Stochastic Fields – Toward High Ku; Random Conductivity

→ So far:

- reviewed theory of Hamiltonian chaos
- derived QL D_M
- derived $\chi_{\perp e}$ due stochastic fields in $Ku < 1$ regime – diffusion
- focused on interaction of scattering (fluctns), collisions, coarse graining
- discussed transport in GC plasma, as example of $\tau_{ac} \to \infty$ regime.

→ Observations

- idea of resonance (small denominator problem) and resonance overlap fundamental to Hamiltonian chaos.
- $Ku \sim \tau_{ac}/\tau_{scatt.}$
- might ask: unified treatment that combines $Ku < 1$, $Ku > 1$ regimes
  ⇒ renormalized response.
- in hydro treatment of $\chi_{\perp eff}$ what of nominal 3rd order contribution vs $\chi_{\perp, colln.}$. See (K+D) !
- is diffusive treatment of high Ku regime (as in Taylor + McNamara) valid?
  See Isichenko, Gruzinov papers.
- $\tau_{ac} \to \infty$ can recover strong $D_\perp$ at modest fluctn. level

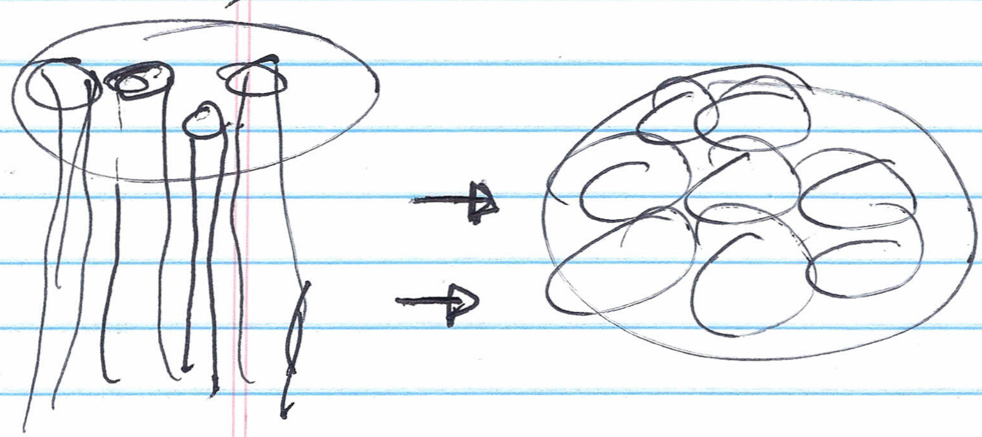
Here:

- general analysis of diffusion
- aspects of percolation, large Ku regime
- Dykhne method → conduction in random media.

⇒ Recall, $Ku \sim l_{ac}\, \delta B/B\Delta_\perp$

- have considered low Ku, with
  - finite l_{ac}
  - ⇒ inhomogeneity in z

- now, consider $Ku \to \infty$ limit, opposite
  - ⇒ random field, in x, y.
  - ⇒ homogeneous in z



i.e.

$$\left\{\begin{aligned} \frac{dx}{dz} &= b_r = \frac{\partial A}{\partial y} \\ \frac{dy}{dz} &= b_\theta = -\frac{\partial A}{\partial x} \end{aligned}\right.$$

from:

$$\frac{dr}{dz} = b_r$$

$$r\frac{d\theta}{dz} = \cancel{\frac{\langle B_\theta \rangle}{B_0}} + b_y$$

equivalent, of course $(L_s \to \infty)$, to G.C. plasma:

$$\left\{\begin{aligned} \frac{dx}{dt} &= -\frac{c}{B}\,\partial_y \phi \\ \frac{dy}{dt} &= \frac{c}{B}\,\partial_x \phi \end{aligned}\right.$$

$\Rightarrow$ motivates study of random media transport!

Formally, can extend D_M calculation to include resonance broadening

i.e.
$$\frac{\partial}{\partial z}\tilde{f} + b \cdot \nabla \tilde{f} = -\tilde{b}_r \frac{\partial \langle f \rangle}{\partial r}$$

⇒

$$D_M = \sum_k |\tilde{b}_{r_k}|^2 \frac{i}{k_{\parallel} + i k_{\perp}^2 D_M}$$

where

$$k_{\perp}^2 D_M / k_{\parallel} \sim K u^2$$

For $Ku < 1$ ⇒

$$D_M \simeq \sum_k |\tilde{b}_{r_k}|^2 \delta(k_{\parallel}) \text{, a la' RSTZ}$$

For $Ku \gg 1$

$$D_M \simeq \sum_k |\tilde{b}_{r_k}|^2 / k_{\perp}^2 D_M$$

a la' Taylor, McNamara.

$$\Rightarrow \quad D_M \cong \left(\sum_{k} |\tilde{A}_k|^2 \right)^{1/2}$$

$$\sim \tilde{b} \, \Delta$$

So

$Ku < 1 \Rightarrow D_M \sim \tilde{b}^2 l_{ac}$

$Ku > 1 \Rightarrow D_M \sim \tilde{b} \, \Delta$

and transport $\sim \langle A^2 \rangle^{1/2}$

But, is $Ku > 1$ regime really diffusive?

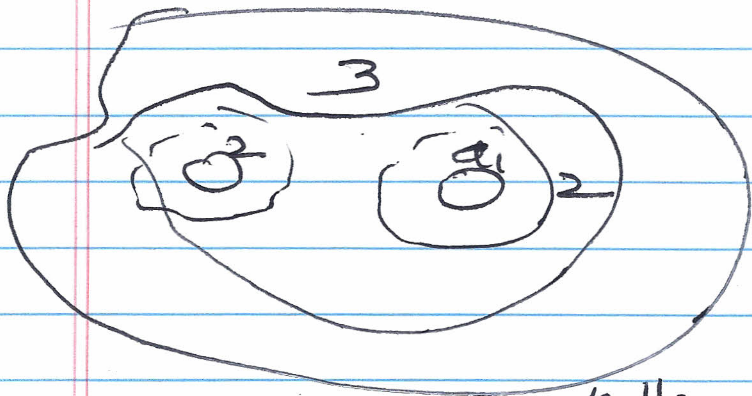
→ recall: $\frac{dx}{dz} = \nabla_\perp A \times \hat{z}$

{ akin 2D random media, for A indep z.

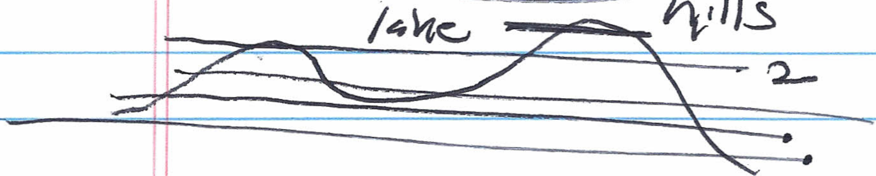
→ Can view physically as:

topographical map

i.e.



Map



levels

Now, as $\frac{dx}{\partial_y A} = \frac{dy}{-\partial_x A} = \frac{dz}{1}$

$\Rightarrow$

$$dy/dx = -\partial_x A / \partial_y A$$

$\Rightarrow$

$$\boxed{\underline{\nabla} A \cdot \underline{dx} = 0}$$

$\Rightarrow$ – Lines traverse const A contours, as on map

– $\langle A \rangle = 0$, $\langle A^2 \rangle = A_0^2$

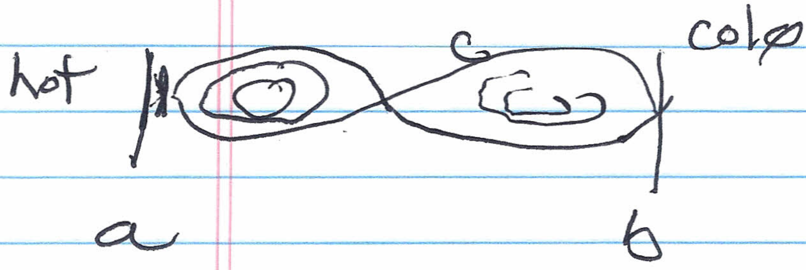
– avg. depth, height of "lakes", "hills" set by A_0

— most contours closed, isolated
⇒ little contribution to transport

— but contours along "passes", i.e. 3, can take on long path lengths.

⇒ transport occurs primarily along these

⇒ <u>percolation</u>, not diffusion



a → b transport isolated along contour c.

~ more like 'lightning bolt' than diffusion

~ signature would be sharply localized strike mark (if b → PFC) and not periodic

~ percolation → extension of mean length as $A \to 0$

$$l_A \sim A^{-\gamma} \qquad , \gamma \sim 2.4.$$

Message:

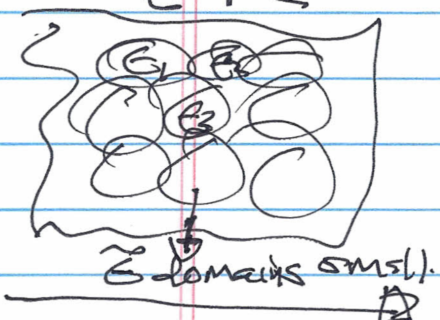
- to understand $Ku > 1$ regime, useful to examine

→ transport in random media

→ percolation theory

– Random Media ⇒ { characterize media with / static ensemble scatterers. → 2D

i) Effective Permittivity – Mean Field Theory (CL/CM)



– what is effective $\underline{D}$, ϵ?

– given local fluctuations

$$\epsilon = \bar{\epsilon} + \tilde{\epsilon}$$

what is effective ϵ?

Now:

– $\bar{\underline{D}} = \bar{\epsilon}\,\bar{\underline{E}}$ Mean field question

($\bar{\underline{D}}$: mean; $\bar{\underline{E}}$: mean; $\bar{\epsilon}$: effective ϵ)

– $\bar{\epsilon} = \frac{1}{V}\int d^3x\, \epsilon$

Now

$$\nabla\cdot(\epsilon\,\underline{E}) = 0$$

proceed by CL/MFT

$$\Rightarrow \nabla\cdot\left((\bar{\epsilon}+\tilde{\epsilon})(\bar{\underline{E}}+\tilde{\underline{E}})\right) = 0$$

where: $\bar{\underline{D}} = \overline{(\bar{\epsilon}+\tilde{\epsilon})(\underline{E}+\tilde{\underline{E}})} = \bar{\epsilon}\,\bar{\underline{E}} + \overline{\tilde{\epsilon}\,\tilde{\underline{E}}}$

↑ $\mathcal{Q}$

so, need $\tilde{E}$:

$$\nabla\cdot(\tilde{\epsilon}\,\tilde{\underline{E}}) + \tilde{\epsilon}\,\underline{\nabla}\cdot\tilde{\underline{E}} + \bar{\underline{E}}\cdot\nabla\tilde{\epsilon} + \not{2}^{nd} = \nabla\cdot\underline{D} = 0$$

to first order:

$$\bar{\epsilon}\,\underline{\nabla}\cdot\tilde{\underline{E}} = -\bar{\underline{E}}\cdot\nabla\tilde{\epsilon}$$

{relates $\tilde{E}$ to $\tilde{\epsilon}$}

Now, avg. over volume of {domains / particles} $\langle\ \rangle$

⇒ {exploits isotropy!}

$\langle\ \rangle$ in principle can be different from $\bar{\ }$.

$$\Rightarrow \partial_x\langle\tilde{E}_x\rangle = \partial_y\langle\tilde{E}_y\rangle = \partial_z\langle\tilde{E}_z\rangle$$

$$= \frac{1}{3}\langle\underline{\nabla}\cdot\tilde{\underline{E}}\rangle$$

so, taking $\bar{\underline{E}}$ to be: $\underline{\bar{E}} = \bar{E}\,\hat{x}$ (general)

$$\langle\underline{\nabla}\cdot\tilde{\underline{E}}\rangle = 3\,\partial_x\langle\tilde{E}_x\rangle$$

so

$$3\bar{\epsilon}\,\partial_x\langle\tilde{E}_x\rangle = -\bar{\underline{E}}\cdot\nabla\tilde{\epsilon}$$

↑ replaced $\tilde{E}_x$ with $\langle\tilde{E}_x\rangle$

$$\langle \tilde{E}_x \rangle = - \frac{\bar{E}_x \tilde{\epsilon}}{3\bar{\epsilon}}$$

then mean $\bar{D}$

$$\overline{\tilde{\epsilon}\tilde{E}} = - \overline{\tilde{\epsilon}\left(\frac{\bar{E}_x \tilde{\epsilon}}{3\bar{\epsilon}}\right)}$$

$$= - \frac{\overline{\tilde{\epsilon}^2}}{3\bar{\epsilon}} \bar{E}_x$$

$$\bar{D} = \bar{\epsilon}\bar{E} + \overline{(\tilde{\epsilon}\tilde{E})}$$

$$= \left(\bar{\epsilon} - \frac{1}{3}\frac{\overline{\tilde{\epsilon}^2}}{\bar{\epsilon}}\right)\bar{E}$$

so

$$\epsilon_{eff} = \bar{\epsilon} - (1/3)\frac{\overline{\tilde{\epsilon}^2}}{\bar{\epsilon}}$$

✓

⇒ effective permittivity of mixture, as calculated in MFT.

iia) Effective Conductivity — See Dykhne paper.

Could ask similar problem:

→ $\tilde{\sigma}$ chunks

→ $\overline{E}$ applied

Current.

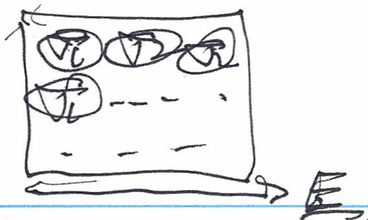
Have: $\underline{J} = \sigma \underline{E}$

$\underline{\nabla} \cdot \underline{J} = 0$

Could approach by P.T. and CLT but do better by considering symmetry.

In particular consider:

- → 2D random medium
- → consists of random assortment of two arbitrary conductivity elements σ_1, σ_2. Note σ need not be "small"
- → consider σ_1, σ_2 as "phases", with domains $l_{1,2} \ll L_0$



conduction path convoluted for $\sigma_2 \neq \sigma_1$!

→ Defines Pblm: Conductivity of 2-Phase System

Assume: – 1:1 mixture
– avg. geometrical config. equal

i.e. – ② equal # domains
– no spatial bias.

$\sigma = \sigma(x, y)$ $\sigma = \begin{cases} \sigma_1 \\ \sigma_2 \end{cases}$ have equal areas.

of course:

$$\begin{cases} \underline{J} = \sigma \underline{E} \\ \nabla \times \underline{E} = 0 \\ \nabla \cdot \underline{J} = 0 \end{cases}$$

seek: σ_{eff}, s/t

$$\langle \underline{J} \rangle = \sigma_{eff} \langle \underline{E} \rangle$$

where $\langle \; \rangle$ represents a volume avg.

$$\langle \underline{J} \rangle = \frac{1}{V} \int d^3x \, \underline{J}$$

$$\langle \underline{E} \rangle = \frac{1}{V} \int d^3x \, \underline{E}$$

Goal is effective proportionality between mean $\underline{J}$, $\langle \underline{J} \rangle$; and mean $\underline{E}$, $\langle \underline{E} \rangle$.

N.B: – σ_{eff} is macroscopic in uniform medium.

– dispersion in σ, i.e. $\langle(\sigma - \bar{\sigma})^2\rangle$ will increase with volume (i.e. # scatts ↑)

Now can exploit the observation; can write:

$\hat{n} \perp$ plane

$$\left.\begin{aligned} \underline{J}' &= (\sigma_1\sigma_2)^{1/2}\,\hat{\underline{n}}\times\underline{E} \\ \underline{E}' &= (\sigma_1\sigma_2)^{-1/2}\,\hat{\underline{n}}\times\underline{J} \end{aligned}\right\} \times \quad \text{rotate \& rescale}$$

and plug into:

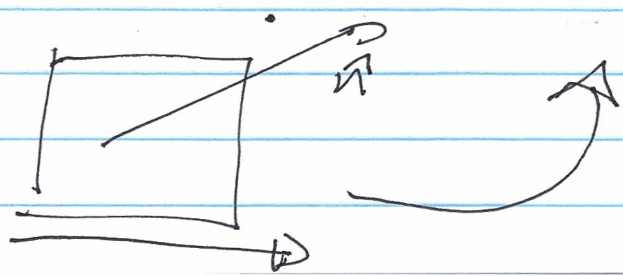
$$\left\{\begin{aligned} \underline{J} &= \sigma\underline{E} \\ \underline{\nabla}\times\underline{E} &= 0 \\ \underline{\nabla}\cdot\underline{J} &= 0 \end{aligned}\right.$$

Now

σ' tbd

$$\underline{J}' = (\sigma_1\sigma_2)^{1/2}\,\hat{n}\times\underline{E}$$
$$= \sigma'\underline{E}' = \left((\sigma_1\sigma_2)^{-1/2}\,\hat{n}\times\underline{J}\right)\sigma'$$

so, simply:



$$\frac{\underline{J}'}{(\sigma_1\sigma_2)^{1/2}} \qquad \hat{n} \times \underline{J} = (\sigma_1\sigma_2)^{1/2}\, \hat{n} \times \underline{E}$$

and

$$\underline{J} = \sigma\, \underline{E}$$

$$\Rightarrow \sigma' = (\sigma_1\sigma_2)^{1/2}$$

but since system statistically homogeneous;

$$\langle \underline{J}' \rangle = \sigma_{eff} \langle \underline{E}' \rangle$$
$$\langle \underline{J} \rangle = \sigma_{eff} \langle \underline{E} \rangle$$
(**)

i.e. effective conductivity same!

now simply averaging (*)

$$\langle \underline{J}' \rangle = (\sigma_1\sigma_2)^{1/2}\, \hat{n} \times \langle \underline{E} \rangle$$

$$\langle \underline{E}' \rangle = (\sigma_1\sigma_2)^{-1/2}\, \hat{n} \times \langle \underline{J} \rangle$$

and noting (**), indeed

$$\boxed{\sigma^* = (\sigma_1\sigma_2)^{1/2}}$$

Note:

- effective conductivity is geometric mean of σ_1, σ_2

- exact result ⇒ { 2D; $l_{1,2} < L_0$; stat. homogeneity }

{ exploits rotation, rescaling, statistical symmetry }

Now can generalize to more general conductivity $\sigma(x, y)$.

Take: $x(x,y) = \ln\sigma - \langle \ln\sigma \rangle$

↓

{ fluctuation in conductivity

$f(1, 2, \ldots | x)$ even

i.e. $\langle x(1)\, x(2) \rangle$ etc }

similar arguments ⇒

$$\sigma_{eff} = \exp\langle \ln\sigma \rangle = \left(\langle \sigma \rangle / \langle 1/\sigma \rangle \right)^{1/2}$$

For Gaussian distribution of flctns.

- $\langle \ln \sigma \rangle = \ln \sigma - \chi$

$$\sigma_{eff} = \left\langle \exp[\ln \sigma - \chi] \right\rangle_\chi$$

$$= \langle \sigma e^{-\chi} \rangle_\chi$$

$$= \langle \sigma \rangle \left\langle \left(1 - \chi - \frac{\chi^2}{2} + \ldots \right) \right\rangle$$

$$\simeq \langle \sigma \rangle e^{-\overline{\chi^2}/2}$$

$$\boxed{\sigma_{eff} = \langle \sigma \rangle e^{-\Delta^2/2}, \quad \Delta^2 = \langle \chi^2 \rangle}$$

→ What of distribution of currents and fields?

- Can calculate distribution of currents fields over phases

⇒ i.e. what are statistical properties of currents, phases?

⇒ what is avg. over a phase?

Observe:

$$\underline{A} \equiv \langle (\sigma - \sigma_1) \underline{E} \rangle$$

$$= \langle \underline{J} \rangle - \sigma_1 \underline{E}$$

but also, $\underline{A} \neq 0$ only in σ_2 phases

$$\underline{A} = \langle (\sigma - \sigma_1) \underline{E} \rangle$$

$$= \left\langle \begin{pmatrix} \sigma_1 & ① \\ \sigma_2 & ② \end{pmatrix} \underline{E} - \sigma_1 \underline{E} \right\rangle$$

cancels in phase ①!

$$\equiv \frac{1}{2} (\sigma_2 - \sigma_1) \langle \underline{E}_2 \rangle$$

$$\underline{E}_2 = \frac{1}{V_{②}} \int_{V_2} d^3x \, \underline{E}$$ → avg over phase 2

↳ note

equating:

$$\langle \underline{J} \rangle - \sigma_1 \underline{E} = \frac{1}{2} (\sigma_2 - \sigma_1) \langle \underline{E}_2 \rangle$$

and re-grouping using:

$$\langle \underline{J} \rangle = \sigma_{eff} \langle E \rangle$$

$$\sigma_{eff} = (\sigma_1 \sigma_2)^{1/2} \langle \underline{E} \rangle$$

Then,

$$\begin{cases} \langle E_2 \rangle = \dfrac{2\sqrt{\sigma_1}}{\sqrt{\sigma_1} + \sqrt{\sigma_2}} \underline{E} \\ \\ \langle E_1 \rangle = \dfrac{2\sqrt{\sigma_2}}{\sqrt{\sigma_1} + \sqrt{\sigma_2}} \underline{E} \end{cases}$$

and $$\langle J \rangle_{1,2} = \frac{2\sqrt{\sigma_{1,2}}}{\sqrt{\sigma_1} + \sqrt{\sigma_2}} \langle \underline{J} \rangle$$

Similarly, can calculate power dissipated in each phase (equal)!

- energy dissipated same in both 1, 2
- Joule heat dist same.

and cranking reveals:

$$\langle E^2\rangle = \frac{1}{2}\left(\sqrt{T_1/T_2} + \sqrt{T_2/T_1}\right)\langle E\rangle^2$$

$$\langle J^2\rangle = \frac{1}{2}\left(\sqrt{T_1/T_2} + \sqrt{T_2/T_1}\right)\langle J\rangle^2$$

$$\frac{\langle J^2\rangle - \langle J\rangle^2}{\langle J\rangle^2} = \frac{1}{2}\left[\left(T_1/T_2\right)^{1/2} - \left(T_2/T_1\right)^{1/2}\right]^2$$

same for E.