Fractional Kinetics

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1 Introduction

"God gave us the integers, all else is the work of man."

— Leopold Kroenecker

1.1 Why Learn this?

These notes are dedicated to the subject of Fractional Kinetics, which is essentially a generalization of a Fokker-Plank theory with fractional derivatives.

A student may question the value of all these fractals and fractional derivatives for physics. After all, the catechesis on theoretical physics by Landau and Lifshitz manages to omit the discussion of fractional derivatives altogether.

One good reason to learn the fractional derivatives business is that it provides a tool for addressing the memory of the dynamics. It is intuitively clear that the fractional derivatives are able to introduce memory since their formal definition also contains an integral. The importance of system memory effects is, of course, far from exotic in physical problems. To build a bridge between the fractional derivatives and the almighty Landau-Lifshitz work, consider the problem of the ball dynamics in viscous fluid from the book on hydrodynamics. The ball excites the fluid which in return affects the ball dynamics, creating a memory effect. The term describing the Boussinesq force corresponds to the fractional order derivative of velocity. Investigation on the impact of system memory seems to be of increasing interest for modern physics, and of significant engineering utility (for example, hidden parameters of a problem often expose themselves as a memory effect in system dynamics). The memory effects are very much present in plasma physics, too. A Kubo number gives the measure of the memory effect on turbulent diffusion processes, with high Kubo number corresponding to non-Markovian processes (see previous class notes for the discussion of $Ku << 1$ weak turbulence case).

It seems reasonable to attempt relaxing the Markovian property of Fokker Plank processes by employing the fractional derivatives methods. The result is the Fractional Kinetics (FK) theory. While being technically challenging to apply for actual problems, FK is a systematic theory that now allows for anomalous scaling

$$< \delta x^2 > \propto t \to < \delta x^2 > \propto t^{\mu}$$

hence introduces anomalous diffusion, and can represent flights and sticking depending on critical exponents.

Question: Come up with a physical example of a process with anomalous diffusion?

Possible answer: Richardson Pair Dispersion, an example of super-diffusive process in turbulence

Figure 1: Elaborating on the concept of a fractal: The eagle on the emblem of Russian Federation holds a Scepter and an Orb. The Scepter itself, however, also has the same emblem on it, resulting in a fractal-like self-similarity, but it is **not** a fractal. **Why** is it not a fractal? The reason is that the surface is only irregular in a vicinity of a certain point, violating the presumption of the equality of statistical character between each parts.

1.2 Comment on Fractals

Since several of this class topics involve the concept of fractals, it is worth giving a brief clarification on the nature of these objects and on to how this concept arises in physics. Fractals can be defined as a curve or geometric figure, each part of which has the same statistical character as the whole. Such structures exhibit self-similarity feature (scale invariance, as in invariance with respect to $x \to \alpha x,\ \alpha \in R^+$ transform), which is present in many physical objects and frequently used in theoretical models. The variability of a fractal-like structures (change from one fractal to the other while preserving the general arrangement) introduces the idea of stochastic fractals and statistical understanding of them. One of the examples of such a fractal is a trajectory of Brownian motion. (It is perhaps important to note here, however, that in case of Brownian motion the fractal trajectory is an idealization, applicable to the spatial scales larger than the characteristic body scales, and time scales larger than the time between consecutive collisions. In other words, the fractal picture is valid within certain intermediate asymptotics.)

Closely related to this discussion is the concept of fractional dimensionality. The most common understanding of dimensions comes from analytical geometry, where the dimension corresponds to a number of integer coordinates, which is restricted to integer values. We know from our daily experience that even such concept of dimension is not uniquely defined and depends on context. For example, consider the SERF building. Is the building one dimensional based on a coordinate

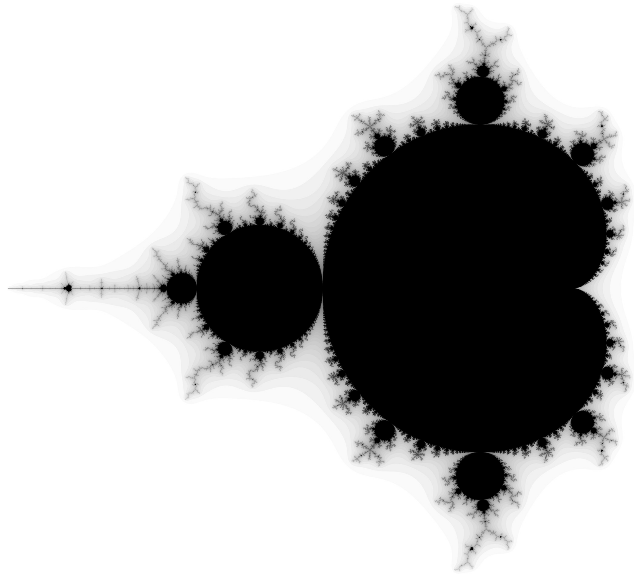

Figure 2: Fractals were popularized in science by Mandelbrot. This picture shows a Mandelbrot set, a classic example of fractal defined by Mandelbrot process $z_{n+1} = z_n + c,\ z, c \in C$. Unlike trajectory of the Wiener Process (describing Brownian motion), this fractal is deterministic.

system based on room numbers, or three dimensional based on map coordinates, or in-between based on something else? Mandelbrot realized that self-similarity allows to introduce the concept of a fractal dimension.

The way of introducing fractal dimension is not unique; for example, there exists a mass dimension

$$m(R) = m(1)R^D, \tag{1}$$

box dimension, etc. Fractals can be defines through concepts of dimensionality, one of the possible definition being the structure with different values of topological and Hausdorff dimension. When we talk about random fractals, such as in the example with Brownian motion, dimensionality Eq. (1) is understood in a sense of an ensemble average.

1.3 Historical Map

"Always historicize."

— Frederic Jameson

To better understand the topic, it is instructive to discuss what has brought Fractional Kinetics into being.

Developments in computer technologies simultaneously lead to the development of the theory of

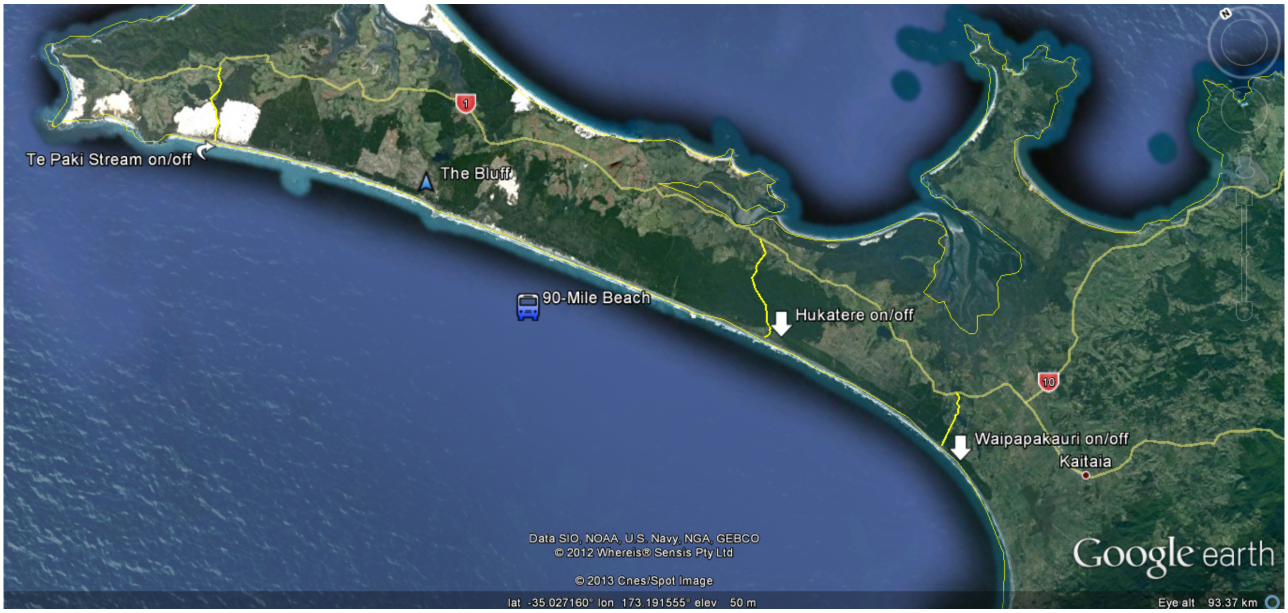

Figure 3: Used as a public highway, the "90 Mile Beach" is reported to be 55 miles long. For a coastline fractionality-aware cartographer, this toponym perhaps makes even less sense. (For more details, see: "How Long Is the Coast of Britain? Statistical Self-Similarity and Fractional Dimension" by Mandelbrot 1963 in Science)

discrete time maps of systems dynamics, and supplemented the research on fractal analysis. One of the better-known map in plasma physics is the Standard Map (the Chiricov-Taylor map)

$$p_{\mathrm{n+1}} = p_n + K\sin(\theta_n)$$
$$\theta_{\mathrm{n+1}} = \theta_n + p_{\mathrm{n+1}}$$

,

where p_n and θ_n are taken $\mod 2\pi$. This is a discrete map of a particle dynamics in a electromagnetic wave, and it captures many features of the dynamics of different physical systems. (ref: Ott's book Ch7). It is area-preserving, relating it to the Vlasov equation

$$\partial_t f + v\partial_x f + q/mE\partial_v f = 0$$

By increasing the value of K beyond the threshold of K_{crit}, the dynamics of Standard Map can be pushed into a chaotic regime. (One can even use the map of a chaotic dynamics to generate random values). Chaotic regime allows for transport, and for certain values of K, for anomalous scaling $\langle p\rangle^2 \propto t_p^{\mu}(K)$, meaning super-diffusive process. For normal diffusion

$$D = D_{\mathrm{QL}} = \frac{K^2}{2}. \tag{2}$$

Defining the diffusion coefficient

$$D = \lim_{n\uparrow}\langle(\theta_{\mathrm{n+1}} - \theta_n)^2\rangle, \tag{3}$$

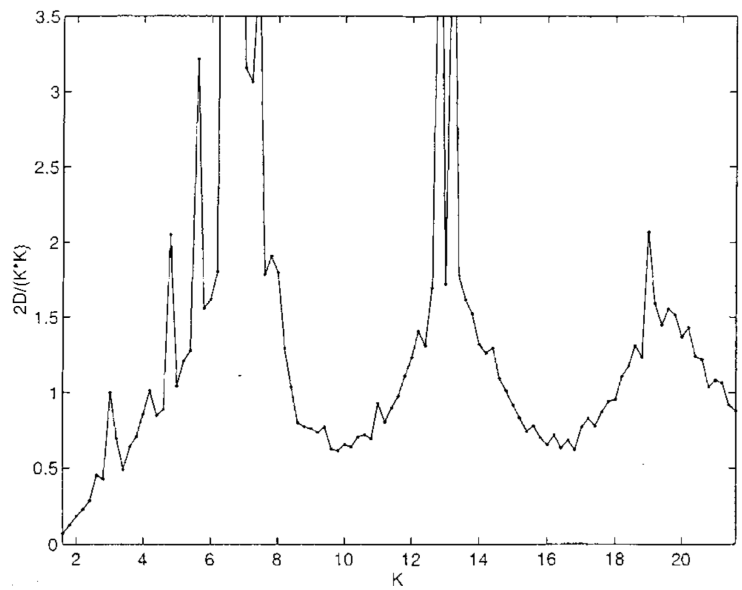

Figure 4: Dependence of (D/D_{QL}) on K for Standard Map, visually resembling fractal structure.

(n grows but not to infinity, due to the upper bound on simulation time), one observes spikes on a (D/D_{QL}) vs K plot figure 4. Explaining the anomalous diffusion of the standard map was perhaps the intellectual drive behind the Fractional Kinetic theory

2 Fractional Calculus

2.1 Geometrical Motivation

As the symbolic notation $\frac{\partial f}{\partial x}$ suggests, the first order derivative relates the increment of the function with the argument increment, $\Delta f = f'\Delta x$. One can however imagine a curve where such a relation is nonlinear. For example, for a Wiener Process, $\delta w_t \propto \Delta t^{1/2}$. This motivates the idea of a fractional derivative, as in trajectory of a Brownian motion has not 1, but 1/2 of a derivative. Fractional derivative does the same as the integer order derivative, but now $\partial_x^\alpha f = \Delta f/\Delta x^\alpha$, where α (called Hölder index) now needs not to be an integer.

Question: What is the order of a spatial derivative of a velocity field in Kolmogorov turbulence?
Answer: $\delta v \propto \delta r^{1/3}$ for Kolmogorov turbulence implies that velocity field has $1/3$ of a spatial derivative

2.2 Negative Integer Derivatives

The usual differentiation operator $D_x = \frac{d}{dx}$ recursively creates from $f(x)$ a sequence of natural order derivatives,

$$f(x),\ f'(x),\ f^{(2)}(x),\ f^{(3)}(x),\ ...,f^{(n)}(x),\ ...$$

Consider now the second sequence of functions, generated from $f(x)$ by integrating from the fixed lower limit a till the argument's value x:

$$f_1(x) = \int_a^x \mathrm{d}\xi_1 f(\xi_1)$$

$$f_n(x) = \int_a^x \mathrm{d}\xi_\mathrm{n} \int_a^{\xi_n} \mathrm{d}\xi_{\mathrm{n}-1} ... \int_a^{\xi_2} \mathrm{d}\xi_1 f(\xi_1)$$

and note that

$$D_x f_n(x) = \frac{d}{dx}\int_a^x f_{n-1}(\xi)\mathrm{d}\xi = f_{n-1}(x),\ n \geq 1$$

Now the operator D_x generates both f_{n+1} and f_{n-1} from f. If we now relabel the elements of the second sequence by the rule $f_n(x) \rightarrow_a f^{-n}(x)$, the combined sequence now reads:

$$...,\ {}_af^{(-n)}(x),\ ...,\ {}_af^{(-2)}(x),\ {}_af^{(-1)}(x),\ f(x),\ f'(x),\ f^{(2)}(x),\ ...,\ f^{(n)}(x),\ ...$$

We can now call ${}_af^{(-n)}(x)$ a derivative of order $-n$ of a function f on the interval $(a,x]$. Simply put, the $(-n)$ order derivative is just an nth order integral of f over $(a,x]$

2.3 Riemann–Liouville fractional derivative

We were able to define derivative of order $n \in Z$. Next step is to expand n to other rational numbers. This can be done in two steps: expand the notion of the order of the integral to fractional values, therefor introducing a fractional order negative derivative, and then use this result and integer positive derivatives to define positive fractional order derivatives.

First step is done with the help of a Cauchy formula:

$${}_af^{-n}(x) = \int_a^x \mathrm{d}\xi_\mathrm{n} \int_a^{\xi_n} \mathrm{d}\xi_{\mathrm{n}-1} ... \int_a^{\xi_2} \mathrm{d}\xi_1 f(\xi_1) = \frac{1}{(n+1)!}\int_a^x (x-\xi)^{n-1} f(\xi)\mathrm{d}\xi$$

that transforms the integral of the integer order n into the first order integral with a power kernel. We now use an analogue of Caushy formula with rational values of $m > 0$:

$$_af^{-m}(x) = \frac{1}{\Gamma(m)} \int_a^x (x-\xi)^{m-1} f(\xi) \mathrm{d}\xi$$

This formula however is not yet enough for the positive fractional derivatives, since the integral diverges for $m < 0$. The way Riemann–Liouville definition overcomes this problem is perhaps explained best with an example.

Suppose you want to calculate $f^{2,4}(x)$ fractional derivative. We already defined positive derivatives of an integer order and negative derivatives of a rational order. Now, to arrive to a point $n = (2.4)$, suppose we make integer steps to the left until we reach negative value. For $n = (2.4)$ that would result in a sequence $1.4; 0.4; -0.6$. Let k denote the amount of such steps ($k = 3$ in the example). Now the $n-k$ order derivative can be calculated with an extended Caushy formula described above:

$$_af^{n-k}(x) = \frac{1}{\Gamma(k-n)} \int_a^x (x-\xi)^{k-m-1} f(\xi) \mathrm{d}\xi$$

now to come back to the m point, apply the D_x operator k times. The resulting formula reads

$$_af^{n-k}(x) = \frac{1}{\Gamma(k-n)} D_x^k \int_a^x (x-\xi)^{k-m-1} f(\xi) \mathrm{d}\xi$$

.

This is the way left ($\int_a^x$) Riemann–Liouville fractional derivatives are defined.

3 Fractional Kinetics

3.1 FKE Derivation

This section presents the derivation of the Fractional Kinetics Equation.

Define the transition probability

$$P(x,t) = W(x, x_0; t - t_0) = W(x, x_0; t), \tag{4}$$

which is a conditional probability for a particle to be in (x,t) position of a phase space provided that this particle was in (x_0, t_0).

The infinitesimal shift of $P(x,t)$ along t by Δt in case of fractal time is

$$\hat{\Delta}_t^\beta P(x,t) = \frac{\partial^\beta P(x,t)}{\partial t^\beta} + O(\Delta t_1^\beta), 0 \le \beta < \beta_1 \le 1 \tag{5}$$

The infinitesimal shift of $P(x,t)$ due to the transitions from other states $P(x',t)$ during the same Δt:

$$\hat{\Delta}_x^{\alpha} P(x,t) = \int dy W(x,y;\Delta t) P(y,t) - P(x,t) + O((\Delta t)_2^{\beta}),\ \beta_2 > \beta \tag{6}$$

where α is a fractal dimension characterizing the local structure of a phase space near x.

The conservation of probability implies

$$\hat{\Delta}_t^{\beta} P(x,t) = \hat{\Delta}_x^{\alpha} P(x,t) + O((\Delta t)^{\min(\beta_1,\beta_2)} \tag{7}$$

which can be rewritten in equivalent form of

$$\lim_{\Delta t \to 0} \frac{\hat{\Delta}_t^{\beta} P(x,t)}{(\Delta t)^{\beta}} = \lim_{\Delta t \to 0} \frac{\hat{\Delta}_x^{\alpha} P(x,t)}{(\Delta t)^{\beta}} \tag{8}$$

Substituting expressions Eq. (5) and Eq. (6) into Eq. (8) results in

$$\frac{\partial^{\beta} P(x,t)}{t^{\beta}} = \lim_{\Delta t \to 0} \frac{1}{(\Delta t)^{\beta}} \left(\int dy W(x,y,\Delta t) P(y,t) - P(x,t) \right) \tag{9}$$

Expanding $W(x,y,\Delta t)$ in the limit $\Delta t \to 0$ (the existence of the expansion is assumed) results in:

$$W(x,y;\Delta t) = \delta(x-y) + A(y,\Delta t)\delta^{\alpha}(x-y) + B(y,\Delta t)\delta^{\alpha_1}(x-y),\ 0 < \alpha < \alpha_1 < 2 \tag{10}$$

where α and α_1 are appropriate fractal dimension characteristics. Truncation of the expansion Eq. (10) is where the treatment becomes approximate. Another assumption made here is that the local transitions are independent of the large time behaviour. This corresponds to stating independence of $A(y,\Delta t)$ and $B(y;\Delta t)$ of the transition probability $P(x,t)$.

Next task is to find the expression for $A(y;\Delta t$ and $B(y;\Delta t)$ in terms of moments of $W(x,y,\Delta t)$. Multiply Eq. (10) $(x-y)^{\alpha_1}$ and integrate over x.

$$\langle\langle|\Delta x|^{\alpha_1}\rangle\rangle = \int dx |x-y|^{\alpha_1} \left(\delta(x-y) + A(y,\Delta t)\delta^{\alpha}(x-y) + B(y,\Delta t)\delta_1^{\alpha}(x-y)\right) \tag{11}$$

Using the integral identity

$$\int g(x)\delta^{(n)}(x)dx = (-1)^n \int \frac{\partial^n g(x)}{\partial x^n}\delta(x)dx \tag{12}$$

from Eq. (11) get:

$$\langle\langle|\Delta x|^{\alpha 1}\rangle\rangle = \int dx |x-y|_1^{\alpha} W(x,y;\Delta t) = \Gamma(1+\alpha_1)B(y;\Delta t). \tag{13}$$

To get an expression for $A(y;\Delta t)$, integrate Eq. (10) over y:

$$\int \mathrm{d}y W(x,y;\Delta t) = \int \delta(x-y) + \int \mathrm{d}y A(y,\Delta t)\delta^{\alpha}(x-y) + \int \mathrm{d}y B(y,\Delta t)\delta\alpha_1(x-y) \tag{14}$$

$$1 = 1 + \int \mathrm{d}y \frac{\partial^{\alpha} A(y,\Delta t)}{\partial y^{\alpha}} \delta^{\alpha}(x-y) + \int \mathrm{d}y \frac{\partial^{\alpha_1} B(y,\Delta t)}{\partial y^{\alpha_1}} \delta(x-y) \tag{15}$$

$$\frac{\partial^{\alpha} A(x;\Delta t)}{\partial(-x)^{\alpha}} + \frac{\partial^{\alpha_1} B(x;\Delta t)}{\partial(-x)^{\alpha_1}} = 0 \tag{16}$$

Defining

$$\mathscr{A}(x) = \lim_{\Delta t \to 0} \frac{A(x;\Delta t)}{(\Delta t)^{\beta}} \tag{17}$$

$$\mathscr{B}(x) = \lim_{\Delta t \to 0} \frac{B(x;\Delta t)}{(\Delta t)^{\beta}} \tag{18}$$

Dividing Eq. (16) by $(\Delta t)^{\beta}$, in the $\Delta t \to 0$ get:

$$\frac{\partial^{\alpha} \mathscr{A}(x)}{\partial(-x)^{\alpha}} + \frac{\partial^{\alpha_1} \mathscr{B}(x)}{\partial(-x)^{\alpha_1}} = 0 \tag{19}$$

rewriting Eq. (9) as

$$\frac{\partial^{\beta} P(x,t)}{\partial t^{\beta}} = \lim_{\Delta t \to 0} \frac{1}{(\Delta t)^{\beta}} \left(\int \mathrm{d}y \left(W(x,y;\Delta t) - \delta(x-y)\right) P(y,t) \right) \tag{20}$$

Plugging Eq. (10) into Eq. (20) gives

$$\frac{\partial^{\beta} P(x,t)}{\partial t^{\beta}} = -\lim_{\Delta t \to 0} \frac{1}{(\Delta t)^{\beta}} \left(\int \mathrm{d}y \left(A(y;\Delta t)\delta^{\alpha}(x-y) + B(y;\Delta t)\delta_1^{\alpha}(x-y)\right) P(y,t) \right)$$

$$\frac{\partial^{\beta} P(x,t)}{\partial t^{\beta}} = -\lim_{\Delta t \to 0} \frac{1}{(\Delta t)^{\beta}} \left(\frac{\partial^{\alpha}}{\partial(-x)^{\alpha}} \left(A(y;\Delta t)P(x,t)\right) + \frac{\partial^{\alpha_1}}{\partial(-x)^{\alpha_1}} \left(B(y,\Delta t)P(x,t)\right) \right)$$

$$\frac{\partial^{\beta} P(x,t)}{\partial t^{\beta}} = \frac{\partial^{\alpha}}{\partial(-x)^{\alpha}} \left((y;\Delta t)P(x,t)\right) + \frac{\partial^{\alpha_1}}{\partial(-x)^{\alpha_1}} \left((y,\Delta t)P(x,t)\right), \tag{21}$$

which is the Fractional Kinetics Equation, and α, α_1 and β are critical exponents. In case $\alpha_1 = \alpha + 1$ Eq. (21) can be simplified into

$$\frac{\partial^\beta P(x,t)}{\partial t^\beta} = \frac{\partial^\alpha}{\partial(-x)^\alpha}\left(\mathscr{B}(x)\frac{\partial P(x,t)}{\partial x}\right), \tag{22}$$

which reduces to FP equation for $\alpha = 1$, $\beta = 1$ and $(x) = D/2$

3.2 Special Cases

Advantage of Fractional Kinetic Equation is that it systematically includes normal and abnormal diffusion processes. In particular, assuming that (x) can be neglected and rewriting Eq. (21) as

$$\frac{\partial^\beta P(x,t)}{\partial t^\beta} = \frac{\partial^\alpha}{\partial |x|^\alpha}\left(\mathscr{A}(x)P(x,t)\right) \tag{23}$$

Case 1: $\alpha = 2$, $\beta = 1$ recovers case of normal diffusion:

$$\frac{\partial P(x,t)}{\partial t} = \frac{\partial^2}{\partial |x|^2}\left(\mathscr{A}(x)P(x,t)\right) \tag{24}$$

Case 2: $\alpha = 2$, $0 < \beta < 1$ recovers equation for fractal Brownian motion:

$$\frac{\partial^\beta P(x,t)}{\partial t^\beta} = \frac{\partial^2}{\partial |x|^2}(\mathscr{A}(x)P(x,t)) \tag{25}$$

Case 3: $1 < \alpha < 2$, $0\beta = 1$ recovers equation for a Levy Process:

$$\frac{\partial P(x,t)}{\partial t} = \frac{\partial^\alpha}{\partial |x|^\alpha}(\mathscr{A}(x)P(x,t)) \tag{26}$$

3.3 Transport

Macroscopic observables of the system correspond to the moments:

$$\langle |x|^\delta \rangle = \int dx |x|^\delta P(x,t), \tag{27}$$

where P(x,t) is the solution to FKE equation. Time dependence of these moments determines macroscopic evolution of the system.

Assume $\mathscr{A}(x) = \text{const}$ and $\mathscr{B}$ is negligeble. Then FK Eq. (21) becomes

$$\frac{\partial^\beta P(x,t)}{\partial t^\beta} = \frac{\partial^\alpha}{\partial |x|^\alpha}(\mathscr{A}(x)P(x,t)) \tag{28}$$

Multiply Eq. (28) by $|x|^\alpha$ and integrate over x:

$$\frac{\partial^\beta \langle |x|^\alpha \rangle}{\partial t^\beta} = \mathscr{A}\int \mathrm{dx}|x|^\alpha \frac{\partial^\alpha P(x,t)}{\partial |x|^\alpha} = \mathscr{A}\int \mathrm{dx}\left(\frac{\partial^\alpha |x|^\alpha}{\partial |x|^\alpha}\right)P(x,t) = \Gamma(1+\alpha)\mathscr{A} \tag{29}$$

Integrate Eq. (29) over t^β to get

$$\langle |x|^\alpha \rangle = \mathscr{A}\frac{\Gamma(1+\alpha)}{\Gamma(1+\beta)}t^\beta \tag{30}$$

Due to self-similarity of FKE solutions,

$$\langle |x| \rangle \propto t^{\beta/\alpha} = t^{\mu/2}, \tag{31}$$

where μ is called transport exponent. $\mu > 1$ and $\mu < 1$ correspond to cases of super- and sub-diffusion, respectively.

3.4 FP vs FKE

Since FKE is a generalization of FP method, it is instructive to list the corresponding parameters of the two. The main original motivation to expand the FP theory was to allow for anomalous scaling. After introducing fractional derivatives anomalous scaling becomes possible, however the physical intuition for A and B terms is now lost.

Table 1: Comparison between parameters of Fokker-Planck and Fractional Kinetics theories.

	FP	FKE
Derivatives	∂_x, ∂_t	∂_x^α, ∂_t^β
Stochastic variable	Δx	Δx, Δt
Time	Fixed	Variable with PDF
Second Moment scaling	$\langle \lvert x \rvert^2 \rangle \propto t$	$\langle \lvert x \rvert^2 \rangle \propto t^\mu$, $\mu \in (0,2)$
Kolmogorov Conditions	$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \langle\langle \Delta x \rangle\rangle = \mathscr{A}(x)$ $\lim_{\Delta t \to 0} \frac{1}{\Delta t} \langle\langle (\Delta x)^2 \rangle\rangle = \mathscr{B}(x)$ $\lim_{\Delta t \to 0} \frac{1}{\Delta t} \langle\langle (\Delta x)^m \rangle\rangle = 0; m > 2$	$\lim_{\Delta t \to 0} \frac{A(x, \Delta t}{(\Delta t)^\beta} = \mathscr{A}(x)$ $\lim_{\Delta t \to 0 \frac{B(x,\Delta t}{(\Delta t)^\beta}} = \mathscr{B}(x)$
$A(y, \Delta t)$	$\langle\langle (\Delta y) \rangle\rangle$	-
$B(y, \Delta t)$	$\langle\langle (\Delta y)^2 \rangle\rangle$	$\frac{\langle\langle \lvert \Delta x \rvert^{\alpha_1} \rangle\rangle}{\Gamma(1+\alpha_1)}$
$\mathscr{A}(x)$ to $\mathscr{B}(x)$ relation	$A(y, \Delta t) = \frac{1}{2} \frac{\partial B(y,\Delta t)}{\partial y}$	$\frac{\partial^\alpha \mathscr{A}(x)}{\partial(-x)^\alpha} = -\frac{\partial^{\alpha_1} \mathscr{B}(x)}{\partial(-x)^{\alpha_1}}$