Notes 7 — Percolation Basics, II

→ Recall:

⇒ 2 questions of interest:

① "Why" — "Any relevance to fluid flow"
(cf J-T) ($\omega \to k\omega \to \infty$)

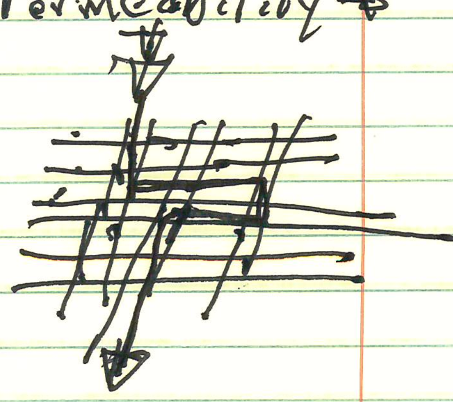
Origins → Broadbent and Hammersly 50's (posted)

- hydrology (like H. E. Hurst)
- interested in transport/flow thru porus media — water seepage in rocks.
- microscopic underpinnings of Darcy's Law and Kozeny equation:

$$\underline{J} = -\frac{k}{\mu}\underline{\nabla}P$$

(k → permeability; $\underline{J}$ → flux)

Permeability → net flow thru random network



Percolation as connection thru random maze

(cf pinball)

- Also, percolation cluster distribution $n_s(p)$, $s n_s(p)$ etc. is measure of emergent order, and its statistical characterization

⇒ simpler problem than avalanche distribution

⇒ SOC originally 'defined' in terms of 'percolation cluster' of single toppling (BTW '87)

⇒ prototype of many body, short range interaction system with universality, scaling etc.

(2) "How do I know it when I see it? specifically, how identify it in a simulation?" (c.f. H. C.)

→ Percolation is intrinsically 'static' concept (i.e. snapshot)

→ suggest analyzing clustering distribution in an image

see Boffetta et. al, posted → Fig 1.

Beautifully shows vorticity clustering in 2D turbulence. Appeals to intuition from percolation.

→ what of time?

- sequence of cluster images? With transport, should manifest avalanches i.e. large clusters discharge across the sytem

- to be continued.

Now describe percolation by:

$n_s(p) \equiv$ avg #/site of s-clusters

population density

and moments:

$\sum_s n_s(p)$ → population

$\sum_s s\, n_s(p)$ → ~~[illegible]~~

probability of a cluster

i.e. $w_s = \dfrac{n_s s}{\sum_s n_s s}$ → probability that cluster, to which an arbitrary site belongs, contains s sites.

see Stauffer

and $\bar{s} = \sum_s s w_s$

avg size

etc.

and $a/L \to 0$

Universality → scaling → power laws

Special focus on $p \sim p_c$

i.e. near percolation threshold, anticipate scalings $\sim |p-p_c|^{\alpha}$, etc.

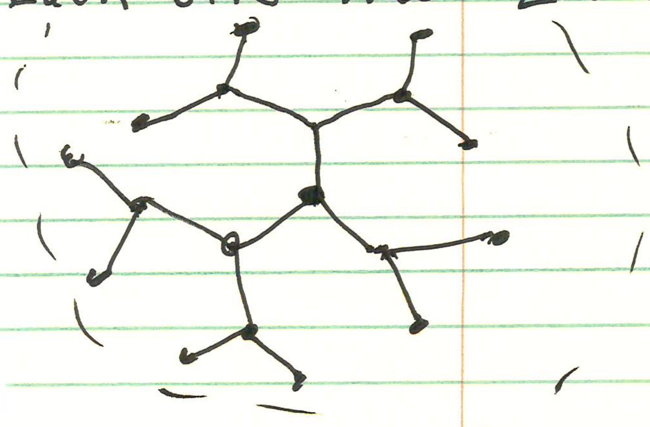
→ structure of scalings

→ relation between critical exponents.

Also – exactly solvable (albeit trivial) 1D model

– Bethe lattice / Cayley tree has z bonds, d dimension

e.g. Each site has $z = 3$ bonds,



surface dense!

(circle ∞ iteration)

also solvable, d dimensions.

i.e. - where do scalings come from?

c.f. → Stauffer, Aharony 2.4 {not disc. here}

$$\boxed{\Rightarrow \text{extract general trends of } n_s(p) \text{ scalings exploiting exact solutions.}}$$

⇒ Toward a Scaling Solution for Cluster Numbers ($n_s(p)$)

Recall: $n_s(p) = (1-p)^2 p^s$

$\Rightarrow s \to \infty \quad n_s(p) \sim e^{-cs}$

For Bethe lattice, need generalize:
→ power law. – could guess from self-similarity at p_c.

$$n_s(p) \sim s^{-\tau} e^{-cs} \quad s \to \infty.$$

$\tau = 5/2$ Bethe lattice

Of course: $c = c(p)$ (not a strict constant)

$s \to \infty$.

For Bethe lattice: $c(p) = (p-p_c)^2$,

more generally,

$$c \sim |p - p_c|^{1/\sigma}$$

($\sigma = 1/2$, Bethe)

Note now have two exponents .....

Observe, then:

$$n_s(p) \sim \frac{1}{s^{\tau}} \exp\left[-|p-p_c|^{1/\sigma} s\right]$$

→ defines effective cut-off on range of cluster sizes

i.e. $s < s_{c.o.} \sim (|p-p_c|)^{-1/\sigma}$

only contribute to cluster averages

there

$n_s(p) \sim s^{-\tau}$ scaling at critical pt.

$s > s_{c.o.}$ → exponentially rare

→ s_{co} defines cross-over from

from critical clusters → contribute
to
non-critical → don't contribute

so : $\boxed{n_s(s) \sim s^{-\tau} \exp\left[-|p-p_c|^{1/\sigma} s\right]}$

→ working model.

Now ⇒ a limitation is validity for large clusters, only

→ improve by examining ratio

$$v_s = n_s(p) / n_s(p_c)$$

so $\boxed{v_s(p) \sim \exp[-cs]}$ → exceedingly simple.

S & A: $\boxed{n_s(p_c) \sim s^{-\tau}}$ ⟵

"You might violate the Official Secrets Act if you now conclude and say loudly this is what theoretical physicists do; make calculations if they are easy irrespective of whether the assumptions are correct or wrong."

Some calculations:

①

→ P_∞ → fraction of sites belonging to infinite network.

↓

P

For P

- site — empty
  — occupied — ∞ cluster
  — finite cluster

- $s = \infty$, $n_s = 0$ ✓

i.e. in ∞ lattice, $\underline{\underline{1}}$ ∞ network,
$\#$ ∞ networks / lattice site $= 0$

- Recall fraction of lattice sites in ∞ network obtained from subtracting from occupied ~~lattice~~ sites those belonging to finite clusters

~ i.e. using $P + \sum_{s \text{ finite}} n_s s = p$ {1D Bethe}

$\therefore$ at $p = p_c$, $P = 0$ above.

$\therefore$ $\boxed{\sum_{s \text{ finite}} n_s s = p_c}$

Now, $n_s \sim s^{-\tau}$ so need

$\tau > 2$ for convergence,

(large powers for convergence.)

So, re-writing:

$$f = p - \sum_{\substack{s \\ \text{fin}}} n_s s$$

$$= \sum_s \left(n_s(p_c) - n_s(p) \right) s + O(p - p_c)$$

$$\simeq \sum_s s^{1-\tau} \left[1 - \exp(-cs) \right]$$

(large s contrib dominated)

so

$$f = \int ds\, s^{1-\tau} \left[1 - \exp(-cs) \right]$$

$z = cs$ and integration by parts:

$$f \approx c \int s^{2-\tau} \exp(-cs)\, ds$$

$$= c^{\tau-2} \underbrace{\int z^{2-\tau} \exp(-z)}_{\Gamma(3-\tau)}$$

and can integrate:

$$f \sim c^{(\tau-2)}$$

but $c \sim (p - p_c)^{1/\tau}$ so

$$P \sim (p-p_c)^{(\tau-2)/\sigma}$$
$$= (p-p_c)^{\beta}$$

$$\boxed{\beta = \frac{\tau-2}{\sigma}} \longrightarrow$$

(i) first relation between scaling exponents

(ii) what we seek,

β ↓ exponent for P

σ ~ exponent for $(p-p_c)$

τ ~ exponent $n_s(s)$.

② How does mean cluster size diverge?

Recall: $$\bar{S} \approx \sum_s s^2 n_s \Big/ \sum_s s n_s$$

but $p \to p_c$, $\sum_s s n_s = p_c$

so,
$$\bar{S} = \sum_s s^2 n_s \Big/ p_c$$
$$\sim \int ds\ s^2 n_s$$
$$\sim \int ds\ s^{2-\tau} e^{-cs}\, ds$$

$$\bar{S} \sim c^{3-\tau} \int z^{2-\tau} e^{-z}\, dz$$

finite

(∞ single cluster neglected)

$$\bar{S} \sim c^{\tau-3}$$

$$\sim (|p-p_c|)^{\tau-3/\sigma}$$

$$\sim |p-p_c|^{-\gamma}$$

$$\boxed{\gamma = \frac{3-\tau}{\sigma}}$$

$$\gamma = \frac{3-\tau}{\sigma}$$

$$\beta = \frac{\tau-2}{\sigma}$$

For $\beta > 0$, $\tau > 0$

$$\boxed{2 < \tau < 3}$$

The cynic: So we have a new exponent for every quantity?

→ No. Have 2 "free", undetermined exponents σ, τ!

recall τ → lead power law of n_s, $n_s \sim s^{-\tau}(\ \ )$

$\sigma \Rightarrow$ scaling of c

$$c \sim (p-p_c)^{1/\sigma}$$

determines $S_{c.o.}$

Can relate all else to these 2

[of course, need σ, τ from simulations, etc.]

i.e. consider general case:

$$M_k = \sum_s s^k n_s$$

$$\sim \sum_s s^{k-\tau} e^{-cs}$$

$$\sim \int ds \; s^{k-\tau} \exp(-cs)$$

$$\sim c^{\tau-1-k} \int dz \; z^{k-\tau} e^{-z}$$

so

$$M \sim c^{\tau-1-k}$$

$$\sim |p-p_c|^{(\tau-1-k)/\sigma}$$

exponent $\sim (\tau-1-k)/\sigma$

Caveat: – not rigorous, find the glitches
– yet, captures essence of scaling game

→ A Quick Look at More General Derivation

"If you have read this far thru the book, it is presumably too late for you to return it for a re-fund".

More general formulation:

→ stretched exponential version of previous

$$n_s(p) = f(z)$$

$$z = (p - p_c)s^{\sigma}$$

$$\boxed{n_s(p) \sim s^{-\tau} f\left[(p-p_c)\, s^{\sigma}\right]} \quad \begin{cases} p \sim p_c \\ s \gg 1 \end{cases}$$

$f(z)$ is TBD from computation.

→ $n_s \sim s^{-\tau} f[(p-p_c)s^{\sigma}]$

now behaves well, all cases

check:

$$\bar{S} \sim \sum s^2 n_s$$

$$\sim |p-p_c|^{(\tau-3)/\sigma}$$

$$\sim |p-p_c|^{-(3-\tau)/\sigma} = |p-p_c|^{\gamma}$$

so

$$\boxed{\gamma = \frac{3-\tau}{\sigma}}$$

Exponents: — β, γ universal

$\beta \leftrightarrow P$

$\gamma \leftrightarrow \bar{S}$

— independent lattice structure

some RG evidence for γ.

→ Can extend the fn to correlation lengths, perimeters (in 2D) etc.

→ cluster perimeter can be fractal. (usually is)

General Message:

- Large scale emergent behavior occurs in systems with local interaction

- system can self-organize hierarchy of clusters, divergent at criticality.
- Universality → scaling → power laws
- scaling theory is really useful phenomenology which links scaling (critical) exponents,
- properties described by scaling exponents i.e. "the answer"

⇒ Emergent critical behavior, $l_c \to \infty$ ⟺ bass?

~~[illegible]~~ Transport Phenomena exist which are not captured by random walk models.

$ku \gg 1$ is good example.

→ Now, return to $ku \to \infty$ magnetic problem.