# Notes 8 – Intermittency 1

→ So far:

Transport (in random media)
- $Ku < 1$ → diffusion
- $Ku > 1$ → percolation

Emphasis: {Irreversibility, Interaction Scattering and Collisions, Emergent Order}

Now, Intermittency

What does it mean?:

→ "patchy", "bursty", in space and time

→ distribution concentrated in widely spaced patches, where local Pdf $\gg \langle$ Pdf $\rangle$

i.e. if $x_i$ → location of intermittent event

$$\mathrm{Pdf}(x_i) \gg \langle \mathrm{Pdf} \rangle$$

$x_i, x_j$ widely spaced.

What Intermittency is not ⇒

the familiar world of:

- Central Limit Theorem
- Law of Large #'s
- Gaussian Statistics

i.e. smooth pdfs

a la Mandelbrot, contrast:

"Mild" vs "Wild", variability

Recall Central Limit Theorem;

Let $x_1, x_2 \ldots$ be a sequence of

- independent
- identically distributed

random variables, each with mean $\mu$ and variance $\sigma$, so:

$$\overline{x_i} = \mu$$

$$\langle (x_i - \overline{x_i})^2 \rangle = \sigma^2$$

then

$$\frac{x_1 + x_2 + \ldots x_n - n\mu}{\sqrt{n}\,\sigma} \text{ is}$$

distributed according to a Gaussian

⇒ sum converges to Gaussian distribution of width $\sqrt{N}\sigma$

Point:

- additive process
- no correlations, step-to-step
- identically distributed steps

i.e. ~ no "special" steps
~ each $x_i$ has same distribution

- $\sigma^2$ exists → no fat tails

cf: Chandrasekhar Review

N.B. Central Limit Theorem underlies:

- Langevin Eqn, additive noise

$$\frac{dv}{dt} = -\mu v + \frac{\tilde{f}(t)}{m}$$

noise

- Fluctuation - Dissipation Thm.

$$T \cong \langle \tilde{f}^2 \rangle \tau_{ac} / \mu$$

- Fokker-Planck Eqn

Now consider <u>multiplicative</u> process:

- define:

$$X = \prod_{i=1}^{N} x_i = x_1 x_2 \ldots x_j \ldots x_N$$

- $x_j = \begin{cases} 0 & p = 1/2 \\ 2 & p = 1/2 \end{cases}$

- what of moments?

Point: $X = 0$ unless all $x_i = 2$, then $X = 2^N$.

$$P(X = 2^N) = 2^{-N}$$

(joint probability)

<u>average</u>

$$\langle X \rangle = \sum_{j} X_j \Big/ 2^N$$

$j$ realizations; $2^N$ = # realizations

$$= 0 + 0 + \ldots + 2^N / 2^N$$

$$= 1$$

variance

$$\langle X^2 \rangle = \sum_j X_j^2 / 2^N$$

$$= 0 + 0 + 0 + \dots 2^{2N} / 2^N \quad = 2^N$$

More generally,

$$\langle X^p \rangle = 2^{(p-1)N}.$$

→ Key signature of intermittency →
higher moments of pdf grow exponentially with $p$.

i.e.

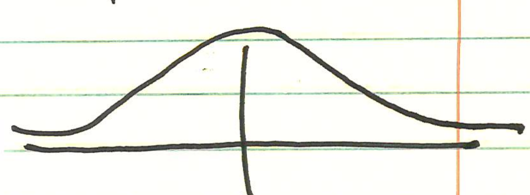
$$\gamma_p = \frac{\log_2 \langle X^p \rangle}{N} = p - 1.$$

growth of moment

$N \to$ time.

→ Observe contrast:

Gaussian pdf:

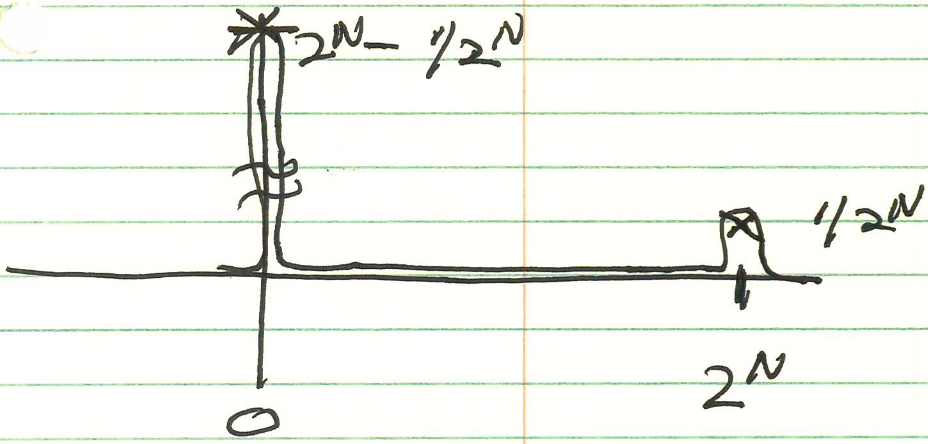


smooth

thin tails

all higher moments converge.

Pdf




Concentrated pdf, with tail.

Tail controls higher moments

N.B.

- N surrogate for time; # steps
- higher moment growth (exponentially) is signature of intermittent random process.

Generally:

- x is an <u>intermittent</u> random quantity

→ intermittent random quantity is a result of <u>multiplying</u>, not adding, many random #'s.

⇒ <u>Multiplicative</u> Processes

Exercises:

Derive Fokker-Planck Equation and stationary pdf for:

i.) $\frac{dx}{dt} = -\gamma x + \tilde{f}(t)$

additive noise

$\tilde{f}$ delta correlated

ii.) $\frac{dx}{dt} = (\bar{\gamma} + \tilde{\gamma}(t))x - \alpha x^2 + \tilde{f}(t)$

multiplicative noise

$\tilde{f}, \tilde{\gamma}$ delta correlated.

- Displays difference between multiplicative and additive noise

- Multiplicative noise is model of nonlinearity

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{q}{m}E\frac{\partial f}{\partial v} = 0$$

vs.

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{q}{m}\tilde{E}\frac{\partial f}{\partial v} = 0$$

How deal with / characterize multiplicative processes?

- Take logarithm!

⇒ To the Lognormal Distribution:

- $x_j \equiv$ random # distributed around unity

- $x = \prod_{j=1}^{N} x_j$

$\Rightarrow \ln x = \ln x_1 + \ln x_2 + \dots \ln x_N$

- Apply CLT to sum of logs:

so, $N \to \infty$

$$P(\ln x) \sim \exp\left[-(\ln x)^2 / N\eta^2\right]$$

$\eta$: # on $(-u\sigma, u\sigma)$

$u \sim 1$, $\sigma \sim$ std deviation

$$\boxed{x \sim \exp(N^{1/2}\eta)}$$

log normal distribution

- $\eta < 0$ $\quad X$ exponentially small

  $\eta > 0$ $\quad X \sim \exp(N^{1/2}\sigma)$

  big.

$\Rightarrow$ spikey distribution.

- see Wikipedia Post.

- Now,

$$\langle X \rangle = \int X(\eta)\, P(\eta)$$

$$= \int d\eta\, \exp\left[N^{1/2}\eta\right] \exp\left[-\eta^2/2\sigma^2\right]$$

$$\sim \exp\left[(N\sigma^2/2)\right]$$

exponentially growing average!

Generally:

$$\langle X^p \rangle \sim \exp\left[p^2 N \sigma^2/2\right]$$

higher moments grow exponentially.

and

$$\gamma_p \sim \lim_{N\to\infty} \frac{\ln \langle X^p \rangle}{N} = \frac{p^2\sigma^2}{2}$$

$\sim p^2$

i.e. symptomatic of concentration of probability.

→ Evolution of Random Quantities

- Multiplicative random quantities arise in evolutionary problems

$N \to$ time.

- Consider simple comparison

i) $\frac{d\psi}{dt} = \varepsilon(t)$

$\Downarrow$

Noise → additive

Gaussian, dispersion $\sigma^2$
delta correlated

$$\langle \varepsilon(t)\varepsilon(t')\rangle = \sigma^2 \tau_{ac}\,\delta(t-t')$$

vs

ii.) $\frac{d\psi}{dt} = \varepsilon(t)\,\psi$ → simple stochastic pde
→ NL model

in detail,

assume $\varepsilon$ has $\tau_{ac}$ → i.e. resets every $\tau_{ac}$

$$\psi = \int \varepsilon(t)\,dt$$

$$= \int_0^{\tau_{ac}} \varepsilon(t)\,dt + \int_{\tau_{ac}}^{2\tau_{ac}} \varepsilon(t)\,dt + \ldots$$

by CLT:

$$\psi = \langle \mathcal{E} \rangle t + \underbrace{\tau_{ac}}_{\mathcal{E}_{rms}} \underbrace{\nabla}_{N^{1/2}} (t/\tau_{ac})^{1/2} \eta$$

$\hookrightarrow$ Gaussian

$\overline{\eta^2} = 1$

taking $\langle \mathcal{E} \rangle \to 0$

$$\langle \psi^2 \rangle \sim \tau_{ac}^2 \nabla^2 \frac{t}{\tau_{ac}} \langle \eta^2 \rangle$$

$$\sim \tau_{ac} \nabla^2 t \quad \to \text{diffusion process}$$

alternatively,

$$\langle \psi(t_1) \psi(t_2) \rangle \simeq \int^{t_1} dt_1' \, \mathcal{E}(t_1') \int^{t_2} \mathcal{E}(t_2') \, dt_2'$$

$\tau, t$

$$\simeq \langle \mathcal{E}^2 \rangle \tau_{ac} t$$

iii.) Multiplicative Case

$\left\{ \begin{array}{l} \text{linear stochastic ode} \\ \text{as model of NL pde} \end{array} \right.$

$$\frac{d\psi}{dt} = \mathcal{E}(t) \psi$$

How deal with ?

Naive → QL

$$\frac{d\psi}{dt} = \varepsilon \psi$$

$$\psi = \langle \psi \rangle + \tilde{\psi}$$

$$\frac{d}{dt} \langle \psi \rangle = \langle \overline{\varepsilon} \tilde{\psi} \rangle$$

as $$\frac{d\tilde{\psi}}{dt} = \varepsilon \langle \psi \rangle$$

$$\frac{d}{dt} \langle \psi \rangle = \left\langle \varepsilon \int \varepsilon \langle \psi \rangle \right\rangle$$

$$\cong \overline{\varepsilon^2} \tau_{ac} \langle \psi \rangle$$

$$\Rightarrow \frac{d\langle \psi \rangle}{dt} = \overline{\varepsilon^2} \tau_{ac} \langle \psi \rangle$$

so $$\langle \psi \rangle \sim \psi_0 \exp\left[\overline{\varepsilon^2} \tau_{ac} t\right]$$

$$\sim \psi_0 \exp\left[\Gamma^2 \tau_{ac} t\right]$$

and could extend

Proper: $$\frac{d\psi}{dt} = \varepsilon(t)\, \psi$$

$$\frac{d}{dt} \ln \psi = \varepsilon(t)$$

$$\psi(t) \sim \psi_0 \exp\left[\int_0^t \varepsilon(s)\, ds\right]$$

$$\sim \prod_{n=0}^{t/\tilde{\tau}} \exp \int_{n\tau_{ac}}^{(n+1)\tilde{\tau}_{ac}} \varepsilon(s)\, ds$$

$\rightarrow$ multiplicative

and <u>log</u> follows CLT:

$$\boxed{\ln \psi \sim \langle \varepsilon \rangle t + \sigma \tilde{\tau}_{ac} (t/\tau_{ac})^{1/2} \eta}$$

$\langle \psi \rangle = 0$

$$\psi \sim \exp\left[\sigma \tilde{\tau} (t/\tilde{\tau})^{1/2} \eta\right]$$

— log normal!
no surprise

$\rightarrow$ $\psi$ grows as $\exp(t^{1/2})$

$\rightarrow$ solution is intermittent, random.

$\rightarrow$ fluctuations grow exponentially

Now, compare (i), (ii):

$$\psi \sim \exp\left[\sigma \tau_{ew} (t/\tau_{ew})^{1/2} \eta\right]$$

$$\langle \psi \rangle \sim \left\langle \exp\left(\sigma \tau (t/\tau)^{1/2} \eta\right) \exp(-\eta^2) \right\rangle$$

$$\sim \int d\eta \, e^{-\eta^2} e^{c\eta}$$

$$\sim \int d\eta \, \exp\left[-(\eta - c/2)^2\right] e^{c^2/4}$$

$$\sim \# \exp\left[\sigma^2 \tau^2 t/\tau\right]$$

agrees!

Point:

→ Mean field / QLT will get average right

→ but won't reveal fundamental nature and structure of higher moments

$$\gamma_p \sim \ln \langle \psi^p \rangle / t \sim (p-1)$$

as before

Bottom Line:

→ Intermittency is occurence of rare but intense peaks in behavior of random quantity

→ Signature of intermittency is large higher order moments.

Another Example: Random Media

What are statistics of particle/density distribution in media with

Random Potential $U(x, \omega)$

at temperature $T$.

$x$ → position

$\omega$ → enumerates realizations

$U$ has $\begin{cases} \text{Gaussian distribution} \\ \text{variance } \sigma^2 \end{cases}$

so, distribution

$$n = n_0 \exp[-U/T]$$

$k_B \equiv 1$

→ Point: Distribution $n$ is non-Gaussian as dependence on $U$ is non-linear!

For most probable concentration

$$P(n(u)) = n(u)\, P(u)$$

$$= n_0 \exp[-u/T]\, e^{-u^2/2\sigma^2}$$

so, for maximum in probability, completing square ⇒

max at $\quad u_{max}/\sigma \sim -\sigma/T$

$$\boxed{P_{max} = n_0 \exp[\sigma^2/2T]}$$

Look at moments:

$$\langle n \rangle = n_0 \exp[\sigma^2/2T^2]$$

$$\langle n^2 \rangle^{1/2} = n_0 \exp[\sigma^2/T^2]$$

$$\langle n^p \rangle^{1/p} = n_0 \exp[p\sigma^2/2T^2]$$

Observe: Fluctuations large

i.e.

* $\langle n^2 \rangle \gg \langle n \rangle^2$ $\qquad$ $n_{rms} \gg \langle n \rangle$

$n_0^2 \exp[2\sigma^2/T^2] > n_0^2 \exp[\sigma^2/T^2]$

$\langle n^4 \rangle \gg \langle n^2 \rangle^2$ ✓ $\qquad$ $\sigma^2/T^2 > 0$ ✓

etc.

→ higher moments larger

→ successive mean not determined by most probable $n$, $\sim \sigma/T$, but by $\boxed{p^{1/2}\,\sigma/T}$

⇒ signature of intermittent distribution of density

~ why the interest in higher moments?

ZRS ⇒

"progressive growth of statistical moments with order can be explained only by a much more pronounced dominance of rare, intense peaks in concentration distribution."