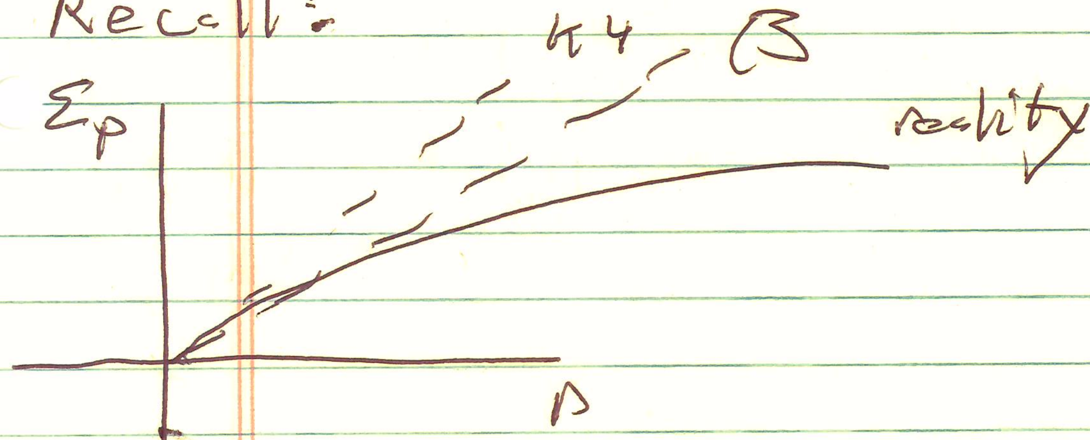
Notes 9 – Multifractals (Supplement)
(Addendum).

→ β model: equivalent to velocity field having:

- scaling exponent h (K41: $h = 1/3$)
- on set of fractal dimension D.

Recall:



How construct ⌒ curve?

(i) Bifractality

Natural extension of β model (unifractal)
→ bifractality

i.e. $$\frac{\delta v_\ell(r)}{v_0} \sim \begin{cases} \left(\frac{\ell}{\ell_0}\right)^{h_1}, & \mathcal{S}_1,\ \dim \mathcal{S}_1 = D_1 \\ \left(\frac{\ell}{\ell_0}\right)^{h_2}, & \mathcal{S}_2,\ \dim \mathcal{S}_2 = D_2 \end{cases}$$

$\mathcal{S}_1 \cup \mathcal{S}_2 =$ space.

i.e. 2 scalings.

So, for p-order structure function:

$$\frac{\langle \delta v_\ell^p \rangle}{v_0^p} = \mu_1 \left(\frac{\ell}{\ell_0}\right)^{ph_1} \left(\frac{\ell}{\ell_0}\right)^{3-D_1} + \mu_2 \left(\frac{\ell}{\ell_0}\right)^{ph_2} \left(\frac{\ell}{\ell_0}\right)^{3-D_2}$$

μ_1, μ_2 $O(1)$ const.

In inertial range, $\ell \ll \ell_0$; ~~[illegible]~~ smaller exponent dominates, so:

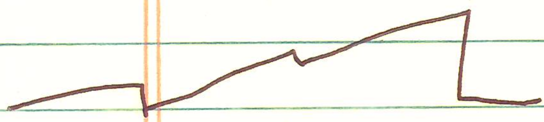
$$\langle \delta v_\ell^p \rangle \sim \ell^{\zeta_p}$$

$$\zeta_p = \min\left(ph_1 + 3 - D_1,\ ph_2 + 3 - D_2\right)$$

→ "Battle of Catastrophes" (after Berry).

→ Examples of Bi-fractality

- Burgers turbulence



ramps: $D=1$

shocks: $D=0$

- K41 + β-model $\left.\right\}$ $D_1=3,\ h_1=1/3$

$0<D_2<3,\ h_2=1/3-(3-D_2)/3$

$$\zeta_p = \min(\quad)$$

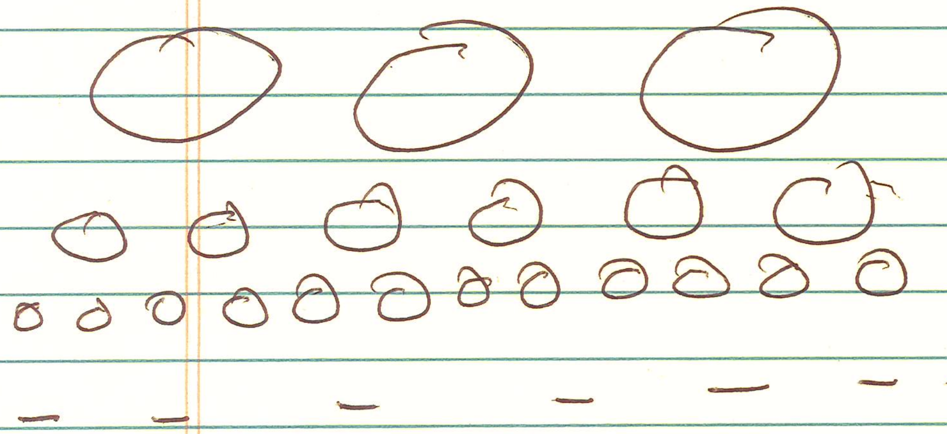
$$= \begin{cases} p/3 & 0\leq p\leq 3 \\ p/3+(3-D_2)\left(1-\frac{p}{3}\right), & p\geq 3 \end{cases}$$

↓

ensures adherence to 4/5 Law.

Asides:

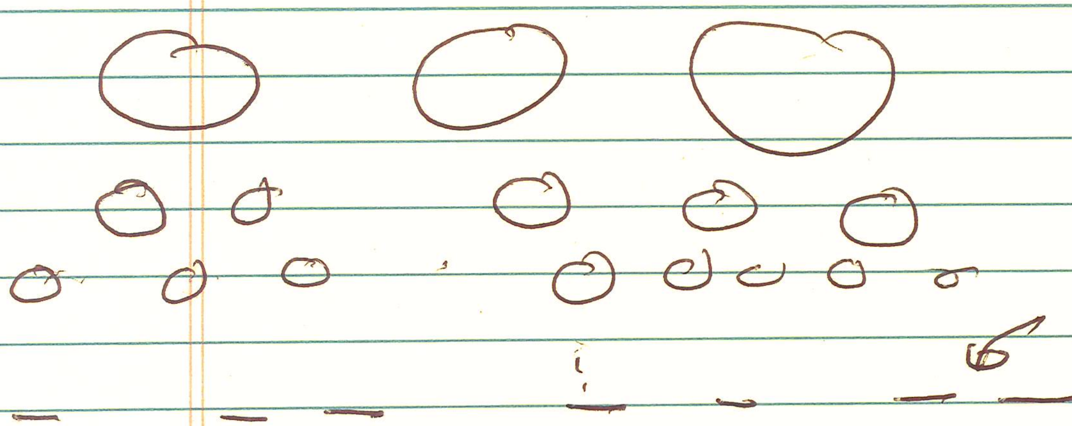
$- D = 3$ (Space Filling $- \beta = 1$)



cascade fills space at all scales

$l_d < l < l_0$

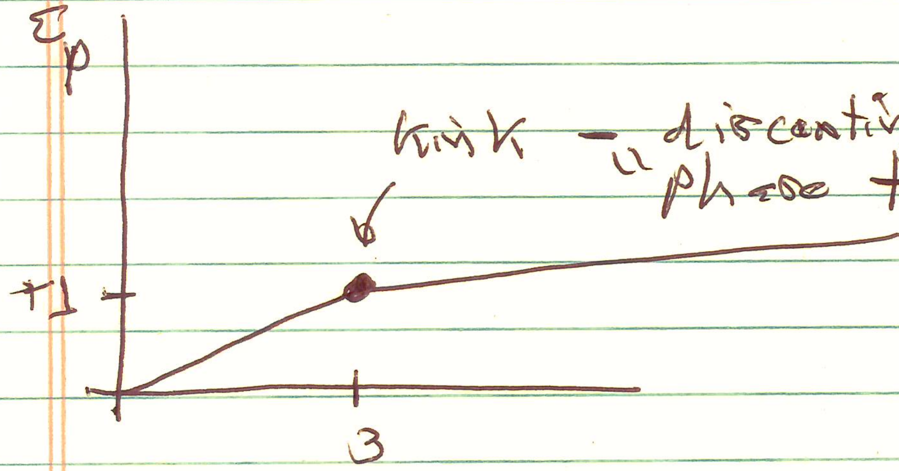
$- D \neq 3$ $(\beta < 1)$



Fractal dissipative structure.

(akin Middle Third Cantor Set)

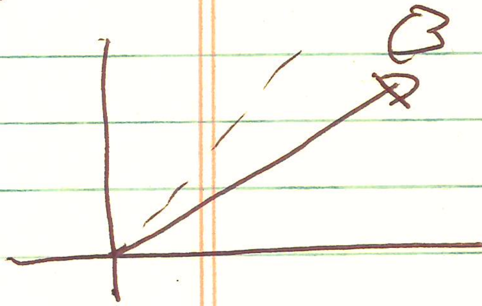
so



kink — discontinuity a la "phase transition" (2nd order)

→ multiple phases for multiple moments!

→ so



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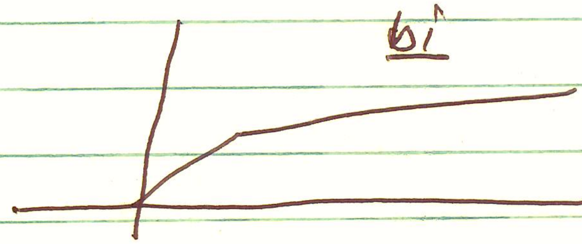

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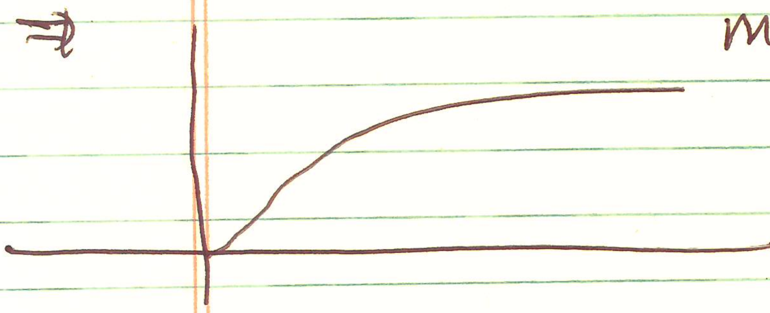

multi ?!

— Multifractality

{ Frisch, '95 after Parisi and Frisch '85 }

- now posit continuous infinity of scaling exponents

- Now, re-visit K 41 foundations!

H 1 → symmetries restored at small scales

H 3 → $\epsilon \neq 0$ → non-vanishing mean dissipation

H 2 → H_{MF}

Turbulent flow assumed to possess a range of scaling exponents $I = (h_{min}, h_{max})$. For each, there is a set $\mathcal{S}_h$ of fractal dimension $D(h)$ s/t as $l \to 0$

$$\frac{\delta v_l(r)}{v_0} \sim (l/l_0)^h$$

h_{min} and h_{max} postulated to be universal (indep. production).

e.g., multi-fractal structure function:

$$\frac{S_p(\ell)}{v_0^p} = \frac{\langle \delta v_\ell^p \rangle}{v_0^p} \sim \int_I d\mu(h) \left(\frac{\ell}{\ell_0}\right)^{ph+3-D(h)}$$

$\ell \to 0$, smallest exp dominates:

$$\boxed{\zeta_p = \inf_h \left[ph + 3 - D(h) \right]}$$

$\rightarrow$ $\zeta(p)$ and $D(h)$ are Legendre Transformation

and of course:

$$\zeta_3 = \inf_h \left[3h + 3 - D(h) \right] = 1$$

and can show:

$$\boxed{D(h) = \inf_p \left[ph + 3 - \zeta_p \right]}$$

inversely.

extract D from scalings!

Can extend ad-infinitum. (cf. Frisch).