Notes 7 – $\left\{ \begin{array}{l} \text{Linear Scalar Equations} \\ \text{Random Potential} \end{array} \right.$
(Addendum)

– in spirit of Reaction-Diffusion Equation, consider stochastic PDE

$$\frac{\partial \psi}{\partial t} = \underbrace{D \nabla^2 \psi}_{\text{diffusion}} + \underbrace{u(x, \overset{\rightarrow \text{realization}}{\omega})}_{\text{stochastic potential}} \psi$$

Here: $u \rightarrow$ Gaussian

and $\langle u(x) u(x') \rangle \rightarrow 0$ range of
$\quad |x - x'| > \ell$ potential limited

N.B. Not a contrived example.

i.e

$$\frac{\partial \psi}{\partial t} = D \nabla^2 \psi + \gamma \psi - \psi^2$$

$\rightarrow$ Fisher Eqn. (NL)
$\left\{ \begin{array}{l} \text{Logistic} \\ + \\ \text{Diffusion} \end{array} \right.$

$\rightarrow$

$$\frac{\partial \psi}{\partial t} = D \nabla^2 \psi + \gamma_0 \psi + \underbrace{\tilde{\gamma}}_{\text{multiplicative noise}} \psi + \ldots \ldots$$

$(\gamma_0 < 0)$

For ψ:

- solve by path integral,

$$\psi(x,t) = M_{\underline{x}}\left[\exp\left(\int_0^t u(\xi_s)\,ds\right)\psi_0(\xi_t)\right]$$

i.e. from QM

$$\frac{\partial\psi}{\partial t} = D\nabla^2\psi + u\psi$$

$$\equiv H\cdot\psi = (H_0 + u)\psi$$

$$\psi = \psi_0\, e^{\int Ht}$$

$$= \psi_0\, e^{\int H_0 dt}\, e^{\int u\,ds}$$

$$= \psi_0\, e^{H_0 t}\, e^{\int u\,ds}$$

diffusion $\Rightarrow$ random trajectories
(known — as in free particle propagator)

so

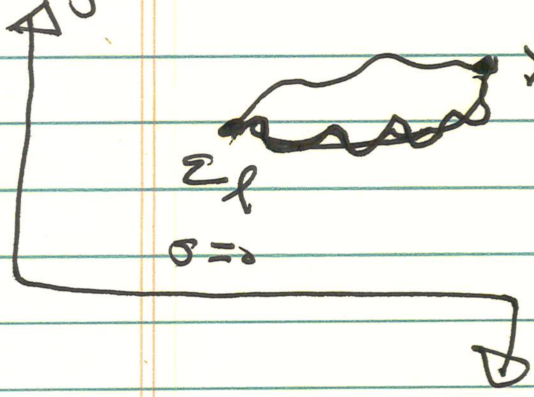
$$\langle\psi\rangle = \langle \psi_0\, e^{\int u\,ds}\rangle_{\substack{\text{traj.}\\ \text{diffus.}}}$$

so:

$$\psi(x,t) = M_x\left\{\left[\exp\left(\int_0^t u(\xi_s)\,ds\right)\right]\psi_0(\xi_s)\right\}$$

$M_x \equiv$ avg. over trajectories of motion (i.e. random walks)

Trajectories:



$\left\{\begin{array}{l}\text{begin at } \xi_\ell \text{ at} \\ s=0, \ x \text{ at } s=t\end{array}\right.$

Aside: Wiener Paths (c.f. N. Wiener)

→ realization (i.e. which path)

$W_t(\omega) \equiv$ random coordinate Brownian particle

↑ time

i.e. $W_t(\omega) \Longleftrightarrow x(t,\omega)$

$W_t(\omega)$ defined by:

– $W_0(\omega) = 0$

– $W_t(\omega)$ Gaussian

$W_{t+\tau}(\omega) - W_t(\omega)$ $\begin{cases} \text{– zero mean} \\ \text{– variance } \tau \end{cases}$

– $W_t(\omega) \to$ trajectory for diffusion

$\rightsquigarrow$ continuous

but

~ not differentiable, "rough"!

like turbulence

how see? heuristically:

$(\Delta W)^2_{\Delta t} \sim \Delta t$

mean square increment

so $\Delta W \sim \sqrt{\Delta t}$

$\frac{d}{dt} \Delta W \sim \frac{1}{\sqrt{\Delta t}}$

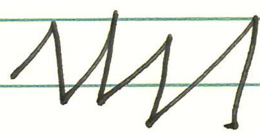
(contrast $(\Delta W) \sim t$)

so

$\lim_{\Delta t \to 0} \frac{d}{dt} \Delta W \to 0$

– Brownian motion:

→ "at any moment the Brownian particle has, so to speak, a new start and increments are independent"



More on W_t:

– correlation:

$$\langle W_t W_s \rangle = \min(t, s)$$

i.e. shorter (common) time, as no memory.

$$W_t W_s = \frac{1}{2}\left[W_t^2 + W_s^2 - (W_t - W_s)^2\right]$$

$$\langle W_t^2 \rangle = t$$

$$\langle W_s^2 \rangle = s$$

$$\langle (W_t - W_s)^2 \rangle = |t - s|$$

i.e. increments independent, so avg product vanishes

- non-stationary

$$\sim t^{1/2}$$

- Wiener processes describe motion of mass-less particles (no inertia)

i.e. $m\frac{dv}{dt} + \mu v = \tilde{F}$

$$\mu = 6\pi\eta R$$

1D, $n = \frac{1}{(2\pi D t)^{1/2}} e^{-x^2/2Dt}$

↓

density $n(x,t)$ i.e. $\delta(x-x_0)$

$$D = (N k_b T / \pi\eta R)^{1/2}$$

no m!

i.e. [ensemble of fictitious random paths.

- how average over paths?
  a la Feynman,

$0 < t_1 < t_2 \ldots < t_n = t$

$\to \quad W_n = (W_{t_1}, W_{t_2}, W_{t_3} \ldots W_{t_n})$

statistical independence

$P(t_1, t_2 - t_1, \ldots t_n - t_{n-1}; x_1, x_2 - x_1, \ldots x_n - x_{n-1})$

$$= \frac{1}{\sqrt{2\pi t}} e^{-x_1^2/2t_1} * \frac{1}{\sqrt{2\pi (t_2 - t_1)}} e^{-\frac{(x_2 - x_1)^2}{2(t_2 - t_1)}}$$

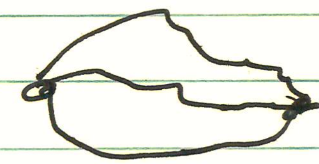
$\to$

$$\equiv \prod_{j=1}^{n} \frac{1}{(2\pi (t_j - t_{j-1}))^{1/2}} e^{-\frac{(x_j - x_{j-1})^2}{2(t_j - t_{j-1})}}$$

$\mu^t \equiv$ Wiener measure w_t $= \int \prod_{j=1}^{\infty}$ ⊙

allows averaging.

→ Functional integral



⇒ explains M_x.

Now,

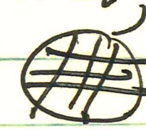
$$\psi(x,t) = M_x\left[\exp\left(\int_0^t U(\xi_s)\,ds\right)\psi_0(\xi_s)\right]$$

- dominant contribution from trajectory which encounters largest value of potential

- (~) concentration pt in trajectory space

Estimate ht maximum!

Region radius R s/t $R \gg \ell$
\# cells in R^3 ball is $(R/\ell)^3$



at maximum:

$$\underbrace{\overset{\text{Volume}}{(R/\ell)^3}}_{\#}\ \underbrace{\overset{\text{Density}}{p}}_{\text{prob}} \sim 1$$

Now, $P \sim \exp\left[-u_0^2/2\sigma^2\right]$

$$(R/\ell)^3 \exp\left[-u_0^2/2\sigma^2\right] \sim 1$$

$$\boxed{\text{Max } u \sim \left(6\sigma^2 \ln(R/\ell)\right)^{1/2}}$$

but

$$R/\ell \sim (Dt)^{1/2}/\ell$$

$$\sim (Dt/\ell^2)^{1/2}$$

<u>so</u>

$$\psi \sim \exp \int u$$

$$\sim \exp\left[t\left(6\sigma^2 \ln (Dt/\ell^2)^{1/2}\right)^{1/2}\right]$$

$$\sim \exp\left[t\left(3\sigma^2 \ln (Dt/\ell^2)\right)^{1/2}\right]$$

$$\boxed{\psi \sim \exp\left[t\left(3\sigma^2 \ln (Dt/\ell^2)\right)^{1/2}\right]}$$

$\Rightarrow$ super-exponential growth.

Observe:

- $D = 0$ higher moments grow super-exponentially
  $\Rightarrow$ concentration

$$\frac{\partial \psi}{\partial t} = D\nabla^2 \psi + u\psi$$

$$\psi = e^{ut}\psi_0$$

$$\langle \psi^p \rangle = \langle e^{put} \rangle \psi_0^{\,p}$$

$$= \psi_0^{\,p} \exp\left[p^2 \frac{\overline{u^2}}{2} t^2 \right]$$

$$= \psi_0^{\,p} \exp\left[p^2 \sigma^2 t^2/2 \right]$$

$\Rightarrow$ super-exponential growth
$\Rightarrow$ faster than $\langle \psi \rangle$!