag{19}$$

Thus the flux of vorticity/polarization charge is equivalent to the Reynolds force and leads to ZF production. The zonal flow production can then be found via a modulational instability calculation. See, for example, [4].

Now, in the presence of a stochastic magnetic field perturbation, we can no longer neglect parallel current, and $\nabla \cdot \mathbf{J} = 0$ now leads to

$$\nabla_\perp \cdot \mathbf{J}_{pol} + \partial_\parallel J_\parallel = 0 \tag{20}$$

with $\partial_\parallel = \partial_\parallel^{(0)} + \frac{\tilde{\mathbf{B}}}{B_0} \cdot \nabla_\perp$.

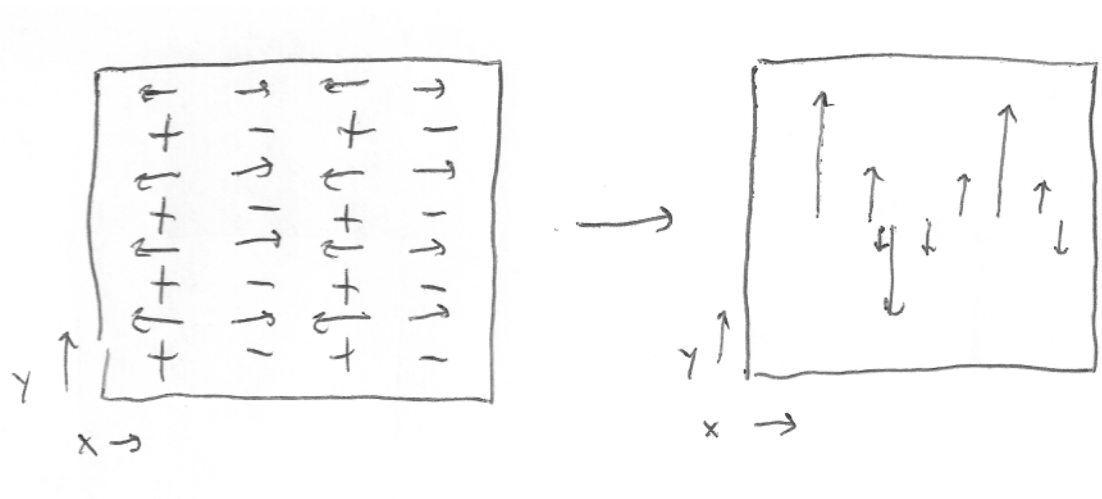

Figure 1: Zonal flow drive by Reynolds stress. Correlations between fluctuating radial flows and charge separation self-amplify and generate a poloidal $E \times B$ flow.

Including this correction in the vorticity evolution now results in the equation (which also follows directly from reduced MHD)

$$\partial_t \langle \partial_r^2 \phi \rangle + \langle \tilde{\mathbf{v}} \cdot \nabla \nabla^2 \tilde{\phi} \rangle = \langle \tilde{\mathbf{B}} \cdot \nabla \tilde{J}_\parallel \rangle. \tag{21}$$

The evolution is again governed by the advective transport of polarization charge (second term on LHS), but now there is also a contribution (RHS) from the current flow along tilted field lines. Note that, invoking the Taylor identity, the former term is the Reynolds stress contribution and the latter is the Maxwell stress contribution:

$$\partial_t \langle \partial_r^2 \phi \rangle = -\partial_r^2 (\Pi_{Reynolds} + \Pi_{Maxwell}). \tag{22}$$

Thus the stochastic fields contribute to ZF production, but note that the sign (flow growth or damping) is not *a priori* clear. In the case of pure Alfvénic fluctuations, it turns out that the fields act to cancel the electrostatic field and damp the ZFs, but this is beyond the scope of these notes. See Figs. 1–2.

4 A word on electron momentum transport

Let us briefly mention the topic of electron momentum transport. In the collisionless regime, it can be argued that stochastic magnetic perturbations leads to transport of electron parallel momentum across magnetic surfaces with the same strength as the heat transport:

$$\partial_t J_\parallel = \nabla_\perp \cdot (\mu \nabla_\perp J_\parallel), \tag{23}$$

where μ is approximately equal to the anomalous heat diffusivity $\chi_\perp \sim D_M v_{th}$. This requires including the electron inertia term in the Ohm's law

$$\mathbf{E} + \mathbf{v} \times \mathbf{B}/c = \underbrace{\eta \mathbf{J}}_{\text{resistivity}} + \underbrace{\frac{m_e}{n_e |e|^2} \partial_t \mathbf{J}}_{\text{electron inertia}}, \tag{24}$$

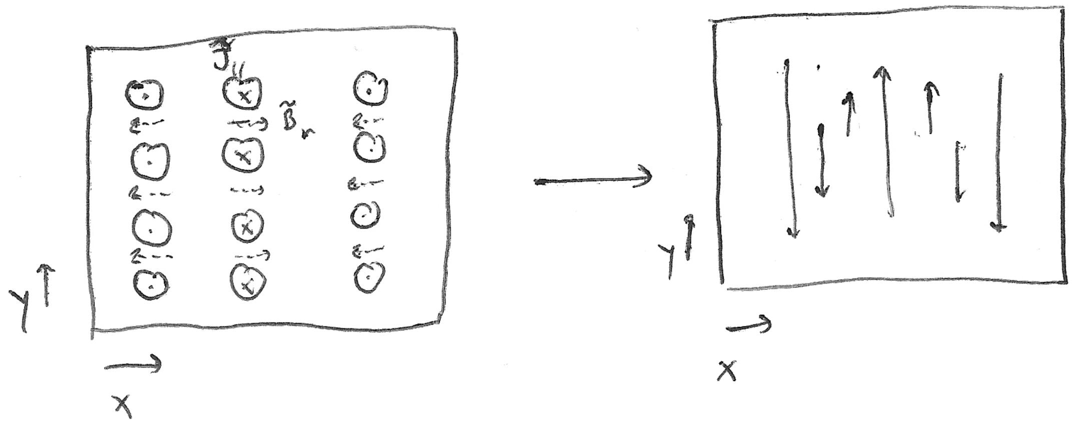

Figure 2: Zonal flow drive by Maxwell stress. Correlations between parallel current and radial magnetic field fluctuations self-amplify and generate a poloidal $E \times B$ flow; this effect tends to oppose that of the Reynolds stress for Alfvènic turbulence.

so that

$$E_{\|} = \eta J_{\|} - \frac{m_e}{n_e|e|^2}\nabla_{\perp} \cdot (\mu \nabla_{\perp} J_{\|}). \tag{25}$$

Refer to [5] for more details.

The coefficient μ is referred to as the "electron viscosity" or the "hyper-resistivity," and it can indeed be important in the collisionless regime. In particular, by modifying the effective resistivity it can have a significant impact on magnetic reconnection.

Recall that the reconnection of magnetic field lines results in the inflow of plasma into the dissipation region where reconnection is occurring (see Fig. reffig:rec). According to the classical Sweet-Parker theory for resistive MHD, the inflow speed is given by $v = v_A/\mathrm{Rm}^{1/2}$, where v_A is the Alfvén speed and $\mathrm{Rm}_{eff} = v_A L/\eta$ is the magnetic Reynolds number (at the Alfvén speed).

The Sweet-Parker result is generally very slow compared to observations of real-world reconnection, a phenomenon known as "fast reconnection." Hyper-resistivity represents one possible avenue to resolving this issue. It can be shown that the inclusion of the electron inertia term modifies this result to $v_{in} \sim v_A/(\mathrm{Rm}_{eff})^{1/4}$, where $\mathrm{Rm}_{eff} = v_A L/\mu$, resulting in significantly stronger reconnection.

To see this, note that the inertia term of the ohmic power dissipation is $\sim \mu J_{\|}^2/\Delta^2$. Equating this to the magnetic energy influx (and applying Ampère's law) yields $v_{in} \sim \mu/\Delta^3$. Finally, employing $v_{in} \sim v_A \Delta/L$ (from pressure balance and mass balance) gives the stated result.

Fast reconnection plays a role in the bane of every confinement physicist, the edge localized mode (ELM) crash. See [6] for results on the impact fast reconnection, induced by hyper-resistivity, has on ELM crashes.

Finally, we mention another prominent perspective on fast reconnection: the formation of intense current layers in the reconnection region can drive instabilities, resulting in a

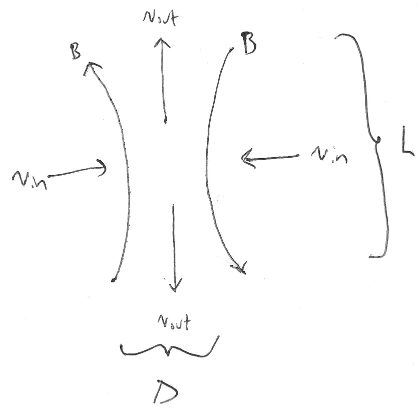

Figure 3: Basic schematic of magnetic reconnection. Reconnection occurs in a dissipation region of width Δ and length L, and plasma flows into and out of this region.

dependence of μ on the current gradient and introducing a nonlinearity into the electron inertia term. See, for example, [7] or [8] for more on this subject.

5 Towards $\mathrm{Ku} > 1$

We have up until this point been concerned with transport in the $\mathrm{Ku} < 1$ regime, with the Kubo number

$$\mathrm{Ku} = \frac{[\Delta(k_\parallel)]^{-1}}{\Delta_r} \frac{\tilde{B}_r}{B_0} \tag{26}$$

set by the magnitude of the magnetic fluctuations, the parallel bandwidth, and the radial correlation length of the scatterer. In this regime, the field experiences many radial kicks per coherence length and thus can be modeled as a diffusion process. The key quantity governing the transport has been the magnetic diffusivity

$$D_M = \int_0^\infty d\ell \, \frac{1}{B_0^2} \langle \tilde{B}_r(0) \tilde{B}_r(\ell) \rangle \tag{27}$$

$$= \int_0^\infty d\ell \sum_{\mathbf{k}} \left| \frac{\tilde{B}_{r,\mathbf{k}}}{B_0} \right|^2 e^{ik_\parallel \ell} \tag{28}$$

$$\simeq \left\langle \left(\frac{\tilde{B}_r}{B_0} \right)^2 \right\rangle \ell_{ac} \tag{29}$$

(recall that the first expression is the Green-Kubo form of the diffusion coefficient). The field lines are scattered diffusively via $k_\parallel = 0$ resonances, and the spectral width of linear wave packets is the controlling spatial lengthscale.

Now, what happens for larger Kubo number? Note that we may rewrite

$$\mathrm{Ku} \sim \frac{\ell_{ac}}{\ell_{NL}}, \tag{30}$$

where $\ell_{NL}^{-1} \sim \frac{\tilde{B}_r}{B_0}\partial_r \sim \frac{\tilde{B}_r}{B_0}\Delta_r^{-1}$ is known as the "nonlinear mixing length."[1] This lengthscale is set by the nonlinear term of the DKE (Eq. (4)) $\sim \frac{\tilde{B}_r}{B_0}\partial_r$. When the parallel lengthscale ℓ_{ac} exceeds the mixing length, the nonlinear term begins to dominate the linear term. Thus, the large Kubo regime is one where spatial/time scales set by *nonlinear* scattering processes become important. Clearly, quasilinear theory is of dubious value here and the above result for D_M is no longer valid.

The effect of the nonlinear processes may be modeled by the entrance of *finite time memory* into the system:

$$D_\perp = \int_0^\infty d\tau \, \langle \tilde{v}(0)\tilde{v}(\tau)\rangle \simeq \int_0^\infty d\tau \, \sum_{\mathbf{k}} |\tilde{v}_{\mathbf{k}}|^2 R(\tau), \tag{31}$$

where $R(\tau)$ is the *memory kernel* or *resonance function.* A typical expression is one of the form

$$R(\tau) = R_\omega(\tau) = e^{i(\omega - k_\parallel v_\parallel)\tau} e^{-\tau/\tau_\perp}, \tag{32}$$

where $\omega - k_\parallel v_\parallel$ is the contribution from unperturbed orbits. The nonlinear effects have *broadened* the linear wave-particle resonance and set the characteristic decoherence time $\tau_\perp$ (though generally there may be multiple nonlinear time scales). We may similarly replace the exponential in Eq. (28) with a more general dependence on ℓ.

In the following section, we will explore this idea by computing the perpendicular diffusivity in the 2D guiding center plasma, recovering the Bohm result $D_\perp \sim 1/B_0$ (even for thermal equilibrium!). This discussion is based on the classic paper by Taylor and McNamara [9]. It will serve as introduction to *resonance broadening theory* (RBT) [10], which is perhaps the simplest approach to attacking the Ku > 1 regime. The calculation that follows will be performed in the limit Ku $\gg 1$ and $k_\parallel \to 0$, $\omega \to 0$.

6 Diffusion in the 2D guiding center plasma

Recall that the 2D guiding center model corresponds to the limit where the magnetic field is so large that the Larmor radius is smaller than the Debye length, so that the plasma may be represented as an ensemble of rods with charge per unit length $\pm e/\ell$ moving in the plane perpendicular to $\mathbf{B}_0$. The particle trajectories are determined by $E \times B$ drift

$$\mathbf{v}_E = \frac{c}{B_0}\hat{z} \times \nabla\phi. \tag{33}$$

We work in an electrostatic approximation where ϕ is determined by the Poisson equation

$$\nabla^2\phi = -4\pi\rho. \tag{34}$$

The charge density evolves according to

$$\partial_t\rho + \mathbf{v}_E \cdot \nabla\rho = D_0\nabla^2\rho. \tag{35}$$

[1] Analogously, for fluid turbulence we may write Ku $\sim \tau_{ac}\tilde{v}/\Delta \sim \tau_{ac}/\tau_{NL}$ with τ_{NL} the eddy circulation time. For collisional drift waves, the autocorrelation time is set by parallel electron motion $\tau_{ac}^{-1} \sim \Delta|\chi_\parallel k_\parallel^2|$.

How does the fluctuating electric field affect particle transport? We seek the diffusion coefficient

$$D_\perp = \int_0^\infty d\tau \, \langle \tilde{v}(0)\tilde{v}(\tau)\rangle \tag{36}$$

$$= \int_0^\infty d\tau \sum_{\mathbf{k}} |\tilde{v}_{\mathbf{k}}|^2 e^{i\mathbf{k}\cdot\mathbf{r}_0} e^{-i\mathbf{k}\cdot\mathbf{r}(\tau)} \tag{37}$$

$$= \int_0^\infty d\tau \sum_{\mathbf{k}} |\tilde{v}_{\mathbf{k}}|^2 e^{-i\mathbf{k}\cdot\delta\mathbf{r}(\tau)}, \tag{38}$$

where we have formally decomposed the particle orbit $\mathbf{r}(\tau)$ as the sum of the unperturbed orbit and a "fluctuating" contribution due to scattering by the spectrum of electric fields

$$\mathbf{r}(\tau) = \mathbf{r}_0 + \delta\mathbf{r}(\tau). \tag{39}$$

This expression for $D_\perp$ is stochastic; how do we make progress? This brings us to the key step of RBT, which is to replace the expression with an ensemble average over the perturbed orbits:

$$D_\perp \simeq \langle D_\perp \rangle = \int_0^\infty d\tau \sum_{\mathbf{k}} |\tilde{v}_{\mathbf{k}}|^2 \langle e^{-i\mathbf{k}\cdot\delta\mathbf{r}(\tau)}\rangle. \tag{40}$$

We have thus replaced the exact propagator with the "effective" propagator that the particle feels due to the ensemble of electric field fluctuations; this is very similar to renormalization methods from field theory.

Of course, actually computing this ensemble average requires further approximation: we assume the distribution of orbits is Gaussian and thus the particles obey a diffusion process. Now:

$$\langle e^{-i\mathbf{k}\cdot\delta\mathbf{r}(\tau)}\rangle = \langle 1 - i\mathbf{k}\cdot\delta\mathbf{r}(\tau) - \frac{(\mathbf{k}\cdot\delta\mathbf{r}(\tau))^2}{2} + \ldots\rangle \simeq 1 - \frac{k_\perp^2\langle \delta r(\tau)^2\rangle}{2}, \tag{41}$$

where the linear term vanishes because it is odd. Finally, we employ the diffusive approximation $\langle \delta r(\tau)^2\rangle \simeq 2D_\perp\tau$ and obtain

$$\langle e^{-i\mathbf{k}\cdot\delta\mathbf{r}(\tau)}\rangle \simeq 1 - k_\perp^2 D_\perp \tau \simeq e^{-k_\perp^2 D_\perp \tau}, \tag{42}$$

so that

$$D_\perp \simeq \int_0^\infty d\tau \sum_{\mathbf{k}} |\tilde{v}_{\mathbf{k}}|^2 e^{-k_\perp^2 D_\perp \tau} = \sum_{\mathbf{k}} |\tilde{v}_{\mathbf{k}}|^2 \frac{1}{k_\perp^2 D_\perp}. \tag{43}$$

We have come upon an interesting result: the scattering process itself sets the decorrelation time $\tau_c^{-1} = k_\perp^2 D_\perp{}^2$, so that the diffusivity $D_\perp$ is expressed *recursively*. This feature is generic to strong scattering processes. Moreover, we see that τ_c becomes large at large spatial scales, which will introduce a dependence on the system size. This behavior is the effect of "slow modes" which originate from the conservation of ρ (the jargon is that ρ is a "conserved order parameter").

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[2]This is closely related to the mixing length estimate of the saturation level for drift wave turbulence, $D_\perp = \gamma_k/k_\perp^2$.

Assuming a symmetric spectrum, we now have

$$D_\perp^2 \simeq \sum_{\mathbf{k}} \frac{|\tilde{v}_{\mathbf{k}}|^2}{k_\perp^2} \simeq \int k_\perp dk_\perp \frac{|\tilde{v}_{\mathbf{k}}|^2}{k_\perp^2} = \int_{k_{min}}^{k_{max}} dk_\perp \frac{|\tilde{v}_{\mathbf{k}}|^2}{k_\perp} \tag{44}$$

where the infrared cutoff is given by the system size $k_{min} \sim L^{-1}$ and the ultraviolet by the Debye length $k_{min} \sim \lambda_D^{-1}$.

We now employ $\tilde{v}_k = \frac{c}{B_0}\mathbf{E_k} \times \hat{z}$, and find

$$D_\perp \simeq \frac{c}{B_0}\left(\int_{k_{min}}^{k_{max}} dk_\perp \frac{|\tilde{E}_{\mathbf{k}}|^2}{k_\perp}\right)^{1/2}. \tag{45}$$

The Bohm-like scaling $\sim 1/B_0$ is now apparent, and the remainder of the problem is reduced to determining the electric field spectrum. If we assume thermal equilibrium, we can compute the spectrum using a test particle model[3]

$$\langle|\tilde{E}_{\mathbf{k}}|^2\rangle = \frac{4\pi k_B T}{\ell}\frac{1}{1+k_\perp^2\lambda_D^2} \tag{46}$$

(note the contribution from Debye screening). As a reminder, ℓ is again the parallel scale length such that each rod has charge per unit length is $\pm e/\ell$. Plugging into Eq. (45) and integrating yields

$$D_\perp \sim \frac{ck_BT}{eB}\left[\frac{1}{n\lambda_D^2}\ln(L/\lambda_D)\right]^{1/2}. \tag{47}$$

This recovers the Bohm result $D_B \propto ck_BT/eB$ (with α a numerical coefficient); surprisingly, this holds even at thermal equilibrium, so the Bohm diffusion is not anomalous in the usual sense. Moreover, we find a (weak) dependence on the system size L, a simple example of non-locality. As mentioned before, the non-locality is due to the presence of slow modes ($\omega \sim k_\perp^2 D_\perp$ is small at large scales), which are associated with the conserved order parameter ρ.

Taylor and McNamara also derive the electric field spectrum for a purely random array of discrete charges/rods (Fig. 4).

We have

$$\nabla\cdot\mathbf{E} = 4\pi\rho = 4\pi\sum_i \frac{q_i}{\ell}\delta(\mathbf{x}-\mathbf{x}_i). \tag{48}$$

Fourier transforming,

$$i\mathbf{k}\cdot\mathbf{E_k} = \frac{4\pi}{\ell}\sum_i q_i \exp(-i\mathbf{k}\cdot\mathbf{x}_i). \tag{49}$$

[3] The test particle picture in the 2D GC plasma actually suffers from some complications compared to the 3D Vlasov plasma. The essence of the difficulty is that the continuity equation yields a trivial linearization; in the absence of interactions, the particles are at rest and unperturbed orbits have zero length, and in the uniform plasma, electric field perturbations fail to produce a charge imbalance. As a result, the calculation of the dielectric function $\epsilon(k,\omega)$ is a bit more involved (see [11, 12, 13] for details). Ultimately, however, it yields the same answer for the spectrum as what Taylor and McNamara wrote down using pure statistical mechanics arguments.

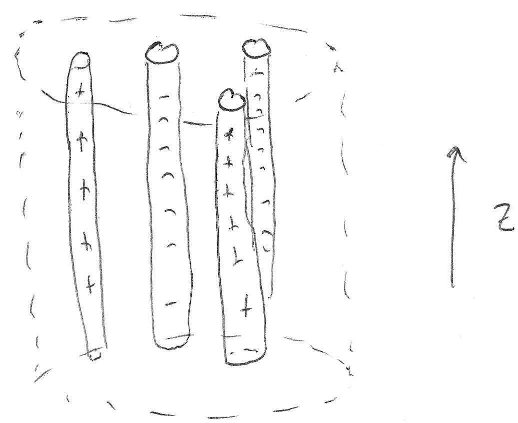

Figure 4: A random array of thin charged rods. This system exhibits strong nonlocality.

Then

$$\langle |\mathbf{E_k}|^2 \rangle = \frac{1}{k_\perp^2} \left(\frac{4\pi}{\ell} \right)^2 \left\langle \sum_{i,j} q_i q_j \exp(i\mathbf{k} \cdot (\mathbf{x}_j - \mathbf{x}_i)) \right\rangle \tag{50}$$

$$= \frac{16\pi^2 n e^2}{k_\perp^2 \ell^2}. \tag{51}$$

This result may also be found by simply taking $T \to \infty$ in the thermal equilibrium expression Eq. (46).

Now, once again plugging this into Eq. (45) and integrating, we obtain

$$D_\perp \simeq \frac{c}{B_0} \frac{4\pi (ne^2)^{1/2}}{\ell} L. \tag{52}$$

Note that the dependence on the system size is now strong (linear instead of logarithmic). Evidently, the random ensemble exhibits strong non-locality!

7 Aside on resonance broadening theory

Let us say a little more about RBT. It rests on a few fundamental assumptions: first, the excursions from unperturbed orbits are assumed to be purely stochastic, so it is only valid in regions of overlap of resonant surfaces. Furthermore, as previously mentioned, it assumes the pdf of δr is Gaussian, so that the scattering is a diffusion process. There is absolutely no reason why this should be true, and indeed no reason even that the pdf should have a finite second moment. Finally, it also employs an approximation known as the "test wave hypothesis" (similar to the test particle model): the ensemble of interacting modes E_k is large and statistically homogenous, so that any one mode may be removed from the ensemble and treated independently without altering the spectrum. This assumption ignores the effects of coherent mode coupling.

To go beyond RBT in a systematic way requires the tools of renormalization (beyond the scope of these notes), which is considerably more involved and necessarily includes uncontrolled approximations. Certainly, the appeal of RBT is its intuitiveness and ease of application.

8 More high Kubo: transport in random media and percolation

The above discussion treats the high Kubo regime diffusively, using resonance broadening theory. A few questions linger:

- Can we connect the Ku <1 and Ku >1 regimes in a unified treatment? This is partly the goal of renormalization theory, though it is unclear how well it works in this context.
- What about higher order corrections to $\chi_\perp$?
- How do ambient collisional diffusion and nonlinear/turbulent scattering synergize?
- **Is the diffusive approximation valid at high Kubo?**

We will now discuss this last question at length (the second to last will be discussed in the next lecture). Some references for this topic are [14, 15, 16, 17, 18].

We have previously considered the strongly magnetized plasma with magnetic perturbations in the limit of small Kubo, wherein ℓ_{ac} is finite, and there is an inhomogeneity in the $\hat{z}$ direction. Let us now consider the opposite extreme, where Ku $\to \infty$. In this limit, the magnetic field perturbations are completely random in the perpendicular plane, and homogenous in $\hat{z}$.

We can then introduce a scalar potential A so that $\mathbf{b} = \tilde{B}_\perp/B_0 = \nabla_\perp A \times \hat{z}$. A field line is described by the equation

$$\frac{dx}{dz} = b_r = \frac{\partial A}{\partial y} \tag{53}$$

$$\frac{dy}{dz} = b_\theta = -\frac{\partial A}{\partial x} \tag{54}$$

This problem (Fig. 5) is structurally equivalent to the guiding center plasma problem of Taylor and McNamara, where we have

$$\frac{dx}{dt} = -\frac{c}{B_0}\frac{\partial \phi}{\partial y} \tag{55}$$

$$\frac{dy}{dt} = \frac{c}{B_0}\frac{\partial \phi}{\partial x}. \tag{56}$$

With this in mind, we might charge forward without fear and compute the magnetic diffusivity with resonance broadening corrections:

$$D_M = \sum_{\mathbf{k}} \pi \left|\frac{\tilde{B}_{r,\mathbf{k}}}{B_0}\right|^2 \delta(k_\parallel) \to \sum_{\mathbf{k}} \left|\frac{\tilde{B}_{r,\mathbf{k}}}{B_0}\right|^2 \frac{i}{k_\parallel + i k_\perp^2 D_M} \tag{57}$$

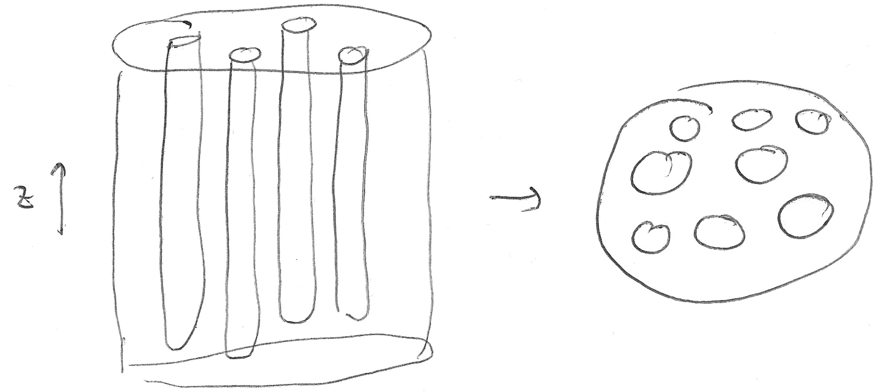

Figure 5: The static, homogeneous system with random perpendicular magnetic field is akin to a system of rods of magnetic potential A, and is structurally identical to the Taylor-McNamara problem.

so that

$$\mathrm{Re}D_M = \sum_{\mathbf{k}} \left| \frac{\tilde{B}_{r,\mathbf{k}}}{B_0} \right|^2 \frac{k_\perp^2 D_M}{k_\parallel^2 + (k_\perp^2 D_M)^2}. \tag{58}$$

For Ku < 1, $k_\perp$ is small and the Lorentzian becomes a delta function $\delta(k_\parallel)$ and we recover the RSTZ result. For Ku $\gg 1$, $k_\parallel \sim 0$ and we attain a result analogous to that of Taylor and McNamara:

$$D_M \simeq \sum_{\mathbf{k}} \left| \frac{\tilde{B}_{r,\mathbf{k}}}{B_0} \right|^2 \frac{1}{k_\perp^2 D_M} \tag{59}$$

and $D_M \simeq \left| \frac{\tilde{B}_{r,\mathbf{k}}}{B_0} \right| \Delta_\perp$.

But now we challenge the notion that the Ku > 1 regime can be treated diffusively like this. For simplicity, take the magnetic field perturbation to be stationary in time (as well as independent of $\hat{z}$). Then from Eqs. (53)-(54)

$$\frac{dy}{dx} = -\frac{\partial_x A}{\partial_y A}. \tag{60}$$

Hence $\nabla A \cdot d\mathbf{x} = 0$, and field lines traverse contours of constant A, as in a topographical map. Now, the subject of "statistical topography" enters the physics. That is, given a random perturbation A, what can we say about the statistical distribution of hills and valleys in the topography of A? We will soon see that this influences the large-scale transport properties.

We may set $\langle A \rangle = 0$ and $\langle A^2 \rangle = A_0^2$, where $\langle \cdots \rangle$ denotes an ensemble average. A_0 then sets the average height of a hill or valley in A.

A typical set of isocontours is illustrated in Fig. (6). Most contours are closed and isolated and do not contribute significantly to transport. However, certain special contours which we might call "passes" take on long path lengths and allow particles to traverse the system

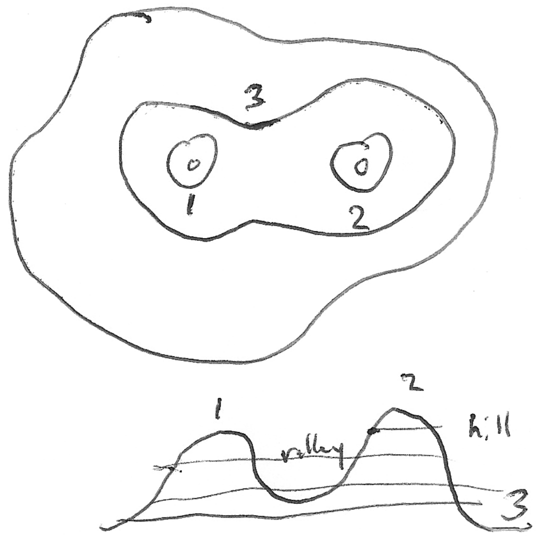

Figure 6: Isocontours of A. One expects most contours to be small and localized, like (1) and (2). However, some — like (3) — are long and allow a particle to traverse the system size.

size. Transport in this system then primarily occurs along these passes. In Fig. (7), this concept is illustrated in a system with a gradient that breaks isotropy. Transport down the gradient is isolated to contour (c), which is the only pass that connects the endpoints. If (b) corresponds to the plasma-facing component in a tokamak, the signature of such behavior would be a strongly localized strike mark on the plasma facing component.

Like a lightning bolt, the transport is sharply localized to a small number of channels/strike marks. This type of transport is clearly *not* diffusive: instead, it is emblematic of *percolation*, which we will discuss in greater detail in coming lectures. Playing the role of the mean free path, we have the mean length of the largest contour

$$\ell_A \sim |A|^{-\gamma}. \tag{61}$$

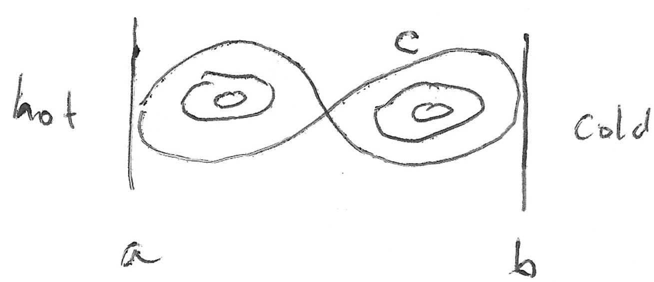

Figure 7: Illustration of isocontours in a system with a temperature gradient. Contour (c) is the only channel for transport from (a) to (b).

Percolation is characterized by the extension of this length as $A \to 0$.

9 Summary

We have discussed anomalous particle transport due to magnetic fluctuations in the low Kubo regime, invoking the Taylor identity to show that the transport is driven by the magnetic stress and the fluctuating parallel ion flow. We have applied this to show that magnetic fluctuations impact zonal flow formation by inducing charge separation. We briefly discussed momentum transport, bringing us to the idea of hyper-resistivity, which can profoundly affect magnetic reconnection. We then turned our attention to the high Kubo regime. We first treated this regime diffusively, introducing the theory of resonance broadening and applying it to the 2D guiding center plasma à la Taylor and McNamara. Finally, we challenged this diffusive treatment by arguing that transport in the high Kubo limit is at least partially governed by topography and exhibits percolation, which will we study in depth in future lectures.

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