

# Transport in Stochastic Fields

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## Contents

<b>1</b>	<b>Motivation</b>	<b>2</b>
<b>2</b>	<b>Basics of Hamiltonian chaos</b>	<b>3</b>
2.1	Hamiltonian mechanics . . . . .	3
2.2	Canonical transformation . . . . .	4
2.3	Integrable systems . . . . .	5
2.4	Effect of Perturbation for Single Orbit . . . . .	8
2.5	The fate of a resonant 2 torus . . . . .	10
2.6	Multiple resonant 2 tori . . . . .	15
<b>3</b>	<b>Stochastic magnetic field lines in a torus</b>	<b>18</b>
3.1	Magnetic field lines in a tokamak . . . . .	18
3.2	Wandering of magnetic field lines . . . . .	22
3.3	Diffusion of magnetic field lines . . . . .	26
3.4	Length scales in diffusion of field lines . . . . .	30
<b>4</b>	<b>Transport in stochastic fields</b>	<b>35</b>
4.1	Collisionless heat transport . . . . .	36
4.2	Collisional heat transport . . . . .	40
4.3	Systematic derivation of transport from hydrodynamics . . . . .	44

# 1 Motivation

How many magnetic field lines are there in our universe?

The answer is probably **only one!** Because of the perturbation to magnetic field in the real world, field lines can wander and diffuse in space, forming a stochastic distribution and chaotic system. Therefore, **even a single field line can fill the volume of the entire universe.** This phenomenon leads to the perpendicular transport of charged particles and energy. Perpendicular transport is a crucial issue for the researches on magnetic confinement for nuclear fusion reaction, and the magnetohydrodynamic (MHD) turbulence in interstellar-medium (ISM). Therefore, it is appealing to study this interesting system. In this lesson, we'll start from reviewing the theory of Hamiltonian chaos, then apply it to the system of magnetic field lines, and finally derive the heat transport rate (thermal diffusivity) in the stochastic fields.

## 2 Basics of Hamiltonian chaos

Hamiltonian mechanics is a formalism of classical mechanics that is different from but equivalent to Newtonian mechanics and Lagrangian mechanics. In the following we review its framework and discuss chaotic systems. This section follows the content in [Edward Ott, *Chaos in dynamical systems* (Cambridge University Press, 2002), Ch. 7].

### 2.1 Hamiltonian mechanics

In Hamiltonian mechanics, a system can be described by a set of generalized coordinate  $\mathbf{q}$  and generalized coordinate  $\mathbf{p}$ , which are not necessarily the real coordinate and momentum of particles. Both  $\mathbf{q}$  and  $\mathbf{p}$  have a dimension  $N$  equal to the degree of freedom of the system. The dynamics of a mechanical system can be solely specified by a single scalar function  $H(\mathbf{p}, \mathbf{q}, t)$  called "Hamiltonian". The equations of motion in Hamiltonian mechanics are the **Hamiltonian equations**:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

$q_i$  and  $p_i$  satisfying the Hamiltonian equations are said to be "canonically conjugate". If the equations defining  $q_i, p_i$  does not explicitly depend on time, and if the forces are derivable from a conservative potential  $V$ ,  $H$  is identical to the total energy  $E = K + V$ . If the Hamiltonian has no explicit time dependence, i.e.  $H = H(\mathbf{p}(t), \mathbf{q}(t))$ , we have

$$\frac{dH}{dt} = \dot{q}_i \frac{\partial H}{\partial q_i} + \dot{p}_i \frac{\partial H}{\partial p_i} = \frac{\partial H}{\partial p_i} \frac{\partial H}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial H}{\partial p_i} = 0$$

Thus we say that  $E = H$  is a "constant of the motion" since it's conserved.

The space described by the canonical coordinates  $(\mathbf{p}, \mathbf{q})$  is called "phase space" ( $2N$ -dimensional). **Liouville theorem** states that Hamiltonian equations preserve the phase-space volume. **Poincaré recurrence theorem** states that a closed volume will eventually return to its original state within long enough but finite time. We can interpret it as stating that any trajectory in phase space must be a closed orbit. However, the theorem doesn't specify how long it is needed to finish one complete cycle.

## 2.2 Canonical transformation

Similar as coordinates in Newtonian mechanics, the choice of  $(\mathbf{p}, \mathbf{q})$  is not unique in Hamiltonian mechanics. We can change from  $(\mathbf{p}, \mathbf{q})$  to another canonical coordinate  $(\bar{\mathbf{p}}, \bar{\mathbf{q}})$  as long as there exists a new Hamiltonian  $\bar{H}(\bar{\mathbf{p}}, \bar{\mathbf{q}}, t)$  such that Hamiltonian equation is satisfied:

$$\dot{\bar{q}}_i = \frac{\partial \bar{H}}{\partial \bar{p}_i}, \quad \dot{\bar{p}}_i = -\frac{\partial \bar{H}}{\partial \bar{q}_i}$$

Such process of changing from  $H(\mathbf{p}, \mathbf{q}, t)$  to  $\bar{H}(\bar{\mathbf{p}}, \bar{\mathbf{q}}, t)$  that preserves the Hamiltonian equation is called a "canonical transformation". A canonical transformation can be achieved by finding the corresponding generating function  $S$  such that

$$\lambda(p_i \dot{q}_i - H) = \bar{p}_i \dot{\bar{q}}_i - \bar{H} + \frac{dS}{dt}$$

for some constant  $\lambda$ .

It is clear that  $S$  is not unique for a transformation. For convenience, we can choose  $S = S(\bar{\mathbf{p}}, \mathbf{q}, t)$ , and Hamiltonian equations require that

$$\boxed{\frac{\partial S}{\partial \bar{p}_i} = \bar{q}_i, \quad \frac{\partial S}{\partial q_i} = p_i}$$

As a consequence, we will have

$$\boxed{\bar{H}(\bar{\mathbf{p}}, \bar{\mathbf{q}}, t) = H(\mathbf{p}, \mathbf{q}, t) + \frac{\partial S}{\partial t}}$$

### 2.3 Integrable systems

We already see that for a time-independent system, the total energy  $H = E$  is a constant of the motion. A time-independent system is called "integrable" if it has  $N$  constants of the motion, where  $N$  is the degree of freedom. It can be shown that for an integrable system, a trajectory  $(\mathbf{p}(t), \mathbf{q}(t))$  in the  $2N$ -dimensional phase space lies on an  $N$ -dimensional torus.

A convenient choice of canonical coordinates for an integrable system is

$$(\bar{\mathbf{p}}, \bar{\mathbf{q}}) = (\mathbf{J}, \boldsymbol{\theta})$$

, where the "action variables"

$$J_i = \frac{1}{2\pi} \oint_{\gamma_i} \mathbf{p} \cdot d\mathbf{q}$$

are chosen to be constants of motion with  $i = 0, 1, 2, \dots, N$ ,  $\gamma_i$  are  $N$  irreducible paths on the torus, and  $\theta_i$  are called the "action angle variables". When one travels through an entire cycle on the irreducible orbit  $\gamma_i$ , value of the corresponding  $\theta_i$  will increase by  $2\pi$ , while values of the other components  $\theta_j$  ( $j \neq i$ ) will return to their original value. That is the reason why they are called "angle" variable. An example for irreducible orbits on a "2 torus" ( $N = 2$ ) is shown in Fig. 1.

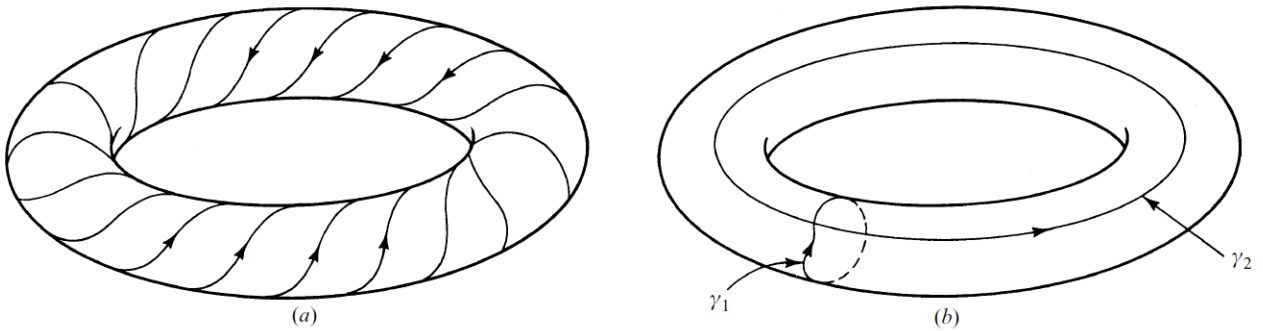


Figure 1: (a) An orbit on a 2 torus (b) Two irreducible orbits  $\gamma_1$  and  $\gamma_2$  on a 2 torus

By construction,  $dJ_i/dt = 0$ . From Hamiltonian equation we have  $\partial \bar{H} / \partial \theta_i = 0$ , thus the new Hamiltonian is only a function of the action variables  $\bar{H} =$

$\bar{H}(\mathbf{J})$ . Hamiltonian equations then becomes:

$$\begin{aligned}\frac{d\mathbf{J}}{dt} &= 0 \\ \frac{d\boldsymbol{\theta}}{dt} &= \frac{\partial \bar{H}(\mathbf{J})}{\partial \mathbf{J}} \equiv \boldsymbol{\omega}(\mathbf{J})\end{aligned}$$

, where  $\boldsymbol{\omega}$  is an angular velocity vector. For given  $\mathbf{J}$ ,  $\boldsymbol{\omega}(\mathbf{J})$  is constant so that  $\boldsymbol{\theta}(t) = \boldsymbol{\theta}_0 + \boldsymbol{\omega}(\mathbf{J})t$ .

Recall Poncairé recurrence theorem, a trajectory on the phase-space torus has "periodicity" in the sense that it will eventually return to its original position if time is long enough. But depending on the choice of torus, some trajectories might have to fill the entire surface before returning back (namely ergodic). A torus such that  $\mathbf{m} = (0, 0, \dots, 0)$  is not the only solution to

$$\boldsymbol{\omega} \cdot \mathbf{m} = 0, m_i \in \text{integer}$$

is called a "resonant" torus, otherwise it's called a "non-resonant" one. We shall see the reason for such naming later by examining the Hamiltonian's response to perturbation. It can be shown that resonant tori are countable, so the probability of finding a point on a non-resonant torus in phase space is 1. But resonant tori are also dense in phase space, so arbitrarily near to any non-resonant torus there exist resonant tori.

Suppose we have a 2-torus ( $N = 2$ ) system where  $\boldsymbol{\omega} = (\omega_1, \omega_2)$ , then a resonant torus has

$$\frac{\omega_1}{\omega_2} = \frac{p}{q}; p, q \in \text{integer}$$

, so it is called a "rational surface". On the other hand, a non-resonant torus has

$$\frac{\omega_1}{\omega_2} \in \text{irrational number}$$

We use Fig. 2 to illustrate their difference between them. When a trajectory pass through the  $\theta_2 = (\text{const.})$  cross section, one crossing point is made. Next time it pass through the cross section, another crossing point is made. Since each trajectory is on a certain toroidal surface, every crossing point is on the same circle. Lets start from a black point, and the crossing points for the following few times are as shown. We can see that for trajectories on

a rational surface, since  $\delta\theta_1 = 2\pi\omega_1/\omega_2$  between two neighbouring crossing points is a rational multiple of  $2\pi$ , the trajectory eventually comes back to its original position and repeats the same motion. Therefore, the crossing points are in a countable set, and the trajectory is a closed orbit on the rational surface. However, if the trajectory is on an irrational surface, since  $\delta\theta_1$  is not a rational multiple of  $2\pi$ , the crossing point never comes back within finite time. That is to say, the crossing points will finally "fill" the whole circle, and the trajectory on an irrational surface is a so-called "ergodic" trajectory that fills the whole surface.

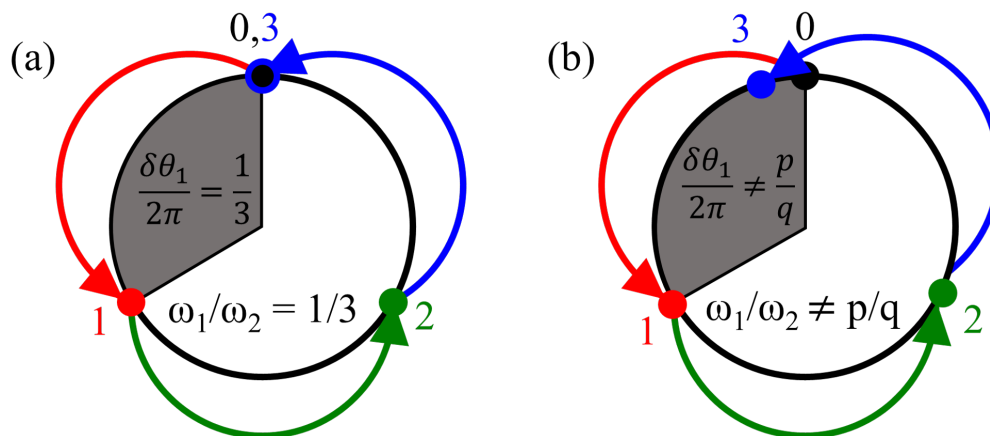


Figure 2: Example of trajectory on the  $\theta_2 = (\text{const.})$  cross section for (a) a rational (resonant) surface (b) a non-resonant surface.

## 2.4 Effect of Perturbation for Single Orbit

Integrable systems are easier to study because in phase space, each torus is a constant-Hamiltonian surface specified by a constant vector  $\mathbf{J}$ . It is our interest to see if integrability still holds if we perturb an integrable system. We will see that the rational surfaces define natural resonances of the system.

Suppose the perturbed system is still time-independent, we can assume the new Hamiltonian to be:

$$H(\mathbf{J}, \boldsymbol{\theta}) = H_0(\mathbf{J}) + \epsilon H_1(\mathbf{J}, \boldsymbol{\theta})$$

, where  $H_0$  is the unperturbed Hamiltonian of the integrable system,  $H_1$  is the perturbation, and  $\epsilon$  is a small number. Integrability of the new system is equivalent to state that there exists a new coordinate  $(\mathbf{J}', \boldsymbol{\theta}')$  such that

$$H(\mathbf{J}, \boldsymbol{\theta}) = H'(\mathbf{J}')$$

Then in terms of the generating function  $S(\bar{\mathbf{p}}, \mathbf{q}) = S(\mathbf{J}', \boldsymbol{\theta})$ , we must have

$$\begin{aligned} H'(\mathbf{J}') &= H(\mathbf{J}, \boldsymbol{\theta}); \quad \frac{\partial S(\mathbf{J}', \boldsymbol{\theta})}{\partial \mathbf{J}'} = \boldsymbol{\theta}'; \quad \frac{\partial S(\mathbf{J}', \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = \mathbf{J} \\ \Rightarrow H'(\mathbf{J}') &= H\left(\frac{\partial S}{\partial \boldsymbol{\theta}}, \boldsymbol{\theta}\right) = H_0\left(\frac{\partial S}{\partial \boldsymbol{\theta}}\right) + \epsilon H_1\left(\frac{\partial S}{\partial \boldsymbol{\theta}}, \boldsymbol{\theta}\right) \end{aligned}$$

For an unperturbed system,  $\epsilon = 0$  and  $\mathbf{J}' = \mathbf{J}$ , we have

$$\mathbf{J}' = \frac{\partial S_0}{\partial \boldsymbol{\theta}} \Rightarrow S_0 = \mathbf{J}' \cdot \boldsymbol{\theta} = \mathbf{J} \cdot \boldsymbol{\theta}$$

Then expand  $S$  near  $S_0$  by  $\epsilon$

$$S = \mathbf{J} \cdot \boldsymbol{\theta} + \epsilon S_1 + \epsilon^2 S_2 + \dots$$

Substitute it into the equation for  $H'$ ,

$$H'(\mathbf{J}') = H_0\left(\mathbf{J} + \epsilon \frac{\partial S_1}{\partial \boldsymbol{\theta}} + \epsilon^2 \frac{\partial S_2}{\partial \boldsymbol{\theta}} + \dots\right) + \epsilon H_1\left(\mathbf{J} + \epsilon \frac{\partial S_1}{\partial \boldsymbol{\theta}} + \dots, \boldsymbol{\theta}\right)$$

Expand it by  $\epsilon$  and keep only up to first-order terms,

$$H'(\mathbf{J}') = H_0(\mathbf{J}) + \epsilon \frac{\partial H_0}{\partial \mathbf{J}} \cdot \frac{\partial S_1}{\partial \boldsymbol{\theta}} + \epsilon H_1(\mathbf{J}, \boldsymbol{\theta})$$



The system has periodicity in  $\boldsymbol{\theta}$  with  $2\pi$  period, thus we can express the perturbation  $H_1$  and  $S_1$  as Fourier series of  $\boldsymbol{\theta}$

$$H_1 = \sum_{\mathbf{m}} H_{1,\mathbf{m}}(\mathbf{J}) \exp(i\mathbf{m} \cdot \boldsymbol{\theta})$$

$$S_1 = \sum_{\mathbf{m}} S_{1,\mathbf{m}}(\mathbf{J}) \exp(i\mathbf{m} \cdot \boldsymbol{\theta})$$

, where  $\mathbf{m}$  is an  $N$  dimensional vector of integers. Substitute it into the equation, we have

$$S_1 = i \sum_{\mathbf{m}} \frac{H_{1,\mathbf{m}}(\mathbf{J})}{\mathbf{m} \cdot \boldsymbol{\omega}_0(\mathbf{J})} \exp(i\mathbf{m} \cdot \boldsymbol{\theta})$$

, where  $\boldsymbol{\omega}_0(\mathbf{J}) \equiv \frac{\partial H_0(\mathbf{J})}{\partial \mathbf{J}}$  is the unperturbed angular frequency vector.

Now it's clear that if the unperturbed torus (defined by  $\mathbf{J}$ ) is chosen to make  $\mathbf{m} \cdot \boldsymbol{\omega}_0 = 0$ , our linearized solution does not work. We then encounter a **small denominator problem**. These tori are called "resonant tori", analogous to the resonant particles in Landau damping phenomena. In a 2-torus system, the condition of a resonant torus is expressed as

$$m\omega_1 + n\omega_2 = 0 \Rightarrow \frac{m}{n} = -\frac{\omega_2}{\omega_1} = \frac{q}{p}$$

, where  $\frac{m}{n}$  is the pitch of the perturbation of Hamiltonian, and  $\frac{q}{p}$  is the pitch of the trajectory.

On the other hand, the non-resonant tori keep their integrability under perturbation. Their perturbed Hamiltonian continuously approaches the unperturbed  $H_0$  as  $\epsilon \rightarrow 0$ . We say that they "survive" the small perturbation, while the resonant tori are "destroyed". Since resonant surfaces are countable in phase space, the Kolmogorov-Arnold-Moser (**KAM**) **theorem** states that for small  $\epsilon$ , "most" tori survive the perturbation. [Edward Ott, *Chaos in dynamical systems* (Cambridge University Press, 2002), Ch. 7] However, we should note that resonant tori are also dense in phase space, so we also have to consider the interaction between multiple nearby resonant tori.

## 2.5 The fate of a resonant 2 torus

What happens to the resonant tori is the central issue in chaos theory. Let's start from studying a single resonant torus. For simplicity, we study the case for a 2 torus.

To label each torus in a simpler way, we take a cross section of phase space that's defined by  $\theta_2 = (\text{const.})$ . Intersection of each constant- $\mathbf{J}$  torus with the cross section is a contour. Without loss of generality, we assume that these contours are concentric circles of constant radii  $r$ , and the angle  $\theta_1 \equiv \phi$  indicates the angular position of a point on each contour. See Fig. 3 for visualization of the coordinate system.

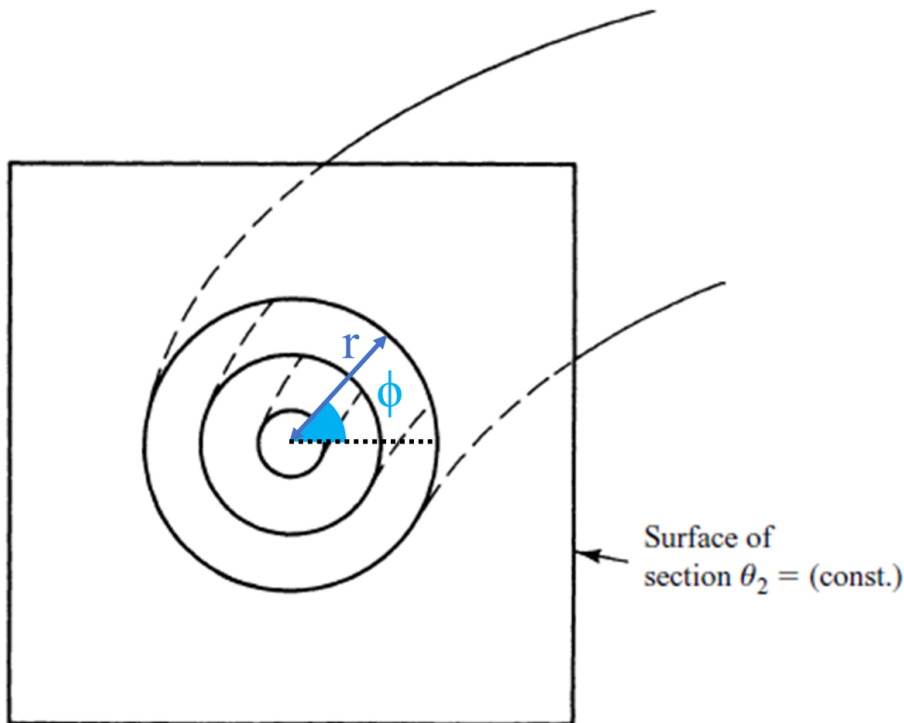


Figure 3: Definition of  $r$  and  $\phi$  on a constant  $\theta_2$  cross section

We can define the rotation rate for each torus

$$R(r) = \frac{\omega_1}{\omega_2}$$

, where  $\omega_1$  is the angular velocity of  $\phi$  for a trajectory on the torus, and  $\omega_2$  that of  $\theta_2$ . For a resonant torus,  $R$  is a rational number  $p/q$  defined by integers  $p$  and  $q$  that have no common factors. The intersection of a trajectory

on a resonant torus with the  $\theta_2 = (\text{const.})$  cross section consists of exactly  $p$  points on the circle, since it travels back to its original position for every  $p$  times passing through the cross section. However, for a non-resonant torus,  $R$  is an irrational number. The intersection of a trajectory on it would be the entire circle.

For convenience, we define a mapping  $\mathbf{M}^{\tilde{q}}$  locally near a resonance circle  $R(\hat{r}) = \tilde{p}/\tilde{q}$ , which projects a point on the cross section to the point its trajectory piercing through the cross section for the  $q^{\text{th}}$  time later. We denote the two points  $(r_n, \phi_n)$  and  $(r_{n+1}, \phi_{n+1})$ . Then the mapping for the unperturbed system (denoted by "0") is:

$$(r_{n+1}, \phi_{n+1}) = \mathbf{M}_0^{\tilde{q}}(r_n, \phi_n)$$

It is obvious that

$$r_n = r_{n+1}$$

for any initial points  $r_n$ , since they travel on the corresponding torus. However,

$$\phi_n = \phi_{n+1}$$

only holds for the resonance circle  $r = \hat{r}$  itself. Suppose  $R(r)$  is a smooth function of  $r$ , we'll have the angular displacement

$$\phi_{n+1} - \phi_n$$

having opposite sign between inside and outside  $r = \hat{r}$ . We can find a circle  $r = r_+$  in the vicinity of  $r = \hat{r}$  where  $\mathbf{M}_0^{\tilde{q}}$  is counter-clockwise rotation, and on the opposite side a circle  $r = r_-$  where  $\mathbf{M}_0^{\tilde{q}}$  is clockwise rotation. See Fig. 4(a) for conceptual picture.

Now we can consider that the system have a sufficiently small perturbation (denoted by  $\epsilon$ ) in the Hamiltonian. Although we don't know the effect of the new mapping  $\mathbf{M}_\epsilon^{\tilde{q}}$  on the resonant circle  $r = \hat{r}$  itself, we do know that the effect on the nearby non-resonant circles are negligibly small. Therefore, in the vicinity of  $r = \hat{r}$  there still exists a circle  $r = r_+$  making  $\mathbf{M}_\epsilon^{\tilde{q}}$  a counter-clockwise rotation, and on the opposite side a circle  $r = r_-$  making  $\mathbf{M}_\epsilon^{\tilde{q}}$  a clockwise rotation. Then use intermediate-value theorem, there must be a circle  $r = \hat{r}_\epsilon$  where  $\mathbf{M}_\epsilon^{\tilde{q}}$  corresponds to no angular rotation, which means its

effect is only in the radial direction. Moreover, Liouville theorem guarantees that  $\mathbf{M}_\epsilon^{\tilde{q}}$  still preserves the area enclosed by the the contour  $r = \hat{r}_\epsilon$ . This leads to the geometry illustrated by Fig. 4(b).

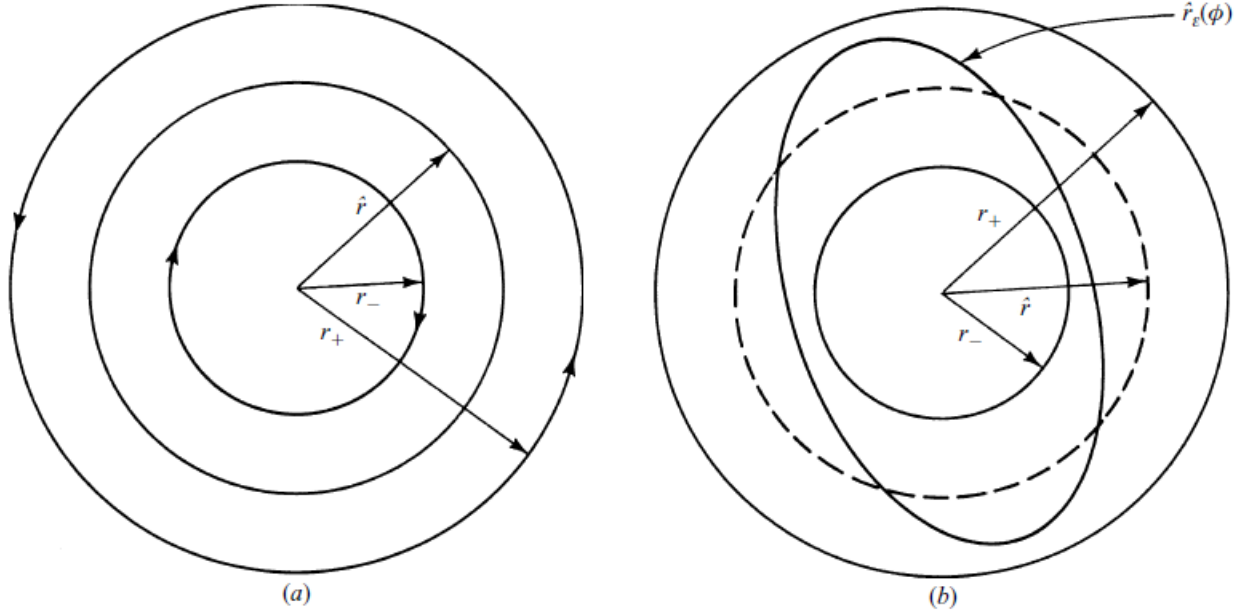


Figure 4: (a) The unperturbed circles (b) the  $r = \hat{r}_\epsilon$  circle

Such geometry of the contour  $r = \hat{r}_\epsilon$  tells us that the mapping  $\mathbf{M}_\epsilon^{\tilde{q}}(\hat{r}_\epsilon) = r'_\epsilon$  makes some sections of the contour move outward, others move inward, and some countable fixed points, as shown in Fig. 5(a). Depending on the relative position of a static point, it can be an "elliptic point" where points around it rotate about it, otherwise a "hyperbolic point" where motion of points around it are like hyperbola, as shown in Fig. 5(b). Poincaré Birkhoff theorem states that there are same number of elliptic and hyperbolic points.

The rotation of points around an elliptic point forms KAM curves surrounding it, hence it's also called an "O point". Between these surrounding curves is the destroyed resonant region, corresponding to the chaotic resonant orbits. On the other hand, a hyperbolic point tends to be heteroclinic intersection points, thus also called an "X point". Then we have the formation of an island chain. Such filamentation effect is illustrated by Fig. 6(a). The

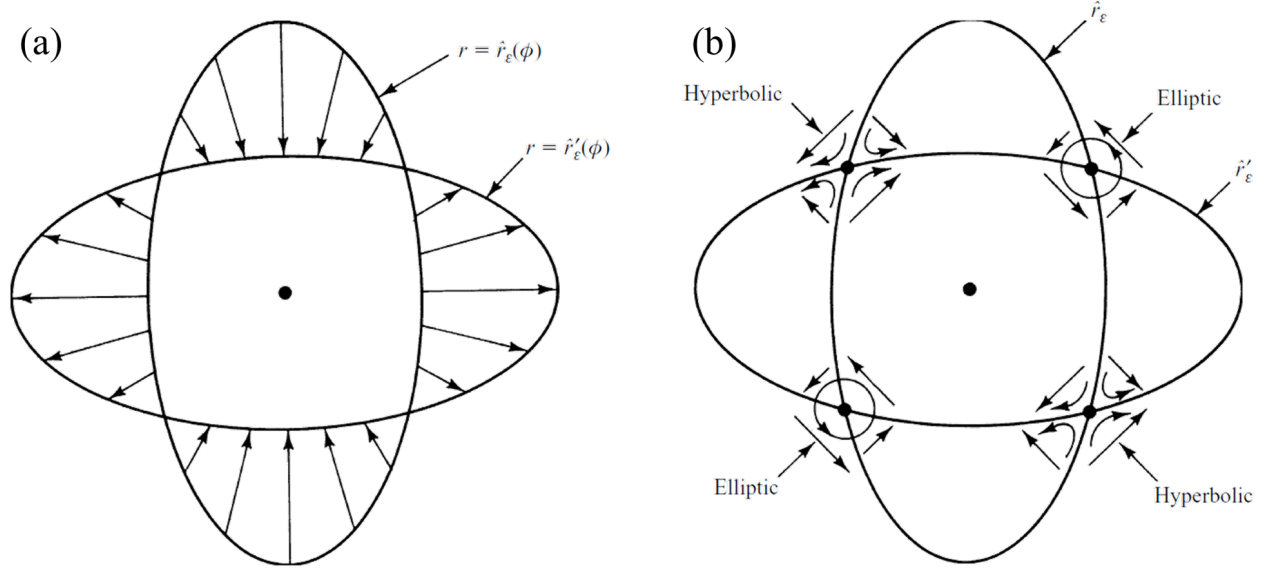


Figure 5: (a) The  $r = \hat{r}_\epsilon$  and  $r = \hat{r}'_\epsilon$  contours and the motion of points, (b) The elliptic and the hyperbolic points

island width  $\Delta J$  can be estimated as

$$\Delta J \approx \sqrt{\frac{\epsilon H_1}{\partial \omega_0 / \partial J}}$$

, where  $\epsilon H_1$  represents the strength of perturbation in Hamiltonian, and  $\partial \omega_0 / \partial J$  represents the unperturbed shear (differential of rotation in phase space). The derivation is given in [M. N. Rosenbluth, et al. Nucl. Fusion **6**, 297 (1966)] for magnetic field lines in torus by Fourier series expansion of the field fluctuation.

We can also derive the result by a simple argument as following. Recall the definition of the generating function  $S$ :

$$\begin{aligned} H &= H_0 + \frac{\partial S}{\partial t} \\ \Rightarrow H_0 + \epsilon H_1 &= H_0 + \frac{\partial S}{\partial t} \\ \Rightarrow \frac{\partial S}{\partial t} &= \epsilon H_1 \end{aligned}$$

, but  $S$  also satisfies that:

$$\frac{\partial S}{\partial J} \approx \theta_0$$

Therefore,

$$\frac{\partial \omega_0}{\partial J} = \frac{\partial}{\partial J} \frac{d\theta_0}{dt} = \frac{\partial}{\partial J} \frac{d}{dt} \frac{\partial S}{\partial J} \approx \frac{\partial^2}{\partial J^2} (\epsilon H_1) \approx \frac{\epsilon H_1}{\Delta J^2}$$

$$\Rightarrow \Delta J \approx \sqrt{\frac{\epsilon H_1}{\partial \omega_0 / \partial J}}$$

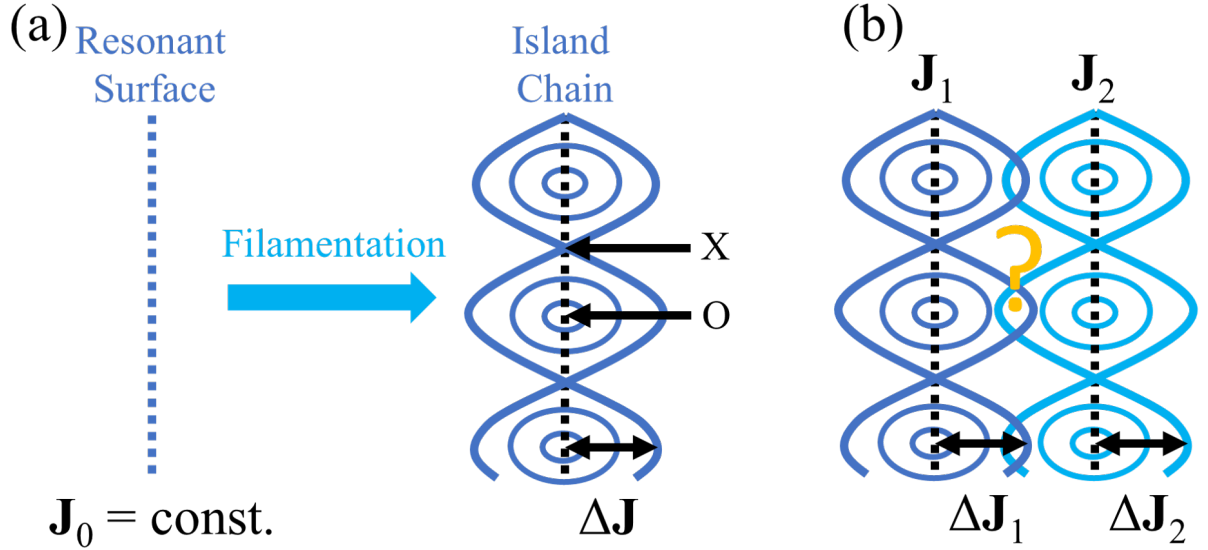


Figure 6: (a) Destruction of a single resonant surface (b) Overlapping multiple resonances

## 2.6 Multiple resonant 2 tori

We've seen the fate of a single resonant surface. Now, as Fig. 6(b) shows, one would inquire what happens when there are two resonant surfaces forming overlapping island chains? It is intuitive that a trajectory in the overlapping region will wander between different radius. It no longer belongs to a certain surface, but fills some volume in phase space. Thus it's a chaotic picture.

Numerical calculation in Fig. 7 gives an example on this concept. With small perturbation, individual island chains still preserves and bounded by some non-resonant KAM curves. But with large perturbation which is comparable with the distance between resonances, the smooth KAM curves are destroyed and becomes stochastic points except for the vicinity of the O points. It reflects that the trajectories in phase space no longer belong to certain surfaces.

In this context, **Chaos** can be understood as that the deviation of the perturbed resonant trajectory from the unperturbed one has linear instability:

$$\Delta \mathbf{J} = \mathbf{J} - \mathbf{J}_0 = \Delta \mathbf{J}_0 \exp(\gamma t); \gamma > 0$$

, or called "Lyapunov instability" where the exponent is "Lyapunov exponent". To understand this, we can take  $\Delta \mathbf{J}_0 \rightarrow 0$ . Then for chaotic system,  $\gamma > 0$  implies that even an infinitesimal difference in initial condition will finally diverge into considerable difference. In other words, the behavior of the system is highly sensitive to the initial condition, which is exactly the definition of chaos. More generally, if there exist multiple resonances, chaos means that there's at least one positive Lyapunov exponent  $\gamma_i > 0$ .

For a chaotic motion, we no longer have a deterministic solution of the Hamiltonian equations. Instead, we need statistical approach to make prediction or characterization of the system. The deterministic trajectory no longer exists, but we can define the "probability density function"  $f$  as an alternative quantity to describe the motion. Then we can use **Fokker-Planck equation**, which is based on a simple model of diffusion processes, to find the evolution of  $f$ .

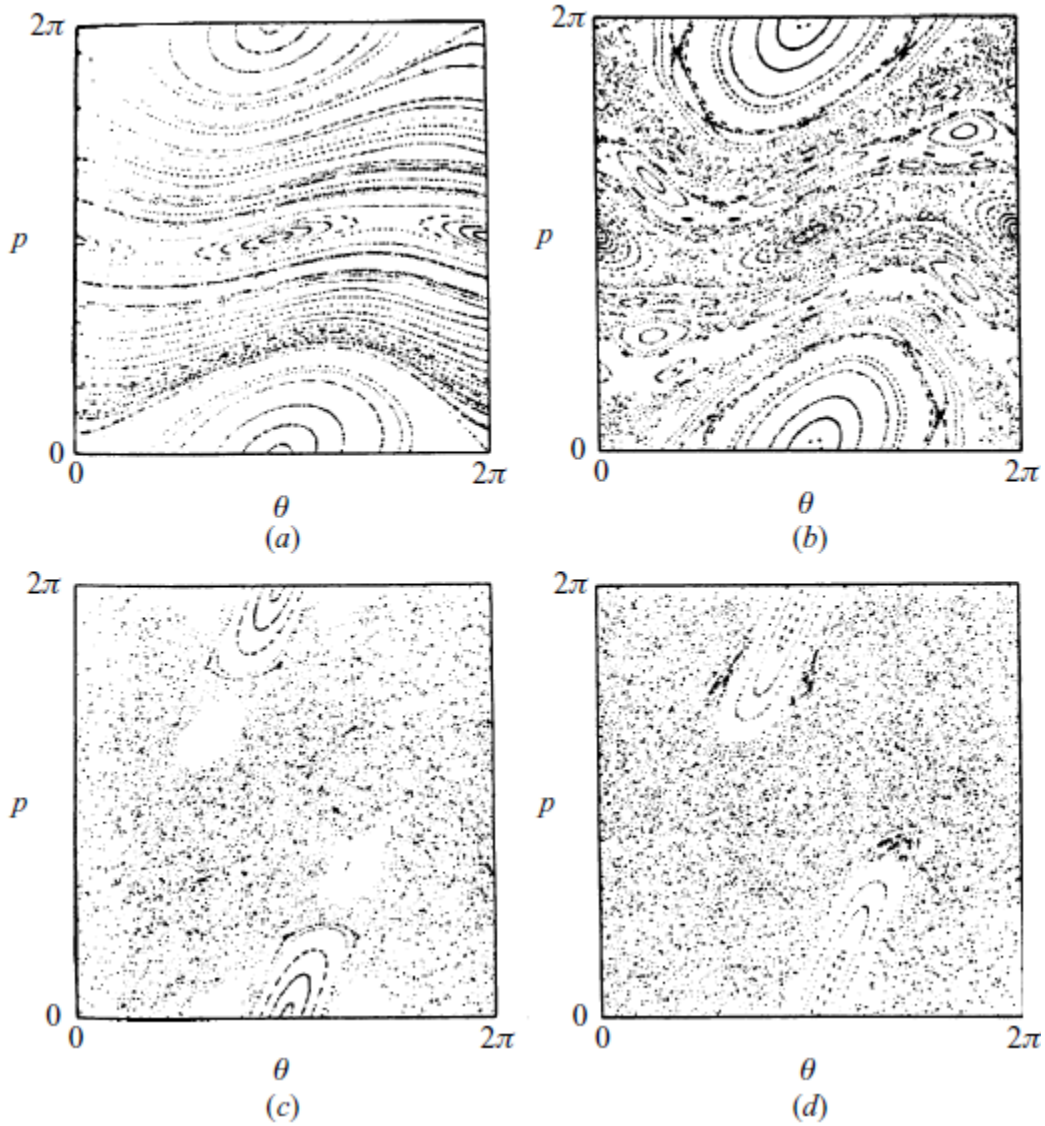


Figure 7: Numerical calculation of multiple resonances by changing a factor  $K$  (a)  $K = 0.5$ , (b)  $K = 1$ , (a)  $K = 2.5$ , (a)  $K = 4$ .  $(p, \theta)$  is analogous to  $(\mathbf{J}, \phi)$  in 2-torus, and  $K$  is analogous to perturbation  $\Delta \mathbf{J}$



To simplify Fokker-Planck problems even more, one convenient way is to use the **quasilinear theory**. It assumes that the unperturbed trajectory is still a good approximation to calculate the diffusion coefficient. However, we should notice that the quasilinear equation is applicable for limited regime. The first criterion is **Chirikov overlap**, which basically states that the island chains of different resonances should "overlap". We already see this point in Fig. 7 that **stochastic** field lines only appear when perturbation is large enough. In terms of the action variable:

$$S_C \equiv \frac{\Delta J_1 + \Delta J_2}{J_1 + J_2} > 1$$

, where 1, 2 denote two neighbouring resonant surfaces, and  $S_C$  is the Chirikov number. In terms of the island width:

$$S_C \equiv \frac{\Delta w_1 + \Delta w_2}{|r_1 + r_2|} > 1$$

, where  $r$  is the radius of circle on the intersection (see Fig. 3), and  $w$  is the island width.

The second criterion to be satisfied for quasilinear regime is small Kubo number ( $K_u$ ):

$$K_u < 1$$

We will give the definition of  $K_u$  for a realistic system, and examine this criterion more carefully later. Naively, it says that the random kicks are so often that the phase-space structure changes before a point has chance to bounce in the structure for once. This criterion is equivalent to stating that using unperturbed trajectories to calculate the diffusion of trajectories is a good approximation, which is an important assumption for quasilinear equations.

The last thing to notice is that while KAM theorem is concerned with the irrational surfaces, the onset of chaos is concerned with the rational (resonant) surfaces.

### 3 Stochastic magnetic field lines in a torus

Our discussion so far is based on the generic description of a Hamiltonian system. For a more realistic discussion on the phenomena of our interest, we now focus on a prime example, the magnetic field lines in a torus.

#### 3.1 Magnetic field lines in a tokamak

”Tokamak” is a toroidal chamber designed for thermal nuclear fusion reaction by magnetically confined plasma. One can imagine it as originally a cylindrical object with radius  $a$ , but two ends connected to form a donut with  $R$  the distance from the center to the symmetry axis of the original cylinder. As shown in Fig. 8, a convenient coordinate system is  $(r, \theta, \phi)$ , where  $r$  is the radial position in the original cylinder,  $\theta$  called ”poloidal” angle is the polar angle in the original cylinder, and  $\phi$  called ”toroidal” angle replaces the original  $z$  coordinate  $Rd\phi = dz$ . For simplicity, we shall still use the cylindrical coordinate  $(r, \theta, z)$  here.

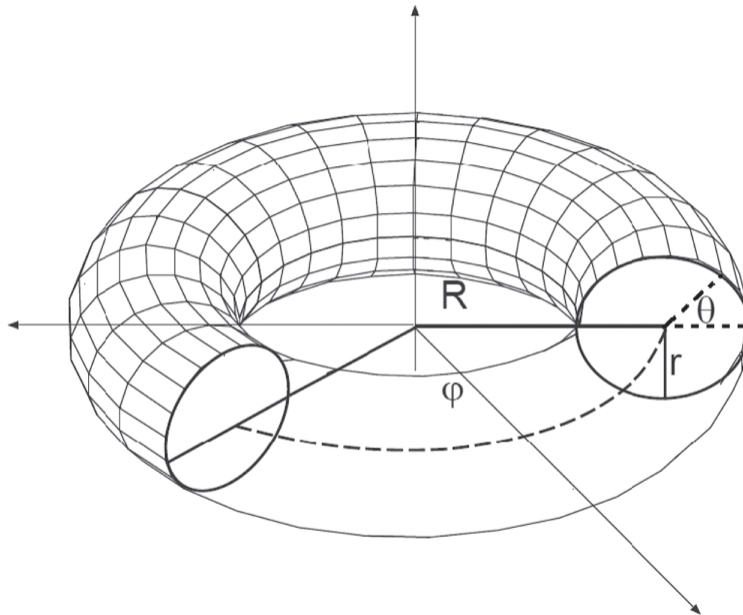


Figure 8: Coordinate system for toroidal objects.

In the unperturbed system, an external coil generates constant toroidal magnetic field  $B_z$ , another coil with time-varying current to induce plasma

current in the toroidal direction, thus generate a poloidal magnetic field  $B_\theta(r)$ , and there's no radial field  $B_r = 0$ . Therefore, unperturbed magnetic field lines are curves winding on toroidal surface with  $r = (\text{const.})$ , which are also called magnetic surfaces. The winding rate of a field line can be found by the ratio between the poloidal and the toroidal field.

$$\frac{rd\theta}{dz} = \frac{B_\theta(r)}{B_z}$$

We can define the "rotational transform"  $q(r)$ , also called "safety factor" when dealing with MHD instability, to represent the winding rate:

$$\boxed{q(r)} \equiv \frac{d\phi}{d\theta} = \frac{dz}{Rd\theta} = \boxed{\frac{rB_z}{RB_\theta}}$$

Now we review the definition of an integrable Hamiltonian system with degree-of-freedom  $N = 1$  (a circle). First, let the action variable be  $x$  and the action angle variable be  $y$ . They should be canonically conjugate (satisfying Hamiltonian equations):

$$\frac{dx}{dt} = -\frac{\partial H}{\partial y}; \quad \frac{dy}{dt} = \frac{\partial H}{\partial x}$$

Secondly, Hamiltonian is not explicitly dependent on time so that it is a constant of motion:

$$H(x, y, t) = H(x(t), y(t)) \Rightarrow \frac{dH}{dt} = 0$$

Lastly, Liouville theorem states that for a Hamiltonian system the phase-space volume is conserved. If we define the probability density function  $f(x, y, t)$  in phase space, Liouville theorem can be expressed by the Liouville equation:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{dx}{dt} \cdot \frac{\partial f}{\partial x} + \frac{dy}{dt} \cdot \frac{\partial f}{\partial y} = \frac{\partial f}{\partial t} - \frac{\partial H}{\partial y} \cdot \frac{\partial f}{\partial x} + \frac{\partial H}{\partial x} \cdot \frac{\partial f}{\partial y} = 0$$

Considering perturbation, if we specify that  $H(x, y) = H_0(x) + \tilde{H}(x, y)$ , then the Liouville equation becomes:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} - \frac{\partial H_0}{\partial x} \frac{\partial f}{\partial y} - \frac{\partial \tilde{H}}{\partial y} \frac{\partial f}{\partial x} + \frac{\partial \tilde{H}}{\partial x} \frac{\partial f}{\partial y}$$

$$= \frac{\partial f}{\partial t} - v_y(x) \frac{\partial f}{\partial y} - (\nabla \tilde{H} \times \hat{z}) \cdot \nabla f = 0$$

Then we check the equations for magnetic field lines in a tokamak. The Maxwell equation gives incompressibility of the field  $\nabla \cdot \mathbf{B} = 0$ . By definition, the magnetic flux density  $\psi$  (or "field line density") in any closed loop is conserved when the loop is convected in the magnetic field. In differential form,

$$(\mathbf{B} \cdot \nabla)\psi = 0$$

We specify the form of the unperturbed field as what we described for a tokamak  $\mathbf{B}_0 = B_0 \hat{z} + B_\theta(r) \hat{\theta}$ . Let's add a perturbation term to it in the perpendicular ( $\perp \hat{z}$ ) direction:

$$\mathbf{B} = B_0 \hat{z} + B_\theta(r) \hat{\theta} + \tilde{\mathbf{B}}_\perp$$

The conservation of magnetic flux in convection reads:

$$\begin{aligned} (\mathbf{B} \cdot \nabla)\psi &= B_0 \frac{\partial \psi}{\partial z} + \frac{B_\theta(r)}{r} \frac{\partial \psi}{\partial \theta} + \tilde{\mathbf{B}}_\perp \cdot \nabla_\perp \psi = 0 \\ \Rightarrow \frac{\partial \psi}{\partial z} + \frac{B_\theta(r)}{r B_0} \frac{\partial \psi}{\partial \theta} + \frac{\tilde{\mathbf{B}}_\perp}{B_0} \cdot \nabla_\perp \psi &= 0 \\ \Rightarrow \frac{\partial \psi}{\partial z} + \frac{B_\theta + \tilde{B}_\theta}{r B_0} \frac{\partial \psi}{\partial \theta} + \frac{\tilde{B}_r}{B_0} \frac{\partial \psi}{\partial r} &= 0 \\ \Rightarrow \frac{\partial \psi}{\partial z} + \frac{r d\theta}{dz} \frac{\partial \psi}{r \partial \theta} + \frac{\tilde{B}_r}{B_0} \frac{\partial \psi}{\partial r} &= 0 \end{aligned}$$

There is clear analogy between the equation and that for the 1-D Hamiltonian:

$$\psi \leftrightarrow f$$

$$z \leftrightarrow t; r \leftrightarrow x; r d\theta \leftrightarrow dy$$

$$\frac{\tilde{B}_r}{B_0} \leftrightarrow -(\nabla \tilde{H} \times \hat{z}) \cdot \nabla$$

Note that  $z$  plays the role of time here. The winding rate corresponds to the angular velocity in an integrable Hamiltonian system:

$$\frac{r d\theta}{dz} = \frac{1}{Rq(r)} \leftrightarrow v_y(x) \leftrightarrow \omega(J)$$

Therefore, the system of magnetic field lines in a tokamak is analogous to an integrable Hamiltonian system of  $N = 1$ .

If we don't treat  $z = R\phi$  as time but use  $\mathbf{q} = (\theta, \phi)$  coordinate system, one can show that the magnetic field lines in tokamak is an  $N = 2$  integrable Hamiltonian system (2 torus). In such system, the condition of resonance is more easily to understand:

$$\begin{aligned}\mathbf{m} \cdot \boldsymbol{\omega} &\equiv m \frac{d\theta}{dt} - n \frac{d\phi}{dt} = 0 \\ \Rightarrow q(r) &= \frac{d\phi}{d\theta} = \frac{m}{n}\end{aligned}$$

for some integers  $m$  and  $n$ . Therefore,

$$\boxed{\text{Resonance} \Leftrightarrow q \in \text{rational number}}$$

### 3.2 Wandering of magnetic field lines

Recall that the magnetic flux density  $\psi$  is equivalent to the density of field lines, or the number of field lines penetrating through an unit area. In the following we denote it as  $f$  because the field line density is analogous to the probability density function  $f$  in Hamiltonian system, which is the number of trajectories penetrating through an unit area in phase space. Now we use the incompressibility of magnetic field  $\nabla \cdot \mathbf{B} = 0$  (analogous to Liouville theorem for Hamiltonian systems),

$$\begin{aligned} & \frac{\partial f}{\partial z} + \frac{B_\theta(r) + \tilde{B}_\theta}{rB_0} \frac{\partial f}{\partial \theta} + \frac{\tilde{B}_r}{B_0} \frac{\partial f}{\partial r} = 0 \\ \Rightarrow & \frac{\partial f}{\partial z} + \frac{B_\theta(r)}{rB_0} \frac{\partial f}{\partial \theta} + \frac{\partial}{\partial \theta} \left( \frac{\tilde{B}_\theta}{rB_0} f \right) + \frac{\partial}{\partial r} \left( \frac{\tilde{B}_r}{B_0} f \right) - f \left( \frac{\partial \tilde{B}_\theta}{r \partial \theta} + \frac{\partial \tilde{B}_r}{\partial r} \right) = 0 \end{aligned}$$

The last term in LHS is just

$$f \left( \frac{\partial \tilde{B}_\theta}{r \partial \theta} + \frac{\partial \tilde{B}_r}{\partial r} \right) = f (\nabla \cdot \tilde{\mathbf{B}}_\perp) = 0$$

Hence, we have

$$\frac{\partial f}{\partial z} + \frac{\partial}{\partial \theta} \left( \frac{B_\theta(r) + \tilde{B}_\theta}{rB_0} f \right) + \frac{\partial}{\partial r} \left( \frac{\tilde{B}_r}{B_0} f \right) = 0$$

Let us assume  $f = \langle f \rangle + \tilde{f}$ , where  $\langle \dots \rangle$  denotes averaging along  $\theta$  direction. Then we take average of the whole equation. Note that the system has periodicity in  $\theta$  direction, so the second term vanishes. However, although the fast perturbation terms ( $\tilde{\dots}$ ) are periodic in  $z$  direction, the averaged density  $\langle f \rangle$  might not. The reason is that  $z$  now plays the role of time here. As one travel through the field lines, some irreversible changes might take place in the system. Such changes are reflected only in the mean profile. Thus we have:

$$\begin{aligned} & \frac{\partial \langle f \rangle}{\partial z} + \frac{\partial}{\partial r} \left\langle \frac{\tilde{B}_r}{B_0} \langle f \rangle \right\rangle + \frac{\partial}{\partial r} \left\langle \frac{\tilde{B}_r}{B_0} \tilde{f} \right\rangle = 0 \\ \Rightarrow & \frac{\partial \langle f \rangle}{\partial z} + \frac{\partial}{\partial r} \left( \frac{\langle f \rangle}{B_0} \langle \tilde{B}_r \rangle \right) + \frac{\partial}{\partial r} \left\langle \frac{\tilde{B}_r}{B_0} \tilde{f} \right\rangle = 0 \end{aligned}$$

The fast perturbation term  $\tilde{B}_r$  must average to zero. Hence,

$$\frac{\partial \langle f \rangle}{\partial z} + \frac{\partial}{\partial r} \left\langle \frac{\tilde{B}_r}{B_0} \tilde{f} \right\rangle = 0$$

Recall that  $f$  is the line density, and  $z$  effectively acts as time here. The structure is exactly a Fick's law:

$$\frac{\partial \langle f \rangle}{\partial z} + \frac{\partial \Gamma_{r,B}}{\partial r} = 0$$

, where

$$\Gamma_{r,B} = \left\langle \frac{\tilde{B}_r}{B_0} \tilde{f} \right\rangle$$

is the flux of field line density.

One can imagine that due to the  $\tilde{B}_r$  perturbation, the field lines can move ("wander") in the  $r$  direction, thus leave their original flux surface. It is tempting to estimate the displacement  $\delta r$ . We notice that:

$$\frac{dr}{dz} = \frac{\tilde{B}_r}{B_z}$$

Hence,

$$\delta r \approx \int_0^l \frac{\tilde{B}_r}{B_0} dz$$

But the fluctuation field  $\tilde{B}_r$  is time-dependent, which means the field lines won't move in  $r$  direction at the same velocity forever. The "auto-correlation time"  $\tau_{ac}$  estimates the time scale in which a field at one point remains the same value. During this time, a point can travel in  $z$  direction for an "auto-correlation length"  $l_{ac}$ . Beyond this length, the trajectory de-coheres from the present fluctuation pattern.  $l_{ac}$  can be simply estimated by the  $z$ -direction spectrum  $k_{\parallel}$  of the fluctuation  $\tilde{B}_r$ :

$$l_{ac} = \frac{1}{|\Delta k_{\parallel}|}$$

, where  $\Delta k_{\parallel}$  is the bandwidth of the spectrum. We can conceptually understand this result by imagining a monochromatic (single-frequency) fluctuation

in space. If the wave does not decay  $\tilde{A} = A_0 \exp(ik_0z)$ , then the correlation length is  $l_{ac} = \infty$  and the spectrum is a delta function  $A(k) \sim \delta(k - k_0)$ . However, if the fluctuation is generated at origin and exponentially decays with characteristic length scale  $l_{ac}$ , namely  $\tilde{A} = A_0 \exp(-|z|/l_{ac}) \exp(ik_0z)$ , the peak start to broaden and becomes a Lorenzian function whose bandwidth is  $\Delta k \sim 1/l_{ac}$ , which is exactly what we got. Then we can express  $\delta r$  by  $l_{ac}$ :

$$\delta r = l_{ac} \frac{\tilde{B}_r}{B_0}$$

is the excursion of field lines in one  $l_{ac}$ .

On the other hand,  $\tilde{B}_r$  also has an  $r$  dependence. This means that even if the structure of  $B_r$  didn't change with time, the motion of field lines in  $r$  direction might be changed due to the change of  $\tilde{B}_r$  in  $r$  direction. Such "radial correlation length"  $\Delta r$  is the radial correlation length of the "scatterer". We then understand that the ratio between  $\delta r$  and  $\Delta r$  is important, since the wandering of magnetic field lines is governed by different mechanisms. This ratio is exactly the **Kubo number** ( $K_u$ ):

$$K_u \equiv \frac{\delta r}{\Delta r} = \frac{\tilde{B}_r l_{ac}}{B_0 \Delta r}$$

If  $K_u < 1$ , namely  $\delta r < \Delta r$ , we would imagine that when moving in one entire coherence length of  $\tilde{B}_r$ ,  $\tilde{B}_r$  have already changed for many times. Since  $\tilde{B}_r$  is a random fluctuation, this means that the field lines have experienced many "random kicks" when travelling in an entire coherence length in radial direction. This leads to a **diffusion** process of radial wandering.

In contrast, if  $K_u > 1$ , or  $\delta r > \Delta r$ , the field lines are influenced by the same  $\tilde{B}_r$  structure without being disturbed by random kicks. Hence, the field lines will experience strong scattering by  $\tilde{B}_r$ . This is a percolation picture, and we won't discuss it in this lesson yet.

Lastly,  $K_u \approx 1$  is the regime that B. B. Kadomtsev called "natural state of EM turbulence" [B. B. Kadomtsev and O. P. Pogutse, *Plasma Phys. Controlled Nucl. Fusion Res.* **1**], and  $K_u = 1$  is the regime of critical balance.



Critical balance indicates that the linear term ( $\frac{\partial}{\partial t}$  or  $B_0 \frac{\partial}{\partial z}$ ) and the non-linear term ( $\tilde{\mathbf{v}} \cdot \nabla$  or  $\tilde{\mathbf{B}} \cdot \nabla$ ) has the same strength. A rough estimation is  $B_0/l_{ac} = \tilde{B}/\Delta r$ , since the fluctuation has characteristic length  $l_{ac}$  in  $z$  direction and  $\Delta r$  in  $r$  direction. This estimation then gives us  $\mathbf{K}_u = \frac{\tilde{B}_r}{B_0} \frac{l_{ac}}{\Delta r} = \mathbf{1}$  as the criterion of critical balance.

We shall restrict ourselves to the  $\mathbf{K}_u < 1$  diffusive regime in the following, and calculate the diffusion coefficient of magnetic field lines.

### 3.3 Diffusion of magnetic field lines

Now we assume  $\mathbf{K}_u < 1$  and  $\mathbf{S}_C > 1$ , which is the quasilinear regime, where field line distribution is stochastic and has no memory about the history. The equation of the radial flux of field line density (denoted  $\Gamma_M$  here):

$$\frac{\partial \langle f \rangle}{\partial z} + \frac{\partial \Gamma_M}{\partial r} = 0; \Gamma_M = \left\langle \frac{\tilde{B}_r}{B_0} \tilde{f} \right\rangle$$

is exactly the form of a quasilinear equation. Quasilinear theory assumes the validity of using a "mean field". That is, the density function has linear response  $f = \langle f \rangle + \tilde{f}$ , where  $\tilde{f} \ll \langle f \rangle$ . This assumption is guaranteed by the low Kubo number, since no strong scattering happens so that the field lines don't deviate from the unperturbed trajectories very much.

We now proceed via quasilinear theory. First recall the Liouville equation:

$$\frac{\partial f}{\partial z} + \frac{B_\theta(r) + \tilde{B}_\theta}{rB_0} \frac{\partial f}{\partial \theta} + \frac{\tilde{B}_r}{B_0} \frac{\partial f}{\partial r} = 0$$

If apply the linear response of  $f$ , and linearize the equation, we have:

$$\frac{\partial \langle f \rangle}{\partial z} + \frac{\partial \tilde{f}}{\partial z} + \frac{B_\theta(r)}{rB_0} \frac{\partial \tilde{f}}{\partial \theta} + \frac{\tilde{B}_\theta}{rB_0} \frac{\partial \langle f \rangle}{\partial \theta} + \frac{\tilde{B}_r}{B_0} \frac{\partial \langle f \rangle}{\partial r} = 0$$

Recall that the unperturbed magnetic field is only a function of  $r$ ,  $\mathbf{B}(r) = B_0 + B_\theta(r)$ . Therefore, the unperturbed flux density must also be only a function of  $r$ , so the first and the fourth term vanish.

$$\frac{\partial \tilde{f}}{\partial z} + \frac{B_\theta(r)}{rB_0} \frac{\partial \tilde{f}}{\partial \theta} + \frac{\tilde{B}_r}{B_0} \frac{\partial \langle f \rangle}{\partial r} = 0$$

Then we take Fourier series expansion of Liouville equation in  $z$  and  $\theta$  direction. Define for any physical quantity  $A$ :

$$A(z, \theta, r) = \sum_{k_z, k_\theta} a_{\mathbf{k}}(r) \exp[i(k_\theta r)\theta - ik_z z]$$

(Note that we take " $-ik_z$ " because  $z$  resembles time here. It won't change the physics.) We have the transformed Liouville equation:

$$-ik_z \tilde{f}_{\mathbf{k}} + \frac{B_\theta(r)}{B_0} ik_\theta \tilde{f}_{\mathbf{k}} + \frac{\tilde{B}_{r,\mathbf{k}}}{B_0} \frac{\partial \langle f \rangle}{\partial r} = 0$$

$$\begin{aligned} &\Rightarrow -i \left( k_z - \frac{B_\theta(r)}{B_0} k_\theta \right) \tilde{f}_{\mathbf{k}_\perp} = -\frac{\tilde{B}_{r,\mathbf{k}}}{B_0} \frac{\partial \langle f \rangle}{\partial r} \\ &\Rightarrow \tilde{f}_{\mathbf{k}}(r) = -i \left( \frac{\tilde{B}_{r,\mathbf{k}}}{B_0} \frac{1}{k_z - \frac{B_\theta(r)}{B_0} k_\theta} \right) \frac{\partial \langle f \rangle}{\partial r} \end{aligned}$$

Plug it into the density flux equation.

$$\begin{aligned} \Gamma_M &= \left\langle \sum_{k'_z, k'_\theta} \frac{\tilde{B}_{r,\mathbf{k}'}}{B_0} \exp[i(k'_\theta r)\theta - ik'_z z] \sum_{k_z, k_\theta} \tilde{f}_{\mathbf{k}}(r) \exp[i(k_\theta r)\theta - ik_z z] \right\rangle \\ &= -i \left\langle \sum_{k_z, k_\theta, k'_z, k'_\theta} \left( \frac{\tilde{B}_{r,\mathbf{k}} \tilde{B}_{r,\mathbf{k}'}}{B_0^2} \right) \frac{1}{k_z - \frac{B_\theta(r)}{B_0} k_\theta} \exp[i(k_\theta + k'_\theta)r\theta - i(k_z + k'_z)z] \right\rangle \frac{\partial \langle f \rangle}{\partial r} \\ &\equiv D_M \frac{\partial \langle f \rangle}{\partial r} \end{aligned}$$

Use Sokhotski–Plemelj theorem with causality (add a small positive imaginary part to  $k_z$ ),

$$\begin{aligned} D_M &= -i \left\langle \sum_{k_z, k_\theta, k'_z, k'_\theta} \left( \frac{\tilde{B}_{r,\mathbf{k}} \tilde{B}_{r,\mathbf{k}'}}{B_0^2} \right) \left[ i\pi \delta \left( k_z - \frac{B_\theta(r)}{B_0} k_\theta \right) \right. \right. \\ &\quad \left. \left. \exp[i(k_\theta + k'_\theta)r\theta - i(k_z + k'_z)z] \right] \right\rangle_{\theta, z} \end{aligned}$$

Then take average along  $\theta$  and  $z$  direction. Remember to divide the value by corresponding periods for normalization.

$$\begin{aligned} D_M &= \sum_{k_z, k_\theta, k'_z, k'_\theta} \left\{ \left( \frac{\tilde{B}_{r,\mathbf{k}} \tilde{B}_{r,\mathbf{k}'}}{B_0^2} \right) \pi \delta \left( k_z - \frac{B_\theta(r)}{B_0} k_\theta \right) \right. \\ &\quad \left. \frac{2\pi \delta[r(k_\theta + k'_\theta)]}{2\pi} \frac{2\pi \delta(k_z + k'_z)}{2\pi R} \right\} \\ &= \frac{\pi}{rR} \sum_{k_z, k_\theta, k'_z, k'_\theta} \left\{ \left( \frac{\tilde{B}_{r,k_z, k_\theta} \tilde{B}_{r,-k_z, -k_\theta}}{B_0^2} \right) \delta \left( k_z - \frac{B_\theta(r)}{B_0} k_\theta \right) \right. \\ &\quad \left. \delta(k_\theta + k'_\theta) \delta(k_z + k'_z) \right\} \end{aligned}$$

And convert summation over  $k'_z, k'_\theta$  to integrations. Remember to divide the value by the smallest unit  $\delta k'_z, \delta k'_\theta$  for normalization

$$\begin{aligned} D_M &= \frac{\pi}{rR} \sum_{k_z, k_\theta} \left| \frac{\tilde{B}_{r, k_z, k_\theta}}{B_0} \right|^2 \delta \left( k_z - \frac{B_\theta(r)}{B_0} k_\theta \right) \frac{1}{\delta k'_z} \frac{1}{\delta k'_\theta} \\ &= \frac{\pi}{rR} \sum_{k_z, k_\theta} \left| \frac{\tilde{B}_{r, k_z, k_\theta}}{B_0} \right|^2 \delta \left( k_z - \frac{B_\theta(r)}{B_0} k_\theta \right) \frac{2\pi R}{2\pi} \frac{2\pi r}{2\pi} \end{aligned}$$

Note that the unperturbed field lines already wind on the magnetic surfaces due to  $B_\theta(r)$ :

$$\frac{B_\theta(r)}{B_0} = \frac{rd\theta}{dz} = \frac{k_z}{k_\theta}$$

This means when we follow the unperturbed field lines:

$$k_z - \frac{B_\theta(r)}{B_0} k_\theta = 0$$

Hence,

$$D_M = \pi \left( \sum_{k_\parallel} \left| \frac{\tilde{B}_{r, k_\parallel}}{B_0} \right|^2 \right) \frac{1}{\Delta k_\parallel}$$

Where  $\Delta k_\parallel$  is conceptually the smallest difference in  $k_\parallel$ . We can use  $2\pi/l_{ac}$  as a reasonable approximation, since  $l_{ac}$  is the largest length within which parallel structures remain coherent.

$$\Rightarrow D_M \approx \left\langle \left( \frac{\delta B_r}{B_0} \right)^2 \right\rangle l_{ac}$$

And the radial displacement of magnetic lines when travelling in  $z$  direction for distance  $Z$  can be estimated by:

$$\langle (\delta r)^2 \rangle \approx D_M Z$$

A more rigorous derivation of  $D_M$  can be found in [M. N. Rosenbluth, et al. Nucl. Fusion **6**, 297 (1966)]

Lastly, we note that the Liouville equation can also be derived from simplifying the drift-kinetic equation (DKE). Assume steady state, no electric field and collisionless:

$$\begin{aligned}
& \cancel{\frac{\partial f}{\partial t}} + v_{\parallel} \hat{\mathbf{n}}_0 \cdot \nabla f + \mathbf{v}_D \cdot \nabla f - \frac{c}{B_0} (\nabla \phi \times \hat{\mathbf{z}}) \cdot \nabla f \\
& + v_{\parallel} \frac{\delta \mathbf{B}_{\perp}}{B_0} \cdot \nabla f - \frac{|e|}{m_e} E_{\parallel} \cancel{\frac{\partial f}{\partial v_{\parallel}}} = \cancel{C(f)} \\
\Rightarrow & \hat{\mathbf{n}}_0 \cdot \nabla f + \frac{\delta \mathbf{B}_{\perp}}{B_0} \cdot \nabla f = 0
\end{aligned}$$

Such method works because from a broader perspective, we derived the equation for the electrons guiding-center (center of gyromotion) motion, which is what DKE is based on.

### 3.4 Length scales in diffusion of field lines

There are different length scales in the diffusion mechanism. The relation between them is important. First we estimate  $l_{ac}$ . Note that we can change the coordinate system from  $(r\theta, z)$  to  $(\theta, \phi)$  by:

$$i[(k_\theta r)\theta - k_z z] = i[(k_\theta r)\theta - (k_z R)\phi] = i[m\theta - n\phi]$$

Along the parallel direction,

$$\begin{aligned} k_z - \frac{B_\theta(r)}{B_0} k_\theta &= 0 \\ \Rightarrow \frac{n}{R} - \frac{B_\theta(r)}{B_0} \frac{m}{r} &= 0 \\ \Rightarrow n &= \frac{RB_\theta(r)}{rB_0} m = \frac{m}{q} \end{aligned}$$

Hence,

$$\begin{aligned} dn &= \left| \frac{m}{q^2} \dot{q} \right| dr \\ \Rightarrow d(Rk_z) &= \left| \frac{k_\theta}{q} \frac{r\dot{q}}{q} \right| dr = \left| \frac{k_\theta}{q} s \right| dr \end{aligned}$$

, where  $s$  is the radial shear of the rotational transform  $q$ . And the magnetic shear length  $L_s$  is defined as:

$$\begin{aligned} L_s &\equiv \frac{Rq}{s} \\ \Rightarrow \left| \frac{dk_z}{dr} \right| &= \left| \frac{k_\theta}{L_s} \right| \end{aligned}$$

, which is the change in  $k_z$  when we move in  $r$  direction due to a shear in the  $B_\theta$ . Recall that the coherence length is  $l_{ac}$  in parallel direction, and  $\Delta$  in  $r$  direction. Therefore,

$$\begin{aligned} \frac{1}{l_{ac}\Delta r} &\approx \left| \frac{k_\theta}{L_s} \right| \\ \Rightarrow l_{ac} &\approx \frac{L_s}{|k_\theta|\Delta r} \end{aligned}$$

This is the parallel correlation length of the fluctuation field. Above  $l_{ac}$  the fluctuation structure changes, so the field lines experience "kicks", or "scattering". Therefore,  $l_{ac}$  can also be viewed as the "memory length" of field

lines.

On the other hand, we can also define a "de-correlation length"  $l_c$  in  $z$  direction, over which the field lines is scattered (deviates) from its unperturbed trajectory. To estimate such scale, we first note that for an unperturbed magnetic line:

$$\frac{rd\theta}{dz} \equiv \frac{dy}{dz} = \frac{B_\theta(r_0)}{B_0}$$

But now  $B_\theta$  has perturbation. We denote the effect of radial wandering by " $\delta$ " so that:

$$\begin{aligned} \frac{dy}{dz} &= \frac{B_\theta(r_0)}{B_0} + \frac{1}{B_0} \left[ \frac{\partial \tilde{B}_\theta(r_0)}{\partial r} \delta r \right] \\ \Rightarrow \frac{d(\delta y)}{dz} &\approx \frac{1}{B_0} \left[ \frac{\partial \tilde{B}_\theta(r_0)}{\partial r} \delta r \right] \\ \Rightarrow \delta y &= \int^Z \frac{\tilde{B}'_\theta(r_0)}{B_0} \delta r dz \\ \Rightarrow \langle \delta y^2 \rangle &= \left\langle \left( \int^Z \frac{\tilde{B}'_\theta(r_0)}{B_0} \delta r dz \right)^2 \right\rangle \\ \Rightarrow \langle \delta y^2 \rangle &= \frac{1}{B_0^2} \tilde{B}'_\theta{}^2 Z^2 \langle \delta r^2 \rangle \end{aligned}$$

Then from the quasilinear diffusion equation:

$$\begin{aligned} \langle \delta r^2 \rangle &\approx D_M Z \\ \Rightarrow \langle \delta y^2 \rangle &\approx \frac{\tilde{B}'_\theta{}^2}{B_0^2} D_M Z^3 \end{aligned}$$

To be more precise, recall that in 1D diffusion:

$$\langle \delta x^2 \rangle = D_v \frac{T^3}{3}$$

Hence,

$$\langle \delta y^2 \rangle = \frac{\tilde{B}'_\theta{}^2}{3B_0^2} D_M Z^3$$

The definition of field line de-correlating from its unperturbed trajectory can be viewed in  $y$  direction ( $dy = r d\theta$ ) as it leaves the local fluctuation structure. That is,

$$\begin{aligned} k_\theta^2 \langle \delta y^2 \rangle &\approx 1 \\ \Rightarrow \frac{1}{k_\theta^2} &\approx \frac{\tilde{B}_\theta'^2}{3B_0^2} D_M l_c^3 \\ l_c(\mathbf{k}) &\approx \left( k_\theta^2 \frac{\tilde{B}_\theta'^2}{3B_0^2} D_M \right)^{-\frac{1}{3}} \end{aligned}$$

Then we use the definition of magnetic shear length:

$$\begin{aligned} \frac{1}{L_s} &= \left| \frac{1}{B_0} \frac{\partial \tilde{B}_\theta}{\partial r} \right| \\ \Rightarrow l_c(\mathbf{k}) &= \left( \frac{k_\theta^2 D_M}{L_s^2 3} \right)^{-\frac{1}{3}} \end{aligned}$$

, which is analogous to shear dispersion of fluid:

$$\left( \frac{k_\theta^2 v^2}{3} D \right)^{\frac{1}{3}}$$

Also, the physical picture of such de-correlation is as Fig. 9 shows, the separation of field lines by "stretching" them in the perpendicular direction. If we have a distance  $d_0$  on the  $z = 0$  plane perpendicular to field lines. Now  $d_0$  is convected by field lines in  $z$  direction. The effect of field line stretching can be expressed as

$$d(z) = d_0 \exp\left(\frac{z}{l_c}\right)$$

, where  $l_c$  is the characteristic length in  $z$  direction. Such effect is also called stochastic instability, exactly the Lyapunov instability which gives chaos to this system. The Lyapunov exponent is  $1/l_c$  here.

As we explained, for quasilinear theory to be valid, we need the unperturbed trajectory to be good approximation. This means that the field lines don't have the chance to deviate from its unperturbed trajectory before being "kicked" by de-coherence in the  $z$  direction. The criterion is, as has been



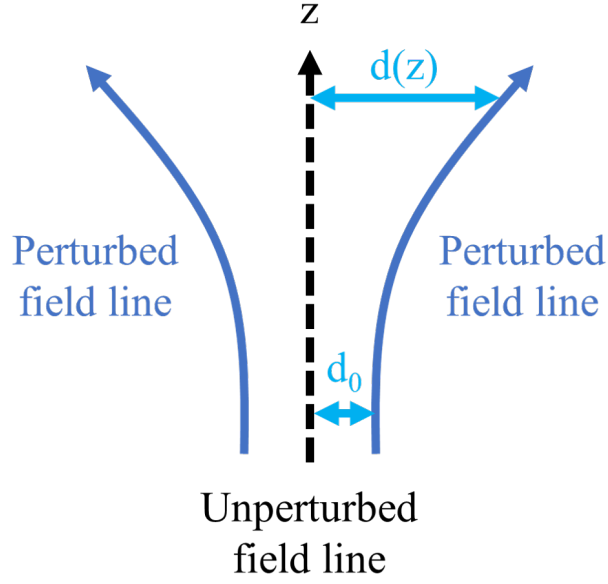


Figure 9: Field lines stretching

stated:

$$K_u \equiv \frac{\delta r}{\Delta r} = \frac{\tilde{B}_r l_{ac}}{B_0 \Delta r} < 1$$

According to diffusion equation, when the field lines deviates from its unperturbed trajectory

$$\langle \delta r^2 \rangle \approx D_M l_c$$

But this radial displacement can also be viewed as the result of field lines wandering in  $r$  direction due to the radial fluctuation  $\tilde{B}_r$ :

$$\begin{aligned} \sqrt{\langle \delta r^2 \rangle} &\approx \frac{\tilde{B}_r}{B_0} l_c \\ \Rightarrow D_M l_c &\approx \left( \frac{\tilde{B}_r}{B_0} \right)^2 l_c^2 \\ \Rightarrow \frac{\tilde{B}_r}{B_0} &\approx \sqrt{\frac{D_M}{l_c}} \end{aligned}$$

Hence, the criterion becomes

$$K_u \approx \sqrt{\frac{D_M}{l_c}} \frac{l_{ac}}{\Delta r} = \sqrt{\frac{D_M}{l_c}} \frac{l_{ac}^2}{L_s} |k_\theta| = \sqrt{\frac{D_M k_\theta^2}{L_s^2 l_c}} l_{ac}^2 < 1$$

Now apply the value of  $l_c$ :

$$l_c^{-3} \approx \frac{D_M k_\theta^2}{L_s^2}$$

Therefore, the criterion becomes:

$$K_u \approx \left( \frac{l_{ac}}{l_c} \right)^2 < 1 \Rightarrow \boxed{l_{ac} < l_c}$$

for the quasilinear theory to be valid.

In the next section, we shall discuss the transport of charged particles and heat in tokamak. In such topic, the length scale is also an important issue. The most common means of transport is, of course, the collisional transport. The characteristic length of collision is the mean-free path  $l_{mfp}$ . However, in tokamak, since charged particles move helically around a field line, the deviation of field lines from their unperturbed trajectory also causes transport. Therefore, the relation between the de-correlation length  $l_c$  and the mean-free path  $l_{mfp}$  is an important criterion for separating regimes. We then have, for transport phenomena:

**Collisionless** regime:  $\boxed{l_{ac} < l_c < l_{mfp}}$

**Collisional** regime:  $\boxed{l_{ac} < l_{mfp} < l_c}$

## 4 Transport in stochastic fields

We introduced the diffusion of field lines in the previous section. However, most people don't really care about just the diffusion of field lines. The reason of studying it is because of its influence on the important physical questions. For example, the transport of (charged) particles, their momentum, and the accompanying heat are what experimental researchers care about.

In the following, we will discuss heat transport of electrons in the perpendicular (to field lines) direction for the collisionless and the collisional regimes separately. The central themes are (1) how irreversibility is generated, and (2) the interacting processes. An article written by Rechester and Rosenbluth [[A. B. Rechester and M. N. Rosenbluth. Phys. Rev. Lett. \*\*40\*\*, 38 \(1978\)](#)] is a must-read one for this topic. In the first and the second subsection following, we will use their idea to estimate the heat transport. Then we will introduce another approach in the third subsection.

## 4.1 Collisionless heat transport

In the collisionless regime,  $l_{ac} < l_c < l_{mfp}$ , we would imagine that the perpendicular electron heat transport is mainly contributed by the wandering of field lines rather than the perpendicular diffusion of electrons. Recall that the thermal diffusivity  $\chi$  is defined by:

$$\frac{\partial T}{\partial t} = \chi \nabla^2 T$$

, where  $T$  is the temperature. It has a dimension of  $[L^2/t]$ . Naively, we would imagine that the collisionless  $\chi_{\perp}$  is directly proportional to the quasilinear diffusion coefficient of field lines  $D_M$ , and the thermal velocity of electrons  $v_{th}$  at which they travel along the field lines. This gives as:

$$\chi_{\perp} \approx v_{th} D_M$$

But is the story so simple? Let's check what happens to our naive story.

Consider a thought experiment. The electrons only move in the parallel direction, so that one electron is always on a certain field line. Besides the influence of field line wandering, collision will take place at a distance of  $l_{mfp}$ . Therefore, in the long run, electron motion along the field lines is "diffusive", which gives us:

$$\delta z^2 \approx D_{\parallel} t \approx \chi_{\parallel} t$$

, where  $D_{\parallel} \approx v_{th}^2 / \gamma_c$  is the parallel particle diffusion coefficient, and  $\chi_{\parallel}$  is the parallel heat diffusivity. For the heat transport in perpendicular direction, we can use the quasilinear diffusion:

$$\langle \delta r^2 \rangle \approx D_M \delta z \approx D_M (\chi_{\parallel} t)^{1/2}$$

Hence, the perpendicular thermal diffusivity is:

$$\chi_{\perp} \equiv \frac{d\langle \delta r^2 \rangle}{dt} \approx D_M \sqrt{\frac{\chi_{\parallel}}{t}}$$

, which will vanish as  $t \rightarrow \infty$ ! Therefore, the naive story won't give us perpendicular thermal transport. But what's the reason?

The problem of this story is that particles get "kicked back" by collision along the line. As Fig. 10 shows, with such kicks, particles' motion on the

perturbed field line is a diffusive process. The probability density function (or distribution) is always centred at the initial point  $\langle \delta z \rangle = 0$ . Electrons don't really "travel" along field lines to anywhere far away. Therefore, the effect of stochastic instability  $d(z) = d_0 \exp(z/l_c)$  is not able to take place, thus no radial wandering of particles. The lesson here is that **we need irreversibility** for the particle motion, so that they won't be kicked back. Irreversibility is controlled by collisions. Rather than collision along field lines, an irreversible collision is that **kicking particles off the field lines!**

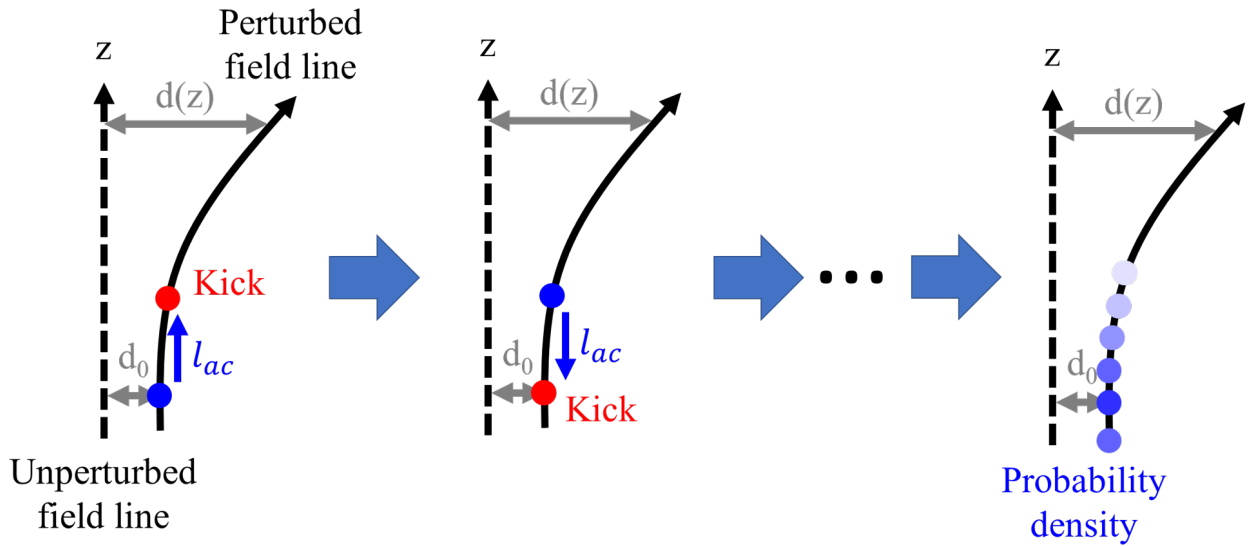


Figure 10: Parallel collisions not contributing to irreversibility.

What is the mechanism of this perpendicular kicking? Recall that isotropically thermalized electrons also have perpendicular velocity, so that they move helically around the field lines. Therefore, there is uncertainty in their perpendicular position due to the gyromotion. It effectively "smear" the electron location in a circle of electron gyro-radius  $\rho_e$  on the perpendicular plane. In other words, this defines the "minimal resolution scale" (or "grain") of electron location on the perpendicular plane. Such phenomenon is called **"coarse graining"**.

To understand its influence, let's consider the following argument. As illustrated in Fig. 11, on the plane perpendicular to the field lines, an electron motion is smeared on a disk with radius  $\rho_e$ . As electron travel in the parallel direction within one  $l_{mfp}$ , which means the longest range without parallel

collision, the field line deviates from its unperturbed trajectory. This leads to the deformation of the disk. The length of it increases due to stochastic instability

$$l \approx \rho_e \exp(l_{mfp}/l_c)$$

But  $\nabla \cdot B = 0$  (analogous to Liouville theorem) preserves the area of it, so the width ( $w$ ) of the area becomes:

$$w \approx \rho_e \exp(-l_{mfp}/l_c)$$

And the original disk is deformed into a more complicated contour as Fig. 11(c) shows.

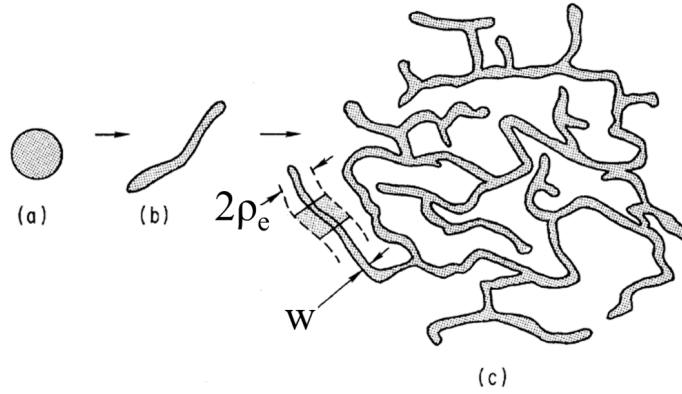


Figure 11: Deformation of the particle motion contour.

However, coarse graining also happens for the new contour. As Fig. 12 shows, if we conceptually divided the whole plane into "cells" (or grids) of minimal resolution, after travelling in one  $l_{mfp}$ , the particle density function is  $f$  re-distributed onto the nearest cells. The total density function should be conserved:

$$A(z)\bar{f}(z) = A_0 f_0$$

, where  $A$  is the total area of cells possessed.

Then comes the parallel collision. Ludwig Boltzmann assures us that there is no memory after collision. The particle contours in different cells will evolve independently, but the density function has successfully spread on the perpendicular plane. Whenever travelling for one  $l_{mfp}$ , particles experience such spread, which is effectively a "kick" for their random-walk motion in the perpendicular direction. Since the step size is exactly one  $l_{mfp}$ , the perpendicular expansion of the area becomes:

$$\langle \delta r^2 \rangle \approx D_M l_{mfp}$$

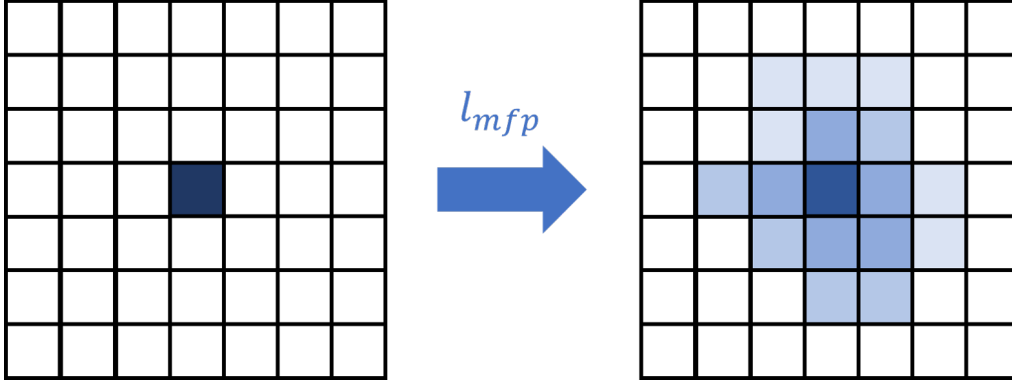


Figure 12: Effect of coarse graining on particle density function.

In other words, **coarse graining interval sets  $\langle \delta r^2 \rangle$  steps!**

Lastly, we estimate the collisionless thermal diffusivity in stochastic field:

$$\chi_{\perp} \approx \frac{\langle \delta r^2 \rangle}{\tau_c} \approx D_M \frac{l_{mfp}}{\tau_c} \approx v_{th} D_M$$

$$\Rightarrow \boxed{\chi_{\perp} \approx v_{th} D_M}$$

, which is exactly the same as our naive estimation. Interestingly, we find that the **diffusivity** is manifestly **independent of collisionality**. However, the **mechanism** is clearly **dependent on collisions and coarse graining**.

To look back on the whole argument, the lesson we learn here is that **coarse graining is essential to irreversibility**. In other words, coarse graining is essential to kick particle off field lines. Otherwise, collisional back-scattering will reverse the wandering.

### [Suggested exercises]

1. Derive the magnetic diffusivity with magnetic drifts. How do these modify  $D_M$ ? Explain why high energy particles (runaways) are confined longer than thermal ones.
2. Formulate the theory of diffusion due to stochastic fields in toroidal geometry using ballooning mode formalism for the fluctuations.
3. What happens to net cross field transport in a standing spectrum of electrons and magnetic perturbations? When might transport vanish? Why?

## 4.2 Collisional heat transport

Now we consider the transport in collisional regime, which is more challenging. Here, we have a short mean-free path  $l_{ac} < l_{mfp} < l_c$ .

We already show that radial transport doesn't happen if there is only collision. Instead, what causes the radial transport is perpendicular spread of particle trajectory due to coarse graining in stochastic field. Therefore, the perpendicular spread still follows quasilinear equation:

$$\langle \delta r^2 \rangle \approx D_M l_{c,\delta}$$

for some length scale  $l_{c,\delta}$ . In the collisionless regime, this length scale is  $l_{mfp}$ , the length scale at which an irreversible "kick" happens. The underlying reason is that  $l_c < l_{mfp}$ , when the parallel collision take places, irreversibility has already been produced by coarse graining within  $l_c$ . However, since  $l_{mfp} < l_c$  in collisional regime, within one  $l_{mfp}$  irreversibility has not formed yet. Then the  $l_{c,\delta}$  that sets the irreversibility here must be a longer length than  $l_{mfp}$ . Moreover, within this length scale particles experience many collisions, so that the motion is diffusive:

$$\langle \delta z^2 \rangle = D_{\parallel} \delta t \approx \chi_{\parallel} \delta t$$

$$\Rightarrow l_{c,\delta}^2 \approx \chi_{\parallel} \delta t_{c,\delta}$$

Thus the perpendicular diffusivity can be found as

$$\chi_{\perp} = \frac{\langle \delta r^2 \rangle}{\delta t_{c,\delta}} \approx D_M \frac{\chi_{\parallel}}{l_{c,\delta}}$$

As we stated,  $l_{c,\delta}$  is the parallel length scale between two irreversible kicks. But what is an irreversible kick? Conceptually, it's the jump of particles from one magnetic field line to another. In other words, that's when the particles originally belonging to one field line are now de-correlated. Therefore, we have to find out (1) the original perpendicular distance between these particles, (2) the perpendicular distance at which they are de-correlated, and (3) the mechanism that makes the distance evolve.

Naively for question (1), one would try to define particles "belonging" to one field line as those on a perpendicular disk with radius  $\rho_e$  centred at



the field line. However, in the collisional regime, there are two processes to deform the shape of this disk. The first mechanism is the shrinking in the contour width  $\delta$  due to stochastic instability of the length and  $\nabla \cdot \mathbf{B} = 0$  that preserves the total area. This is addressed in the collisionless regime that:

$$\begin{aligned}\delta(z) &= \delta_0 \exp(-z/l_c) \\ \Rightarrow \frac{d\delta}{dz} &= -\frac{\delta}{l_c}\end{aligned}$$

Another mechanism is from collision. As Fig. 13 illustrates, when a particle is scattered by another particle, its perpendicular velocity changes direction, so that the guiding center of the gyromotion changes. Hence, the guiding center's motion is a random walk with the step length around one gyro-radius  $\Delta \approx \rho_e$ , and collision frequency  $\gamma_{ee} = 1/\tau_c$ . Its influence on the width  $\delta$  is a classical perpendicular diffusion:

$$\langle d\delta^2 \rangle \approx D_{\perp} dt$$

, where

$$D_{\perp} = \frac{\rho_e^2}{\tau_c} = \frac{\rho_e^2 v_{th}}{l_{mfp}}$$

Hence,

$$d\delta \approx \sqrt{\langle \delta^2 \rangle} \approx \sqrt{D_{\perp} dt}$$

Recall that the motion in parallel direction is also diffusive, so we have:

$$\begin{aligned}\langle dz^2 \rangle &\approx \chi_{\parallel} dt \approx v_{th} l_{mfp} dt \\ \Rightarrow d\delta &\approx \sqrt{\frac{D_{\perp}}{\chi_{\parallel}}} dz \\ \Rightarrow \frac{d\delta}{dz} &= \sqrt{\frac{D_{\perp}}{\chi_{\parallel}}}\end{aligned}$$

The balance between the two processes sets the equilibrium size of  $\delta$ :

$$\begin{aligned}-\frac{\delta_0}{l_c} + \sqrt{\frac{D_{\perp}}{\chi_{\parallel}}} &= 0 \\ \Rightarrow \delta_0 &= l_c \sqrt{\frac{D_{\perp}}{\chi_{\parallel}}}\end{aligned}$$

Thus we find the answer to question (1). Note that this length scale can also be derived from thermal energy conservation:

$$\begin{aligned} \frac{dT}{dt} &= \frac{\partial T}{\partial t} - \chi_{\parallel} \nabla_{\parallel}^2 T - D_{\perp} \nabla_{\perp}^2 T = 0 \\ \Rightarrow \frac{\chi_{\parallel}}{l_c^2} &\approx \frac{D_{\perp}}{\delta_0^2} \\ \Rightarrow \delta_0 &\approx l_c \sqrt{\frac{D_{\perp}}{\chi_{\parallel}}} \end{aligned}$$

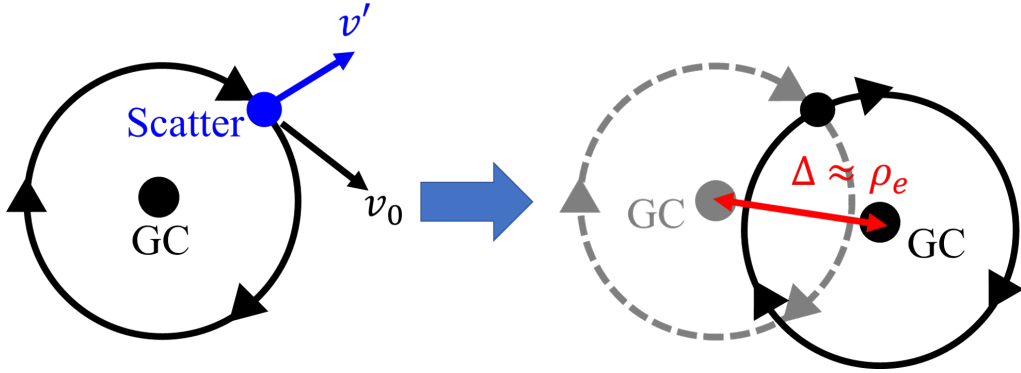


Figure 13: Guiding center (GC) random walk by scattering.

Next, question (2) is the perpendicular correlation length  $\delta_c$ . As illustrated in the collisionless case, the correlation length is between which one observes significant change of the magnetic field structure. That is to say, we can roughly estimate this length by:

$$\begin{aligned} k_{\theta}^2 \langle \delta_c^2 \rangle &\approx 1 \\ \Rightarrow \delta_c &\approx \frac{1}{|k_{\theta}|} \end{aligned}$$

Lastly, question (3) is what makes the perpendicular width to grow from  $\delta_0$  to  $\delta_c$ . The answer is, of course, the stochastic instability. Hence, we can estimate the corresponding parallel length scale  $l_{c,\delta}$  between two "kicks" by:

$$\begin{aligned} \frac{\delta_c}{\delta_0} &= \exp\left(\frac{l_{c,\delta}}{l_c}\right) \\ \Rightarrow l_{c,\delta} &= l_c \ln\left(\frac{\delta_c}{\delta_0}\right) = l_c \ln\left(\frac{1}{|k_{\theta}| l_c} \sqrt{\frac{\chi_{\parallel}}{D_{\perp}}}\right) \end{aligned}$$

Since log function is not a strong dependence function, we can also drop it to have:

$$l_{c,\delta} \approx l_c$$

This is as expected because  $l_c$  is the de-correlation length in collisionless regime, at which irreversibility is generated.

Now plugging this into the equation for  $\chi_{\perp}$ , we finally find out:

$$\begin{aligned} \chi_{\perp} &= D_M \frac{\chi_{\parallel}}{l_{c,\delta}} \approx D_M \frac{\chi_{\parallel}}{l_c} = D_M \frac{v_{th} l_{mfp}}{l_c} \\ &\Rightarrow \boxed{\chi_{\perp} \approx v_{th} D_M \left( \frac{l_{mfp}}{l_c} \right)} \end{aligned}$$

Compare this result with the collisionless thermal diffusivity:

$$\frac{\chi_{\perp, \text{collisional}}}{\chi_{\perp, \text{collisionless}}} \approx \left( \frac{l_{mfp}}{l_c} \right)_{\text{collisional}} < 1$$

The lessons we learned here are:

1. **Collisions reduce  $\chi_{\perp}$**  by  $(l_{mfp}/l_c)$  from the "collisionless" case. But we should remember that even the **collisionless transport also requires collisions**.
2. The collisional heat transport is the **interplay of perpendicular and parallel diffusions**.
3. Same as the collisionless case, it is critical to **knock particles off field lines** to produce **irreversibility**.

### 4.3 Systematic derivation of transport from hydrodynamics

So far we have successfully estimated the thermal diffusivity in the collisionless and the collisional regimes. However, although such arguments only require simple calculation, they do rely on deep insight into the problems. Is there a more systematic approach to derive the diffusivity more mindlessly? Kadomtsev and Pogutse gave an approach using hydrodynamics in [B. B. Kadomtsev and O. P. Pogutse, *Plasma Phys. Controlled Nucl. Fusion Res.* **1**, 649-662 (1978)]. In the following we briefly introduce it.

Firstly, the heat transport in hydrodynamic system is the about the heat flux  $\mathbf{q}$ . We now consider the heat flux along wiggling fields.

$$\mathbf{q} = -\chi_{\parallel} \nabla_{\parallel} T \hat{\mathbf{b}} - \chi_{\perp} \nabla_{\perp} T$$

, where  $\hat{\mathbf{b}}$  is the unit vector along field direction. The first term in RHS is the parallel heat conduction, and the second term is the perpendicular heat conduction. Due to gyro-motion of particles in perpendicular direction, we would imagine that

$$\chi_{\parallel} \gg \chi_{\perp}$$

Then we include perturbation to the field:

$$\mathbf{b} = \mathbf{b}_0 + \tilde{\mathbf{b}}$$

, where the first term is the unperturbed term, and the second term is the perturbed one. Now define  $z$  axis to be along the unperturbed field direction  $\hat{\mathbf{z}} \equiv \tilde{\mathbf{b}}_0$ . We then have:

$$\nabla_{\parallel} = \frac{\partial}{\partial z} + \tilde{\mathbf{b}} \cdot \nabla_{\perp}$$

Therefore, the (perpendicular) wiggling of field lines contributes to the perpendicular transport by the second term in the new  $\nabla_{\parallel}$ . Plugging this into equation of  $\mathbf{q}$ , and taking average of the radial heat flux, we have four contributions:

$$\langle q_r \rangle = -\underbrace{\chi_{\parallel} \left\langle \tilde{b}_r^2 \right\rangle \frac{\partial \langle T \rangle}{\partial r}}_{(a)} - \underbrace{\chi_{\parallel} \left\langle \tilde{b}_r \frac{\partial \tilde{T}}{\partial z} \right\rangle}_{(b)} - \underbrace{\chi_{\parallel} \left\langle \tilde{b}_r \tilde{b}_r \frac{\partial \tilde{T}}{\partial r} \right\rangle}_{(c)} - \underbrace{\chi_{\perp} \nabla_r \langle T \rangle}_{(d)}$$

Term (a) and term (b) are the usual quadratic terms of perturbation, term (d) is just the perpendicular heat conduction, but now a cubic term (c) arises. To see its influence, we take:

$$\frac{(c)}{(b)} \approx \frac{\chi_{\parallel} \tilde{b}_r \tilde{b}_r \tilde{T} / \Delta r}{\chi_{\parallel} \tilde{b}_r \tilde{T} / l_{ac}} = \tilde{b}_r \frac{l_{ac}}{\Delta r} = \frac{\tilde{B}_r l_{ac}}{B_0 \Delta r} = \frac{\delta r}{\Delta r} = K_u$$

Therefore, the cubic non-linearity dominates for  $K_u > 1$ . But since we assume small Kubo number  $K_u < 1$ , term (c) is negligible, and we will drop it.

To compute  $\langle q_r \rangle$ , we shall retain term (a) (as usual) and term (b), and then iterate for  $\tilde{T}$  using  $\nabla \cdot \mathbf{q} = 0$  via quasilinear theory. Now think about the expression for radial heat flux:

$$\langle q_r \rangle \approx -\chi_{\parallel} \left[ \left\langle \tilde{b}_r^2 \right\rangle \frac{\partial T}{\partial r} + \left\langle \tilde{b}_r \frac{\partial \tilde{T}}{\partial z} \right\rangle \right] - \chi_{\perp} \nabla_r \langle T \rangle$$

From linearization,

$$\widetilde{\mathbf{b} \cdot \nabla T} \approx \tilde{b}_r \frac{\partial T}{\partial r} + \frac{\partial \tilde{T}}{\partial z}$$

Hence,

$$\langle q_r \rangle \approx -\chi_{\parallel} \left[ \left\langle \tilde{b}_r \widetilde{\mathbf{b} \cdot \nabla T} \right\rangle \right] - \chi_{\perp} \nabla_r \langle T \rangle$$

The point here is that we need

$$\widetilde{\mathbf{b} \cdot \nabla T} \neq 0$$

to drive net heat flux  $\langle q_r \rangle \neq 0$ . That is to say, to drive parallel heat flux, **temperature can't be constant along field line**. But we still need  $\nabla \cdot \mathbf{q} = 0$ . Hence, the result must imply  $\chi_{\perp}$  dependence to balance the heat flux, which is seen here:

$$\langle q_r \rangle \approx -\chi_{\parallel} \left[ \left\langle \tilde{b}_r^2 \right\rangle \frac{\partial T}{\partial r} + \left\langle \tilde{b}_r \frac{\partial \tilde{T}}{\partial z} \right\rangle \right] - \chi_{\perp} \nabla_r \langle T \rangle$$

We proceed to consider the total heat flux:

$$\nabla \cdot \mathbf{q} = 0$$

$$\Rightarrow \nabla_{\parallel} \tilde{q}_{\parallel} + \nabla_{\perp} \cdot \tilde{\mathbf{q}}_{\perp} = -\chi_{\parallel} \frac{\partial}{\partial z} \left( \tilde{b}_r \frac{\partial \langle T \rangle}{\partial r} \right)$$

, where

$$\mathbf{q} = -\chi_{\parallel} \left[ \left( \frac{\partial}{\partial z} + \tilde{\mathbf{b}} \right) (T_0 + \tilde{T})(\mathbf{b}_0 + \tilde{\mathbf{b}}) \right] - \chi_{\perp} \nabla_{\perp} T$$

Hence,

$$-\chi_{\parallel} \frac{\partial^2 \tilde{T}}{\partial z^2} - \chi_{\perp} \nabla_{\perp}^2 \tilde{T} = -\chi_{\parallel} \frac{\partial}{\partial z} \left( \tilde{b}_r \frac{\partial \langle T \rangle}{\partial r} \right)$$

Take spatial Fourier series expansion of the equation:

$$\Rightarrow \tilde{T}_{\mathbf{k}} = -\frac{\chi_{\parallel} i k_z \tilde{b}_{\mathbf{k}} \frac{\partial \langle T \rangle}{\partial r}}{\chi_{\parallel} k_z^2 + \chi_{\perp} k_{\perp}^2}$$

Then term (a) and term (b) can be written as:

$$\begin{aligned} & -\chi_{\parallel} \langle \tilde{b}^2 \rangle \frac{\partial \langle T \rangle}{\partial r} - \chi_{\parallel} \left\langle \tilde{b}_r \frac{\partial \tilde{T}}{\partial z} \right\rangle \\ &= -\chi_{\parallel} \sum_{\mathbf{k}} \left( -\frac{\chi_{\parallel} k_{\parallel}^2 |\tilde{b}_{\mathbf{k}}|^2}{\chi_{\parallel} k_z^2 + \chi_{\perp} k_{\perp}^2} + |\tilde{b}_{\mathbf{k}}|^2 \right) \frac{\partial \langle T \rangle}{\partial r} \\ &= -\chi_{\parallel} \frac{\partial \langle T \rangle}{\partial r} \sum_{\mathbf{k}} \left( -\frac{\chi_{\parallel} k_{\parallel}^2}{\chi_{\parallel} k_z^2 + \chi_{\perp} k_{\perp}^2} + \frac{\chi_{\parallel} k_z^2 + \chi_{\perp} k_{\perp}^2}{\chi_{\parallel} k_z^2 + \chi_{\perp} k_{\perp}^2} \right) |\tilde{b}_{\mathbf{k}}|^2 \end{aligned}$$

Thus we have:

$$\langle q_r \rangle_{NL} = -\chi_{\parallel} \frac{\partial \langle T \rangle}{\partial r} \sum_{\mathbf{k}} \frac{\chi_{\perp} k_{\perp}^2 |b_{\mathbf{k}}|^2}{\chi_{\parallel} k_{\parallel}^2 + \chi_{\perp} k_{\perp}^2}$$

Note that it is proportional to  $\chi_{\perp}$ ! We've already seen that the mechanism behind  $\chi_{\perp}$  is the irreversibility caused by coarse graining. Therefore, this expression of  $\langle q_r \rangle_{NL}$  directly tells us the importance of coarse graining in perpendicular heat transport.

Now replace the summation by integration,

$$\langle q_r \rangle_{NL} \approx -\chi_{\parallel} \frac{\partial \langle T \rangle}{\partial r} \int d\mathbf{k}_{\perp} \int dk_z \frac{\chi_{\perp} k_{\perp}^2 |b_{\mathbf{k}}|^2}{\chi_{\parallel} \left( k_{\parallel}^2 + \frac{\chi_{\perp}}{\chi_{\parallel}} k_{\perp}^2 \right)}$$

$$= -\frac{\partial\langle T \rangle}{\partial r} \int d\mathbf{k}_\perp \int dk_z \frac{\chi_\perp k_\perp^2 |b_{\mathbf{k}}|^2}{\left(\frac{k_z^2}{(\chi_\perp/\chi_\parallel)k_\perp^2} + 1\right) \left(\frac{\chi_\perp k_\perp^2}{\chi_\parallel}\right)}$$

Then we consider normalization and integrate over  $k_z$ :

$$\langle q_r \rangle_{NL} = -\frac{\partial\langle T \rangle}{\partial r} \int d\mathbf{k}_\perp \frac{k_\perp^2 (\chi_\parallel \chi_\perp)^{1/2}}{\sqrt{k_\perp^2}} |\tilde{b}_{\mathbf{k}}|^2 l_{ac}$$

Note that the auto-correlation length enters here via normalization of  $k_z$  integration by bandwidth ( $l_{ac} = 1/\Delta k_z$ ) Then

$$\boxed{\langle q_r \rangle_{NL} \approx -\sqrt{\chi_\parallel \chi_\perp} \langle \tilde{b}^2 \rangle l_{ac} \left\langle \sqrt{k_\perp^2} \right\rangle \frac{\partial\langle T \rangle}{\partial r}}$$

Note that,

1. We need  $\nabla_\parallel \tilde{T} \neq -\tilde{b}_r$  (namely,  $\widetilde{\mathbf{B} \cdot \nabla T} \neq 0$ ) to drive net perpendicular heat flux  $\mathbf{q}_\perp$ .
2.  $\langle \tilde{b}^2 \rangle l_{ac} \approx D_M$
3.  $\sqrt{k_\perp^2} \approx 1/\Delta_\perp$

Hence,

$$\langle q_r \rangle \approx -\chi_{\perp,\text{eff}} \frac{\partial\langle T \rangle}{\partial r} - \chi_\perp \frac{\partial\langle T \rangle}{\partial r}$$

, where

$$\chi_{\perp,\text{eff}} \approx \sqrt{\chi_\parallel \chi_\perp} \frac{D_M}{\Delta_\perp}$$

and

$$\chi_\parallel \chi_\perp \approx \frac{v_{the}^2}{\gamma} \rho_e^2 \gamma \approx D_B$$

, where  $D_B$  is the Bohm diffusion coefficient. Plug this into thermal diffusivity:

$$\boxed{\chi_{\perp,\text{eff}} \approx \frac{D_B}{\Delta_\perp} D_M}$$

Here we found that:

1.  $\chi_{\perp,\text{eff}}$  scales with Bohm diffusion coefficient rather than Spitzer diffusion ( $\chi_\parallel$ )

2. Kicking particles off the field lines is again seen as important.

To compare this result with what we estimate from Rechester and Rosenbluth's method, we use:

$$\chi_{\perp} \approx \sqrt{\chi_{\parallel} \chi_{\perp}} \frac{\langle \tilde{b}^2 \rangle}{\Delta_{\perp}} l_{ac}$$

But what is the length scale  $\Delta_{\perp}$ ? It is the thickness of a small layer that enters the spectrum. Now we have

$$\frac{\chi_{\parallel}}{l_c^2} \approx \frac{\chi_{\perp}}{\Delta_{\perp}^2}$$

$$\Delta_{\perp} \approx l_c \sqrt{\chi_{\perp} / \chi_{\parallel}}$$

Thus  $\Delta_{\perp}$  is set by diffusion. Then we can plug it into  $\chi_{\perp}$ :

$$\chi_{\perp} \approx \sqrt{\chi_{\parallel} \chi_{\perp}} \frac{\langle \tilde{b}^2 \rangle l_{ac}}{l_c (\chi_{\parallel} / \chi_{\perp})^{1/2}}$$

$$\Rightarrow \chi_{\perp} \approx \frac{\chi_{\parallel}}{l_c} D_M = v_{th} D_M \left( \frac{l_{mfp}}{l_c} \right)$$

So we found that

1. Using modulo  $k_{\perp}$ ,  $\Delta_{\perp}$ , the result here (fluid approach) agrees with R.&R.'s result (single-particle approach) to within a logarithmic factor
2.  $\chi_{\perp} \approx v_{th} D_M \frac{l_{mfp}}{l_c}$

This covers diffusion in the quasilinear regime ( $K_u < 1$ ,  $S_C > 1$ ). The lesson we learned here is that we should take care of **coarse graining**, since it is crucial to **irreversibility**!