Notes 9 – Intermittency 2

Recall: **Intermittency**

→ probability measure concentrated in small chunks of phase space

→ growth of higher moments (with N, t) a symptom, i.e. $\langle \psi^n \rangle$ vs $\langle \psi \rangle$, $\langle \psi^2 \rangle$.

[QL, etc ok for low order moments, fail for higher]

→ often associated with multiplicative process

Now → Intermittency in Space

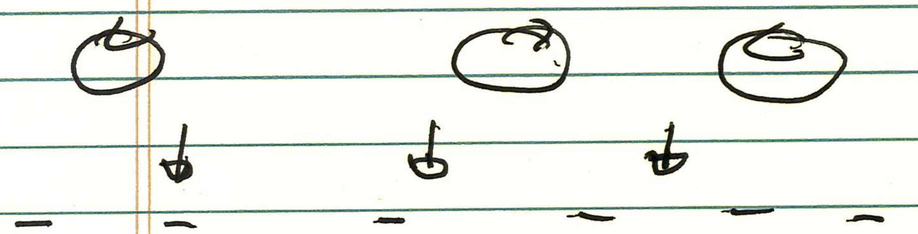
⇒ Turbulence ⇅ Fractals, Multi-Fractals

How describe { inertial range "patchiness" / fluctuations in dissipation / → deviations from K41.

Recall K41:

→ ϵ indep ν ⇒ sing
→ Richardson orbit separation
→ inertial range ; 4/5 Law
→ dissipation scale
→ 2D.

– K41 phenomenology suggests uniform distribution of dissipation



l_d

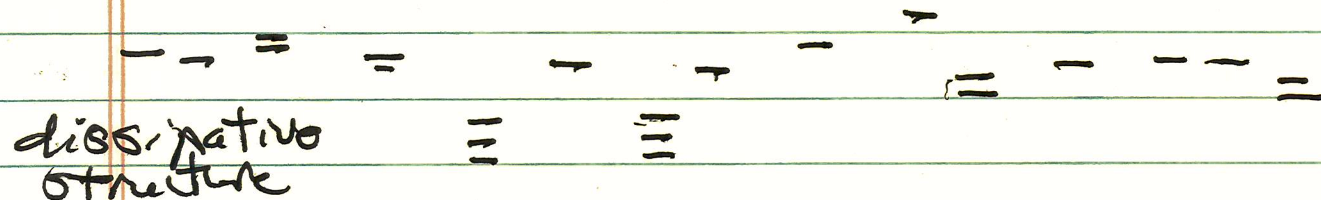
dissipation on l_d, filling space

in reality:

→ distribution of dissipation is variable in intensity

→ patchy, intermittent, not space filling i.e. $\downarrow$

$\Rightarrow$



some departure from K41 spectrum concomitant.

→ How characterize? → Phenomenology?

∴ Characterize geometry of dissipative structure?

→ effective dimensionality $\left\{ \begin{array}{l} \text{ribbon} \\ \text{vs} \\ \text{blob} \end{array} \right.$

<u>so</u> Fractal Intermittency Model
(β-model)

* → cf Frisch, Sulem, Nelkin → a <u>must</u>
('78)

→ Frisch, book ('95)
"Turbulence - The Legacy of A.N. Kolmogorov"

→ Mandelbrot: "Multifractals and 1/f Noise" - Wild Self-Affinity in Physics (1998).

- fractal? why?

⇒ - intermittency / patchiness → effective dimension
- fractals enable geometric phenomenology

Dimension

- consider structure embedded in Cartesian space

- covering, N-D cubes, Cartesian size ϵ

if $\tilde{N}(\epsilon)$ → # cubes to cover set (structure)

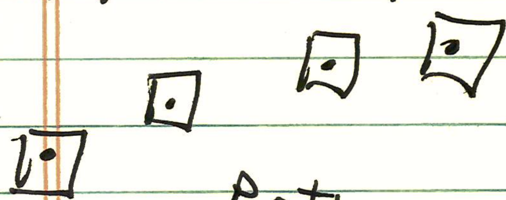
$$D_0 = \lim_{\epsilon \to 0} \ln \tilde{N}(\epsilon) / \ln (1/\epsilon)$$

Box Counting Dimension:

$$\boxed{D_0 = \lim_{\epsilon \to 0} \frac{\ln \tilde{N}(\epsilon)}{\ln (1/\epsilon)}}$$

check:

(a) finite # pts

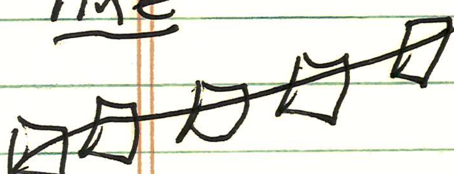


P pts

$$D_0 = \lim_{\epsilon \to 0} \frac{\ln P}{\ln(1/\epsilon)}$$

$\to 0$ ✓

(b) line



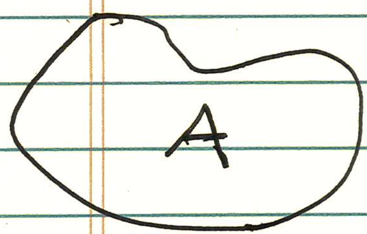
— length ℓ

$$D_0 = \lim_{\epsilon \to 0} \frac{\ln \tilde{N}(\epsilon)}{\ln(1/\epsilon)}$$

$$= \lim_{\epsilon \to 0} \frac{\ln(\ell/\epsilon)}{\ln(1/\epsilon)}$$

$$= \frac{\ln(\ell) + \ln(1/\epsilon)}{\ln(1/\epsilon)} \to$$ ✓

(c) Area A enclosed by closed curve



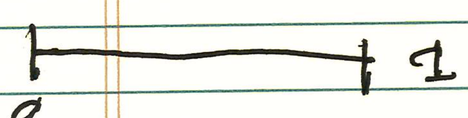
$$D_0 = \lim_{\epsilon \to 0} \frac{\ln \tilde{N}(\epsilon)}{\ln(1/\epsilon)}$$

$$= \lim_{\epsilon \to 0} \frac{\ln(A/\epsilon^2)}{\ln(1/\epsilon)}$$

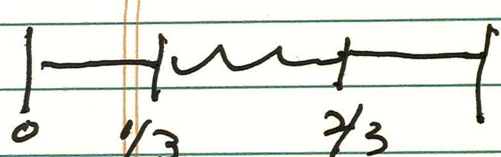
$$= 2 \checkmark$$

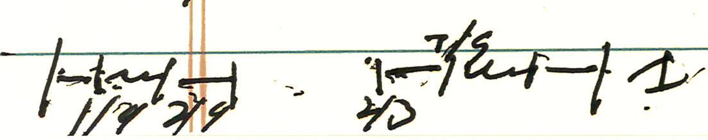
Now, juicier examples:

→ Middle Third Cantor Set



chop middle 1/3





For each n cover with 2^n pieces, $(1/3)^n$ length ℓ

→ n.b. multiplicative process

$$D_0 = \lim_{\substack{n \to \infty \\ \epsilon \to 0}} \frac{\ln 2^n}{\ln(1/(1/3)^n)}$$

↓ box cntg dim.

$$= \lim_{n \to \infty} \frac{n \ln 2}{n \ln 3} = \frac{\ln 2}{\ln 3}$$

$D_0 = .63 \ldots$

- fractal dimension
- $0 < D_0 < 1$
- embedded in $D = 1$

$D_0 < D_{embed.}$

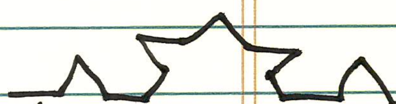
- $D \Rightarrow$ power law

$$N(\epsilon) \sim \epsilon^{-D_0}$$

Fractals - self-similar.

~ other canonical example: Koch curve (cf "Coast of Britain")

4



1/3

16

every

l extend $\quad$ $4/3\, l$

$$D_0 = \lim_{\epsilon \to 0} \frac{\tilde{N}(\epsilon)}{\ln(1/\epsilon)}$$

$$= \lim_{\substack{\epsilon \to 0 \\ n \to \infty}} 4^n \Big/ \ln\left[1/(1/3)^n\right]$$

$$= \lim_{n \to \infty} \left[\frac{2n \ln 2}{n \ln 3}\right] = \frac{2 \ln 2}{\ln 3}$$

$$D_0 = 2 \ln 2 / \ln 3 = 2 (\text{MTCS})$$

~ 1.2.

- example of a fractal which thickens, $1 < D < 2$.
  Need embedd in 2D.

- C-O-B : increased resolution reveals longer, more convoluted coastline

- rougher on smaller scales (N increases with n).

Why Fractals and Intermettency (Mandelbrot) ?

- self-similar structures with dimension < dimension of embedding space (i.e. 3)

- natural candidates to describe
  - spatially intermittent dissipation events
  - geometry of dissipative structures.

⇒ Intermittent cascade is due to hierarchical, embedded process. Dissipative structure does not fill space.

$D_0 \to$ intermittency correction to K41 spectrum.

Fractal intermittency idea:

- picture / phenomenology
- predicts / fits everything, explains nothing. (where is N-S Eqn?).
- D_0 not predicted.

which brings us to:

→ **β-model** (Frisch*, Sulem, Nelkin)

→ basic idea: (Mandelbrot)

- cascade active region is self-similar fractal structure, with $D < 3$

so

- dissipation events are 'patchy'
- forces correction to K41

→ **Analysis**

- why intermittency? ⇒ physics of cascade.

⇒ vortex stretching is very nonlinear

isentropic
$\underline{\nabla} \cdot \underline{V} = 0$

enthalpy

$$\partial_t \underline{V} + \underline{V} \cdot \underline{\nabla} \underline{V} = -\nabla w + \nu \nabla^2 \underline{V}$$

$$\partial_t \underline{V} = -\nabla \left(w + \frac{\underline{V}^2}{2} \right) + \underline{V} \times \underline{\omega} + \nu \nabla^2 \underline{V}$$

$\boxed{\underline{\omega} \equiv \underline{\nabla} \times \underline{V} \rightarrow \text{vorticity}}$ (key physics)

$\Rightarrow$ $\nabla\times$ $\Rightarrow$

$$\boxed{\partial_t \underline{\omega} = \nabla\times(\underline{v}\times\underline{\omega}) + \nu\nabla^2\underline{\omega}}$$

vorticity advection eq'n.

$\nabla\cdot\underline{v}=0$

$$\frac{d\underline{\omega}}{dt} = \underline{\omega}\cdot\nabla\underline{v} + \nu\nabla^2\underline{\omega}$$

$\hookrightarrow$ vortex stretching

Kelvin Thm.

$\oint \underline{v}\cdot d\underline{\ell} = \text{const.}$

$$\sim \omega^2 \pm \nu\nabla^2\underline{\omega}$$

heuristic only

$\Rightarrow$ fast (nearly explosive) growth of vorticity, enstrophy $\langle\omega^2\rangle$ produced to dissp'n.

$\Rightarrow$ bursts, etc.

$\Rightarrow$ vortex stretching feeds on self $\Leftrightarrow$ localized process.

$\rightarrow$ so, "patchy" "embedded" cascade:

$\hookrightarrow$ occupation factor

$$\bar{\epsilon} \sim \beta_n \frac{V_n^3}{\ell_n}$$

~~mean~~ dissp'n. rate

$\beta_n \equiv$ { fraction of space active in nth step cascade.

N.B.

- $\oint \underline{v} \cdot d\underline{l} = \int \underline{\omega} \cdot d\underline{a} = \text{const}$

$\omega_1 r_1^2 \sim \omega_2 r_2^2 \Rightarrow$ vorticity increases on small scale

- n.b. analogy:

$$\underline{E} + \frac{\underline{v} \times \underline{B}}{c} = \eta \underline{J}$$

$$\underline{\nabla} \times \underline{E} = -\frac{1}{c} \frac{\partial \underline{B}}{\partial t}$$

$$\Rightarrow \quad \partial_t \underline{B} = \underline{\nabla} \times \left(\frac{\underline{v} \times \underline{B}}{c} \right) + \eta \nabla^2 \underline{B}$$

etc.

$\boxed{\beta_n\ ? \rightarrow \text{fraction of volume active in } n^{th} \text{ step of cascade}}$

N.B. – if each eddy scale $\ell \rightarrow \ell/2$ per step

then # "children" to fill space per step

is $\underline{2^3 = 8}$ i.e. 

$\Rightarrow \quad \beta = N/2^3 < 1$

N: # off-spring

occupation reduction factor → intermittency

$$\beta_n = (\beta)^n = (N/2^3)^n \quad \rightarrow n \text{ steps.}$$

→ now, interpretation only

$$N \equiv 2^D \qquad D < 3$$

D: box counting dimension

[simply an interpretation]

$$\boxed{\beta_n = \left(2^{D-3}\right)^n} \quad \leftrightarrow \text{ multiplicative}$$

So, taking mean energy balance:

$$\bar{\epsilon} = \beta_n \frac{V_n^3}{\ell_n} \qquad \beta_n = 2^{n(D-3)} = \left(\frac{\ell_0}{\ell_n}\right)^{(D-3)}$$

$$= \left(\frac{\ell_n}{\ell_0}\right)^{3-D} \frac{V_n^3}{\ell_n}$$

$\Rightarrow$

$$V(\ell_n) \sim (\bar{\epsilon}\ell_n)^{1/3} (\ell_n/\ell_0)^{-1/3(3-D)}$$

Intermittency → memory
have ℓ_0

correction due NEO
⇒ induces explicit ℓ_0

Fraction of active

$$\boxed{E_n \sim \beta_{\ell_n} V(\ell_n)^2} \sim \bar{\epsilon}^{2/3} \ell_n^{2/3} (\ell_n/\ell_0)^{-2/3(3-D)} \left(\frac{\ell_n}{\ell_0}\right)^{(3-D)}$$

Velocity
in active region

$$\sim \boxed{\bar{\epsilon}^{2/3} \ell_n^{2/3} (\ell_n/\ell_0)^{(3-D)/3}}$$

and so

$$\boxed{\begin{aligned} E(l_n) &\simeq \bar{\epsilon}^{2/3}\, l_n^{2/3} \left(l_n/l_0\right)^{(3-D)/3} \\ E(k) &\simeq \bar{\epsilon}^{2/3}\, k^{-5/3} \left(k\, l_0\right)^{-\frac{1}{3}(3-D)} \end{aligned}}$$

→ correction to K41, in proportion $3-D$

* → slight steepening of spectrum

can deduce effective dimension from fit to spectral data (2.7, 2.8).

Finally, dissipation scale changes:

i.e. $$\frac{\nu}{l_d^2} = \frac{V(l_d)}{l_d}$$ R_e [redacted] $\sim \frac{l_0 V_0}{\nu}$

[redacted]

but

$$V(l_d) \sim \bar{\epsilon}^{1/3}\, l_d^{1/3} \left(l_d/l_0\right)^{-\frac{(3-D)}{3}}$$ $\bar{\epsilon} \sim V_0^3/l_0$

$$\Rightarrow \boxed{l_d \sim l_0 \left(Re\right)^{-3/(1+D)}}$$

$$Re = \frac{l_0 V_0}{\nu} = \frac{\bar{\epsilon}^{1/3} l_0^{4/3}}{\nu}$$

<u>$D = 3$</u>

$$l_d \sim l_0 \left(\frac{\bar{\epsilon}^{1/3} l_0^{4/3}}{\nu}\right)^{-3/4}$$

$$\sim \frac{\cancel{l_0}}{\cancel{l_0}} \epsilon^{-1/4} \nu^{+3/4}$$

$$l_d \sim \nu^{3/4} / \epsilon^{1/4} \quad , \; D = 3.$$

modified for $D < 3$.

[redacted]

More Intermittency [redacted]

A.)

→ Why the Fractology and what do we get from β-Model?

- $\boxed{\text{higher moments are a more severe probe of small scale structure of turbulence than energy is!}}$

Recall K41 ⇒ $\delta v(\ell) \sim \epsilon^{1/3} \ell^{1/3}$

$$\therefore \langle \delta v(\ell)^p \rangle \sim \underline{\epsilon^{p/3} \ell^{p/3}}$$

so normalizing:

$$\langle \delta v(\ell)^p \rangle \Big/ \langle \delta v(\ell)^2 \rangle^{p/2} \sim \underline{1}$$

$\Big[$ normalized moments all independent of scale! → Testable Prediction $\Big]$

So, what of higher moments, i.e. $p>2$?

Special interest (v):

→ $p = 3$ — skewness S (Measure of symmetry)

why? turbulence → statistical approach/picture

naively: Gaussian distribution (i.e. random)

so $S \to 0$.

but:

$$\partial_t E \sim \partial_t V^2 \sim V^3 \quad ; \quad \underline{\text{so}}$$

net energy transfer in cascade, and

$\langle V^3 \rangle \neq 0$, necessarily.

Similarly, $p = 4$ — kurtosis K

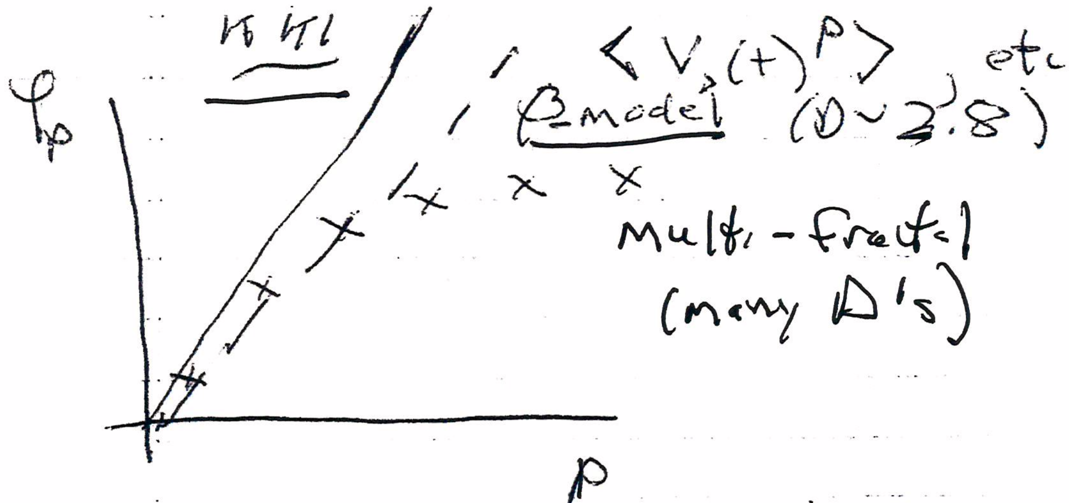
$$K = \langle \delta V^4 \rangle / \langle \delta V^2 \rangle^2 \to 3 \quad \text{for Gaussian}$$

process. ⇒ measure of importance/weight of <u>tails</u> of distribution.

⇒

$K \gg 1$ is indicative of strong correlations and non-Gaussian behavior, and fat tails.

→ The Data (mostly in time, i.e. $\langle V_r(t)^p \rangle$ etc.)



⇒ reality departs K41!

→ What does β-model predict?

$$\langle |\delta v(l_n)|^p \rangle \sim \underbrace{\beta_n}_{\text{volume factor}} \, [\delta v(l_n)]^p$$

↳ set by cascade scaling

$$\sim \bar{\epsilon}^{p/3} \, l_n^{p/3} \, (l_n / l_0)^{\zeta_p}$$

exponent of intermittency correction

wt factor

$$\boxed{\zeta_p = \frac{1}{3}(3-D)(3-p)}$$

δv^p

so, normalization $\Rightarrow$

$$a_p(\ell_n) \sim \langle \delta v(\ell_n)^p \rangle / \langle \delta v(\ell_n)^2 \rangle^{p/2}$$

plugging in $\Rightarrow$

$$\boxed{\begin{aligned} a_p(\ell_n) &\sim (\ell_n/\ell_o)^{\varepsilon_p} \\ \varepsilon_p &= 1/2\,(3-D)(2-p) \end{aligned}}$$

n.b. $p>2$
$\varepsilon_p < 0$

In particular:

$$S \sim \langle \psi^3 \rangle / \langle \psi^2 \rangle^{3/2}$$

$$\sim Re^{3(3-D)/2(1+D)}$$

$\psi \sim \nabla v$

taking $\ell_n \sim \ell_d$ as effect maximal

$$\boxed{K \sim S^2}$$

Note:

- departure from K41 strongest at smallest scales
  $\Rightarrow$ 'focus' of cascade strongest

- β model ⇒ stages in cascade have "memory" of initial scale l_0

  ⇒ l_0 explicit, beyond ϵ.

- $D \approx 2.8$ is reasonable data fit

  ⇒ [dissipative structure is highly convoluted sheets.

- ζ_p departs β-model as $p \uparrow$

  ⇒ Multi-fractal model; i.e.

  β-Model → single dissipative structure of dimension D.

  Multi-fractal → multiple dissipative structures, different.

⇒ [connection to Navier-Stokes equation and dynamics is increasingly obscure ----

[redacted]

— a natural question:

→ have argued that intermittency

⇔ departure from simple, self-similarity scaling

⇒ manifested as $\ln(\ell/\ell_0)$ "memory" in structure function

---

→ have also stressed analogy between

self-similarity in space (Blast wave)

vs self-similarity in scale (K41).

∴

→ so, what is analogue of intermittency for space-time similarity, i.e.

K41 ⟷ β- model

as

Spatio-temporal self-similarity ⟷ ?

⇒ Memory of initial condition!

i.e. $f \to f(\underbrace{r/\rho(t)}_{\text{self-sim. variable}},\ r_0)$ ⇓ .

Note for Sedov-Taylor effectively ignored initial radius of blast!

⇒ { See Barenblatt, "Scaling" Chapter 3 }

Now, one can go further, and calculate:

$\langle \tilde{\epsilon}^2 \rangle$ → mean square fluctuations in dissipation

but $\epsilon \sim \nu \langle (\nabla \tilde{v})^2 \rangle$ $\quad \langle \tilde{\epsilon}^2 \rangle \Leftrightarrow$ kurtosis

so $\langle \tilde{\epsilon}^2 \rangle \sim \nu^2 \langle (\nabla \tilde{v})^2 (\nabla \tilde{v})^2 \rangle \sim \nu^2 \beta_{\ell_d} \dfrac{v(\ell_d)^4}{\ell_d^4}$

if normalize:

$\langle \tilde{\epsilon}^2 \rangle / \langle \epsilon \rangle^2 \sim \cancel{\nu^2} K$
$\sim \cancel{\nu^2} Re^{3(3-D)/(1+D)}$ } kurtosis!

Can also address:

$$\boxed{\langle \epsilon(\underline{r})\, \epsilon(\underline{r}+\underline{\ell}) \rangle \rightarrow \underline{\text{dissipation correlation}}}$$

Now,

$$\langle \epsilon(\underline{r})\, \epsilon(\underline{r}+\underline{\ell}) \rangle \sim \langle \epsilon \rangle^2 \, \text{Prob.}\left(\underline{r}, \underline{r}+\underline{\ell} \ \underline{\text{belong to } m\text{-eddy}}\right)$$

$$\langle \epsilon \rangle \sim v_m^3/\ell_m$$

$$\sim v(\ell_m)^3/\ell_m$$

⇒ allowing for:
- packing
- if correlated by ℓ_m, then correlated by all larger eddys.

⇒

$$\langle \epsilon(r)\, \epsilon(r+\ell) \rangle \sim \sum_{m=0}^{\overset{\ell_n \lesssim}{n}} \left(\frac{v_m^3(\ell_m)}{\ell_m} \right)^2 \underline{P_m}$$

$$\sim \bar{\epsilon}^2 (\ell/\ell_0)^{-(3-D)}$$

In particular,

$$\boxed{\langle \epsilon(r)\,\epsilon(r+l_d)\rangle \sim \bar{\epsilon}^2 (l_d/l_0)^{(D-3)}}$$

$\Rightarrow$ strong correlation in dissipation at dissipn. scale.